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A rigid triple of conjugacy classes in G_2

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Abstract. We produce a rigid triple of classes in the algebraic group G_2 in characteristic 5, and use it to show that the finite groups $G_2(5^n)$ are not $(2,5,5)$-generated.

1 Introduction

Let G be a connected simple algebraic group over an algebraically closed field K, and let $C_1,\dots,C_s$ be conjugacy classes in G. Let $\mathbf{C}$ denote the s-tuple $(C_1,\dots,C_s)$, and define

$$\mathbf{C}_0 = \{(x_1,\dots,x_s) \in C_1 \times \cdots \times C_s : x_1x_2\dots x_s = 1\}.$$

Then G acts on $\mathbf{C}_0$ by componentwise conjugation. Following [8], we say that the s-tuple $\mathbf{C} = (C_1,\dots,C_s)$ is *rigid* in G if $\mathbf{C}_0$ is non-empty and G is transitive on $\mathbf{C}_0$.

For G a classical group, there are many known examples of rigid tuples of classes, such as Belyi triples and Thompson tuples, as defined in [9]. However we are not aware of many examples in the literature for exceptional algebraic groups. In this paper we produce a rigid triple of classes in the algebraic group G_2 in characteristic 5, and use it to answer a question raised in [5] concerning the generation of the finite groups $G_2(5^n)$.

Let $K = \bar{\mathbb{F}}_5$, the algebraic closure of the field $\mathbb{F}_5$ of five elements, and let $G = G_2(K)$. The conjugacy classes of G can be read off from [1]. We pick out two of the classes. The first is the unique involution class: letting $t \in G$ be an involution, we have

$$C_G(t) = A_1\tilde{A}_1,$$

a central product of commuting SL_2's, where A_1 (resp. $\tilde{A}_1$) is generated by long (resp. short) root elements of G. The class t^G has dimension 8.

Adopting the notation of [4, Table B, p. 4130], we see that G has three classes of elements of order 5: the long and short root elements, and the class labelled $G_2(a_1)$, with representative

$$u = x_b(1)x_{3a+b}(1)$$

where a, b are simple roots with a short and b long. The centralizer $C_G(u)$ has connected component U_4, a unipotent group of dimension 4, and the component group $C_G(u)/C_G(u)^0 \cong S_3$. The class u^G has dimension 10.

Here is our main result. In part (iii), by a $(2,5,5)$-group we mean a group which is generated by elements x, y, z of orders 2,5,5 satisfying $xyz = 1$.

Theorem. (i) *The triple of classes* $\mathbf{C} = (t^G, u^G, u^G)$ *is rigid in* $G = G_2(K)$.

(ii) *Every triple of elements* $(x_1, x_2, x_3) \in \mathbf{C}_0$ *generates a subgroup of* G *isomorphic to the alternating group* Alt_5.

(iii) *None of the groups* $G_2(5^n)$ *is a* $(2,5,5)$*-group for any* n. *Neither are* $\mathrm{SL}_3(5^n)$ *or* $\mathrm{SU}_3(5^n)$.

Remarks. (1) Notice that $\dim t^G + 2\dim u^G = 28 = 2\dim G_2$. This agrees with [8, Corollary 3.2], which states that for any rigid tuple $\mathbf{C} = (C_1, \ldots, C_s)$ in G, such that $C_{L(G)}(x_1, \ldots, x_s) = 0$ for $(x_1, \ldots, x_s) \in \mathbf{C}_0$, we have

$$\sum_{i=1}^{s} \dim C_i = 2\dim G.$$

(We shall see that the subgroup Alt_5 in (ii) of the theorem has zero centralizer in $L(G)$.)

(2) Part (iii) of the theorem answers one case of the conjecture posed in [5]. This conjecture asserts that if (p_1, p_2, p_3) is a 'rigid' triple of primes for a simple algebraic group X in characteristic p (meaning that the varieties of elements of orders dividing p_1, p_2, p_3 have dimensions adding up to $2\dim X$), then there are only finitely many values of n such that $X(p^n)$ is a (p_1, p_2, p_3)-group. The only rigid triple of primes for exceptional groups is $(2,5,5)$ for G_2. For this case part (iii) verifies the conjecture in characteristic $p = 5$.

2 Proof of the theorem

Let $G = G_2(K)$ with $K = \overline{\mathbb{F}}_5$, and let $t, u \in G$ be as defined in the previous section. If σ is the Frobenius morphism of G induced by the map $x \mapsto x^5$ on K, then

$$G = \bigcup_{n=1}^{\infty} G_{\sigma^n} = \bigcup_{n=1}^{\infty} G_2(5^n).$$

To begin the proof, observe that $u = x_b(1)x_{3a+b}(1)$ is a regular unipotent element in the subgroup $A_2 \cong \mathrm{SL}_3(K)$ of G generated by the long root groups $X_{\pm b}$, $X_{\pm(3a+b)}$. Hence u lies in an orthogonal subgroup $\Omega_3(5) \cong \mathrm{Alt}_5$ of this A_2. Write A for this Alt_5, so

$$u \in A < A_2 < G. \tag{1}$$

Also $N_{A_2}(A) = \mathrm{SO}_3(5) \cong S_5$.

