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Elementary abelian subgroups in p-groups with a cyclic derived subgroup

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ABSTRACT

Let p be an arbitrary prime number and let P be a finite p-group. Let $\mathcal{A}_p(P)$ be the partially ordered set (poset for short) of all non-trivial elementary abelian subgroups of P ordered by inclusion and let $\mathcal{A}_p(P)_{\geqslant 2}$ be the poset of all elementary abelian subgroups of P of rank at least 2. In [Serge Bouc, Jacques Thévenaz, The poset of elementary abelian subgroups of rank at least 2, Monogr. Enseign. Math. 40 (2008) 41–45], Bouc and Thévenaz proved that $\mathcal{A}_p(P)_{\geqslant 2}$ has the homotopy type of a wedge of spheres (of possibly different dimensions). The general objective of this paper is to obtain more refined information on the homotopy type of the posets $\mathcal{A}_p(P)$ and $\mathcal{A}_p(P)_{\geqslant 2}$. We give three different kinds of results in this direction. Firstly, we compute exactly the homotopy type of $\mathcal{A}_p(P)_{\geqslant 2}$ when P is a p-group with a cyclic derived subgroup, that is we give the number of spheres occurring in each dimension in $\mathcal{A}_p(P)_{\geqslant 2}$.
Secondly, we compute a sharp upper bound on the dimension of the spheres occurring in $\mathcal{A}_p(P)_{\geqslant 2}$ and give information on the p-groups for which this bound is reached.
Thirdly, we determine explicitly for which of the p-groups with a cyclic derived subgroup the poset $\mathcal{A}_p(P)$ is homotopically Cohen–Macaulay.

1. Introduction

If G is a finite group and p is an arbitrary prime number, it is standard to denote by $\mathcal{A}_p(G)$ the partially ordered set (*poset* for short) of all elementary abelian p-subgroups of G ordered by inclusion. As Quillen [12] pointed out, if G is a finite Chevalley group then $\mathcal{A}_p(G)$ (or rather its associated geometric realization) has the homotopy type of the Tits building of G, and hence has the homotopy

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type of a wedge of spheres. A natural related question, attributed to Thévenaz by Pulkus and Welker in [11], is the following.

Question 1.1. If G is a finite group, does $\mathcal{A}_p(G)$ have the homotopy type of a wedge of spheres?

A negative answer to this question was given by Shareshian [13], who proved that $\tilde{H}_2(\mathcal{A}_3(S_{13}))$ is not torsion-free (note, however, that a part of his proof relies on computer calculations). In [6], Fumagalli claimed that Question 1.1 has a positive answer if the group G is solvable. Unfortunately, his proof relies on a result that turns out to be false (see [2]), so that to our knowledge the following question seems to remain opened.

Question 1.2. If G is solvable, does $\mathcal{A}_p(G)$ have the homotopy type of a wedge of spheres (of possibly different dimensions)?

In [11], Pulkus and Welker showed that for solvable groups G, the study of $\mathcal{A}_p(G)$ can be reduced to the study of upper intervals $\mathcal{A}_p(C_G(A))_{>A}$. According to [11], the homotopy type of these upper intervals is not clear, even for p-groups. A first result in this direction was given by Bouc and Thévenaz in [4], where they study the poset $\mathcal{A}_p(P)_{\geqslant 2}$ of all elementary abelian subgroups of P of rank at least 2. They proved that for any p-group P, the poset $\mathcal{A}_p(P)_{\geqslant 2}$ has the homotopy type of a wedge of spheres. This is related to upper intervals by the fact that for any $A \in \mathcal{A}_p(P)$ with $|A| = p$, there is a homotopy equivalence $\mathcal{A}_p(P)_{>A} \simeq \mathcal{A}_p(C_P(A))_{\geqslant 2}$. Computer calculations led Bouc and Thévenaz to raise the following questions.

Question 1.3 *(Bouc, Thévenaz).* Let P be a p-group. Do the spheres occurring in $\mathcal{A}_p(P)_{\geqslant 2}$ all have the same dimension if p is odd? Does one get only two consecutive dimensions if $p = 2$?

In Section 4, we review the results of Bouc and Thévenaz and show how they can be used effectively in some cases to determine the homotopy type of $\mathcal{A}_p(P)_{\geqslant 2}$. More precisely, we compute the number of spheres occurring in each dimension in the homotopy type of $\mathcal{A}_p(P)_{\geqslant 2}$ when P is a p-group with a cyclic Frattini subgroup. As a consequence, we obtain that Question 1.3 has a positive answer if P has a cyclic derived subgroup.

In Section 5, we give first a sharp upper bound on the dimension of the spheres occurring in the homotopy type of $\mathcal{A}_p(P)_{\geqslant 2}$. We consider then the problem of describing the p-groups for which this bound is reached. Although we obtain a complete satisfactory answer if the p-valuation of the order of the group is odd, the even case seems to be more difficult.

Upper intervals in $\mathcal{A}_p(G)$ also play a key role in determining whether $\mathcal{A}_p(G)$ is homotopically Cohen–Macaulay. This is roughly speaking a recursive sphericity condition, in the sense that not only the poset itself, but also the link of each k-simplex, must have the homotopy type of a wedge of spheres (of prescribed dimensions). In Section 6, we determine for which of the p-groups with a cyclic derived subgroup, the poset $\mathcal{A}_p(P)$ is homotopy Cohen–Macaulay. We will also show more precisely at the end of this section how our results improve former results on this question.

Most of the time, in order to prove results concerning $\mathcal{A}_p(P)$ or $\mathcal{A}_p(P)_{\geqslant 2}$ for p-groups with a cyclic derived subgroup, it is enough to consider p-groups with a cyclic Frattini subgroup. The advantage lies in the fact that these groups are classified and their structure is easy to describe. We recall in Section 3 the parts of the classification that are relevant for our purpose.

We briefly recall some notation and terminology in Section 2.

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2. Notation

For the convenience of the reader, we introduce in this section the terminology we will use in this paper. We begin with some notation in group theory. Most of our notation is standard and the reader can refer to [8] if something is left unexplained.

If G is a group, we denote by $r_p(G)$ the p-rank of G and if P is a p-group, we simply write $r(P)$ for $r_p(P)$.

Let P be a p-group, we denote by $\Omega_1(P)$ the subgroup of P generated by elements of order at most p, by P' the derived subgroup of P and by $\Phi(P)$ the Frattini subgroup of P, that is the subgroup of P generated by P' and all p-th powers of elements in P.

Remark 2.1. The key property to keep in mind is that the quotient $P/\Phi(P)$ is an elementary abelian p-group and hence can be viewed as a vector space over the field $\mathbb{F}_p$ with p elements. We will frequently use this fact implicitly in this paper.

For two p-groups P_1 and P_2 with cyclic centers, we denote by $P_1 * P_2$ the central product of P_1 and P_2. The amalgamation is performed by identifying the two unique central subgroups of order p of the centers. In general, two different identifications will yield non-isomorphic central products. However, if Z is a central subgroup of a group P such that any automorphism of Z is the restriction of an automorphism of the whole group P, then the central products performed over Z are all isomorphic. This condition will always be satisfied for the groups studied in this work.

For $\ell \geqslant 1$, we use the notation $P^{*\ell}$ for the iterated central product defined by $P^{*\ell} = P * P^{*(\ell-1)}$ with $P^{*1} = P$. We also make the convention $P^{*0} = 1$.

For $k \geqslant 1$, we denote by C_{p^k} the cyclic p-group of order p^k. As usual, we denote by D_8, respectively Q_8, the dihedral group, resp. quaternion group, of order 8. For $m > 1$, we write respectively $D_{2^{m+2}}$, $SD_{2^{m+2}}$ and $Q_{2^{m+2}}$ for the dihedral, respectively semi-dihedral and quaternion group of order 2^{m+1}.

