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The Hölder-Poincaré duality for $L_{q,p}$-cohomology

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Abstract We prove the following version of Poincaré duality for reduced $L_{q,p}$-cohomology: For any $1 < q, p < \infty$, the $L_{q,p}$-cohomology of a Riemannian manifold is in duality with the interior $L_{p',q'}$-cohomology for $1/p + 1/p' = 1/q + 1/q' = 1$.

Keywords $L_{q,p}$-cohomology · Poincaré duality · Parabolicity

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1 Introduction and statement of the results

The main goal of this article is to describe the dual space of the reduced $L_{q,p}$-cohomology of an oriented Riemannian manifold (M, g).

Let us denote by $\mathcal{D}^k = C_0^\infty(M, \Lambda^k)$ the vector space of smooth differential k-forms with compact support in M and by $L^p(M, \Lambda^k)$ the Banach space of p-integrable differential k-forms. The authors also consider the space $\Omega_{q,p}^k(M)$ of q-integrable differential k-forms whose weak exterior differentials are p-integrable

$$\Omega_{q,p}^k(M) = \left\{ \omega \in L^q(M, \Lambda^k) \mid d\omega \in L^p(M, \Lambda^{k+1}) \right\}.$$

We now define the basic objects of investigation.

Definition 1.1 The reduced $L_{q,p}$-*cohomology* of the Riemannian manifold (M, g) is defined as

$$\overline{H}_{q,p}^k(M) = Z_p^k(M)/\overline{B}_{q,p}^k(M), \tag{1.1}$$

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where $Z^k_p(M)$ is the set of weakly closed forms in $L^p(M,\Lambda^k)$ and $\overline{B}^k_{q,p}$ is the closure in $L^p(M,\Lambda^k)$ of the set of weakly exact forms having a q-integrable primitive:

$$\overline{B}^k_{q,p} = \overline{\mathrm{d}(\Omega^{k-1}_{q,p}(M))}^{L^p(M,\Lambda^k)}.$$

(we shall use the notation $\overline{A}^E$ for the closure of the subset set A in the Banach space E.)

In the special case $q = p$, the space defined in (1.1) is simply denoted by $\overline{H}^k_p(M) = \overline{H}^k_{q,p}(M)$ and is called the reduced L_p*-cohomology* of the manifold.

The reduced $L_{q,p}$-cohomology is naturally a Banach space. Two closed forms ω,ω' in $Z^k_p(M)$ represent the same $L_{q,p}$-reduced cohomology class if one can find a sequence $\theta_j \in L^q(M,\Lambda^{k-1})$ such that $\|(\omega-\omega') - \mathrm{d}\theta_j\|_{L^p(M,\Lambda^k)} \to 0$.

The subject-matter of L_p-cohomology is now an important and well-established subject, see, e.g., the books [3,14,24] as well as the papers [22,23] and the references therein for a more recent update on the subject. The more general $L_{q,p}$-cohomology has been the object of a number of investigations in recent years: the paper [10] contains some foundational material and shows how Sobolev inequalities for differential forms can be interpreted in the framework of $L_{q,p}$-cohomology, see also [20]. The paper [11] gives some applications to quasi-conformal geometry and [12] relates the $L_{q,p}$-cohomology to more general classes of mappings. The paper [12] contains some computations for negatively curved Riemannian manifolds and the papers [19,28] study the relation between the $L_{q,p}$-cohomology of a manifold and the L_p-Hodge decomposition on that manifold. In [16,17], some computations for warped product manifolds and the general Heisenberg group are developed.

In order to describe the dual space to $\overline{H}^k_{q,p}(M)$, we introduce another type of cohomology which we shall call the *interior reduced* $L_{r,s}$*-cohomology*. This cohomology captures the idea of cohomology relative to the (ideal) boundary of the manifold.

Definition 1.2 The *interior reduced* $L_{r,s}$*-cohomology* of the Riemannian manifold (M,g) is the Banach space defined as

$$\overline{H}^k_{r,s;0}(M) = Z^k_{r,s;0}(M)/\overline{\mathrm{d}\mathcal{D}^{k-1}}, \tag{1.2}$$

where $\overline{\mathrm{d}\mathcal{D}^{k-1}}$ is the closure of $\mathrm{d}\mathcal{D}^{k-1}$ in $L^r(M,\Lambda^k)$ and $Z^k_{r,s;0}(M) \subset \Omega^k_{r,s}(M)$ is defined as:

$$Z^k_{r,s;0}(M) = \ker(d) \cap \overline{\mathcal{D}^k(M)}^{\Omega^k_{r,s}}.$$

A form α belongs thus to $Z^k_{r,s;0}(M)$ if and only if α is a weakly closed form in $L^r(M,\Lambda^k)$ such that there exists a sequence $\theta_j \in \mathcal{D}^k(M)$ such that $\theta_j \to \alpha$ in $\Omega^k_{r,s}(M)$, i.e.,

$$\|\theta_j - \alpha\|_r \to 0 \quad \text{and} \quad \|\mathrm{d}\theta_j\|_s \to 0.$$

The main result is the following

Theorem 1.1 *Let (M,g) be an oriented n-dimensional Riemannian manifold. If $1 < p,q < \infty$, then $\overline{H}^k_{q,p}(M)$ is isomorphic to the dual of $\overline{H}^{n-k}_{p',q';0}(M)$ where $\frac{1}{p'}+\frac{1}{p} = \frac{1}{q'}+\frac{1}{q} = 1$. The duality is induced by the natural pairing $\overline{H}^k_{q,p}(M) \times \overline{H}^{n-k}_{p',q';0}(M) \to \mathbb{R}$ given by integration:*

$$\langle[\omega],[\varphi]\rangle = \int_M \omega \wedge \varphi. \tag{1.3}$$

It is useful to introduce a third, auxiliary, cohomology which we called the *semi-compact* or the (comp, p)-cohomology of (M, g). This object turns out to be more manageable than the reduced $L_{q,p}$-cohomology and its interior version.

Definition 1.3 The *reduced* (comp, p)*-cohomology* of (M, g) is the Banach space defined as

$$\overline{H}^k_{\mathrm{comp},p}(M) = Z^k_p(M)/\overline{\mathrm{d}\mathcal{D}^{k-1}},$$

where $\overline{\mathrm{d}\mathcal{D}^{k-1}}$ is the L^p closure of $\mathrm{d}\mathcal{D}^{k-1}$.

Observe that the following inclusions hold

$$Z^k_{p,q;0}(M) \subset Z^k_p(M) \quad \text{and} \quad \overline{\mathrm{d}\mathcal{D}^{k-1}} \subset \overline{B}^k_{q,p}.$$

This implies that the interior reduced $L_{q,p}$-cohomology embeds in the reduced (comp, p)-cohomology. On the other hand, the reduced $L_{q,p}$-cohomology is a quotient of the reduced (comp, p)-cohomology:

$$\overline{H}^k_{p,q;0}(M) \hookrightarrow \overline{H}^k_{\mathrm{comp},p}(M) \quad \text{and} \quad \overline{H}^k_{\mathrm{comp},p}(M) \twoheadrightarrow \overline{H}^k_{q,p}(M).$$

It follows in particular that $\dim(\overline{H}^k_{q,p}(M)) \leq \dim(\overline{H}^k_{\mathrm{comp},p}(M))$ for any q.

We have the following duality result for the (comp, p)-cohomology.

Theorem 1.2 *Let (M, g) be an oriented n-dimensional Riemannian manifold. If $1 < p < \infty$, then $\overline{H}^k_{\mathrm{comp},p}(M)$ is isomorphic to the dual of $\overline{H}^{n-k}_{\mathrm{comp},p'}(M)$ where $\frac{1}{p'} + \frac{1}{p} = 1$. The duality is induced by the integration pairing (1.3).*

We now give a sufficient condition for the reduced (comp, p)-cohomology to coincide with the reduced $L_{q,p}$-cohomology. Recall that a Riemannian manifold (M, g) is said to be s*-parabolic*, $1 \leq s \leq \infty$ if one can approximate the unity by functions with small s-energy, i.e., if there exists a sequence of smooth functions with compact support $\{\eta_j\} \subset C^\infty_0(M)$ such that $\eta_j \to 1$ uniformly on every compact subset of M and $\lim_{j\to\infty} \int_M |\mathrm{d}\eta_j|^s = 0$.

Theorem 1.3 *Suppose that M is s-parabolic for some $1 \leq s \leq \infty$, and assume that $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$. Then we have*

$$\overline{H}^k_{q,p}(M) = \overline{H}^k_{\mathrm{comp},p}(M) \quad \textit{and} \quad \overline{H}^k_{q,p;0}(M) = \overline{H}^k_{\mathrm{comp},q}(M).$$

This result gives us in particular conditions under which the reduced $L_{q,p}$-cohomology and the interior $L_{q,p}$-cohomology coincide.

The case $s = \infty$ is important, because a manifold is ∞-parabolic if and only if it is complete. It follows that for complete manifolds, we have $\overline{H}^k_p(M) = \overline{H}^k_{p,p;0}(M) = \overline{H}^k_{\mathrm{comp},p}(M)$ and the pairing (1.3) induces a duality

$$\overline{H}^k_p(M)' = \overline{H}^{n-k}_{p'}(M),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. The paper [13] contains a short proof of this special case.

