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Real-time MPC – Stability through Robust MPC design

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***Abstract*— Recent results have suggested that online Model Predictive Control (MPC) can be computed quickly enough to control fast sampled systems. High-speed applications impose a hard real-time constraint on the solution of the MPC problem, which generally prevents the computation of the optimal controller. In current approaches guarantees on feasibility and stability are sacrificed in order to achieve a real-time setting. In this paper we develop a real-time MPC scheme based on robust MPC design that recovers these guarantees while allowing for extremely fast computation. We show that a simple warm-start optimization procedure providing an enhanced feasible solution guarantees feasibility and stability for arbitrary time constraints. The proposed method can be practically implemented and efficiently solved for dynamic systems of significant problem size. Implementation details for a real-time robust MPC method are provided that achieves computation times equal to those reported for methods without guarantees. A 12-dimensional problem with 3 control inputs and a prediction horizon of 10 time steps is solved in 2msec with a performance deterioration less than 1% and thereby allows for sampling rates of 500Hz.**

I. Introduction

In online Model Predictive Control (MPC) approaches, a constrained optimal control problem is solved at each time instant, which has restricted the applicability of MPC to slow dynamic processes. In recent years it was shown that the optimal solution to this type of problem can be solved offline and the so-called explicit solution can then be used as a control look-up table online [1], [2]. Whereas this enables MPC to be used for fast sampled systems its application is strongly limited by the problem size. This work is motivated by recent results showing that the computation times for solving an MPC problem can be pushed into a range where an online solution becomes a reasonable alternative for the control of high-speed systems. Significant reduction of the computational complexity can be achieved by exploiting the particular structure and sparsity of the optimization problem given by the MPC problem using tailored solvers [23], [24].

High speed systems impose a strict real-time constraint on the problem which generally prevents the computation of the optimal controller. The goal is then to provide a suboptimal control action within the time constraint that still guarantees stability of the closed-loop system and achieves acceptable performance. A method providing these guarantees by combining online and explicit MPC was introduced in [25], which is however limited to smaller problem dimensions. Available methods for fast online MPC do not give guarantees on either feasibility or stability of the applied control action in a real-time implementation. In this paper we develop a real-time MPC scheme that guarantees stability for all time constraints and allows for fast online computation. The a-priori stability guarantee then allows one to trade the performance of the suboptimal controller for lower online computation times. We show that the method can be *practically* implemented and efficiently solved for systems of significant size.

A standard warm-start procedure is applied in which the optimization problem for the current state is initialized with the shifted suboptimal control sequence computed at the previous time instance. The optimization is terminated early when a specified time constraint τ is hit returning an enhanced feasible solution. First, asymptotic stability of the so-called τ-real-time control law resulting from this procedure is established for the nominal system. The approach is then extended to the case of uncertain systems that are subject to bounded additive disturbances. We show that the use of a robust MPC design guarantees constraint satisfaction as well as input-to-state stability of the uncertain system under the proposed τ-real-time control law. The presented results are based on existent stability theory in the MPC literature, e.g. [15], [16], [20] and the references therein, that is studied and emphasized in the context of real-time MPC for linear systems. In particular we exploit the fact that a suboptimal rather than optimal solution to the (robust) MPC problem is sufficient for stability if it satisfies the constraints and has a lower cost than the shifted sequence from the last sample. We point out that this theory offers a stability guarantee for the nominal and the uncertain case under a standard and simple online procedure.

A real-time MPC procedure for uncertain linear systems is developed using the tube based robust MPC approach in [17]. We present the computational details for implementing the proposed method and emphasize that the required sets can be computed or approximated for all problem sizes allowing the real-time MPC scheme to be applied to dynamic systems of 30 dimensions or more. We show that the structure and sparsity of the optimization problem is maintained in the robust case and can be solved efficiently using solvers tailored for the solution of MPC problems. A custom solver was developed for this paper using a primal barrier interior-point method [4] that achieves computation times that are equal or even faster compared to existing methods with no guarantees. For a 12-dimensional example system the MPC problem with a limit of 5 interior-point iterations was solved in 2msec with an average performance deterioration of less than 1% allowing for sampling rates of 500Hz. The corresponding computation times for a 30-dimensional system were 10msec allowing for sampling rates of 100Hz.

The outline of the paper is as follows: In Section III the nominal real-time control procedure is introduced and asymptotic stability is shown repeating important results from the literature. Section IV extends these results to the uncertain case. The concept of robust MPC is described and input-to-state stability is shown for the proposed method. The computational details necessary to apply the proposed procedure are provided in Section V. Finally, in Section VI we illustrate our approach and its advantages using numerical examples and provide a comparison with the literature.