Let $m > 1$. Following [1], we denote by $D^+_{2^{m+3}}$ and $Q^+_{2^{m+1}}$ the 2-groups of order 2^{m+3} defined by the following presentations.

$$D^+_{2^{m+3}} = \langle a, b, u \mid a^2 = b^2 = u^{2^{m+1}} = 1,\ [a,b] = 1,\ [a,u] = u^{2^m},\ [b,u] = u^{-2}\rangle,$$
$$Q^+_{2^{m+3}} = \langle a, b, u \mid a^2 = u^{2^{m+1}} = 1,\ b^2 = u^{2^m},\ [a,b] = 1, [a,u] = u^{2^m},\ [b,u] = u^{-2}\rangle.$$

For an odd prime number p and $\ell \geqslant 1$, we denote by $X_{p^{2\ell+1}}$ the extraspecial p-group of order $p^{2\ell+1}$ and exponent p. One has in particular $X_{p^{2\ell+1}} = X^{*\ell}_{p^3}$, where X_{p^3} is the non-abelian p-group of order p^3 and exponent p.

For an arbitrary prime number $p \geqslant 2$ and $\ell \geqslant 1$, we define the extraspecial p-group of type I and order $p^{2\ell+1}$ as the group $X_{p^{2\ell+1}}$ if p is odd and as the central product $D^{*\ell}_8$ if $p = 2$.

Let P be an extraspecial p-group of type I and order $p^{2\ell+1}$ for some prime number $p \geqslant 2$. The center of P is cyclic of order p and given a generator z of $Z(P)$, one can find generators $x_1, y_1, \ldots, x_\ell, y_\ell$ of P satisfying the following conditions.

$$x_i^p = y_i^p = 1, \quad \text{for } 1 \leqslant i \leqslant \ell,$$
$$[x_i, x_j] = [x_i, y_j] = [y_i, y_j] = 1, \quad \text{for } 1 \leqslant i \neq j \leqslant \ell,$$
$$[x_i, y_i] = z, \quad \text{for } 1 \leqslant i \leqslant \ell.$$

Such generators $x_1, y_1, \ldots, x_\ell, y_\ell$ of P will be called symplectic generators of P. Our choice of terminology is motivated by the fact that such a set of generators induces a symplectic basis on $P/Z(P)$ relatively to the non-degenerate alternating form induced by taking commutators.

Let us recall also some poset-related definitions and notation. Our terminology is standard and more details can be found in Quillen's paper [12] or Wachs' survey [14].

We will use the same notation for a poset, its order complex and its geometric realization. If we say that a poset has a certain topological property, we mean that its geometric realization has this property.

We will say that a poset is discrete if its partial order is given by the identity.

For an element x of a poset $\mathcal{Q}$ we denote by $\mathcal{Q}_{\leqslant x}$ the subposet of $\mathcal{Q}$ defined by

$$\mathcal{Q}_{\leqslant x} = \{x' \in \mathcal{Q} \mid x' \leqslant x\}.$$

The posets $\mathcal{Q}_{<x}$, $\mathcal{Q}_{\geqslant x}$ and $\mathcal{Q}_{>x}$ are defined similarly. For $x_1 < x_2$, the open interval (x_1, x_2) is the poset $\mathcal{Q}_{>x_1} \cap \mathcal{Q}_{<x_2}$.

A poset $\mathcal{Q}$ is said to be conically contractible if there is a poset map $f : \mathcal{Q} \to \mathcal{Q}$ and an element $x_0 \in \mathcal{Q}$ such that

$$x \leqslant f(x) \geqslant x_0, \quad \text{for all } x \in \mathcal{Q}.$$

Recall that for two elements x, y of a poset $\mathcal{Q}$, the join $x \vee y$ of x and y in $\mathcal{Q}$ is an element of $\mathcal{Q}$ greater than or equal to both x and y that is less than all other such elements. This element $x \vee y$ may not exist, but if it exists it is unique. An element $x_0 \in \mathcal{Q}$ is a conjunctive element if for each $x \in \mathcal{Q}$ the join $x \vee x_0$ exists in $\mathcal{Q}$.

Let us emphasize here the fact that if a poset $\mathcal{Q}$ has a conjunctive element, then $\mathcal{Q}$ is conically contractible, hence contractible. This is a direct consequence of Quillen's fiber lemma [12] and the fact that the geometric realization of a poset with a top element is a cone, hence is contractible.

The suspension of a poset $\mathcal{Q}$ is the poset $\Sigma\mathcal{Q} = \mathcal{Q} \cup \{o_1, o_2\}$, where o_1 and o_2 are smaller than every element of $\mathcal{Q}$ but there is no order relation between o_1 and o_2.

3. p-groups with a cyclic Frattini subgroup

Let P be a p-group. The elementary abelian subgroups of P have exponent p, so that

$$\mathcal{A}_p(P) = \mathcal{A}_p\big(\Omega_1(P)\big).$$

This allows one to assume that $P = \Omega_1(P)$, i.e. that P is generated by elements of order p. In this case, the abelian group P/P' is also generated by elements of order p, hence has exponent p. It follows that P/P' is elementary abelian and thus $\Phi(P)$ is equal to P'.

This fact is particularly useful in order to describe $\mathcal{A}_p(P)$ when P has a cyclic derived subgroup. Indeed, one can then assume that P is a p-group with a cyclic Frattini subgroup, the advantage being that these groups are classified and can be rather easily described. This knowledge of the structure of these groups is essential to most of the calculations performed in this paper, especially in Section 4 and Section 6. There is however no need for a deep understanding of how this classification can be performed and we will mostly state the results without further details. The interested reader can refer to [3] for a complete account of this classification.

Lemma 3.1. *If P is p-group with a cyclic Frattini subgroup, then $P = Q \times E$, where E is elementary abelian and both $\Phi(Q)$ and $Z(Q)$ are cyclic.*

Proof. Let $Z = \Omega_1(Z(P))$ and let $U = \Phi(P) \cap Z$. Since Z is elementary abelian, we can choose E such that $Z = U \times E$. Since $P/\Phi(P)$ is elementary abelian, we can choose a subgroup Q of P such that

$$P/\Phi(P) = \Phi(P)Z/\Phi(P) \times Q/\Phi(P).$$

It is not difficult to see that Q and E have the desired properties. $\square$

We wish to describe the p-groups with a cyclic Frattini subgroup. In view of the preceding lemma, we may assume that the center itself is cyclic. When $\Phi(P)$ is further assumed to be central, these groups behave very similarly to extraspecial p-groups, in the sense that there is a natural geometry on $P/Z(P)$ induced by taking commutators and p-th powers. The situation is however very different when $\Phi(P)$ is not central. Note that the following result, which can be found in [1], shows that this happens only for $p=2$.

Lemma 3.2. *Let p be and odd prime number and let P be a p-group. If $\Phi(P)$ is cyclic, then $\Phi(P)$ is also central.*

Calculations are indeed far more complicated when the Frattini subgroup is not central and this case will occupy most of this paper. For this reason, we will frequently distinguish between these two cases, as in the next proposition.

Proposition 3.3. *Let P be a non-abelian p-group with both $\Phi(P)$ and $Z(P)$ cyclic.*

1. *If $\Phi(P)$ is central and $\Omega_1(P)=P$, then P is isomorphic either to $X_{p^3}^{*\ell}$, $D_8^{*\ell}$, $D_8^{*\ell} * C_4$ or $D_8^{*\ell} * Q_8$, for some $\ell \geqslant 1$.*
2. *If $\Phi(P)$ is not central, then $p=2$ and P is isomorphic to a central product $D_8^{*\ell} * S$, for some $\ell \geqslant 0$ and with S is isomorphic to one of the following groups, all with $m>1$.*

$$D_{2^{m+2}},\, Q_{2^{m+2}},\, SD_{2^{m+2}},\, D_{2^{m+2}} * C_4,\, SD_{2^{m+2}} * C_4,\, D^{+}_{2^{m+3}},\, Q^{+}_{2^{m+3}},\, D^{+}_{2^{m+3}} * C_4.$$

Moreover, $\Omega_1(P)=P$ unless P is isomorphic to $SD_{2^{m+2}}$ or $Q_{2^{m+2}}$.