Theorem 1.3 implies that if M is s-parabolic for every s, then $\overline{H}^n_{q,p}(M)$ is independent of the choice of q and $\overline{H}^n_{q,p;0}(M)$ is independent of the choice of p provided $p \leq q$.

In the finite volume case, we have a stronger result. If (M, g) is complete with finite volume, then it is s-parabolic for any $1 \leq s \leq \infty$. For such manifolds, we then have

$$\overline{H}^k_{q,p}(M) = \overline{H}^k_{\mathrm{comp},p}(M) = \overline{H}^k_p(M)$$

for any $q \geq p$. Combining this result with [10, Proposition 7.1], we obtain that $\overline{H}^k_{q,p}(M) = 0$ if and only if $\overline{H}^k_p(M) = 0$ (without the restriction $q \geq p$).

Other results relating the $L_{q,p}$-cohomology and parabolicity can be found in [2,25].

The article is organized as follows: In the next section we recall some basic notions and facts and in Sect. 3, we prove the duality Theorem 1.2 for the (comp, p)-cohomology. Sect. 4 is devoted to the proof of Theorem 1.1 and it occupies the largest part of this article. We reformulate the problem in the convenient language of Banach complexes. This section also contains a description of the dual space of $\Omega^k_{q,p}(M)$, see Corollary 4.7. In Sect. 5 we recall the notion of s-parabolic manifolds and we give a proof of Theorem 1.3. In Sect. 6 we give a relation of the top dimensional reduced $L_{q,p}$-cohomology to the classic Sobolev inequality and in Sect. 7 we deduce from the main theorem a result on the Poincaré duality of the conformal cohomology. In the last section, we derive some consequences and applications of the duality theorems.

2 Some background

For any Riemannian n-manifold (M, g), the Riemannian metric induces a norm (in fact a scalar product) on $\Lambda^k T_x M^*$ at any point $x \in M$. Using this norm, one can define the space $L^p(M, \Lambda^k)$ of measurable differential k-forms ω such that

$$\|\omega\|_p = \left(\int_M |\omega|_x^p \, d\,\mathrm{vol}_g(x) \right)^{1/p} < \infty,$$

if $1 \leq p < \infty$ and $\|\omega\|_\infty = \operatorname{esssup} |\omega| < \infty$ if $p = \infty$. If $p' = p/(p-1)$, one can define a pairing

$$\langle , \rangle : L^p(M, \Lambda^k) \times L^{p'}(M, \Lambda^{n-k}) \to \mathbb{R}$$

by integration, that is

$$\langle \omega, \varphi \rangle = \int_M \omega \wedge \varphi \tag{2.1}$$

for $\omega \in L^p(M, \Lambda^k)$ and $\varphi \in L^{p'}(M, \Lambda^{n-k})$. This is well-defined because we have at (almost) every point $x \in M$

$$|\omega \wedge \varphi|_x \leq |\omega|_x \, |\varphi|_x \, ,$$

and by Hölder's inequality we have

$$|\langle \omega, \varphi \rangle| = \left| \int_M \omega \wedge \varphi \right| \leq \int_M |\omega| \, |\varphi| \, d\mu \leq \|\omega\|_{L^p(M,\Lambda^k)} \, \|\varphi\|_{L^{p'}(M,\Lambda^{n-k})} \, .$$

The bilinear function $\langle , \rangle$ allows us to define for any $\varphi \in L^{p'}(M, \Lambda^{n-k})$ a bounded linear functional $F_\varphi(\omega) = \int_M \omega \wedge \varphi$, and the familiar Hölder duality between p and p' extends to differential forms, see, e.g., [8]:

Theorem 2.1 *For any $1 \leq p < \infty$ the correspondence $\varphi \to F_\varphi$ is an isometric isomorphism from the Banach spaces $L^{p'}(M, \Lambda^{n-k})$ to the dual space of $L^p(M, \Lambda^k)$, and the duality is explicitly given by the pairing (2.1).*

Corollary 2.2 *$\mathcal{D}^k = C_0^\infty(M, \Lambda^k)$ is dense in $L^p(M, \Lambda^k)$.*

Proof Here is a short proof: Suppose there exists a k-form $\omega \in L^p(M, \Lambda^k)$ such that $\omega \notin \overline{\mathcal{D}}^k$. By the Hahn–Banach Theorem, there exists a continuous linear form $\lambda \colon L^p(M, \Lambda^k) \to \mathbb{R}$ such that $\lambda = 0$ on $\overline{\mathcal{D}}^k$ and $\lambda(\omega) \neq 0$. By the previous Theorem, there exists $\psi \in L^{p'}(M, \Lambda^{n-k})$ such that $\lambda(\theta) = \int_M \theta \wedge \psi$ for any $\theta \in L^p(M, \Lambda^k)$. In particular $\lambda(\theta) = \int_M \theta \wedge \psi = 0$ for any $\theta \in \mathcal{D}^k$. This implies that $\psi = 0$ and therefore $\lambda(\omega) = 0$. Therefore, no such ω exists and we conclude that $\overline{\mathcal{D}}^k = L^p(M, \Lambda^k)$. □

We now define the notion of weak exterior differential (see, e.g., [6]):

Definition 2.1 Assume M to be oriented. One says that a differential form $\theta \in L^p(M, \Lambda^{k+1})$ is the *weak exterior differential* of the form $\phi \in L^p(M, \Lambda^k)$ and one writes $d\phi = \theta$ if one has

$$\int_M \theta \wedge \omega = (-1)^{k+1} \int_M \phi \wedge \mathrm{d}\omega$$

for any $\omega \in \mathcal{D}^{n-k}(M) = C_0^\infty(M, \Lambda^{n-k})$.

We then introduce the space (see, e.g., [6]):

$$\Omega^k_{q,p}(M) = \left\{ \omega \in L^q(M, \Lambda^k) \mid \mathrm{d}\omega \in L^p(M, \Lambda^{k+1}) \right\}.$$

This is a Banach space for the graph norm

$$\|\omega\|_{q,p} = \|\omega\|_{L^q} + \|\mathrm{d}\omega\|_{L^p}. \tag{2.2}$$

The space $\Omega^k_{q,p}(M)$ is a reflexive Banach space for any $1 < q, p < \infty$, this can be proved using standard arguments of functional analysis (see, e.g., [1]), but is also follows from Corollary 4.7 below.

Lemma 2.3 *For any $1 \leq q, p < \infty$, $C^\infty(M, \Lambda^k) \cap \Omega^k_{q,p}(M)$ is dense in $\Omega^k_{q,p}(M)$.*

Proof This follows from the regularization theorem, see [10]. □

We now (re)define the basic ingredients, for $p, q, r \in [1, \infty]$.

Definition 2.2 The closure of $\mathcal{D}^k = C_0^\infty(M, \Lambda^k)$ in $\Omega^k_{q,p}(M)$ is denoted by $\Omega^k_{q,p;0}(M)$. We also define the following subspaces:

(a) $Z^k_{p,r}(M) = \ker[d : \Omega^k_{p,r}(M) \to L^r(M, \Lambda^{k+1})]$.
(b) $B^k_{q,p}(M) = \mathrm{im}[d : \Omega^{k-1}_{q,p}(M) \to L^p(M, \Lambda^k)]$.
(c) $Z^k_{p,r;0}(M) = \ker[d : \Omega^k_{p,r;0}(M) \to L^r(M, \Lambda^{k+1})]$.
(d) $B^k_{q,p;0}(M) = \mathrm{im}[d : \Omega^{k-1}_{q,p;0}(M) \to L^p(M, \Lambda^k)]$.

Lemma 2.4 *The previously defined spaces satisfy the following properties*

(i) $Z^k_{p,r}(M)$ *does not depend on* r *and it is a closed subspace of* $L^p(M,\Lambda^k)$.
(ii) $d(\mathcal{D}^{k-1})$ *is dense in* $B^k_{r,p;0}(M)$ *for the* L^p*-topology.*

Proof (i) $Z^k_{p,r}(M)$ is a closed subspace of $\Omega^k_{p,r}(M)$ since it is the kernel of the bounded operator d. It is also a closed subspace of $L^p(M,\Lambda^k)$ since for any $\alpha \in Z^k_{p,r}(M)$, we have $||\alpha||_{\Omega^k_{p,r}(M)} = ||\alpha||_{L^p(M,\Lambda^k)}$. Now $Z^k_{p,r}(M)$ does not depend on r because it coincides with the space of weakly closed k-forms in $L^p(M,\Lambda^k)$.