II. Notation & Preliminaries

A *polyhedron* is the intersection of a finite number of halfspaces $P = \{x | Ax \leq b\}$ and a *polytope* is a bounded polyhedron. If $A \in \mathbb{R}^{m\times n}$ then $A_i \in \mathbb{R}^n$ is the vector formed by the i-th row of A. If $b \in \mathbb{R}^m$ is a vector then b_i is the i-th element of b. Given two sets $S_1, S_2 \subseteq \mathbb{R}^n$ the Minkowski sum is defined as $S_1 \oplus S_2 \triangleq \{s_1 + s_2 | s_1 \in S_1, s_2 \in S_2\}$ and the Pontryagin difference as $S_1 \ominus S_2 \triangleq \{s_1 | s_1 + s_2 \in S_1, s_2 \in S_2\} = \{s_1 | s_1 \oplus S_2 \subseteq S_1\}$. For a collection of sets $\{S_i \subset \mathbb{R}^n, i \in [a, a+1, \cdots, b]\}$, $\bigoplus_{i=a}^{b} S_i \triangleq S_a \oplus S_{a+1} \oplus \cdots \oplus S_b$. Given a sequence $\mathbf{u} \triangleq [u_0, \cdots, u_{N-1}]$, $\mathbf{u}(j)$ denotes the j-th element of $\mathbf{u}$. If a sequence depends on a parameter denoted by $\mathbf{u}(x)$, $\mathbf{u}(j, x)$ denotes its j-th element. If the elements $\mathbf{u}(j) \in \mathbb{U}$ then $\mathbf{u} \in \mathbb{U}^N$, where $\mathbb{U}^N \triangleq \mathbb{U} \times \cdots \times \mathbb{U}$.

Consider the discrete-time linear system

$$x^+ = Ax + Bu + w \tag{1}$$

and the corresponding nominal system

$$x^+ = Ax + Bu \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input and w is a bounded disturbance that is contained in a convex and compact set $W \subset \mathbb{R}^n$ that contains the origin. The solution of the nominal system (2) at sampling time k for the initial state $x(0)$ and a sequence of control inputs $\mathbf{u}$ is denoted as $\phi(k, x(0), \mathbf{u})$. Consider that system (1) is controlled by the control law $u = \kappa(x)$ denoted by

$$x^+ = Ax + B\kappa(x) + w = A_\kappa(x) + w \tag{3}$$

The solution of the controlled uncertain system for a sequence of disturbances $\mathbf{w}$ is denoted as $\bar{\phi}_\kappa(k, x(0), \mathbf{w})$.

Definition II.1 (Positively invariant (PI) set). A set $S \subseteq \mathbb{R}^n$ is a positively invariant (PI) set of system $x^+ = A_\kappa(x)$, if $A_\kappa(x) \in S$ for all $x \in S$.

Definition II.2 (Robust positively invariant (RPI) set). A set $S \subseteq \mathbb{R}^n$ is a robust positively invariant (RPI) set of (3) if $A_\kappa(x) + w \in S$ for all $x \in S, w \in W$.

The smallest RPI set that is contained in every closed RPI set of (3) is called a minimal RPI (mRPI) set, and the biggest PI set that contains every closed PI set of $x^+ = A_\kappa(x)$ is called a maximal PI (MPI) set.

Definition II.3 (Regional ISS [7], [21]). Given an RPI set $\Gamma \subseteq \mathbb{R}^n$ with $0 \in \Gamma$, system (3) is Input-to-State Stable (ISS) in Γ w.r.t. w if there exists a $\mathcal{KL}$-function [22] β and a $\mathcal{K}$-function [22] γ such that for all initial states $x(0) \in \Gamma$ and for all disturbance sequences $\mathbf{w} = [w_0, \cdots, w_j] \in W^{j+1}$: $\|\phi_\kappa(j, x(0), \mathbf{w})\|_2 \leq \beta(\|x(0)\|_2, j) + \gamma(\|\mathbf{w}_{[j-1]}\|), \;\; \forall j \geq 0$, where $\|\mathbf{w}_{[j-1]}\| \triangleq \sup_{0\leq k\leq j-1}\{\|w_k\|_2\}$.

Remark II.4. Note that the condition for input-to-state stability reduces to that for asymptotic stability if $w = 0$. A system $x^+ = Ax + w$ is ISS if the nominal system $x^+ = Ax$ is asymptotically stable and the effect of the disturbance on the evolution of the system is bounded.

III. Real-time MPC

Consider the nominal linear discrete-time system (2). The goal is to regulate the state of the system to the origin while respecting constraints on inputs and states, which can be formulated as the following MPC problem $\mathbb{P}_N(x)$:

$$\min_{\mathbf{u}} \; V_N(x, \mathbf{u}) \triangleq \sum_{i=0}^{N-1} l(x_i, u_i) + V_f(x_N) \tag{4}$$

$$\begin{aligned} \text{s.t.} \quad & x_{i+1} && = Ax_i + Bu_i, \;\; i = 0, \ldots, N-1 \;, \\ & (x_i, u_i) && \in \mathbb{X} \times \mathbb{U}, \;\; i = 0, \ldots, N-1 \;, \\ & x_N && \in X_f \;, \\ & x_0 && = x \;, \end{aligned} \tag{5}$$

where $\mathbf{u} = [u_0, \cdots, u_{N-1}] \in \mathbb{U}^N$ denotes the input sequence, $\mathbb{X}$ and $\mathbb{U}$ are polytopic constraints on the states and inputs, the stage cost is defined as $l(x_i, u_i) := \frac{1}{2}x_i^T Q x_i + \frac{1}{2}u_i^T R u_i$, $V_f(x) = \frac{1}{2}x^T P x$ is a terminal penalty function, Q, R and P are positive definite matrices and $X_f \subseteq \mathbb{X}$ is a compact terminal target set with properties as defined in Assumption III.1.