4. The poset $\mathcal{A}_p(P)_{\geqslant 2}$

If P is a p-group, we denote by $\mathcal{A}_p(P)_{\geqslant 2}$ the poset of all elementary abelian subgroups of P of rank at least 2. In this section, we describe the homotopy type of $\mathcal{A}_p(P)_{\geqslant 2}$ when P is a p-group with a cyclic derived subgroup. Our main tool is a wedge decomposition of $\mathcal{A}_p(P)_{\geqslant 2}$ due to Bouc and Thévenaz [4]. We begin this section by reviewing this result and some of its consequences on the structure of $\mathcal{A}_p(P)$.

Proposition 4.1 *(Bouc, Thévenaz). Let P be a p-group with a cyclic center and let $Z=\Omega_1(Z(P))$. Suppose that P contains a normal elementary abelian subgroup E_0 of rank 2 and let $M=C_P(E_0)$. There is then a homotopy equivalence*

$$\mathcal{A}_p(P)_{\geqslant 2} \simeq \bigvee_{F\in\mathcal{F}} \Sigma \mathcal{A}_p\big(C_M(F)\big)_{\geqslant 2}, \tag{1}$$

where $\mathcal{F}=\{F\in\mathcal{A}_p(P)_{>Z} \mid M\cap F=Z\}$ and Σ is the suspension operator. In particular, for all $k\geqslant 0$ there is an isomorphism

$$\tilde{H}_k\big(\mathcal{A}_p(P)_{\geqslant 2}\big) \cong \bigoplus_{F\in\mathcal{F}} \tilde{H}_{k-1}\big(\mathcal{A}_p\big(C_M(F)\big)_{\geqslant 2}\big). \tag{2}$$

Proof. See [4] for the original proof and see also [2] for a slight generalization. □

Remark 4.2. In the context of the preceding proposition, considering the action of the p-group P on the elementary abelian normal subgroup E_0, one can see that $M=C_P(E_0)$ has index p in P.

Let us mention here for further reference that the existence of the normal subgroup E_0 in the previous proposition is guaranteed in almost all cases by the following standard result.

Lemma 4.3. *If P is a p-group with no non-cyclic abelian normal subgroups, then either P is cyclic, or $p=2$ and P is isomorphic to $D_{2^{m+2}}$, $m>1$, $Q_{2^{m+2}}$, $m\geqslant 1$, or $SD_{2^{m+2}}$, $m>1$.*

Proof. See for example Theorem 5.4.10 in [8]. □

Proposition 4.1 is the key ingredient in the proof of the following general result on the homotopy type of $\mathcal{A}_p(P)_{\geqslant 2}$. See [4] for details.

Proposition 4.4 *(Bouc, Thévenaz). For any p-group P, the poset $\mathcal{A}_p(P)_{\geqslant 2}$ has the homotopy type of a wedge of spheres.*

To relate this result with upper intervals in $\mathcal{A}_p(P)$, one can use the following standard fact.

Lemma 4.5. *Let P be a p-group and let A be a central elementary abelian subgroup of P of rank r. There is a homotopy equivalence*

$$\mathcal{A}_p(P)_{>A} \simeq \mathcal{A}_p(P)_{\geqslant r+1}.$$

Proof. This follows from Quillen's fiber lemma [12]. To be a little more precise, the inclusion map $\mathcal{A}_p(P)_{>A} \hookrightarrow \mathcal{A}_p(P)_{\geqslant r+1}$ has a homotopy inverse given by the map sending $B \in \mathcal{A}_p(P)_{\geqslant r+1}$ to $BA \in \mathcal{A}_p(P)_{>A}$. □

Corollary 4.6. *If A is a subgroup of order p of a p-group P, then $\mathcal{A}_p(P)_{>A}$ has the homotopy type of a wedge of spheres.*

Proof. By Lemma 4.5, the poset $\mathcal{A}_p(P)_{>A} = \mathcal{A}_p(C_P(A)_{>A})$ is homotopy equivalent to the poset $\mathcal{A}_p(C_P(A))_{\geqslant 2}$. □

Recall that the objective of this section is to describe the homotopy type of the poset $\mathcal{A}_p(P)_{\geqslant 2}$ for all p-groups with a cyclic derived subgroup. In view of the reductions made at the beginning of Section 3, we will restrict our attention to p-groups generated by elements of order p and with a cyclic Frattini subgroup. We will also assume that $Z(P)$ is cyclic, since otherwise $Z = \Omega_1(Z(P))$ is a conjunctive element in the poset $\mathcal{A}_p(P)_{\geqslant 2}$, which is then contractible.

When p is odd, these assumptions are only satisfied if P is extraspecial of type I (see Proposition 3.3). In this situation, $\mathcal{A}_p(P)_{\geqslant 2}$ is homotopy equivalent to $\mathcal{A}_p(P)_{>Z(P)}$ and the homotopy type of this later poset can be determined by a standard argument using buildings (see for example [12]). We give here an alternative proof based on the wedge decomposition provided in Proposition 4.1.

Proposition 4.7. *Let p be an odd prime number and let $P = X_{p^{2\ell+1}}$ be an extraspecial p-group of type I and order $2^{2\ell+1}$, $\ell \geqslant 1$. Then $\mathcal{A}_p(P)_{\geqslant 2}$ has the homotopy type of a wedge of p^{ℓ^2} spheres of dimension $\ell - 1$.*

Proof. If $\ell = 1$, i.e. $P = X_{p^3}$, then $\mathcal{A}_p(P)_{\geqslant 2}$ is a discrete poset consisting of $p+1$ points, hence has the homotopy type of a wedge of p spheres of dimension 0.

Suppose now $\ell > 1$. As a consequence of Lemma 4.3, there exists in P a normal elementary abelian subgroup E_0 of rank 2. Let $M = C_P(E_0)$, then Proposition 4.1 gives a homotopy equivalence

$$\mathcal{A}_p(P)_{\geqslant 2} \simeq \bigvee_{F \in \mathcal{F}} \Sigma \mathcal{A}_p\big(C_M(F)\big)_{\geqslant 2}, \tag{3}$$

where $\mathcal{F} = \{F \in \mathcal{A}_p(P)_{>Z(P)} \mid M \cap F = Z(P)\}$. Since P has exponent p, the set $\mathcal{F}$ corresponds bijectively to the set of all 1-dimensional complements of $M/Z(P)$ in $P/Z(P)$ and hence $|\mathcal{F}| = p^{2\ell-1}$.

Furthermore, the group E_0F is isomorphic to X_{p^3} (consider the action of F on E_0 by conjugation) and $P = E_0F * C_M(F)$. It follows that $C_M(F)$ is extraspecial of type I and order $p^{2\ell-1}$. By a recursive argument, we have now that $\mathcal{A}_p(C_M(F))_{\geqslant 2}$ has the homotopy type of a wedge of $p^{(\ell-1)^2}$ spheres of dimension $\ell-2$.

Putting this information in (3), we see that $\mathcal{A}_p(P)_{\geqslant 2}$ has the homotopy type of a wedge of $p^{2\ell-1} \cdot p^{(\ell-1)^2} = p^{\ell^2}$ spheres of dimension $\ell-1$ (the dimension of the spheres is increased by the suspension operator) and the proposition is proved. □

In view of the reductions made above, the preceding proposition closes the discussion for p odd. This is the content of the following corollary.

Corollary 4.8. *Let p be an odd prime number. If P is a p-group with a cyclic derived subgroup, then $\mathcal{A}_p(P)_{\geqslant 2}$ is contractible, unless $\Omega_1(P)$ is extraspecial of type I, say $\Omega_1(P) = X_{p^{2\ell+1}}$, in which case $\mathcal{A}_p(P)_{\geqslant 2}$ is homotopy equivalent to a wedge of p^{ℓ^2} spheres of dimension $\ell-1$.*

When $p=2$ the situation is a little bit more complicated. In similarity with the case p odd, an argument using buildings and the Solomon–Tits theorem can be used when $\Phi(P)$ is central in P. Here also Proposition 4.1 can be used instead. The proof is very similar to the proof of Proposition 4.7. We have however to be more careful with the order of the elements.