Statement (ii) is almost obvious. Fix $\omega \in B^k_{r,p;0}(M)$ and choose $\theta \in \Omega^{k-1}_{r,p;0}(M)$ such that $d\theta = \omega$. By definition of $\Omega^{k-1}_{r,p;0}(M)$, there exists a sequence $\theta_j \in \mathcal{D}^{k-1}$ such that $\lim_{j\to\infty} \|\theta - \theta_j\|_{r,p} = 0$. This means that $\theta = \lim_{j\to\infty}\theta_j$ in L^r and $\omega = \mathrm{d}\theta = \lim_{j\to\infty}\mathrm{d}\theta_j$ in L^p. □

The Banach space $Z^k_{p,r}(M)$ will then simply be denoted by $Z^k_p(M)$. We will also identify the closed subspaces

$$\overline{B}^k_{r,p;0}(M) = \overline{\mathrm{d}(\mathcal{D}^{k-1})}^{L^p(M,\Lambda^k)}.$$

Our reduced cohomologies are then naturally defined as the following quotients of Banach spaces:

$$\overline{H}^k_{q,p}(M) = Z^k_p(M)/\overline{B}^k_{q,p}(M), \qquad \overline{H}^k_{q,p;0}(M) = Z^k_{q,p;0}(M)/\overline{\mathrm{d}(\mathcal{D}^{k-1})(M)}$$

and

$$\overline{H}^k_{\mathrm{comp},p}(M) = Z^k_p(M)/\overline{\mathrm{d}(\mathcal{D}^{k-1})(M)}.$$

Let us finally mention that there is a notion of *unreduced* $L_{q,p}$-cohomology, which is defined as

$$H^k_{q,p}(M) = Z^k_p(M)/B^k_{q,p}(M).$$

This is generally not a Banach space as $B^k_{q,p}(M) \subset Z^k_p(M)$ need not be a closed subspace. We may also define an unreduced interior cohomology, but this space will depend on three indices instead of two:

$$H^k_{r,p,q;0}(M) = Z^k_{p,q;0}(M)/B^k_{r,p;0}(M).$$

3 The duality theorem for the (comp, p)-cohomology

In this section, we prove Theorem 1.2. The proof is based on the following lemma from functional analysis.

Lemma 3.1 *Let* $I : X_0 \times X_1 \to \mathbb{R}$ *be a duality (non-degenerate pairing) between two reflexive Banach spaces. Let* B_0, A_0, B_1, A_1 *be linear subspaces such that*

$$B_0 \subset A_0 = B_1^\perp \subset X_0 \quad \text{and} \quad B_1 \subset A_1 = B_0^\perp \subset X_1.$$

Then the pairing $\overline{I} : \overline{H}_0 \times \overline{H}_1 \to \mathbb{R}$ *of* $\overline{H}_0 = A_0/\overline{B}_0$ *and* $\overline{H}_1 = A_1/\overline{B}_1$ *is well-defined and induces duality between* $\overline{H}_0$ *and* $\overline{H}_1$.

Here $B_1^{\perp}$ and $B_0^{\perp}$ are the annihilators of B_1 and B_0 for the duality I.

Proof We first prove the Lemma under the assumption that $A_0 \neq \overline{B}_0$ and $A_1 \neq \overline{B}_1$. Observe that A_i is a closed subspace of the Banach spaces X_i since it is the annihilators of B_i $(i = 1, 2)$. The bounded bilinear map $I : A_0 \times A_1 \to \mathbb{R}$ is defined by restriction and it gives rise to a well-defined bounded bilinear map

$$\overline{I} : A_0/\overline{B}_0 \times A_1/\overline{B}_1 \to \mathbb{R}$$

because we have the inclusions $B_0 \subset A_0 \subset B_1^{\perp}$ and $B_1 \subset A_1 \subset B_0^{\perp}$. (Indeed, let $\alpha_0 \in A_0$, $\alpha_1 \in A_1$, $b_0 \in \overline{B_0}$, $b_1 \in \overline{B_1}$, then we have

$$I(\alpha_0 + b_0, \alpha_1 + b_1) = I(\alpha_0, \alpha_1) + I(\alpha_0 + b_0, b_1) + I(b_0, \alpha_1).$$

By the definition of annihilators the second and third terms vanish and the duality $\overline{I}$ is thus well-defined.)

We show that $\overline{I}$ is non-degenerate: let $a_0 \in A_0$ be such that $[a_0] \neq 0 \in A_0/\overline{B}_0$; i.e., $a_0 \notin \overline{B}_0$. By the Hahn–Banach Theorem and the fact that X_1 is dual to X_0, there exists an element $y \in X_1$ such that $I(a_0, y) \neq 0$ and $I(b, y) = 0$ for all $b \in \overline{B}_0$. Thus, $y \in B_0^{\perp} = A_1$ and we have found an element $[y] \in A_1/\overline{B}_1$ such that $\overline{I}([a], [y]) \neq 0$.

The same argument shows that for any $[\alpha] \neq 0 \in A_1/\overline{B}_1$, we can find an element $[x] \in A_0/\overline{B}_0$ such that $\overline{I}([x], [\alpha]) \neq 0$. The proof is thus complete in the case $A_i \neq \overline{B}_i$.

If $A_0 = \overline{B}_0$, then we have from the hypothesis of the Lemma:

$$\overline{B}_1 \subset A_1 = B_0^{\perp} = A_0^{\perp} = (B_1^{\perp})^{\perp} = \overline{B}_1,$$

and it follows that $A_1 = \overline{B}_1$. The argument shows in fact that

$$A_0 = \overline{B}_0 \quad \Leftrightarrow \quad A_1 = \overline{B}_1.$$

In that case both $\overline{H}_0 = A_0/\overline{B}_0$ and $\overline{H}_1 = A_1/\overline{B}_1$ are null spaces and the Lemma is trivial. □

Remark With a little extra work, we can show that if the pairing I in the previous lemma is an isometric duality, then $\overline{H}_1$ is isometric and not only isomorphic to the dual of $\overline{H}_0$, see [13].

Proof of Theorem 1.2 Let $\alpha \in L^p(M, \Lambda^k)$, then by definition of the weak exterior differential, we have $d\alpha = 0$ if and only if

$$\int_M \alpha \wedge d\omega = 0$$

for any $\omega \in \mathcal{D}^{n-k-1}$. This precisely means that $Z_p^k(M) \subset L^p(M, \Lambda^k)$ is the annihilator of $d\mathcal{D}^{n-k-1}$ for the integration pairing, we thus have the following relations with respect to the pairing (2.1):

$$dD^{k-1} \subset Z_p^k(M) = (d\mathcal{D}^{n-k-1})^{\perp} \subset L^p(M, \Lambda^k).$$

Likewise, we have $dD^{n-k-1} \subset Z_{p'}^{n-k}(M) = (d\mathcal{D}^{k-1})^{\perp} \subset L^{p'}(M, \Lambda^{n-k})$, and the previous Lemma implies that (2.1) induces a duality between

$$\overline{H}^k_{\mathrm{comp},p}(M) = Z_p^k(M)/\overline{d\mathcal{D}^{k-1}(M)}^{L^p(M,\Lambda^k)}$$

and

$$\overline{H}^{n-k}_{\mathrm{comp},p'}(M) = Z_{p'}^{n-k}(M)/\overline{d\mathcal{D}^{n-k-1}(M)}^{L^{p'}(M,\Lambda^k)}.$$

□

4 The main duality Theorem

The goal of this section is to prove Theorem 1.1. It will be convenient to use the language of Banach complexes, and for that we need to reformulate the definition of the cohomology in a new language.

4.1 The L_π-cohomology of M

In order to define a Banach complex, we fix an $(n+1)$-tuple of real numbers

$$\pi = \{p_0, p_1, \dots, p_n\} \subset [1, \infty] \tag{4.1}$$

and define

$$\Omega^k_\pi(M) = \Omega^k_{p_k, p_{k+1}}(M).$$

Observe that $\Omega^n_\pi(M) = L^{p_n}(M, \Lambda^n)$ and $\Omega^0_{p,p}(M)$ coincides with the Sobolev space $W^{1,p}(M)$. Since the weak exterior differential is a bounded operator $d : \Omega^{k-1}_\pi \to \Omega^k_\pi$, we have constructed a Banach complex $\{\Omega^*_\pi(M), d\}$:

$$0 \to \Omega^0_\pi \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}_\pi \xrightarrow{d} \Omega^k_\pi \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_\pi \to 0.$$

Definition 4.1 *The (reduced) L_π-cohomology of M is the (reduced) cohomology of the Banach complex* $\{\Omega^k_\pi(M), d_k\}$.

By Lemma 2.4 the L_π-cohomology space $H^k_\pi(M)$ depends only on p_k and p_{k-1} and we have in fact

$$H^k_\pi(M) = H^k_{p_{k-1}, p_k}(M) = Z^k_{p_k}(M)/B^k_{p_{k-1}, p_k}(M)$$

and the reduced L_π-cohomology is

$$\overline{H}^k_\pi(M) = \overline{H}^k_{p_{k-1}, p_k}(M) = Z^k_{p_k}(M)/\overline{B}^k_{p_{k-1}, p_k}(M).$$

We also introduce a notion of interior L_π-cohomology. Let us denote by $\Omega^k_{\pi;0}(M)$ the closure of $\mathcal{D}^k(M)$ in $\Omega^k_\pi(M)$. This is an another Banach complex

$$0 \to \Omega^0_{\pi;0}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}_{\pi;0}(M) \xrightarrow{d} \Omega^k_{\pi;0}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\pi;0} \to 0.$$

Definition 4.2 The *interior L_π-cohomology* of (M, g) is the cohomology of this Banach complex, i.e.,

$$H^k_{\pi;0}(M) = H^k_{p_{k-1}, p_k, p_{k+1};0}(M) = Z^k_{p_k, p_{k+1}:0}(M)/B^k_{p_{k-1}, p_k:0}(M)$$

and by Lemma 2.4 the interior reduced L_π-cohomology is

$$\overline{H}^k_{\pi;0}(M) = \overline{H}^k_{p_k, p_{k+1};0}(M) = Z^k_{p_k, p_{k+1}:0}(M)/\overline{dD^{k-1}(M)}^{L^{p_k}(M,\Lambda^k)}.$$

Definition 4.3 The *dual* of the $(n+1)$-tuple $\pi = \{p_0, p_1, ..., p_n\} \subset [1, \infty]$ Is the $(n+1)$-tuple $\pi' = \{r_0, r_1 ..., r_n\}$ such that $\frac{1}{r_k} + \frac{1}{p_{n-k}} = 1$.