The associated state trajectory to a given control sequence $\mathbf{u}(x)$ at state x is $\mathbf{x}(x) \triangleq [x_0, x_1, \cdots, x_N]$, where $x_0 = x$ and for each i, $x_i = \phi(i, x, \mathbf{u}(x))$. Problem $\mathbb{P}_N(x)$ implicitly defines the set of feasible control sequences $\mathcal{U}_N(x) = \{\mathbf{u}(x) \,|\, \mathbf{u}(x) \in \mathbb{U}^N, \mathbf{x}(x) \in \mathbb{X}^N \times X_f\}$ and feasible initial states $\mathcal{X}_N \triangleq \{x \,|\, \mathcal{U}_N(x) \neq \emptyset\}$. For a given state $x \in \mathcal{X}_N$ the solution of $\mathbb{P}_N(x)$ yields the optimal control sequence $\mathbf{u}^*(x)$. The implicit optimal MPC control law is then given in a receding horizon fashion by $\kappa(x) \triangleq \mathbf{u}^*(0, x)$.

Assumption III.1. In the following it is assumed that $V_f(\cdot)$ is a Lyapunov function in X_f and X_f is a PI set for system (2) under the control law $\kappa_f(x) = Kx$, given by the following conditions:

A1: $X_f \subseteq \mathbb{X}$, $(A + BK)X_f \subseteq X_f$, $KX_f \subseteq \mathbb{U}$
A2: $V_f((A + BK)x) - V_f(x) \leq -l(x, Kx) \, \forall x \in X_f$.

Theorem III.2 (Stability under $\kappa(x)$, [16]). *Consider Problem $\mathbb{P}_N(x)$ fulfilling Assumption III.1. The closed-loop* *system $x^+ = Ax + B\kappa(x)$ is asymptotically stable with region of attraction $\mathcal{X}_N$.*

For a given value $x \in \mathcal{X}_N$, we can write (4) as a Quadratic Program (QP) of the following form :

$$\min_{\mathbf{z}} \ \mathbf{z}^T H \mathbf{z} \quad \text{s.t. } P\mathbf{z} \le g \ , \ \ E\mathbf{z} = c \ , \tag{6}$$

where $\mathbf{z} \triangleq [u_0, x_1, \cdots, u_{N-1}, x_N] \in \mathbb{R}^{N(n+m)}$ is a vector containing the sequence of states and control inputs. See e.g. [2] for details on the conversion and [24] for details on the structure of H, P, g, E and c.

Most real applications impose a real-time constraint on the solution of the MPC problem, i.e. a limit on the computation time that is available to compute the control input, at each time instance. This often prevents the computation of the optimal solution to (6). The introduction of a τ-real-time constraint can lead to the loss of feasibility and more importantly stability when using a general optimization solver. A suboptimal control input therefore has to be provided within the real-time constraint that ensures stability and a minimal performance deterioration. In the following a control law is called τ-real time (τ-RT) if it is computed in τ seconds.

Various approaches trying to reduce the computation time in online MPC have been recently proposed. Many methods are based on the development of custom solvers that take advantage of the particular sparse structure in an MPC problem (e.g. [23], [24]). In [23], for example, an infeasible start newton method is applied that is terminated after a fixed number of steps. A tailored solver was developed that exploits the sparse structure of the MPC problem resulting in computation times in the range of milliseconds. The authors in [5] develop a warm-start based homotopy approach that is terminated early in case of a time constraint. Most available approaches however sacrifice feasibility and/or stability in order to achieve a real-time guarantee. In [12] a relation between the level of suboptimality and the stability guarantee is derived. These results can however not be applied to the considered case of real-time MPC since it is currently not possible to determine the level of suboptimality that a given solver will achieve in a fixed amount of time.

We show that in the nominal case a τ-RT input sequence that is guaranteed to be feasible and stabilizing can be easily constructed for any time constraint τ. A standard warm-start procedure is employed where the input sequence computed at state x is used to initialize the QP (6) for the next state x^+ in (2) (note that the vector $\mathbf{z}$ can be directly constructed from the current state and a given input sequence). The QP is then solved using an optimization routine that is iteratively improving the solution by taking feasible steps and is terminated after time τ. The described procedure returns a τ-RT control law.

Definition III.3 (τ-RT optimizer). Assume $\widetilde{\mathbf{u}}(x) = [\widetilde{u}_0, \cdots, \widetilde{u}_{N-1}]$ is a feasible control sequence for x, $\widetilde{\mathbf{x}}(x)$ is the associated state sequence and $x^+ = Ax + B\widetilde{\mathbf{u}}(0, x)$ is the current state. The shifted sequence is given by

$$\mathbf{u}_{\text{shift}}(x) = [\widetilde{u}_1, \cdots, \widetilde{u}_{N-1}, K\widetilde{\mathbf{x}}(N, x)] \tag{7}$$

We define $\mathbf{u}_{\text{RT}}(x^+, \tau)$ to be the control variables of $\mathbb{P}_N(x)$ after time τ, with $\tau \ge 0$. A τ-RT optimizer computes a control sequence in τ seconds with the following properties:

- $\mathbf{u}_{\text{RT}}(x^+, \tau)$ is feasible for x^+
- $\mathbf{u}_{\text{RT}}(x^+, 0) = \mathbf{u}_{\text{shift}}(x)$
- $V_N(x^+, \mathbf{u}_{\text{RT}}(x^+, \tau)) \le V_N(x^+, \mathbf{u}_{\text{RT}}(x^+, 0))$