Proposition 4.9.

a) *If $P = D_8^{*\ell}$ with $\ell \geqslant 1$, then $\mathcal{A}_2(P)_{\geqslant 2}$ has the homotopy type of a wedge of $2^{\ell(\ell-1)}$ spheres of dimension $\ell-1$.*
b) *If $P = D_8^{*\ell} * C_4$ with $\ell \geqslant 0$, then $\mathcal{A}_2(P)_{\geqslant 2}$ has the homotopy type of a wedge of 2^{ℓ^2} spheres of dimension $\ell-1$.*
c) *If $P = D_8^{*\ell} * Q_8$ with $\ell \geqslant 0$, then $\mathcal{A}_2(P)_{\geqslant 2}$ has the homotopy type of a wedge of $2^{\ell(\ell+1)}$ spheres of dimension $\ell-1$.*

Proof. Suppose first $P = D_8^{*\ell}$. If $\ell = 1$, i.e. $P = D_8$ the result is clear. Suppose $\ell > 1$ and let z be a generator of $Z(P)$ and $x_1, y_1, \dots, x_\ell, y_\ell$ be symplectic generators of P. Recall that they are elements of order 2 satisfying the following conditions.

$$[x_i, x_j] = [x_i, y_j] = [y_i, y_j] = 1, \quad \text{for } 1 \leqslant i \neq j \leqslant \ell,$$
$$[x_i, y_i] = z, \quad \text{for } 1 \leqslant i \leqslant \ell.$$

Let E_0 be the elementary abelian subgroup of P generated by z and y_1. It is clear from the above conditions that E_0 is normal in P. Note that $M = C_P(E_0)$ has the following generators.

$$M = \langle y_1, x_2, y_2, \dots, x_\ell, y_\ell \rangle.$$

Proposition 4.1 gives a homotopy equivalence

$$\mathcal{A}_2(P)_{\geqslant 2} \simeq \bigvee_{F \in \mathcal{F}} \Sigma \mathcal{A}_2\big(C_M(F)\big)_{\geqslant 2}, \tag{4}$$

where $\mathcal{F} = \{F \in \mathcal{A}_p(P)_{>Z(P)} \mid M \cap F = Z(P)\}$. A subgroup $F \in \mathcal{F}$ is generated by z and an element $x_1 y_1^k x$ with $k \in \{0, 1\}$ and x in the subgroup D of P generated by the elements x_j, y_j for $j \geqslant 2$. Since F is elementary abelian, we have

$$1 = \big(x_1 y_1^k x\big)^2 = z^k x^2.$$

We must have therefore $k=0$ if x has order 2 and $k=1$ if x has order 4. It follows that the elements $x_1 y_1^k x$ of order 2 are in bijection with the elements of D. Furthermore, two such elements $x_1 y_1^k x$ and $x_1 y_1^{k'} x'$ define the same subgroup in $\mathcal{F}$ if and only if they differ by an element of $Z(P)$. We have therefore $|\mathcal{F}| = |D/Z(D)| = 2^{2(\ell-1)}$.

It is not difficult to see that in all cases the centralizer $C_M(F)$ is isomorphic to $D_8^{*(\ell-1)}$.

Putting this in Eq. (4) shows that the poset $\mathcal{A}_2(D_8^{*\ell})_{\geqslant 2}$ has the homotopy type of a wedge of $2^{2(\ell-1)}$ copies of the suspension of the poset $\mathcal{A}_2(D_8^{*(\ell-1)})_{\geqslant 2}$. The homotopy type follows then from an induction argument.

The proofs for $P = D_8^{*\ell} * C_4$ and $P = D_8^{*\ell} * Q_8$ follow exactly the same pattern and (technical) details are left to the reader. Let us however mention that for $P = D_8^{*\ell} * C_4$, there are $2^{2\ell-1}$ subgroups in $\mathcal{F}$ all with a centralizer isomorphic to $D_8^{*(\ell-1)} * C_4$. For $P = D_8^{*\ell} * Q_8$, there are $2^{2\ell}$ subgroups in $\mathcal{F}$, all with a centralizer isomorphic to $D_8^{*(\ell-1)} * Q_8$. $\square$

If the Frattini subgroup of P is not central, then it follows from the reductions made above and Proposition 3.3 that P is isomorphic to a central product $D_8^{*\ell} * S$, where $\ell \geqslant 0$ and S is one of the following groups.

$$D_{2^{m+2}},\ SD_{2^{m+2}},\ Q_{2^{m+2}},\ D^+_{2^{m+3}},\ Q^+_{2^{m+3}},\ D^+_{2^{m+3}} * C_4,\ D_{2^{m+2}} * C_4,\ SD_{2^{m+2}} * C_4. \tag{5}$$

Note that in order to be consistent with our assumption that $P = \Omega_1(P)$, we should not consider the two cases $P = SD_{2^{m+2}}$ and $P = Q_{2^{m+2}}$. We have however chosen to include them in the statement of the following proposition for convenience and clarity.

Proposition 4.10. *Let $m > 1$ and $\ell \geqslant 0$. Let $P = D_8^{*\ell} * S$, where S is one of the groups in the list (5).*

1. *If $S = SD_{2^{m+2}}$, then:*
   - *If $\ell = 0$, $\mathcal{A}_2(P)_{\geqslant 2}$ has the homotopy type of a wedge of $2^{m-1} - 1$ spheres of dimension 0.*
   - *If $\ell \geqslant 1$, $\mathcal{A}_2(P)_{\geqslant 2}$ has the homotopy type of a wedge of $2^{\ell^2}(2^{m-2}(2^\ell + 1) - 1)$ spheres of dimension ℓ and $2^{\ell^2+m-2}(2^\ell - 1)$ spheres of dimension $\ell - 1$.*
2. *If $S = D^+_{2^{m+3}} * C_4$, then $\mathcal{A}_2(P)_{\geqslant 2}$ is contractible.*
3. *Otherwise, $\mathcal{A}_2(P)_{\geqslant 2}$ has the homotopy type of a wedge of N spheres of dimension d, where N and d take the following values.*
   - *$N = 2^{\ell^2}(2^{m-1}(2^\ell + 1) - 1)$ and $d = \ell$ if $S = D_{2^{m+2}}$.*
   - *$N = 2^{\ell^2}(2^{m-1}(2^\ell - 1) + 1)$ and $d = \ell - 1$ if $S = Q_{2^{m+2}}$.*
   - *$N = 2^{(\ell+1)^2+m-2}$ and $d = \ell$ if $S = D^+_{2^{m+3}}$.*
   - *$N = 2^{(\ell+1)^2+m-2}$ and $d = \ell$ if $S = Q^+_{2^{m+3}}$.*
   - *$N = 2^{(\ell+1)^2+m-1}$ and $d = \ell$ if $S = D_{2^{m+2}} * C_4$.*
   - *$N = 2^{(\ell+1)^2+m-1}$ and $d = \ell$ if $S = SD_{2^{m+2}} * C_4$.*

Proof. The proof follows exactly the same pattern as the proof of Proposition 4.9. The main difference is in the fact that there may be centralizers of different isomorphism types. To illustrate this, we will give some hints on how to find the centralizers when $P = D_8^{*\ell} * SD_{2^{m+2}}$. The choice of this group is motivated by the fact that it is the only case in which we will obtain spheres of different dimensions.

Let $P = D_8^{*\ell} * S$ with $S = SD_{2^{m+2}}$. The case $\ell = 0$ is easy and we assume from now $\ell \geqslant 1$. Let z be a generator of $Z(P)$ and let D denote the subgroup $D_8^{*\ell}$ of P. Let $x_1, y_1, \dots, x_\ell, y_\ell$ be symplectic generators of D.

Let E_0 be the subgroup of P generated by y_1 and z and let $M = C_P(E_0)$. The subgroup E_0 is elementary abelian and normal in P and Proposition 4.1 gives a homotopy equivalence

$$\mathcal{A}_2(P)_{\geqslant 2} \simeq \bigvee_{F\in\mathcal{F}} \Sigma\mathcal{A}_2\big(C_M(F)\big)_{\geqslant 2}, \tag{6}$$

where $\mathcal{F} = \{F \in \mathcal{A}_p(P)_{>Z} \mid M \cap F = Z\}$.