In the sequel, we will use the notation $a < \pi < b$ or $a \le \pi \le b$ if these inequalities hold for all $p_0, p_1, \dots, p_n$.

4.2 The complex of pairs of forms on M

Because the space $\Omega_\pi^k(M)$ is equipped with the graph norm, it is useful to investigate the structure of the space where this graph lives, that is the following Cartesian product:

Definition 4.4 Given a n-dimensional Riemannian manifold (M, g) and a sequence π as in (4.1), we introduce the vector space

$$\mathcal{P}_\pi^\star(M) = L_\pi(M, \Lambda^\star) \oplus L_\pi(M, \Lambda^\star).$$

We introduce a grading of this vector space by defining

$$\mathcal{P}_\pi^k(M) = L_\pi(M, \Lambda^k) \oplus L_\pi(M, \Lambda^{k+1}).$$

Remark We have $L_\pi(M, \Lambda^{n+1}) = L_\pi(M, \Lambda^{-1}) = \{0\}$, hence $\mathcal{P}_\pi^n(M) \cong L^{p_n}(M, \Lambda^n)$ and $\mathcal{P}_\pi^{-1}(M) \simeq L^{p_0}(M, \Lambda^0)$.

An element in $\mathcal{P}_\pi^k(M)$ will be denoted as a column vector $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ with $\alpha \in L_\pi(M, \Lambda^k)$ and $\beta \in L_\pi(M, \Lambda^{k+1})$. This space is a Banach space for the norm

$$\left\| \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \right\|_{\mathcal{P}_\pi^k(M)} = \|\alpha\|_{p_k} + \|\beta\|_{p_{k+1}}, \tag{4.2}$$

and it can be turned as a Banach complex for the "differential" $d_{\mathcal{P}} : \mathcal{P}_\pi^k \to \mathcal{P}_\pi^{k+1}(M)$ defined by

$$\mathrm{d}_{\mathcal{P}} \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\beta\\ 0\end{pmatrix}.$$

Lemma 4.1 *The complex $(\mathcal{P}_\pi^k(M), d_{\mathcal{P}})$ has trivial cohomology.*

Proof Suppose that $\mathrm{d}_{\mathcal{P}} \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$, this means that $\beta = 0$. But then it is clear that

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ 0\end{pmatrix} = \mathrm{d}_{\mathcal{P}} \begin{pmatrix}0\\ \alpha\end{pmatrix}.$$

□

Proposition 4.2 *Let π be a sequence as in the Definition (4.1) and π' be the dual sequence. If M is oriented and $1 \le \pi < \infty$, then $\mathcal{P}_\pi^k(M)$ and $\mathcal{P}_{\pi'}^{n-k-1}(M)$ are in duality for the following pairing*

$$\left\langle \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\varphi\\ \psi\end{pmatrix} \right\rangle = \int_M ((-1)^k \alpha \wedge \psi + \beta \wedge \varphi). \tag{4.3}$$

Proof By Theorem 2.1, we have the following isomorphims (in fact isometries):

$$\left(L^{p_k}(M, \Lambda^k)\right)' \cong L^{p_k'}(M, \Lambda^{n-k}), \quad \left(L^{p_{k+1}}(M, \Lambda^{k+1})\right)' \cong L^{p_{k+1}'}(M, \Lambda^{n-k-1}).$$

Hence

$$\begin{aligned}\left(\mathcal{P}^k_\pi(M)\right)' &= \left(L^{p_k}(M,\Lambda^k)\oplus L^{p_{k+1}}(M,\Lambda^{k+1})\right)'\\ &= \left(L^{p_k}(M,\Lambda^k)\right)'\oplus\left(L^{p_{k+1}}(M,\Lambda^{k+1})\right)'\\ &= L^{p'_{k+1}}(M,\Lambda^{n-k-1})\oplus L^{p'_k}(M,\Lambda^{n-k})\\ &= \mathcal{P}^{n-k-1}_{\pi'}(M).\end{aligned}$$

□

Lemma 4.3 *The operator $d_{\mathcal{P}}:\mathcal{P}^{k-1}_{\pi'}\to\mathcal{P}^k_{\pi'}$ and $(-1)^k d_{\mathcal{P}}:\mathcal{P}^{n-k-1}_{\pi}\to\mathcal{P}^{n-k}_{\pi}$ are adjoint for the duality* (4.3).

Proof Let $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\in\mathcal{P}^{k-1}_{\pi'}$ and $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\in\mathcal{P}^{n-k-1}_{\pi}$, then

$$\left\langle d_{\mathcal{P}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix},\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\rangle=\left\langle\begin{pmatrix}\beta\\ 0\end{pmatrix},\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\rangle=\int_M(-1)^k\beta\wedge\psi.$$

On the other hand

$$\left\langle\begin{pmatrix}\alpha\\ \beta\end{pmatrix},d_{\mathcal{P}}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\rangle=\left\langle\begin{pmatrix}\alpha\\ \beta\end{pmatrix},\begin{pmatrix}\psi\\ 0\end{pmatrix}\right\rangle=\int_M\beta\wedge\psi.$$

□

4.3 The annihilator of $\Omega^\star_{\pi;0}(M)$ in $\mathcal{P}^\star_\pi(M)$

We will investigate $\Omega^\star_\pi(M)$ as a closed subspace of $\mathcal{P}^\star_\pi(M)$.

Definition 4.5 We denote by $\Sigma^\star_\pi(M)\subset\mathcal{P}^\star_\pi(M)$ the set of pairs of the form $\begin{pmatrix}\omega\\ d\omega\end{pmatrix}$, where $\omega\in\Omega^\star_\pi(M)$. We also denote by $\Sigma^\star_{\pi;0}(M)$ the subspace of those elements $\begin{pmatrix}\omega\\ d\omega\end{pmatrix}$ such that $\omega\in\Omega^\star_{\pi;0}(M)$.

It is clear that $\Sigma^\star_\pi(M)$ and $\Sigma^\star_{\pi;0}(M)$ are closed subspaces of $\mathcal{P}^\star_\pi(M)$, they are subcomplexes and are isomorphic (as Banach complexes) to $\Omega^\star_{\pi;0}(M)$ and $\Omega^\star_\pi(M)$.

Lemma 4.4 *The subspaces $\Sigma^k_\pi(M)\subset\mathcal{P}^k_\pi(M)$ and $\Sigma^{n-k-1}_{\pi';0}(M)\subset\mathcal{P}^{n-k-1}_{\pi'}(M)$ are orthogonal for the duality pairing* (4.3).

Proof This is clear by the definition of weak exterior differential and the density of $\mathcal{D}^{n-k-1}$ in $\Omega^{n-k-1}_{\pi';0}(M)$. □

In fact we have a stronger result:

Proposition 4.5 *The subspace $\Sigma^k_\pi(M)$ in $\mathcal{P}^k_\pi(M)$ is the annihilator of $\Sigma^{n-k-1}_{\pi;0}(M)$:*

$$\Sigma^k_\pi(M)=(\Sigma^{n-k-1}_{\pi';0}(M))^\perp. \tag{4.4}$$

We also have

$$\Sigma^k_{\pi;0}(M)=(\Sigma^{n-k-1}_{\pi'}(M))^\perp. \tag{4.5}$$

Proof By density of $\mathcal{D}^{n-k-1}$ in $\Omega^{n-k-1}_{\pi;0}(M)$, an element $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathcal{P}^k_\pi(M)$ belongs to the annihilator of $\mathcal{D}^{n-k-1}$ if and only if

$$(-1)^k \int_M \alpha \wedge \mathrm{d}\varphi + \int_M \beta \wedge \varphi = 0$$

for any $\mathcal{D}^{n-k-1}$, but this means by definition that $\beta = \mathrm{d}\alpha$ in the weak sense, i.e., that $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \Sigma^k_\pi(M)$. Therefore,

$$\Sigma^k_\pi(M) = \left(\mathcal{D}^{n-k-1}\right)^\perp = \left(\Sigma^{n-k-1}_{\pi';0}(M)\right)^\perp,$$

this proves (4.4). To prove (4.5), we use the fact that $\Sigma^{n-k-1}_{\pi';0}(M) \subset \mathcal{P}^{n-k-1}_{\pi'}(M)$ is a closed subspace together with (4.4) and the following property of annihilator $(A^\perp)^\perp = \overline{A}$ to deduce that

$$\Sigma^k_{\pi;0}(M) = \left(\left(\Sigma^k_{\pi;0}(M)\right)^\perp\right)^\perp = \left(\Sigma^{n-k-1}_{\pi'}(M)\right)^\perp.$$

□

4.4 The dual of $\Omega^\star_\pi(M)$

We introduce the following quotient of $\mathcal{P}^\star_{\pi'}(M)$:

$$\mathcal{A}^\star_{\pi'} = \mathcal{P}^\star_{\pi'}(M)/\Sigma^\star_{\pi';0}(M), \tag{4.6}$$

This space inherits a grading from that of $\mathcal{P}^\star_{\pi'}(M)$.