Definition III.4 (Nominal τ-RT control law). Let $\mathbf{u}_{\text{RT}}(\cdot, \tau)$ be as defined in III.3. The τ-RT control law is

$$\kappa_\tau(x) = \mathbf{u}_{\text{RT}}(0, x, \tau), \text{ for } x \in \mathcal{X}_N \ . \tag{8}$$

It is well-known that in the nominal case the shifted sequence in (7) is feasible and stabilizes the system in (2) [20]. The fact that any improved feasible solution of the shifted sequence also offers guaranteed stability has been pointed out previously in several places (see e.g. [16], [20]) but its importance for real-time MPC has not been previously studied. We therefore state this result for completeness.

Theorem III.5 (Stability under $\kappa_\tau(x)$, [20]). *The closed-loop system $x^+ = Ax + B\kappa_\tau(x)$ is asymptotically stable for all $\tau \ge 0$ with region of attraction $\mathcal{X}_N$.*

Remark III.6. The critical requirement in order to guarantee stability of the τ-RT control law is feasibility of the shifted sequence in (7) or so-called recursive feasibility. This guarantees stability of the 0-RT control law. In order to guarantee stability for all subsequent times $\tau \ge 0$ the optimization has to maintain feasibility and ensure that the cost function is not increased with respect to the cost at time $\tau = 0$. These requirements are not automatically fulfilled by all standard optimization routines in which case they have to be explicitly enforced, see Section V-B for details.

IV. Real-time robust MPC

In practice, model inaccuracies or disturbances usually cause violation of the nominal system dynamics in (2) which can lead to the loss of (recursive) feasibility. Stability of the nominal optimal MPC controller as well as the proposed τ-RT control law can then not be guaranteed. This issue is addressed in robust MPC schemes that recover recursive feasibility by changing the problem formulation. As mentioned previously, this is the crucial item in order to prove stability of the proposed real-time MPC method. Note that in the considered case stability can therefore not be achieved by the approach described in [13], where a constraint on the Lyapunov decrease in the first step is introduced, since the solutions are not recursively feasible. In this section the results for the nominal case are extended to the uncertain case using a robust MPC method. We first describe the idea of robust MPC and then develop a robust τ-RT control law that guarantees ISS of the closed-loop uncertain system.

A. Robust MPC

Consider the discrete-time uncertain system in (1). The goal of robust MPC is to provide a controller that satisfies the state and input constraints and achieves some form of stability despite disturbances that are acting on the system. Asymptotic stability of the origin can not be achieved in the presence of persistent disturbances. It can, however be shown that under certain conditions the trajectories converge to an RPI set $\mathcal{Z}$, which can be seen as the 'origin' for the uncertain system. This is captured in the concept of ISS in Definition II.3, requiring the nominal system to be asymptotically stable and the influence of the disturbance on the evolution of the states to be bounded [21].

There is a vast literature on the synthesis of robust MPC controllers, see e.g. [15], [16] and the references therein for an overview. The crucial property of recursive feasibility is guaranteed by all available robust MPC methods and could be used to derive a real-time MPC controller for the uncertain system (1). In order to allow for fast computation we use the tube based robust MPC approach for linear systems described in [17] in this work. The main steps of the procedure are outlined in the following (see [17] for details).

The method is based on the use of a feedback policy of the form $u = \bar{u} + K(x - \bar{x})$ that bounds the effect of the disturbances and keeps the states x of the uncertain system in (1) close to the states $\bar{x}$ of the nominal system $\bar{x}^+ = A\bar{x} + B\bar{u}$. Loosely speaking, the controlled uncertain system will stay within a so-called tube with constant section $\mathcal{Z}$ and centers $\bar{x}(i)$, where $\mathcal{Z}$ is an RPI set for system $x^+ = (A + BK)x + w$. The robust MPC problem can therefore be reduced to the control of the tube centers, which are steered to the origin by choosing a sequence of control inputs $\bar{u}$ and the initial tube center $\bar{x}(0)$. It can be shown that if the initial center is chosen according to the constraint $x = x(0) \in \bar{x}(0) \oplus \mathcal{Z}$ for a given initial state x, then the trajectory of the uncertain system remains within the described tube (in fact for all $\bar{u}, x(i) \in \bar{x}(i) \oplus \mathcal{Z}$ if $x(0) \in \bar{x}(0) \oplus \mathcal{Z}$). This can be formulated as a standard MPC problem with the only difference that the first state $\bar{x}_0$ is also an optimization variable representing the tube center for the current state x. In order to guarantee that the uncertain system does not violate the constraints in (5) the constraints for the tube centers must be tightened in the following way: $\bar{\mathbb{X}} = \mathbb{X} \ominus \mathcal{Z}$, $\bar{\mathbb{U}} = \mathbb{U} \ominus K\mathcal{Z}$. This results in the following robust MPC problem $\bar{\mathbb{P}}_N(x)$:

$$\min_{\bar{x}_0, \bar{\mathbf{u}}} \; \bar{V}_N(x, \bar{x}_0, \bar{\mathbf{u}}) \triangleq \sum_{i=0}^{N-1} l(\bar{x}_i, \bar{u}_i) + V_f(\bar{x}_N) \tag{9}$$

$$\begin{aligned} \text{s.t.} \quad & \bar{x}_{i+1} && = A\bar{x}_i + B\bar{u}_i, \;\; i = 0, \dots, N-1 \;\;, \\ & (\bar{x}_i, \bar{u}_i) && \in \bar{\mathbb{X}} \times \bar{\mathbb{U}}, \;\; i = 0, \dots, N-1 \;\;, \\ & \bar{x}_N && \in X_f \;\;, \\ & x && \in \bar{x}_0 \oplus \mathcal{Z} \;\;, \end{aligned} \tag{10}$$