The group $S = SD_{2^{m+2}}$ can be generated by two elements a, u, with u of order 2^{m+1}, a of order 2 and $aua^{-1} = u^{-1+2^m}$. The group M has then the following generators

$$M = \langle y_1, x_2, y_2, \dots, x_\ell, y_\ell, u, a\rangle.$$

A subgroup F in $\mathcal{F}$ is generated by z and an element g of order 2 which does not commute with y_1. The element g has the form $g = x_1 y_1^{\varepsilon} xs$, for some $\varepsilon \in \{0, 1\}$, and where x is in the subgroup generated by the x_i, y_i's for $i \geqslant 2$ and s is in the subgroup S. It is not difficult to describe all the possible choices of such an element g and hence all the subgroup in $\mathcal{F}$. It remains then to describe for each $F \in \mathcal{F}$ the centralizer of F in M. We will not do this in full details, but we will give examples of the three different situations that can arise.

If F is generated by $g = x_1$ and z, then $C_M(F) = C_M(g)$ is isomorphic to $D_8^{*\ell-1} * SD_{2^{m+2}}$. More precisely, $C_M(F)$ has the following generators

$$C_M(F) = \langle x_2, y_2, \dots, x_\ell, y_\ell, u, a\rangle,$$

where $w = u^{2^{m-1}}$. If F is generated by $g = x_1 a$ and z, then $C_M(F)$ is isomorphic to $D_8^{*\ell}$. More precisely, $C_M(F)$ has the following generators

$$C_M(F) = \langle x_2, y_2, \dots, x_\ell, y_\ell, y_1 w, a\rangle.$$

If F is generated by $g = x_1 au$ and z, then $C_M(F)$ is isomorphic to $D_8^{*\ell-1} * Q_8$. More precisely, $C_M(F)$ has the following generators

$$C_M(F) = \langle x_2, y_2, \dots, x_\ell, y_\ell, y_1 w, au\rangle$$

and the subgroup generated by $y_1 w$ and au is isomorphic to Q_8.

Altogether, we find $2^{2\ell-1}(2^{m-1}+1)$ subgroups in $\mathcal{F}$ and among them $2^{2\ell-1}$ have a centralizer in M isomorphic to $D_8^{*\ell-1} * SD_{2^{m+2}}$, $2^{2\ell+m-3}$ have a centralizer in M isomorphic to $D_8^{*\ell-1} * D_8$ and $2^{2\ell+m-3}$ have a centralizer in M isomorphic to $D_8^{*\ell-1} * Q_8$. Putting all this information in Eq. (6), the result follows now by an induction argument and Proposition 4.9.

If S is isomorphic to $D_{2^{m+2}}$ or $Q_{2^{m+2}}$, the proof is very similar. The only difference is in the number of subgroups in $\mathcal{F}$ and the isomorphism types of their centralizers in M. More precisely, if S is isomorphic to $D_{2^{m+2}}$, then one obtains that $2^{2\ell-1}$ subgroups in $\mathcal{F}$ have a centralizer in M isomorphic to $D_8^{*\ell-1} * D_{2^{m+2}}$ and $2^{2(\ell-1)+m}$ subgroups in $\mathcal{F}$ have a centralizer in M isomorphic to $D_8^{*\ell}$. If S is isomorphic to $Q_{2^{m+2}}$, then one obtains that $2^{2\ell-1}$ subgroups in $\mathcal{F}$ have a centralizer in M isomorphic to $D_8^{*\ell-1} * Q_{2^{m+2}}$ and $2^{2(\ell-1)+m}$ subgroups in $\mathcal{F}$ have a centralizer in M isomorphic to $D_8^{*\ell-1} * Q_8$.

If S is isomorphic to $D^+_{2^{m+3}}$ or $Q^+_{2^{m+3}}$, the calculations are eased by the choice of a different subgroup E_0. Recall from the definition of these groups (see Section 3) that S has a generator of u of order 2^{m+1} and a generator a of order 2 acting on u by $[a, u] = u^{2^m}$. We let E_0 be the subgroup of P generated by a and z. With this choice, we find in both cases that there are $2^{2\ell+m-1}$ subgroups in $\mathcal{F}$, all with a centralizer in M isomorphic to $D_8^{*\ell} * C_4$.

If S is isomorphic to $D_{2^{m+2}} * C_4$, $SD_{2^{m+2}} * C_4$ or $D^+_{2^{m+3}} * C_4$, then we can choose E_0 to be the subgroup of P generated by wc and z, where c is a generator of the central summand C_4 and w is an element of order 4 in $\Phi(S)$. In the first two cases, we obtain that there are $2^{2\ell+m}$ subgroups in $\mathcal{F}$, all with a centralizer isomorphic to $D_8^{*\ell} * C_4$. In the third case, namely $S = D^+_{2^{m+3}} * C_4$, we obtain that all centralizers in M of elements of $\mathcal{F}$ have central elementary abelian subgroup of rank 2, hence $\mathcal{A}_2(C_M(F))_{\geqslant 2}$ is contractible. $\square$

Remark 4.11. The programming language GAP [7] was particularly useful to check the numerical results of Proposition 4.10 for some (small) values of ℓ and m.

As a corollary to all these computations, we obtain that p-groups with a cyclic derived subgroup give a positive answer to Question 1.3 raised by Bouc and Thévenaz.

Corollary 4.12. *Let P be a p-group with cyclic derived subgroup. Then*:

a) *If p is odd, $\mathcal{A}_p(P)_{\geqslant 2}$ is homotopy equivalent to a wedge of spheres of the same dimension.*
b) *If $p = 2$, then $\mathcal{A}_p(P)_{\geqslant 2}$ is homotopy equivalent to a wedge of spheres of the same dimension, or of two consecutive dimensions.*

5. Maximal dimension of spheres in $\mathcal{A}_p(P)_{\geqslant 2}$

Let P be a p-group of order p^n for some $n \geqslant 1$. As Bouc and Thévenaz showed in [4] (see Proposition 4.4), the poset $\mathcal{A}_p(P)_{\geqslant 2}$ is homotopy equivalent to a wedge of spheres. Of course, the dimension of these spheres cannot be greater than the dimension of the simplicial complex $\mathcal{A}_p(P)_{\geqslant 2}$, that is $r(P) - 2$. The dimension of $\mathcal{A}_p(P)_{\geqslant 2}$ is in particular bounded by $n - 2$. But even though the dimension of $\mathcal{A}_p(P)_{\geqslant 2}$ can reach the bound $n - 2$ (if P is elementary abelian), the dimension of the spheres has actually a much smaller bound. Indeed, as the next proposition shows, the dimension of the spheres cannot be greater than $\lfloor \frac{n-1}{2} \rfloor$, where for any positive real number r we denote by $\lfloor r \rfloor$ the greatest integer n such that $n \leqslant r$.

Proposition 5.1. *Let p be an arbitrary prime number. If P is a p-group of order p^n, $n \geqslant 1$, then $\tilde{H}_k(\mathcal{A}_p(P)_{\geqslant 2}) = 0$ if $k \geqslant \lfloor \frac{n-1}{2} \rfloor$.*

Proof. The proof goes by induction on n. It is clear for $n = 1, 2, 3$ and we assume from now on $n \geqslant 4$.

If $|\Omega_1(Z(P))| > p$, then $\mathcal{A}_p(P)_{\geqslant 2}$ is contractible and the result holds trivially. If either P is cyclic, or $p = 2$ and P is isomorphic to one of the groups

$$D_{2^n}, \ Q_{2^n} \text{ or } SD_{2^n} \tag{7}$$

with $n \geqslant 4$, then $\tilde{H}_k(\mathcal{A}_p(P)_{\geqslant 2}) = 0$ for $k \geqslant 1$, so that the result holds for these groups.

Suppose now that $Z = \Omega_1(Z(P))$ has order p and that P is neither cyclic nor one of the groups in (7). By Lemma 4.3, we have that P contains a normal elementary abelian subgroup E_0 of rank 2 containing Z. Let $M = C_P(E_0)$, Proposition 4.1 gives an isomorphism

$$\tilde{H}_k\big(\mathcal{A}_p(P)_{\geqslant 2}\big) \cong \bigoplus_{F \in \mathcal{F}} \tilde{H}_{k-1}\big(\mathcal{A}_p\big(C_M(F)\big)_{\geqslant 2}\big), \tag{8}$$

where $\mathcal{F} = \{F \in \mathcal{A}_p(P)_{\geqslant 2} \mid F \cap M = Z\}$.