Theorem 4.6 *The graded vector space $\mathcal{A}^\star_{\pi'}$ has the following properties:*

(a) *$\mathcal{A}^k_{\pi'}$ is a Banach space for the norm*

$$\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|_{\mathcal{A}} = \inf\left\{\left(\|\varphi+\rho\|_{p'_k} + \|\psi + d\rho\|_{p'_{k+1}}\right) \mid \rho \in \Omega^k_{\pi';0}(M)\right\}.$$

(b) *$\mathcal{A}^k_{\pi'}$ is dual to $\Sigma^{n-k-1}_\pi(M)$ for the pairing given by* (4.3).

(c) *The differential $d_{\mathcal{P}} : \mathcal{P}^k_{\pi'} \to \mathcal{P}^{k+1}_{\pi'}$ induces a differential $d_{\mathcal{A}} : \mathcal{A}^k_{\pi'} \to \mathcal{A}^{k+1}_{\pi'}$ and $\left(\mathcal{A}^\star_{\pi'}, d_{\mathcal{A}}\right)$ is a Banach complex.*

(d) *The operator $d_{\mathcal{A}} : \mathcal{A}^{k-1}_{\pi'} \to \mathcal{A}^k_{\pi'}$ and $d : \Sigma^{n-k-1}_\pi \to \Sigma^{n-k}_\pi$ are adjoint (up to sign) for the duality* (4.3).

Proof The statement (a) is a standard fact on quotient of Banach spaces and (b) is a consequence of the orthogonality relation (4.5). The statement (c) follows directly from the fact that $\Sigma^*_{\pi';0}$ is a subcomplex of $\mathcal{P}^*_{\pi'}$, i.e., $\mathrm{d}_{\mathcal{P}}(\Sigma^k_{\pi';0}) \subset (\Sigma^{k+1}_{\pi';0})$. Finally, the last statement follows directly from Lemma 4.3. □

Corollary 4.7 *The dual of the space $\Omega^k_\pi(M)$ is isomorphic to the completion of $\mathcal{D}^{n-k}$ with respect to the following norm:*

$$\|\sigma\| = \inf\left\{\left(\|\rho\|_{p'_{n-k-1}} + \|\sigma + d\rho\|_{p'_{n-k}}\right) \mid \rho \in \Omega^{n-k-1}_{\pi';0}(M)\right\}.$$

Proof We know by the Corollary 2.2 that the space $\mathcal{S}^m = \mathcal{D}^m \oplus \mathcal{D}^{m+1}$ is a dense subspace of $\mathcal{P}^m_{\pi'}$. The image of $\mathcal{S}^m$ in $\mathcal{A}^m_{\pi'}$ is thus also dense. Observe that $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in \mathcal{S}^m$ and $\begin{pmatrix} 0 \\ \psi - \mathrm{d}\varphi \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \varphi \\ \mathrm{d}\varphi \end{pmatrix} \in \mathcal{S}^m$ represent the same element in $\mathcal{A}^m_{\pi'}$; this implies that the map

$$j : \mathcal{D}^{m+1} \to \mathcal{A}^m_{\pi'}, \quad j(\sigma) = \begin{pmatrix} 0 \\ \sigma \end{pmatrix}$$

has a dense image. This map is furthermore injective, indeed, if $\sigma \in \ker j$, then $\begin{pmatrix} 0 \\ \sigma \end{pmatrix} = \begin{pmatrix} \rho \\ \mathrm{d}\rho \end{pmatrix}$ for some $\rho \in \Sigma^{m-1}_{\pi';0}(M)$, but then $\rho = 0$ and therefore $\sigma = \mathrm{d}\rho = 0$. It follows that $\mathcal{A}^m_{\pi'}$ is the completion of $j(\mathcal{D}^{m+1})$ for the natural norm in $\mathcal{A}^m_{\pi'}$. This norm is given by

$$\|j(\sigma)\| = \left\| \begin{pmatrix} 0 \\ \sigma \end{pmatrix} \right\|_{\mathcal{A}} = \inf \left\{ \left(\|\rho\|_{p'_{m-1}} + \|\sigma + \mathrm{d}\rho\|_{p'_m} \right) \,\middle|\, \rho \in \Omega^{m-1}_{\pi';0}(M) \right\}.$$

We now deduce from statement (b) of the previous Theorem that the above completion of $\mathcal{D}^{m+1}$ is isomorphic to the dual of $\Omega^{n-m-1}_{\pi}(M)$. Setting $m = n - k - 1$ completes the proof of the Corollary. □

4.5 The duality theorem in L_π-cohomology

In this subsection, we prove a duality statement between the reduced L_π-cohomology of M and the reduced interior $L_{\pi'}$-cohomology. We begin by investigating the cohomology of the Banach complex $\mathcal{A}^\star_{\pi'}$. Let us define the spaces

$$Z^k(\mathcal{A}^\star_{\pi'}) = \ker[\mathrm{d}_{\mathcal{A}} : \mathcal{A}^k_{\pi'} \to \mathcal{A}^{k+1}_{\pi'}] \quad \text{and} \quad B^k(\mathcal{A}^\star_{\pi'}) = \mathrm{im}[\mathrm{d}_{\mathcal{A}}(\mathcal{A}^{k-1}_{\pi'})].$$

We also denote by $\overline{B}^k(\mathcal{A}^\star_{\pi'})$ the closure of $B^k(\mathcal{A}^\star_{\pi'})$. Observe that $Z^k(\mathcal{A}^\star_{\pi'}) \subset \mathcal{A}^\star_{\pi'}$ is a closed subspace and that $B^k(\mathcal{A}^\star_{\pi'}) \subset \overline{B}^k(\mathcal{A}^\star_{\pi'}) \subset Z^k(\mathcal{A}^\star_{\pi'})$. The cohomology and reduced cohomology of $\mathcal{A}^\star_{\pi'}$ are defined as

$$H^k(\mathcal{A}^\star_{\pi'}) = Z^k(\mathcal{A}^\star_{\pi'})/B^k(\mathcal{A}^\star_{\pi'}) \quad \text{and} \quad \overline{H}^k(\mathcal{A}^\star_{\pi'}) = Z^k(\mathcal{A}^\star_{\pi'})/\overline{B}^k(\mathcal{A}^\star_{\pi'}).$$

Proposition 4.8 *The pairing given by* (4.3) *induces a duality between the reduced cohomology of* $\mathcal{A}^\star_{\pi'}$ *and that of* $\Sigma^\star_\pi(M)$*:*

$$\overline{H}^{k-1}(\mathcal{A}^\star_{\pi'}) \times \overline{H}^{n-k}(\Sigma^\star_\pi(M)) \to \mathbb{R}.$$

Proof Observe first that

$$B^{k-1}(\mathcal{A}^\star_{\pi'}) \subset Z^{k-1}(\mathcal{A}^\star_{\pi'}) = (B^{n-k}(\mathcal{A}^\star_{\pi'}))^\perp \subset \mathcal{A}^{k-1}_{\pi'}. \tag{4.7}$$

Indeed, both inclusions are trivial and the above equality follows from the fact that d_Σ and $\mathrm{d}_{\mathcal{A}}$ are adjoint operators.

Let us recall the (classic) argument. Fix $a \in \mathcal{A}^{k-1}_{\pi'}$, if $a \in \ker[\mathrm{d}^{k-1}_{\mathcal{A}}]$, then $\langle a, \mathrm{d}_\Sigma s\rangle = \pm\langle \mathrm{d}_{\mathcal{A}} a, s\rangle = 0$ for any $s \in \Sigma^{n-k-1}_\pi(M)$, i.e., $a \in (\mathrm{im}[\mathrm{d}^{n-k-1}_\Sigma])^\perp$. Conversely, if $a \in (\mathrm{im}[\mathrm{d}^{n-k-1}_\Sigma])^\perp$, then $\langle \mathrm{d}_{\mathcal{A}} a, s\rangle = \pm\langle a, \mathrm{d}_\Sigma s\rangle = 0$ for any $s \in \Sigma^{n-k-1}_\pi(M)$. Because $\Sigma^{n-k-1}_\pi(M)$ is dual to $\mathcal{A}^k_{\pi'}$, this means that $\mathrm{d}_{\mathcal{A}} a = 0$.