Problem $\bar{\mathbb{P}}_N(x)$ implicitly defines the set of feasible initial states $\bar{\mathcal{X}}_N \subseteq \mathcal{X}_N$ and feasible control sequences $\bar{\mathcal{U}}_N(x)$.

Remark IV.1. Note that the re-optimization of the tube center at every time step introduces feedback to the disturbance. A feasible and stable controller could however also be obtained by computing the center trajectory and control sequence once for the intial state $x(0)$ and then running the system with the obtained control sequence and after that with the auxiliary control law.

Assumption IV.2. It is assumed that $Q, R, P, V_f(\cdot), X_f$ fulfill Assumption III.1 with $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ replacing $\mathbb{X}$ and $\mathbb{U}$ and $\kappa_f(x) = Kx$.

Remark IV.3. The set $X_f \oplus \mathcal{Z}$ is an RPI set for system $x^+ = (A + BK)x + w$.

The resulting robust MPC control law is given by:

$$\bar{\kappa}(x) = \bar{\mathbf{u}}^*(0, x) + K(x - \bar{x}_0^*(x)) \;\;, \tag{11}$$

where $\bar{\mathbf{u}}^*(x)$ and $\bar{x}_0^*(x)$ is the optimal solution to $\bar{\mathbb{P}}_N(x)$. Note that the optimal initial center $\bar{x}_0^*(x)$ is not necessarily equal to the current state x.

Theorem IV.4 (Stability under $\bar{\kappa}(x)$). *Consider Problem $\bar{\mathbb{P}}_N(x)$ fulfilling Assumption IV.2. The closed-loop system $x^+ = Ax + B\bar{\kappa}(x) + w$ is ISS in $\bar{\mathcal{X}}_N$ w.r.t. w.*

Proof. Robust stability of the set $\mathcal{Z}$ with region of attraction $\bar{\mathcal{X}}_N$ was proven in [17] which corresponds to ISS in $\bar{\mathcal{X}}_N$ [21]. ∎

B. Real-time robust MPC

The previously described robust MPC scheme separates the effect of the uncertainty from the nominal system behavior and thereby allows us to directly extend the results for the nominal case in Section III to the case of uncertain systems. We show in the following that a τ-RT control law for the uncertain system (1) can be obtained by applying the proposed real-time MPC scheme to the robust MPC problem $\bar{\mathbb{P}}_N(x)$ and prove that the resulting robust τ-RT control law is ISS. Our results are based on the fact that recursive feasibility of the MPC control law is recovered by means of a robust MPC design. This is a well-known fact in the robust MPC literature and has been stated in various places (e.g. [15], [17]). Although it was previously remarked e.g. in [15] that this result can be used to show that an enhanced solution of $\bar{\mathbb{P}}_N(x)$ rather than the optimal one is sufficient for stability, this result has not been exploited in the context of real-time MPC to our knowledge.

The same standard optimization routine as described in Section III is applied to solve $\bar{\mathbb{P}}_N(x)$. The optimization is initialized with the input sequence and tube centers computed at the previous time step. The procedure improves the initial solution and is terminated early after a fixed time τ determined by the real-time constraint. The resulting feasible suboptimal controller together with the static feedback in (11) is applied to the system.

Definition IV.5 (Robust τ-RT optimizer). We define $\bar{\mathbf{u}}_{\text{RT}}(x^+, \tau)$ to be the control variables and $\bar{x}_{0,\text{RT}}(x^+, \tau)$ to be the corresponding state variable $\bar{x}_0$ after solving $\bar{\mathbb{P}}_N(x)$ for time τ. A robust τ-RT optimizer computes a control sequence and an initial center in τ seconds with properties as defined in Assumption III.3 and the additional property:

- $x^+ \in \bar{x}_{0,\mathrm{RT}}(x^+, \tau) \oplus \mathcal{Z}$ for x^+ in (1).

Definition IV.6 (Robust τ-RT control law). Let $\bar{\mathbf{u}}_{\mathrm{RT}}(\cdot, \tau)$ and $\bar{x}_{0,\mathrm{RT}}(\cdot, \tau)$ be as defined in IV.5. The robust τ-RT control law is

$$\bar{\kappa}_\tau(x) = \bar{\mathbf{u}}_{\mathrm{RT}}(x, \tau) + K(x - \bar{x}_{0,\mathrm{RT}}(x, \tau)), \text{ for } x \in \bar{\mathcal{X}}_N. \quad (12)$$

Theorem IV.7 (Stability under $\bar{\kappa}_\tau(x)$). *The closed-loop system $x^+ = Ax + B\bar{\kappa}_\tau(x) + w$ is ISS in $\bar{\mathcal{X}}_N$ w.r.t. w for all $\tau \geq 0$.*