For any $F \in \mathcal{F}$ we have $C_M(F) < M$, since otherwise F would be central in P (see Remark 4.2) and this would contradict our assumption that $|Z| = p$. We have thus that $|C_M(F)| = p^r$ for some $r \leqslant n - 2$.

By induction we have then $\tilde{H}_{k-1}(\mathcal{A}_p(C_M(F))_{\geqslant 2}) = 0$ if $k - 1 \geqslant \lfloor \frac{r-1}{2} \rfloor$. But since $r \leqslant n - 2$, we have in particular that $\tilde{H}_{k-1}(\mathcal{A}_p(C_M(F))_{\geqslant 2}) = 0$ if $k - 1 \geqslant \lfloor \frac{n-3}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor - 1$. Using this information in (8) shows that $\tilde{H}_k(\mathcal{A}_p(P)_{\geqslant 2}) = 0$ if $k \geqslant \lfloor \frac{n-1}{2} \rfloor$ and the proposition is proved. $\square$

The preceding proposition tells us that the dimension of the spheres occurring in $\mathcal{A}_p(P)_{\geqslant 2}$ is bounded by $\lfloor \frac{n-3}{2} \rfloor$. It is not difficult to see that this bound is actually sharp (see Proposition 5.2 and the examples following Proposition 5.3). We will call this value $\lfloor \frac{n-3}{2} \rfloor$ the *maximal dimension of spheres* in $\mathcal{A}_p(P)_{\geqslant 2}$ and we will say that $\mathcal{A}_p(P)_{\geqslant 2}$ *has spheres in maximal dimension* if there is at least one sphere of dimension $\lfloor \frac{n-3}{2} \rfloor$ in its homotopy type.

If P has order p^n with n odd, say $n = 2\ell + 1$, the maximal dimension of the spheres in $\mathcal{A}_p(P)_{\geqslant 2}$ is then equal to $\ell - 1$. In view of Proposition 4.7 and Proposition 4.9, we already know that if P is extraspecial of type I and order $p^{2\ell+1}$ then $\mathcal{A}_p(P)_{\geqslant 2}$ has spheres in maximal dimension. What is maybe more surprising is that the converse also holds as the next proposition shows.

Proposition 5.2. *Let p be an arbitrary prime number and let P be a p-group of order $p^{2\ell+1}$ with $\ell \geqslant 1$. Then $\tilde{H}_k(\mathcal{A}_p(P)_{\geqslant 2}) = 0$ if $k \geqslant \ell$. Furthermore, $\tilde{H}_{\ell-1}(\mathcal{A}_p(P)_{\geqslant 2}) \neq 0$ if and only if P is extraspecial of type I.*

Proof. The first assertion, namely $\tilde{H}_k(\mathcal{A}_p(P)_{\geqslant 2}) = 0$ for $k \geqslant \ell$, follows directly from Proposition 5.1.

Let P be an extraspecial p-group of type I and order $p^{2\ell+1}$, with $\ell \geqslant 1$. We know from Proposition 4.7 and Proposition 4.9 that the poset $\mathcal{A}_p(P)_{\geqslant 2}$ has the homotopy type of a wedge of p^{ℓ^2} spheres, respectively $2^{\ell(\ell-1)}$ spheres if $p = 2$, of dimension $\ell - 1$. Since $\ell \geqslant 1$, this implies $\tilde{H}_{\ell-1}(\mathcal{A}_p(P)_{\geqslant 2}) \neq 0$.

The proof of the converse goes by induction on ℓ. It is clear for $\ell = 1$ and we suppose from now on $\ell \geqslant 2$. The hypothesis $\tilde{H}_{\ell-1}(\mathcal{A}_p(P)_{\geqslant 2}) \neq 0$ implies in particular that P is not cyclic, dihedral, quaternion or semidihedral and that $\Omega_1(Z(P))$ is cyclic of order p. It follows that P has a normal elementary abelian subgroup E_0 of rank 2 and Proposition 4.1 gives then an isomorphism

$$\tilde{H}_{\ell-1}\big(\mathcal{A}_p(P)_{\geqslant 2}\big) \cong \bigoplus_{F \in \mathcal{F}} \tilde{H}_{\ell-2}\big(\mathcal{A}_p\big(C_M(F)\big)_{\geqslant 2}\big),$$

where $M = C_P(E_0)$ and $\mathcal{F} = \{F \in \mathcal{A}_p(P)_{\geqslant 2} \mid F \cap M = Z\}$. By assumption $\tilde{H}_{\ell-1}(\mathcal{A}_p(P)_{\geqslant 2}) \neq 0$, so that there exists a subgroup $F_0 \in \mathcal{F}$ such that $\tilde{H}_{\ell-2}(\mathcal{A}_p(C_M(F_0))_{\geqslant 2}) \neq 0$. By Proposition 5.1 we must have then $\ell - 2 < \lfloor \frac{r-1}{2} \rfloor$, where $|C_M(F_0)| = p^r$. Since $C_M(F_0) < M$ (otherwise F_0 would be central in P), we also have $r \leqslant 2\ell - 1$ and these two conditions together force r to be equal to $2\ell - 1$.

Now, $C_M(F_0)$ has order $p^{2(\ell-1)+1}$ and $\tilde{H}_{\ell-2}(\mathcal{A}_p(C_M(F_0))_{\geqslant 2}) \neq 0$, so that by the induction hypothesis $C_M(F_0)$ is extraspecial of type I. Since P is a central product $P = C_M(F_0) * (E_0F_0)$ and E_0F_0 is extraspecial of type I, we have that P is extraspecial of type I. □

The proof of Proposition 5.2 can be adapted when the p-valuation of the order of the group is even. The obtained result is the content of the following proposition.

Proposition 5.3. *Let p be an arbitrary prime number and let P be a p-group of order $p^{2\ell}$, with $\ell \geqslant 2$. Then $\tilde{H}_k(\mathcal{A}_p(P)_{\geqslant 2}) = 0$ if $k \geqslant \ell - 1$. Moreover, if $\tilde{H}_{\ell-2}(\mathcal{A}_p(P)_{\geqslant 2}) \neq 0$, then P has a maximal subgroup which is extraspecial of type I.*

Proof. The first assertion, namely $\tilde{H}_k(\mathcal{A}_p(P)_{\geqslant 2}) = 0$ for $k \geqslant \ell - 1$, follows directly from Proposition 5.1 since $\lfloor \frac{2\ell-1}{2} \rfloor = \ell - 1$.

Let now P be such that $\tilde{H}_{\ell-2}(\mathcal{A}_p(P)_{\geqslant 2}) \neq 0$. Since $\ell \geqslant 2$, we have in particular that $Z = \Omega_1(Z(P))$ has order p (otherwise $\mathcal{A}_p(P)_{\geqslant 2}$ would be contractible). The proof goes by induction on ℓ and we treat first the case $\ell = 2$, i.e. $|P| = p^4$. We have thus by assumption that

$$\tilde{H}_0\big(\mathcal{A}_p(P)_{\geqslant 2}\big) \neq 0. \tag{9}$$

This condition (9) implies in particular that P is not cyclic and that P is not quaternion if $p = 2$. If $P = D_{16}$ or SD_{16}, then P contains a maximal subgroup isomorphic to D_8 so that the result will always hold for these groups. Note that (9) holds in these two cases.

We assume now that P is not cyclic and furthermore that P is neither dihedral, semi-dihedral nor quaternion if $p=2$. It follows then from Lemma 4.3 that P has a normal elementary abelian subgroup E_0 of rank 2 with $Z \leqslant E_0$. Let $M = C_P(E_0)$, Proposition 4.1 gives an isomorphism

$$\tilde{H}_0\big(\mathcal{A}_p(P)_{\geqslant 2}\big) \cong \bigoplus_{F\in\mathcal{F}} \tilde{H}_{-1}\big(\mathcal{A}_p\big(C_M(F)\big)_{\geqslant 2}\big),$$

where $\mathcal{F} = \{F \in \mathcal{A}_p(P)_{\geqslant 2} \mid F \cap M = Z\}$. Condition (9) implies that there exists an elementary abelian subgroup $F \neq E_0$ of rank 2 containing $Z = \Omega_1(Z(P))$ and such that $\mathcal{A}_p(C_M(F))_{\geqslant 2} = \emptyset$. It follows that E_0 does not centralize F and hence E_0F is an extraspecial p-group of type I and order p^3. Since $|P| = p^4$, it follows that E_0F is maximal in P so that the result holds for $\ell = 2$.