Similarly, by Theorem 4.6 we have

$$\mathrm{im}[\mathrm{d}_\Sigma^{n-k-1}] \subset \ker[\mathrm{d}_\Sigma^{n-k}] = (\mathrm{im}[\mathrm{d}_\mathcal{A}^{k-2}])^\perp \subset \Sigma_\pi^{n-k}(M). \tag{4.8}$$

The Proposition follows now from Lemma 3.1 for $X_0 = \mathcal{A}_{\pi'}^{k-1}$, $X_1 = \Sigma_\pi^{n-k}(M)$ and equations (4.7) and (4.8) since by definition we have

$$\overline{H}^{k-1}(\mathcal{A}_{\pi'}^\star) = \ker[\mathrm{d}_\mathcal{A}^{k-1}]/\overline{\mathrm{im}[\mathrm{d}_\mathcal{A}^{k-2}]}$$

and

$$\overline{H}^{n-k}(\Sigma_\pi^\star(M)) = \ker[\mathrm{d}_\Sigma^{n-k}]/\overline{\mathrm{im}[\mathrm{d}_\Sigma^{n-k-1}]}.$$

□

Proposition 4.9 *The reduced and non-reduced cohomology of the Banach complex* $\left(\mathcal{A}_{\pi'}^\star, d_\mathcal{A}\right)$ *are isomorphic to the interior* $L_{\pi'}$*-cohomology of* M *up to a shift:*

$$H_{\pi';0}^k(M) = H^{k-1}(\mathcal{A}_{\pi'}^\star) \quad \textit{and} \quad \overline{H}_{\pi';0}^k(M) = \overline{H}^{k-1}(\mathcal{A}_{\pi'}^\star). \tag{4.9}$$

These isomorphisms are induced from the map $j : Z_{\pi';0}^k(M) \to \mathcal{P}_{\pi'}^{k-1}$ *defined by* $j(\beta) = \begin{pmatrix} 0 \\ \beta \end{pmatrix}$.

Because we have a short exact sequence of complexes

$$0 \to \Omega_{\pi';0}^\star(M) \to \mathcal{P}_{\pi'}^\star \to \mathcal{A}_{\pi'}^\star \to 0$$

and $\mathcal{P}_{\pi'}^\star$ has trivial cohomology, the result follows from general principles (see, e.g., Theorem 1b in [18]). However, we give below a more informative, explicit proof.

Proof It will be convenient to describe the cohomology $H^{k-1}(\mathcal{A}_{\pi'}^\star)$ as a quotient of $\mathcal{P}_{\pi'}^{k-1}$. An element in $\mathcal{A}_{\pi'}^{k-1}$ is represented by an element $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathcal{P}_{\pi'}^{k-1}$ modulo $\Sigma_{\pi';0}^{k-1}$, i.e., the equality

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + \varphi \\ \beta + \mathrm{d}\varphi \end{pmatrix}$$

holds in $\mathcal{A}_{\pi'}^{k-1}$ if and only if $\varphi \in \Sigma_{\pi';0}^{k-1}$.

Now $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ represents an element in $Z^{k-1}(\mathcal{A}_{\pi'}^\star)$ if $\mathrm{d}_\mathcal{A} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ 0 \end{pmatrix} \in \Sigma_{\pi';0}^k$. This means that $\beta \in \Omega_{\pi';0}^k(M)$ and $\mathrm{d}\beta = 0$, i.e., $\beta \in Z_{\pi';0}^k(M)$, in other words we have

$$Z^{k-1}(\mathcal{A}_{\pi'}^\star) = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathcal{P}_{\pi'}^{k-1} \,\middle|\, \beta \in Z_{\pi';0}^k(M) \right\} \Big/ \Sigma_{\pi';0}^{k-1}.$$

Likewise, $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ represents an element in $B^{k-1}(\mathcal{A}_{\pi'}^\star)$ if there exists $\begin{pmatrix} \gamma \\ \delta \end{pmatrix} \in \mathcal{P}_{\pi'}^{k-2}$ such that $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \mathrm{d}_\mathcal{A} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \delta \\ 0 \end{pmatrix}$ modulo $\Sigma_{\pi';0}^{k-1}$, and this means that $\beta = \mathrm{d}\varphi \in B_{\pi';0}^k(M)$. We thus have

$$B^{k-1}(\mathcal{A}^\star_{\pi'}) = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathcal{P}^{k-1}_{\pi'} \,\middle|\, \beta \in B^k_{\pi';0}(M) \right\} \Big/ \ \Sigma^{k-1}_{\pi';0},$$

and, by completion

$$\overline{B}^{k-1}(\mathcal{A}^\star_{\pi'}) = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathcal{P}^{k-1}_{\pi'} \,\middle|\, \beta \in \overline{B}^k_{\pi';0}(M) \right\} \Big/ \ \Sigma^{k-1}_{\pi';0}.$$

The above equalities allows us to give the following explicit description of the cohomology of $\mathcal{A}^\star_{\pi'}$:

$$H^{k-1}(\mathcal{A}^\star_{\pi'}) = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathcal{P}^{k-1}_{\pi'} \,\middle|\, \beta \in Z^k_{\pi';0}(M) \right\} \Big/ \left\{ \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \in \mathcal{P}^{k-1}_{\pi'} \,\middle|\, \mu \in B^k_{\pi';0}(M) \right\}.$$

But this quotient can also be described as

$$H^{k-1}(\mathcal{A}^\star_{\pi'}) = \left\{ \begin{pmatrix} 0 \\ \beta \end{pmatrix} \in \mathcal{P}^{k-1}_{\pi'} \,\middle|\, \beta \in Z^k_{\pi';0}(M) \right\} \Big/ \left\{ \begin{pmatrix} 0 \\ \mu \end{pmatrix} \in \mathcal{P}^{k-1}_{\pi'} \,\middle|\, \mu \in B^k_{\pi';0}(M) \right\}.$$

In short, we have established that the embedding $j : Z^k_{\pi';0}(M) \to \mathcal{P}^{k-1}_{\pi'}$ defined by $j(\beta) = \begin{pmatrix} 0 \\ \beta \end{pmatrix}$ induces an algebraic isomorphism $j : H^k_{\pi';0}(M) \cong H^{k-1}(\mathcal{A}^\star_{\pi'})$. We also have

$$\overline{H}^{k-1}(\mathcal{A}^\star_{\pi'}) = \left\{ \begin{pmatrix} 0 \\ \beta \end{pmatrix} \in \mathcal{P}^{k-1}_{\pi'} \,\middle|\, \beta \in Z^k_{\pi';0}(M) \right\} \Big/ \left\{ \begin{pmatrix} 0 \\ \mu \end{pmatrix} \in \mathcal{P}^{k-1}_{\pi'} \,\middle|\, \mu \in \overline{B}^k_{\pi';0}(M) \right\}.$$

This quotient is equipped with its natural quotient norm and the homomorphism j clearly induces an isometric isomorphism $j : \overline{H}^k_{\pi';0}(M) \cong \overline{H}^{k-1}(\mathcal{A}^\star_{\pi'})$. □

From the propositions 4.9 and 4.8, we now deduce the following duality result:

Theorem 4.10 *Let M be an arbitrary smooth n-dimensional oriented Riemannian manifold. For any $1 < \pi < \infty$, the Banach spaces $\overline{H}^k_\pi(M)$ and $\overline{H}^{n-k}_{\pi';0}(M)$ are in duality for the pairing $\langle \beta, \omega \rangle = \int_M \beta \wedge \omega$. where $\beta \in Z^k_\pi(M)$ and $\omega \in Z^{n-k}_{\pi';0}(M)$.*

Proof By Propositions 4.8, we know that $\overline{H}^{k-1}(\mathcal{A}^\star_{\pi'})$ is isomorphic to the dual of $\overline{H}^{n-k}(\Sigma^\star_\pi(M))$ for the pairing (4.3):

$$\left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \omega \\ \mathrm{d}\omega \end{pmatrix} \right\rangle = \int_M ((-1)^k \alpha \wedge \mathrm{d}\omega + \beta \wedge \omega).$$

By Propositions 4.9, we have an isomorphism $\overline{H}^k_{\pi';0}(M) \cong \overline{H}^{k-1}(\mathcal{A}^\star_{\pi'})$ induced by the map $\beta \mapsto \begin{pmatrix} 0 \\ \beta \end{pmatrix}$, and we trivially have an isomorphism $\overline{H}^{n-k}_{\pi';0}(M) \cong \overline{H}^{n-k}(\Sigma^\star_\pi(M))$ given by $\omega \mapsto \begin{pmatrix} \omega \\ 0 \end{pmatrix}$. It follows that $\overline{H}^k_\pi(M)$ and $\overline{H}^{n-k}_{\pi';0}(M)$ are in duality for the pairing

$$\left\langle \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \begin{pmatrix} \omega \\ 0 \end{pmatrix} \right\rangle = \int_M \beta \wedge \omega.$$

□

Observe that Theorem 1.1 is now also proved, since it suffices to apply the previous Theorem to any sequence $\pi = \{p_0, p_1, \ldots, p_n\} \subset (1, \infty)$ such that $q = p_{k-1}$ and $p = p_k$.

5 Parabolicity

In this section, we prove Theorem 1.3. Recall the definition of an s-*parabolic* manifold.

Definition 5.1 The Riemannian manifold (M, g) is said to be s-*parabolic*, $1 \leq s \leq \infty$ if there exists a sequence of smooth functions with compact support $\{\eta_j\} \subset C_0^\infty(M)$ such that

(i) $0 \leq \eta_j \leq 1$;
(ii) $\eta_j \to 1$ uniformly on every compact subset of M;
(iii) $\lim_{j\to\infty} \int_M |d\eta_j|^s = 0$.

A non s-parabolic manifold is called s-*hyperbolic*.

It is easy to check that M is ∞-parabolic if and only if it is complete. If the manifold is complete with finite volume, then it is s-parabolic for any $s \in [0, \infty]$. There are numerous characterizations of s-parabolicity as well as a number of geometric interpretations. See [9,15,27] and references therein for more on this subject.