Proof. Recursive feasibility was proven in [17]. Since $\bar{V}_N(x, \bar{x}_{0,\mathrm{RT}}(x, \tau), \mathbf{u}) = V_N(\bar{x}_{0,\mathrm{RT}}(x, \tau), \mathbf{u})$ and $\bar{\kappa}_\tau(\bar{x}_{0,\mathrm{RT}}(x, \tau)) = \kappa_\tau(\bar{x}_{0,\mathrm{RT}}(x, \tau))$, asymptotic stability of the center trajectory $\bar{x}_{0,\mathrm{RT}}(\phi_{\bar{\kappa}_\tau}(j, x(0), \mathbf{w}), \tau)$ is shown by Theorem III.5. ISS then follows from the fact that $\phi_{\bar{\kappa}_\tau}(j, x(0), \mathbf{w}) \in \bar{x}_{0,\mathrm{RT}}(\phi_{\bar{\kappa}_\tau}(j, x(0), \mathbf{w}), \tau) \oplus \mathcal{Z}$ [17]. ■

Theorem IV.7 guarantees stability of the uncertain system (1) in a real-time MPC implementation by using the robustified problem formulation $\bar{\mathbb{P}}_N(x)$. The solution of (9) can be stopped after an arbitrary available time τ and even a 0-time feasible and stabilizing solution is available with the shifted sequence.

V. Computational Methods

In order to apply the proposed suboptimal control scheme to high-speed systems, problem $\bar{\mathbb{P}}_N(x)$ has to be solved very quickly. The following sections describe the computational details necessary for the problem setup and show that it can be efficiently solved even for higher dimensional systems.

A. Tube based robust MPC

Method [17] described in Section IV requires the following elements to be computed: the RPI set $\mathcal{Z}$, the PI set X_f satisfying Assumption A1 in IV.2 and the tightened constraints $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$. Ideally, $\mathcal{Z}$ is taken as the minimal RPI (mRPI) set and X_f as the maximal PI (MPI) set. An explicit representation of these sets can however generally not be computed except in special cases [6], [8]. It is however always possible to compute an invariant outer approximation of the mRPI set and an invariant inner approximation of the MPI set of predefined shape.

The details for computing ellipsoidal and polytopic approximations for $\mathcal{Z}$ and X_f as well as the tightened constraint $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ are outlined in the following. For simplicity, the unconstrained infinite horizon optimal cost is taken as the terminal cost $V_f(x) = \frac{1}{2}x^T Px$ and the corresponding optimal LQ controller is used for K in (11), there are however different ways of choosing a stabilizing affine controller [9], [19]. We denote $A_K \triangleq A + BK$. The polyhedral state and control constraints are $\mathbb{X} = \{x \,|\, Gx \leq f\}$ and $\mathbb{U} = \{u \,|\, Cu \leq d\}$. For simplicity we assume that $W = \{w \,|\, \|w\|_\infty \leq \delta\}$ in the polytopic case and $W = \{w \,|\, \|w\|_2 \leq \delta\}$ in the ellipsoidal case. The results can however be extended to the case where W is a general bounded polytope or a general ellipse, respectively.

1) Polytopic approximation of X_f: The MPI set corresponds to the output admissible set [6]. Using the LQ controller the output admissible set can be finitely determined by $O_\infty = \{x \in \mathbb{R}^n \,|\, GA^t x \leq f, \forall t \in [1, \cdots, t^*]\}$ which can be computed using [10]. The exact calculation might however be prohibitively complex in higher dimensions. An approximation of the MPI set is described in [11] where a positively invariant polytope is derived as the separator of two ellipsoidal sets resulting in a number of QPs, which will still be limited to relatively small dimensions (< 10).

2) Ellipsoidal approximation of X_f: Most approaches use the level set of a quadratic Lyapunov function to derive an invariant ellipsoidal inner approximation of the MPI set [3], [9]. In the considered case a Lyapunov function is readily available with $V_f(x) = x^T Px$. An ellipsoidal approximation of the MPI set can be computed as the biggest level set fulfilling the state and control constraints: $\mathcal{E}_{X_f} = \{x \in \mathbb{R}^n \,|\, x^T Px \leq \gamma_{\max}\}$, where $\gamma_{\max} = \arg\min\{-\gamma \,|\, x^T Px \leq \gamma, x \in \bar{\mathbb{X}}, Kx \in \bar{\mathbb{U}}\}$. This results in a simple LP that can be solved for all dimensions.

3) Polytopic approximation of $\mathcal{Z}$: The method described in [18] can be used to compute an ϵ-outer approximation of the mRPI set. An inner approximation is obtained using a series of projections and is then scaled by a suitable amount: $F(\alpha, s) = (1-\alpha)^{-1} \bigoplus_{i=0}^{s-1} A^i W$. For a given value of s the best scaling parameter α is such that $F(\alpha, s)$ is the smallest outer approximation. The parameters s and α are chosen by means of an iterative procedure that requires the solution of simple algebraic equations for the case where W is defined by box constraints. The computation of the Minkowski sum however limits this method to small dimensions (< 10).

Another variant for the computation of an approximate polyhedral mRPI set that simultaneously optimizes for a piece-wise affine auxiliary control law is presented in [19], which may potentially cause a large increase in the number of optimization variables.