We assume from now on $\ell \geqslant 3$ and we have by assumption that

$$\tilde{H}_{\ell-2}\big(\mathcal{A}_p(P)_{\geqslant 2}\big) \neq 0. \tag{10}$$

This condition (10) implies in particular that P is not cyclic and that P is not quaternion if $p=2$. Since $\ell \geqslant 3$, this implies also that P is not dihedral, nor semi-dihedral. It follows now that P has a normal elementary abelian subgroup E_0 of rank 2 with $Z \leqslant E_0$. Let $M = C_P(E_0)$, Proposition 4.1 gives an isomorphism

$$\tilde{H}_{\ell-2}\big(\mathcal{A}_p(P)_{\geqslant 2}\big) \cong \bigoplus_{F\in\mathcal{F}} \tilde{H}_{\ell-3}\big(\mathcal{A}_p\big(C_M(F)\big)_{\geqslant 2}\big), \tag{11}$$

where $\mathcal{F} = \{F \in \mathcal{A}_p(P)_{\geqslant 2} \mid F \cap M = Z\}$.

Since the left-hand term of (11) is not trivial by assumption, there exists $F_0 \in \mathcal{F}$ such that $\tilde{H}_{\ell-3}(\mathcal{A}_p(C_M(F_0))_{\geqslant 2}) \neq 0$. To ease notation, let $C_0 = C_M(F_0)$ and let p^a be the order of C_0. Since C_0 is strictly contained in M, we have $a \leqslant 2\ell - 2$. Furthermore, Proposition 5.1 implies $\ell - 3 < \lfloor \frac{a-1}{2} \rfloor$. These two conditions imply $a = 2\ell - 2$ or $a = 2\ell - 3$.

Let us see first what happens if $a = 2\ell - 2$. In this situation, C_0 is maximal in M and $P = C_0 * E_0F_0$. But since the order of C_0 is $p^{2(\ell-1)}$ and $\tilde{H}_{\ell-3}(\mathcal{A}_p(C_M(F_0))_{\geqslant 2}) \neq 0$, we have by induction that C_0 has a maximal subgroup N which is extraspecial of type I. The subgroup $N * E_0F_0$ is then extraspecial of type I and maximal in P.

Let us see now what happens if $a = 2\ell - 3 = 2(\ell - 2) + 1$. Since we have $\tilde{H}_{\ell-3}(\mathcal{A}_p(C_M(F_0))_{\geqslant 2}) \neq 0$, it follows from Proposition 5.2 that C_0 is extraspecial of type I. Therefore $C_0 * E_0F_0$ is extraspecial of type I and is maximal in P.

In both cases P has a maximal subgroup which is extraspecial of type I and the proposition is proved. $\square$

To our knowledge, there is no classification of p-groups with a maximal subgroup extraspecial of type I. Therefore Proposition 5.3 does not give enough information to describe all p-groups of order $p^{2\ell}$, $\ell \geqslant 2$, for which $\mathcal{A}_p(P)_{\geqslant 2}$ has spheres in maximal dimension. We would like to end this section with some examples.

Example 5.4. Let $\ell \geqslant 2$.

a) If either $P = (X_{p^3})^{*(\ell-1)} * C_{p^2}$ or $P = D_8^{*\ell-1} * C_4$, then $|P| = p^{2\ell}$ and $\tilde{H}_{\ell-2}(\mathcal{A}_p(P)_{\geqslant 2}) \neq 0$.
b) If $p=2$ and $P = D_8^{*(\ell-2)} * S$, where S is either D_{16} or SD_{16}, then $|P| = 2^{2\ell}$ and $\tilde{H}_{\ell-2}(\mathcal{A}_p(P)_{\geqslant 2}) \neq 0$.

If P is a p-group of order $p^{2\ell}$ with a cyclic Frattini subgroup and spheres in maximal dimension, then it follows from the classifications given in Section 3 and the calculations performed in Section 4, that P is one of the groups given in Example 5.4. There are however p-groups of order $p^{2\ell}$ with spheres in maximal dimension and a non-cyclic Frattini subgroup. We end this section with two examples of such groups.

Example 5.5. Let p be an odd prime number and let $X = X_{p^3}$ with generators x, y. Let P be the semi-direct product $P = X \rtimes C_p$ with respect to the automorphism of X order p fixing x and sending y to xy. This group P has order p^4 and $\Phi(P) = \langle x, z\rangle$ is not cyclic. It is not difficult to see that $\mathcal{A}_p(P)_{\geqslant 2}$ is a discrete poset and that $\tilde{H}_{\ell-2}(\mathcal{A}_p(P)_{\geqslant 2}) = \tilde{H}_0(\mathcal{A}_p(P)_{\geqslant 2}) \neq 0$.

Example 5.6. Let p be an arbitrary prime number and let $X = (X_{p^3})^{*p}$ be a central product of p copies of the group X_{p^3}. We choose a generator z of $Z(X)$ and for each $1 \leqslant i \leqslant p$ we choose symplectic generators x_i, y_i of the corresponding central summand X_{p^3} such that $[x_i, y_i] = z$.

Let P be the semi-direct product $P = X \rtimes C_p$ with respect to the automorphism of X that permutes cyclically the generators $\{x_1, \dots, x_p\}$ and the generators $\{y_1, \dots, y_p\}$. More precisely, the automorphism is defined on the generators of X by the following.

$$\alpha(x_i) = x_{i+1}, \quad \text{for } i = 1, \dots, p-1, \quad \text{and} \quad \alpha(x_p) = x_1;$$

$$\alpha(y_i) = y_{i+1}, \quad \text{for } i = 1, \dots, p-1, \quad \text{and} \quad \alpha(y_p) = y_1.$$

The group P has an order $p^{2p+2} = p^{2\ell}$ with $\ell = p+1$. Let $x = x_1 \cdots x_p$ and let $E_0 = \langle x, z\rangle$. The subgroup E_0 is normal in P and let $M = C_P(E_0)$. The subgroup $F = \langle y_1, z\rangle$ is elementary abelian of rank 2 and $F \cap M = \langle z\rangle$. Furthermore, $C_M(F)$ is isomorphic to $X_{p^{2p-1}}$, so that $\tilde{H}_{p-2}(\mathcal{A}_p(C_M(F))_{\geqslant 2}) \neq 0$. It follows now from Proposition 4.1 that $\tilde{H}_{\ell-2}(\mathcal{A}_p(P)_{\geqslant 2}) = \tilde{H}_{p-1}(\mathcal{A}_p(P)_{\geqslant 2}) \neq 0$.

These last two examples seem to suggest that classifying all p-groups of order $p^{2\ell}$ such that $\tilde{H}_{\ell-2}(\mathcal{A}_p(P)_{\geqslant 2}) \neq 0$, may be a difficult task.

6. hCM property for $\mathcal{A}_p(P)$

Let $\mathcal{Q}$ be a poset and let d be its dimension. Following Quillen [12], we will say that $\mathcal{Q}$ is spherical of dimension d, or just spherical, if $\mathcal{Q}$ has the homotopy type of a wedge of spheres of dimension d. We would like to emphasize here the fact that in this definition the dimension of the spheres is always equal to the dimension of the poset.

If P is a non-trivial p-group, it is well known that $\mathcal{A}_p(P)$ is contractible. Consequently, one cannot distinguish between p-groups by looking at the homotopy type of $\mathcal{A}_p(P)$. The homotopy Cohen–Macaulay property (hCM for short) is more accurate, since it takes all intervals into consideration.

The aim of this section is to study the hCM property of $\mathcal{A}_p(P)$ when P is a p-group with a cyclic derived subgroup. We recall first the definition of hCM posets and some of their properties.