Proposition 5.1 *If M is s-parabolic and $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$, then $\mathcal{D}^\ell$ is dense in $\Omega^\ell_{q,p}(M)$.*

Proof Let $\alpha \in \Omega^\ell_{q,p}(M)$. We know by Lemma 2.3 that there exists a sequence $\alpha_j \in C^\infty(M, \Lambda^\ell)$ converging to α in $\Omega^\ell_{q,p}(M)$. Since M is s-parabolic, there exists a sequence $\eta_j \in C_0^\infty(M)$ satisfying the conditions (i)–(iii) above. It is then clear that $\beta_j = \eta_j\alpha_j \in \mathcal{D}^\ell$, and we claim that β_j converges to α in $\Omega^\ell_{q,p}(M)$. Indeed, we have

$$\begin{aligned}\|\alpha - \beta_j\|_{L^q} &\leq \|\alpha - \alpha_j\|_{L^q} + \|\alpha_j - \beta_j\|_{L^q}\\ &= \|\alpha - \alpha_j\|_{L^q} + \|(1-\eta_j)\alpha_j\|_{L^q}.\end{aligned}$$

Since $\|\alpha - \alpha_j\|_{L^q} \to 0$ by hypothesis and $\|(1-\eta_j)\alpha_j\|_{L^q} \to 0$ by the Lebesgue Dominated convergence theorem, we have $\|\alpha - \beta_j\|_{L^q} \to 0$. We also have

$$\begin{aligned}\|\mathrm{d}\alpha - \mathrm{d}\beta_j\|_{L^p} &\leq \|\mathrm{d}\alpha - \mathrm{d}\alpha_j\|_{L^p} + \|\mathrm{d}\alpha_j - \mathrm{d}\beta_j\|_{L^p}\\ &= \|\mathrm{d}\alpha - \mathrm{d}\alpha_j\|_{L^p} + \|\mathrm{d}\left((1-\eta_j)\alpha_j\right)\|_{L^p}\\ &= \|\mathrm{d}\alpha - \mathrm{d}\alpha_j\|_{L^p} + \|(1-\eta_j)\mathrm{d}\alpha_j\|_{L^p} + \|\mathrm{d}(1-\eta_j)\wedge\alpha_j\|_{L^p}\\ &= \|\mathrm{d}\alpha - \mathrm{d}\alpha_j\|_{L^p} + \|(1-\eta_j)\mathrm{d}\alpha_j\|_{L^p} + \|\mathrm{d}(1-\eta_j)\|_{L^s}\|\alpha_j\|_{L^q},\end{aligned}$$

which converges to 0. We have shown that for any $\alpha \in \Omega^\ell_{q,p}(M)$, there exists a sequence $\beta_j \in \mathcal{D}^\ell$ such that

$$\lim_{j\to 0} \|\alpha - \beta_j\|_{q,p} = \|\alpha - \beta_j\|_{L^q} + \|\mathrm{d}\alpha - \mathrm{d}\beta_j\|_{L^p} = 0.$$

This means that $\mathcal{D}^\ell \subset \Omega^\ell_{q,p}(M)$ is dense. □

Proof of Theorem 1.3 The previous Proposition implies that if M is s-parabolic for $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$, then the set of smooth exact k-forms with compact support is dense in $B^k_{q,p}(M)$. In particular, $\overline{B}^k_{q,p}(M)$ is the closure of $d\mathcal{D}^{k-1}(M)$ in $L^p(M, \Lambda^k)$:

$$\overline{B}^k_{q,p}(M) = \overline{\mathrm{d}\mathcal{D}^{k-1}(M)}^{L^p}.$$

Therefore

$$\overline{H}^k_{q,p}(M) = Z^k_p(M)/\overline{B}^k_{q,p}(M) = Z^k_p(M)/\overline{\mathrm{d}\mathcal{D}^{k-1}(M)} = \overline{H}^k_{\mathrm{comp},p}(M).$$

The Proposition (5.1) also implies that any closed k-form in $Z^k_q(M)$ can be approximated in the $\Omega^k_{q,p}(M)$-topology by a sequence of smooth k-forms with compact support. Hence

$$Z^k_q(M) = \ker(\mathrm{d}) \cap \overline{\mathcal{D}^k(M)}^{\Omega^k_{q,p}} = Z^k_{q,p;0}(M),$$

and we have

$$\overline{H}^k_{q,p;0}(M) = Z^k_{q,p;0}(M)/\overline{B}^k_{q;0}(M) = Z^k_q(M)/\overline{\mathrm{d}\mathcal{D}^{k-1}(M)} = \overline{H}^k_{\mathrm{comp},q}(M).$$

The Theorem is proved. □

6 Sobolev inequality and reduced $L_{q,p}$-cohomology in degree n

Using the duality Theorem 1.1 one can prove the following result on the top degree $L_{q,p}$-cohomology.

Proposition 6.1 *If the n-dimensional Riemannian manifold satisfies the Sobolev inequality*

$$\|f\|_{p'} \leq C \cdot \|df\|_{q'} \tag{6.1}$$

for any smooth function f with compact support then $\overline{H}^n_{q,p}(M) = 0$, where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

Proof By the duality theorem, we have to show that the Sobolev inequality (6.1) implies

$$Z^0_{p',q';0}(M) = \overline{H}^0_{p',q';0}(M) = 0.$$

Choose an arbitrary element α in $Z^0_{p',q';0}(M)$. By definition it means that $\alpha \in L^p(M)$ is a locally constant function ($\mathrm{d}\alpha = 0$) such that there exists a sequence $f_j \in \mathcal{D}^0(M)$ satisfying

$$\|f_j - \alpha\|_{p'} \to 0 \quad \text{and} \quad \|df_j\|_{q'} \to 0.$$

Using the Sobolev inequality, we have

$$\begin{aligned} \|\alpha\|_{p'} &\leq \|\alpha - f_j\|_{p'} + \|f_j\|_{p'} \\ &\leq \|\alpha - f_j\|_{p'} + C \cdot \|df_j\|_{q'}. \end{aligned}$$

Therefore $\alpha = 0$ and it follows that $\overline{H}^0_{p',q';0}(M) = 0$. □

Remark The previous Proposition implies that for an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$ satisfying the cone condition, we have $\overline{H}^n_{q,p}(\Omega) = 0$ provided $1/p - 1/q \leq 1/n$. Indeed the Sobolev inequality (6.1) holds in Ω since

$$\frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}.$$

For more general unbounded and/or irregular domains, there is a vast literature on geometric conditions implying the Sobolev inequality, see, e.g., [5,21,26] and references therein. Each such result gives us a corresponding vanishing of the top degree $L_{q,p}$-cohomology.

7 The conformal cohomology

Definition The *conformal de Rham complex* of the n-dimensional Riemannian manifold (M, g) is defined as

$$\Omega^k_{\mathrm{conf}}(M) = \{\omega \in L^{n/k}(M, \Lambda^k) \,\big|\, d\omega \in L^{n/(k+1)}(M, \Lambda^{k+1})\}$$

It is thus simply the complex $\Omega^k_\pi(M)$ associated to the sequence π defined by $p_k = n/k$. It is proved in [11] that $\Omega^k_{\mathrm{conf}}(M)$ is a differential graded Banach algebra and is invariant under quasi-conformal maps. Moreover, a homeomorphism $f : (M, g) \to (N, h)$ between two Riemannian manifolds. is a quasiconformal map if and only if the pull-back of differential forms defines an isomorphism of Banach differential algebras $f^* : \Omega^\bullet_{\mathrm{conf}}(N) \longrightarrow \Omega^\bullet_{\mathrm{conf}}(M)$.

The conformal sequence $\pi = (p_k)$ defined by $p_k = n/k$ is its own dual, because $\frac{1}{p_k} + \frac{1}{p_{n-k}} = \frac{k}{n} + \frac{n-k}{n} = 1$. Theorem (1.1) can thus be restated as follow in the particular case of the conformal cohomology:

Theorem 7.1 *Let M be an oriented Riemannian manifold of dimension n and $2 \le k \le (n-2)$. Then $\overline{H}^k_{\mathrm{conf}}(M)$ is isomorphic to the dual of $\overline{H}^{n-k}_{\mathrm{conf};0}(M)$ The duality is induced by the integration pairing (1.3).*

8 Applications

In this section, we give some consequences of the previously established results.

Proposition 8.1 *Let M be a connected oriented Riemannian manifold (M, g) whose isometry group admits an unbounded orbit, i.e. there exists a point $x_0 \in M$ and a sequence of isometries $f_j : M \to M$ such that $d(x_0, f_j(x_0)) \to \infty$ as $j \to \infty$. Then for any $1 < q, p < \infty$, the reduced cohomology $\overline{H}^k_{q,p}(M)$ is finite dimensional if and only if $\overline{H}^k_{q,p}(M) = 0$. The same holds for $\overline{H}^k_{\mathrm{comp},p}(M)$.*

Gromov made this observation for L^p-cohomology assuming M to be complete and of bounded geometry, see [14, p. 220].