We next calculate $C_G(A)$. Certainly this contains the centre $\langle z\rangle$ of A_2, and it also contains an outer involution τ in $N_G(A_2) = A_2.2$ (since such an involution centralizes an orthogonal group $SO_3(K)$ in A_2). We claim that

$$C_G(A) = \langle z, \tau\rangle \cong S_3. \tag{2}$$

To see this, take a Klein 4-subgroup $E = \langle t_1, t_2\rangle < A$. By viewing E inside $C_G(t_1) = A_1\tilde{A}_1$, we see that E lies in a maximal torus T_2 of G, and

$$C_G(E) \leqslant N_G(T_2) = T_2.W(G_2).$$

Since $W(G_2) \cong D_{12}$ has order coprime to $p = 5$, it follows that $C_G(E)$ consists of semisimple elements. But also $C_G(A) \leqslant C_G(u) = U_4.S_3$, where U_4 is a connected unipotent group. Consequently $C_G(A)$ is isomorphic to a subgroup of S_3, and hence (2) holds.

Call a triple of elements (a_1, a_2, a_3) in A^3 a $(2, 5, 5)$-triple if a_1, a_2, a_3 have orders 2,5,5 respectively, and $a_1a_2a_3 = 1$. A simple calculation using the character table of Alt_5 shows that the number of $(2, 5, 5)$-triples in A^3 is 120, and these are permuted transitively by $N_{A_2}(A) \cong S_5$.

Now let $\mathbf{C}$ denote the triple of classes (t^G, u^G, u^G), and define $\mathbf{C}_0$ as in the Introduction. Fix any $q = 5^n$, so $G_{\sigma^n} = G_2(q)$, and let $\mathbf{C}_0(q) = \mathbf{C}_0 \cap G_2(q)^3$. Next we show that

$$|\mathbf{C}_0(q)| = |G_2(q)|. \tag{3}$$

To prove this we require the character table of $G_2(q)$, given in [2]. Since $C_G(u) = U_4.S_3$, Lang's theorem shows that $u^G \cap G_2(q)$ splits into three $G_2(q)$-classes with representatives denoted in [2] by u_3, u_4, u_5 and having respective centralizer orders $6q^4$, $2q^4$, $3q^4$. For $x, y, z \in G_2(q)$ let $a_{x,y,z}$ be the class algebra constant of the classes with representatives x, y, z. From the character table (and using CHEVIE [3] to assist with the calculations) we find that for $i, j \in \{3, 4, 5\}$,

$$a_{t,u_i,u_j} = \begin{cases} q^4 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

It follows that

$$|\mathbf{C}_0(q)| = \sum_{i=3}^{5} |u_i^{G_2(q)}|.a_{t,u_i,u_i} = q^4|G_2(q)|\left(\frac{1}{2q^4} + \frac{1}{3q^4} + \frac{1}{6q^4}\right) = |G_2(q)|,$$

proving (3).

At this point we can complete the proof of the theorem. Define $\mathbf{C}_0'$ to be the set of triples $(x_1, x_2, x_3) \in \mathbf{C}_0$ such that $\langle x_1, x_2, x_3\rangle$ is a G-conjugate of A. Since σ

centralizes A, it acts on $\mathbf{C}_0'$. Moreover, G acts transitively on $\mathbf{C}_0'$: for if (x_1, x_2, x_3) and (y_1, y_2, y_3) are triples in $\mathbf{C}_0'$, with $\langle x_1, x_2, x_3 \rangle = A$, $\langle y_1, y_2, y_3 \rangle = A^g$, then (x_1, x_2, x_3) and $(y_1, y_2, y_3)^{g^{-1}}$ are $(2,5,5)$-triples in A^3, and hence by the observation two paragraphs above, they are conjugate by an element of $N_G(A)$.

Now we apply Lang's theorem in the form of [7, (I, 2.7)] to the transitive action of G on $\mathbf{C}_0'$. By (2), a point stabilizer is $C_G(A) = S_3$. Hence Lang's theorem shows that the set $\mathbf{C}_0'(q) = \mathbf{C}_0' \cap G_2(q)^3$ splits into three $G_2(q)$-orbits, of sizes $|G_2(q)|/r$ for $r = 2, 3, 6$, and so

$$|\mathbf{C}_0'(q)| = |G_2(q)| \cdot \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) = |G_2(q)|.$$

It follows by (3) that $\mathbf{C}_0'(q) = \mathbf{C}_0(q)$. Hence

$$\mathbf{C}_0 = \bigcup_{n=1}^{\infty} \mathbf{C}_0(5^n) = \bigcup_{n=1}^{\infty} \mathbf{C}_0'(5^n) = \mathbf{C}_0'.$$

Therefore G is transitive on $\mathbf{C}_0$ and every triple in $\mathbf{C}_0$ generates a conjugate of A.

This completes the proof of parts (i) and (ii) of the theorem. Finally, for part (iii), suppose that $G_2(5^n)$, $\mathrm{SL}_3(5^n)$ or $\mathrm{SU}_3(5^n)$ is $(2,5,5)$-generated, with corresponding generators x_1, x_2, x_3. Now $L(G_2) \downarrow A_2$ is the sum of $L(A_2)$ and two irreducible 3-dimensional A_2-modules (see for example [6, (1.8)]), and hence $C_{L(G_2)}(x_1, x_2, x_3) = 0$. (It also follows that $C_{L(G_2)}(\mathrm{Alt}_5) = 0$; see Remark 1 in the Introduction.) At this point, the first argument in the proof of [8, (3.2)] shows that the generators x_1, x_2, x_3 must lie in classes of dimensions summing to at least $2 \dim G_2 = 28$, hence in the classes t^G, u^G, u^G. But this is impossible by part (ii) of the theorem.

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