4) Ellipsoidal approximation of $\mathcal{Z}$: An RPI ellipsoidal outer approximation of the mRPI set can be determined using a level set of $V_f(\cdot)$ similar to V-A.2. An extra constraint is added enforcing that the ellipsoid is in fact an RPI set. The minimal ellipsoidal RPI set is then given by $\mathcal{E}_{\mathcal{Z}} = \{x \in \mathbb{R}^n \,|\, x^T Px \leq \gamma_{\min}\}$, where $\gamma_{\min} = \arg\min\{\gamma \,|\, x^T Px \leq \gamma,\ x \in \bar{\mathbb{X}},\ Kx \in \bar{\mathbb{U}},\ x^+ Px^+ \leq \gamma,\ \forall x \text{ s.t. } x^T Px \leq \gamma$ with x^+ in (3)$\}$. This problem can be tranformed into an LMI using the S-procedure [3] and can be efficiently solved for all dimensions.

5) Constraint tightening: In the case that polyhedral approximations are used, the Minkowski differences $\bar{\mathbb{X}} = \mathbb{X} \ominus \mathcal{Z}$, $\bar{\mathbb{U}} = \mathbb{U} \ominus K\mathcal{Z}$ were computed using [10] which requires the computation of a series of LPs. In the case of ellipsoidal approximations the Minkowski differences can be computed similarly by solving a series of LMIs. The robustified constraints can hence be computed rapidly for all dimensions. The tightened constraints $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ are polytopic in both cases and of the same complexity as the origional constraints $\mathbb{X}$ and $\mathbb{U}$.

The question is then of course, which approximation to use, a polyhedral or an ellipsoidal one. Since the above described computations are performed offline the computation times are not crucial. In general however polyhedral approximations can only be computed for smaller systems, approximately 6-7 dimensions. Whereas ellipsoidal approximations might be slightly more conservative in this range, they represent the better if not the only choice for higher dimensions. If the considered system is in the range where a polyhedral approximation can be computed, an explicit solution [1] of the MPC problem should as well be considered since it allows for extremely fast computation times in lower dimensions. Another advantage of ellipsoidal invariant sets is the fact that the number of constraints introduced by the condition $x \in \bar{x}_0 \oplus \mathcal{Z}$ in (10) is fixed, whereas polytopic invariant sets may add a large number of constraints leading to slower computation times and excessive memory requirements. It is hence *always* possible to compute the invariant sets and tightened constraints using ellipsoidal approximations even for significant problem sizes. The offline set computations were carried out using the YALMIP toolbox [14].

B. Optimization

In the case that $\mathcal{Z}$ and X_f are polyhedral sets, the robust MPC problem $\bar{\mathbb{P}}_N(x)$ can be written as a QP of the form (6), where the vector **z** also includes the initial state $\bar{x}_0$. If $\mathcal{Z}$ and X_f are represented by ellipsoidal sets, problem $\bar{\mathbb{P}}_N(x)$ can be transformed into a quadratically constrained QP (QCQP), that is the QP (6) with two extra quadratic constraints on the initial and the terminal state: $(\bar{x}_0 - x)^T P(\bar{x}_0 - x) \leq \gamma_{\text{min}}$ and $\bar{x}_N^T P \bar{x}_N \leq \gamma_{\text{max}}$. Both problems, the QP as well as the QCQP, can be efficiently solved using interior-point methods. A feasible start primal barrier interior-point method [4] was chosen to realize the τ-RT control law in (12). Standard interior-point methods maintain feasibility but since they use a modified cost for the optimization the actual MPC cost in (4) could increase during the iteration steps. An extra constraint enforcing a non-increasing MPC cost $V_N(x^+, \mathbf{u}_{\text{RT}}(x^+, \tau)) \leq V_N(x^+, \mathbf{u}_{\text{RT}}(x^+, 0))$ and thereby ensuring the controller properties defined in Definition III.3 has to be explicitly added to the optimization problem. The time constraint is realized by performing a fixed number of optimization steps.

The QP as well as the QCQP result in very sparse problem structures similar to those described e.g. in [23], [24] with a dense band from the decrease constraint, forming a so-called arrow structure. A solver exploiting this particular structure can solve the QP as well as the QCQP extremely quickly. The authors developed a simple custom solver written in C++, based closely on that given in [23], for the real-time method proposed in this paper that results in computation times in the range of milliseconds (see results in Section VI). This offers the possibility to apply real-time robust MPC to high-speed systems with the big advantage that stability is always guaranteed and the available computation time is used to improve the solution and increase the performance. The simulations were executed on a 2.8GHz AMD Opteron running Linux using a single core.