Proof Because $\lim_{j\to\infty} d(x_0, f_j(x_0)) = \infty$, the balls $f_j(B(x_0, R))$ and $B(x_0, R)$ are disjoint for any fixed $R > 0$ and sufficiently large $j \in \mathbb{N}$. It follows that

$$\lim_{j\to\infty} \|f_j^*\omega\|_{L^p(B(x_0,R))} = 0$$

for any $\omega \in Z^k_p(M)$ and any $R > 0$. Using Hölder's inequality, we then have for any $\varphi \in Z^{n-k}_{p',q';0}(M)$

$$\left| \int_{B(x_0,R)} f_j^*\omega \wedge \varphi \right| \le \lim_{j\to\infty} \|f_j^*\omega\|_{L^p(B(x_0,R))} \cdot \|\varphi\|_{L^{p'}(B(x_0,R))},$$

and therefore $\lim_{j\to\infty} \int_{B(x_0,R)} f_j^*\omega \wedge \varphi = 0$. We thus have

$$\lim_{j\to\infty} \int_M f_j^*\omega \wedge \varphi = 0$$

for any $\varphi \in Z^{n-k}_{p',q';0}(M)$. From the duality Theorem 1.1, this implies that the sequence $\{[f_j^*\omega]\} \subset \overline{H}^k_{q,p}(M)$ converges weakly to zero. Since $\overline{H}^k_{q,p}(M)$ has finite dimension, this sequence converges strongly to zero and because f_j is an isometry, we have

$$\|[\omega]\| = \lim_{j\to\infty} \|[f_j^*\omega]\| = 0$$

for any element $[\omega] \in \overline{H}^k_{q,p}(M)$. The proof for $\overline{H}^k_{\mathrm{comp},p}(M)$ is the same. □

The next result gives us an explicit criterion implying that a given cohomology class does not vanish.

Proposition 8.2 *Let (M, g) be an oriented Riemannian manifold of dimension n and let $\alpha \in Z^k_p(M)$. Then the following conditions are equivalent.*

(a) $[\alpha] \neq 0$ *in* $\overline{H}^k_{q,p}(M)$.

(b) *There exists $\omega \in Z^{n-k}_{p',q';0}(M)$ such that*

$$\int_M \alpha \wedge \omega \neq 0. \tag{8.1}$$

(c) *There exists a sequence $\{\gamma_i\} \subset \mathcal{D}^{n-k}$ such that*

  (i) $\limsup_{i\to\infty} \int_M \alpha \wedge \gamma_i > 0$.
  (ii) $\lim_{i\to\infty} \|d\gamma_i\|_{q'} = 0$ *where* $q' = \frac{q}{q-1}$.
  (iii) $\|\gamma_i\|_{p'}$ *is a bounded sequence for* $p' = \frac{p}{p-1}$.

Proof (a) ⇒ (b). If $[\alpha] \neq 0$ in $\overline{H}^k_{q,p}(M)$, then by the duality Theorem 1.1, there exists $[\omega] \in \overline{H}^{n-k}_{p',q';0}(M)$ such that (8.1) holds. The cohomology class $[\omega]$ is represented by an element $\omega \in Z^{n-k}_{p',q';0}(M)$, which is the desired differential form.

(b) ⇒ (c) Let $\omega \in Z^{n-k}_{p',q';0}(M)$ be an element such that (8.1) holds. By definition of $Z^k_{p',q';0}(M)$, there exists a sequence

$$\gamma_j \in \mathcal{D}^k(M)$$

such that $\|\gamma_j - \omega\|_{p'} \to 0$ and $\|d\gamma_j\|_{q'} \to 0$. This sequence clearly satisfies the conditions (i), (ii), and (iii).

The implication (c) ⇒ (a) has been proved in [10, Sect. 8]. We repeat the proof for convenience. Suppose that $\alpha \in \overline{B}^k_{q,p}(M)$. Then $\alpha = \lim_{j\to\infty} \mathrm{d}\beta_j$ for $\beta_j \in L^q(M, \Lambda^{k-1})$ with $\mathrm{d}\beta_j \in L^p(M, \Lambda^k)$. We have for any i, j

$$\int_M \gamma_i \wedge \alpha = \int_M \gamma_i \wedge \mathrm{d}\beta_j + \int_M \gamma_i \wedge (\alpha - \mathrm{d}\beta_j).$$

For each $j \in \mathbb{N}$, we can find $i = i(j)$ large enough so that $\|\mathrm{d}\gamma_{i(j)}\|_{q'} \|\beta_j\|_q \leq 1/j$, we thus have

$$\left|\int_M \gamma_{i(j)} \wedge \mathrm{d}\beta_j\right| \leq \left|\int_M \mathrm{d}\gamma_{i(j)} \wedge \beta_j\right| \leq \|\mathrm{d}\gamma_{i(j)}\|_{q'} \|\beta_j\|_q \leq \frac{1}{j}.$$

On the other hand,

$$\lim_{j\to\infty}\left|\int_M \gamma_{i(j)} \wedge (\alpha - \mathrm{d}\beta_j)\right| \le \lim_{j\to\infty} \|\gamma_{i(j)}\|_{p'} \, \|(\alpha - \mathrm{d}\beta_j)\|_p = 0$$

since $\|\gamma_{i(j)}\|_{p'}$ is a bounded sequence and $\|(\alpha-\mathrm{d}\beta_j)\|_p \to 0$. It follows that $\int_M \gamma_{i(j)} \wedge \alpha \to 0$ in contradiction to the hypothesis. □

The previous criterion is not always very practical because we cannot always produce useful closed forms with compact support. The next result is thus convenient in the case of complete manifolds.

Corollary 8.3 *Assume that M is complete. Let $\alpha \in Z_p^k(M)$, and assume that there exists a closed $(n-k)$-form $\gamma \in Z_{p'}^{n-k}(M) \cap Z_{q'}^{n-k}(M)$, where $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$, such that*

$$\int_M \gamma \wedge \alpha > 0,$$

then $\alpha \notin \overline{B}_{q,p}^k(M)$ where $q' = \frac{q}{q-1}$. In particular, $\overline{H}_{q,p}^k(M) \neq 0$.

This result is Proposition 8.4 from [10], an example of its usefulness can be found in that paper where it is used to prove the nonvanishing of the $L_{q,p}$-cohomology of the hyperbolic plane. We give a short proof.

Proof Let γ be a differential form as in the statement, by density we may assume γ to be smooth. Let $\{\eta_j\} \subset C_0^\infty(M)$ be a sequence of smooth function with compact support uniformly converging to 1 and such that $\|\eta_j\|_{L^\infty} \to 0$. Then the sequence $\gamma_j = \eta_j \cdot \gamma$ satisfies the condition (c) of the Proposition. □

If $p = q$, then the previous condition is not only sufficient, but also necessary.

Corollary 8.4 *Let M be a complete oriented Riemannian manifold. Then an element $\alpha \in Z_p^k(M)$ is not zero in reduced L_p-cohomology if and only if there exists $\omega \in Z_{p'}^{n-k}(M)$ such that*

$$\int_M \alpha \wedge \omega \neq 0.$$

Proof This is a special case of the previous Proposition, since for complete manifolds, we have $Z_{p'}^{n-k}(M) = Z_{p',p';0}^{n-k}(M)$. □

Based on the previous Corollary, we can prove the following vanishing result which has been first observed by Gromov, see also [22, Proposition 15].

Proposition 8.5 *Assume that the Riemannian manifold is complete and admits a complete and proper Killing vector field ξ, then $\overline{H}_{\mathrm{comp},p}^k(M) = 0$ and (therefore) $\overline{H}_{q,p}^k(M) = 0$ for any $1 < p < \infty$ and $1 \le q < \infty$.*

In particular, if N is a complete Riemannian manifold, then $\overline{H}_{q,p}^k(N \times \mathbb{R}) = 0$ for any $1 < p < \infty$ and $1 \le q < \infty$.

Remark The hypothesis $p \neq 1$ is a necessary condition. For instance $\overline{H}^k_{q,1}(\mathbb{R}) \neq 0$ for any $1 \leq q < \infty$ (see [10, prop. 9.3]).

Proof Because M is complete, we have $\overline{H}^k_{\mathrm{comp},p}(M) = \overline{H}^k_p(M)$. We thus only need to prove the vanishing of the L_p-cohomology of M. Since ξ is a Killing vector field, we have $\|i_\xi \alpha\|_{L^p} \leq \|\alpha\|_{L^p}$ for any differential form α. Let f_t be the flow of ξ. Using Cartan's formula $\mathcal{L}_\xi \alpha = \mathrm{d}i_\xi \alpha + i_\xi \mathrm{d}\alpha$, we see that for any $\alpha \in Z^k_p(M)$ and $\varphi \in Z^{n-k}_{p'}(M)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M f_t^* \alpha \wedge \varphi = \int_M \mathcal{L}_\xi \alpha \wedge \varphi = \int_M \mathrm{d}i_\xi \alpha \wedge \varphi = 0$$

because $[\mathrm{d}i_\xi \alpha] = 0$ in $\overline{H}^{k-1}_p(M)$. On the other hand, because the flow f_t is proper, we have

$$\lim_{t \to \infty} \int_M f_t^* \alpha \wedge \varphi = 0.$$

It follows that $\int_M \alpha \wedge \varphi = 0$ for any $\alpha \in Z^k_p(M)$ and $\varphi \in Z^{n-k}_{p'}(M)$ and therefore $\overline{H}^k_p(M) = 0$. □

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