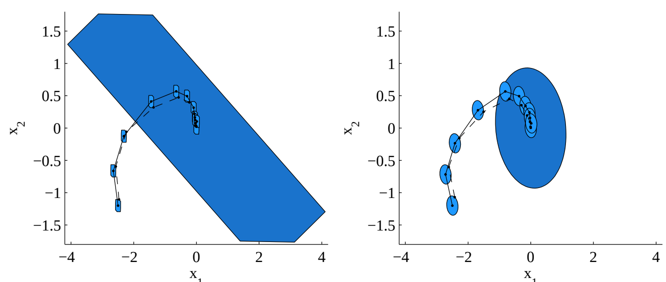
VI. Results & Examples

A. Illustrative Example

We first illustrate the method and its components using the following 2D system:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u + w \ , \tag{13}$$

with a prediction horizon $N = 5$ and the constraints $\|x\|_\infty \leq 5$ and $\|u\|_\infty \leq 1$ on the states and control inputs, $Q = I$ and $R = 1$. The disturbance is assumed to be bounded in $W = \{w \| \|w\|_{2/\infty} \leq 0.025\}$. The terminal cost function $V_f(x)$ is taken as the unconstrained infinite horizon optimal value function for the nominal system with $P = \left[\begin{smallmatrix} 1.8085 & 0.2310 \\ 0.2310 & 2.6489 \end{smallmatrix}\right]$ and $\kappa_f(x) = Kx$ is the corresponding optimal LQ controller.



(a) Polyhedral invariant sets (b) Ellipsoidal invariant sets

Fig. 1. State trajectories for example (13). The dash-dotted line is the actual trajectory $x(i)$ and the solid line represents the trajectory of tube centers $\bar{x}_0(x(i))$. The terminal set X_f is shown as well as the sets $\bar{x}_0(x(i)) \oplus \mathcal{Z}$.

Polytopic and ellipsoidal approximations of $\mathcal{Z}$ and X_f were calculated as described in Section V. In Figure 1 a state trajectory $x(j)$ starting from $x(0) = [-2.5, 1.2]^T$ is illustrated for a sequence of extreme disturbances for the ellipsoidal and the polytopic case together with the corresponding trajectory of tube centers $\bar{x}_0(x(j))$ and sets $\bar{x}_0(x(i)) \oplus \mathcal{Z}$ and X_f. The ellipsoidal terminal set is significantly smaller and the ellipsoidal set $\mathcal{Z}$ slightly bigger than in the polytopic case which is due to the fact that the shape of the ellipsoid was fixed. Despite the different set sizes, the two approaches however result in a very similar region of attraction.

B. Oscillating masses

The oscillating masses example described in [23] is chosen to examine our real-time method and evaluate it against that proposed in [23]. The considered model has $n = 12$ states and $m = 3$ inputs. Ellipsoidal invariant sets were computed for X_f and $\mathcal{Z}$, polytopic approximations

Tab. I

CLOSED-LOOP PERFORMANCE DETERIORATION IN %

$k_{\max}$	1	2	3	4	5	6	7	8
$\triangle J_{cl}$	1.39	1.32	1.10	0.88	0.70	0.55	0.44	0.33

cannot be computed for this problem size. For a horizon of $N = 30$ this results in a QCQP with 462 optimization variables and 1238 constraints. A random disturbance sequence with $\|w\|_2 \leq 0.25$ is acting on the system, which corresponds to 20% of the actuators control range. The method was run with the same optimization parameters given in [23] and a fixed number of optimization steps $k_{\max} = 5$ in order to have a direct comparison with the reported results. Our solver was able to compute 5 Newton steps in 6msec (averaged over 100 runs) and hereby achieves timings that are essentially equal to those reported in [23]. We can hence achieve the same fast sampling rates using the robust MPC design and achieve guaranteed feasibility and stability. Both methods provide a closed-loop performance deterioration $\triangle J_{cl} < 1\%$ taken over a large number of sample points, where $\triangle J_{\text{cl}} = \frac{\sum_{i=0}^{\infty}(l(x_i,\widehat{\kappa}(x_i)) - l(x_i,\kappa(x_i)))}{\sum_{i=0}^{\infty} l(x_i,\kappa(x_i))}$, $\widehat{\kappa}(x)$ denotes the suboptimal controller obtained after $k_{\max}$ iterations and $\kappa(x)$ the optimal controller of the considered method. $\triangle J_{\text{cl}}$ is estimated by simulating the trajectory for a long time period.

After establishing that the proposed approach performs equally well for the particular example it is important to note that one would choose the optimization parameters differently for our method. A long horizon was taken in [23] since no stability guarantee is provided. This is however not necessary using the presented approach due to its a-priori stability guarantee. We therefore reduce the horizon to $N = 10$ resulting in a QCQP with 162 decision variables and 398 constraints and investigate the effect of the number of allowed iterations on the closed-loop performance deterioration, reported in Table I. It is important to note that the performance as well as the region of attraction are not affected by the reduction of the horizon to $N = 10$. One Newton step can now be computed in 0.3msec. Consequently the real-time MPC method with $k_{\max} = 5$ iterations can be implemented with a sampling time of 2msec resulting in a controller rate of 500Hz. It is remarkable that the one step solution still shows considerably low performance loss. Since stability is guaranteed at all times one could therefore choose $k_{\max} = 1$ in order to achieve extremely low computation times of 0.3msec in trade for lower performance.

C. Large Example

A random example with $n = 30, m = 8$ and $N = 10$ was generated resulting in an optimization problem with 410 optimization variables and 1002 constraints. Ellipsoidal invariant sets were computed for X_f and $\mathcal{Z}$. We recorded the computation time for the invariant sets and tightened constraints which were computed offline in only 19 seconds. The robust MPC problem with $k_{\max} = 5$ Newton iterations was solved in 10msec allowing for an implementation of the MPC controller at a sampling rate of 100Hz.

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