Definition 6.1. A poset $\mathcal{Q}$ of dimension d is said to be homotopically Cohen–Macaulay (hCM for short) if the following conditions are satisfied.

- $\mathcal{Q}$ is spherical of dimension d.
- $\mathcal{Q}_{>q}$ is spherical of dimension $d - h(q) - 1$, for all $q \in \mathcal{Q}$.
- $\mathcal{Q}_{<q}$ is spherical of dimension $h(q) - 1$, for all $q \in \mathcal{Q}$.
- (q, q') is spherical of dimension $h(q') - h(q) - 2$, for all $q < q' \in \mathcal{Q}$.

Remark 6.2. Originally, the hCM posets were simply called CM posets by Quillen who introduced them in [12]. Since then, other related notions have been introduced and the term ‘homotopically’ has been added to emphasize on the fact that one is looking at the homotopy type of the intervals. See for example Wachs’ survey [14] for more details.

We list next some general results concerning the hCM property for posets of the form $\mathcal{A}_p(G)$. Most of them are due to Quillen [12]. We note first that groups such that $\mathcal{A}_p(G)$ is hCM are somewhat special.

Lemma 6.3. *(See [12, Remark 10.2].) If $\mathcal{A}_p(G)$ is hCM, then all maximal elementary abelian p-subgroups of G have rank $r_p(G)$.*

The next result asserts that $\mathcal{A}_p(E)$ is hCM if E is an elementary abelian p-group. As Quillen showed, this follows from the theory of buildings and the Solomon–Tits theorem (see [12]). The reader not familiar with buildings can refer to the discussion following [12, Proposition 8.6] or to [9, Proposition 3.6] for alternative arguments.

Lemma 6.4. *If E is an elementary abelian p-group of rank r, then $\mathcal{A}_p(E)$ is hCM of dimension $r-1$.*

As a consequence, for any group G we have that $\mathcal{A}_p(G)_{<A} = \mathcal{A}_p(A)_{<A}$ is spherical. This is also true for any interval (A, B) in $\mathcal{A}_p(G)$ since this interval is isomorphic to $\mathcal{A}_p(B/A)_{<B/A}$. We have thus the following characterization of the hCM property for posets of the form $\mathcal{A}_p(G)$.

Proposition 6.5. *(See [12, Proposition 10.1].) The poset $\mathcal{A}_p(G)$ is hCM if and only if $\mathcal{A}_p(G)$ is spherical of dimension $r_p(G)-1$ and $\mathcal{A}_p(G)_{>A}$ is spherical of dimension $r_p(G)-r_p(A)-1$ for any $A \in \mathcal{A}_p(G)$.*

Suppose now that $G = P$ is a p-group. Then $\mathcal{A}_p(P)$ is spherical since it is contractible, so that, by the previous proposition, only upper intervals need to be considered. Moreover, thanks to the recursive nature of the definition of the hCM property, it is not always necessary to compute the homotopy type of all upper intervals. This rather trivial fact appears along the lines of [12, Remark 10.4] and we give the version of it we will need later.

Lemma 6.6. *Let P be a p-group and let $Z = \Omega_1(Z(P))$. Suppose that $\mathcal{A}_p(P)_{>Z}$ is spherical of dimension $r(P)-r(Z)-1$ and that $\mathcal{A}_p(C_P(A))$ is hCM of dimension $r(P)-1$ for each A minimal in $\mathcal{A}_p(P)_{>Z}$. Then $\mathcal{A}_p(P)$ is hCM.*

Proof. By Proposition 6.5, we have to show that $\mathcal{A}_p(P)_{>B}$ is spherical of dimension $r(P)-r(B)-1$ for each $B \in \mathcal{A}_p(P) \cup \{1\}$. Since P is a p-group, this holds for $B = 1$, so assume from now on $B > 1$. If Z is not contained in B, the subgroup BZ is a conjunctive element in $\mathcal{A}_p(P)_{>B}$, so that $\mathcal{A}_p(P)_{>B}$ is contractible. If $B = Z$, then this holds by one of our assumptions, so we may suppose $Z < B$. There exists then $A \in \mathcal{A}_p(P)_{>Z}$ of rank 2 such that $Z < A \leqslant B$ and we have $\mathcal{A}_p(P)_{>B} = \mathcal{A}_p(C_P(A))_{>B}$. By assumption, $\mathcal{A}_p(C_P(A))$ is hCM of dimension $r(P)-1$, so that $\mathcal{A}_p(C_P(A))_{>B}$ is spherical of dimension $r(P)-r(B)-1$ and the lemma is proved. $\square$

Lemma 6.7. *(See Proposition 10.3 in [12].) If $\mathcal{A}_p(G_1)$ and $\mathcal{A}_p(G_2)$ are hCM of dimension d_1 and d_2 respectively, then $\mathcal{A}_p(G_1 \times G_2)$ is hCM of dimension $d_1 + d_2 + 1$.*

Similarly to the case of elementary abelian p-groups, the case of extraspecial p-groups can be deduced from arguments using buildings. We will provide here an alternative proof which should clarify the usefulness of Lemma 6.6. This proof is a prototype for all other proofs in this section.

Proposition 6.8. *(See Example 10.4 in [12].) Let p be an odd prime number. If P is an extraspecial p-group, then $\mathcal{A}_p(P)$ is hCM.*

Proof. Suppose first that $P = X_{p^{2\ell+1}}$ is extraspecial of type I. Since $Z = \Omega_1(Z(P)) = Z(P)$ has order p, Lemma 4.5 implies that there is a homotopy equivalence $\mathcal{A}_p(P)_{>Z} \simeq \mathcal{A}_p(P)_{\geqslant 2}$. We know moreover from Proposition 4.7 that $\mathcal{A}_p(P)_{\geqslant 2}$ has the homotopy type of a wedge of spheres of dimension $\ell - 1 = \dim \mathcal{A}_p(P)_{>Z}$. Therefore, $\mathcal{A}_p(P)_{>Z}$ is spherical of dimension $\ell - 1 = r(P) - r(Z) - 1$.

Let A be minimal in $\mathcal{A}_p(P)_{>Z}$. We choose a generator z of $Z = Z(P)$ and symplectic generators $x_1, y_1, \dots, x_\ell, y_\ell$ of P. We may assume without loss of generality that $A = \langle x_1, z\rangle$, so that $C_P(A)$ has the following generators.

$$C_P(A) = \langle x_1, x_2, y_2, \dots, x_\ell, y_\ell\rangle.$$

We have in particular that $C_P(A) = \langle x_1\rangle \times Q$, where $Q = \langle x_j, y_j,\ j = 2, \dots, \ell\rangle$ is isomorphic to $X_{p^{2\ell-1}}$. We have then

$$\mathcal{A}_p(P)_{>A} = \mathcal{A}_p\big(C_P(A)\big)_{>A} = \mathcal{A}_p\big(\langle x_1\rangle \times Q\big)_{>(\langle x_1\rangle \times \langle z\rangle)} \cong \mathcal{A}_p(Q)_{>Z(Q)}.$$

Note that the last isomorphism holds since x_1 is central in P. A recursive argument can be used to conclude that $\mathcal{A}_p(P)_{>A}$ is hCM, once we know that $\mathcal{A}_p(X_{p^3})$ is hCM. But this is easy to check, since $\mathcal{A}_p(X_{p^3})_{>Z(X_{p^3})}$ is a discrete poset. The result for $P = X_{p^{2\ell+1}}$ follows now from Lemma 6.6.

If P is extraspecial of type II, that is of exponent p^2, then $\Omega_1(P)$ is isomorphic to $X \times C_p$ with X extraspecial of type I. It follows then from the previous case and Lemma 6.7 that $\mathcal{A}_p(P)$ is hCM. $\square$

Recall that the aim of this section is to study the hCM property for a larger class of p-groups, namely those with a cyclic derived subgroup. Recall from the reductions made at the beginning of Section 3 that we may well assume that $\Phi(P)$ is cyclic.

If p is odd, then $\Phi(P)$ is central by Lemma 3.2. Moreover, Lemma 3.1 and Proposition 3.3 imply that P is a direct product $P = Q \times E$, where E is elementary abelian and Q is extraspecial of type I. The following result follows now directly from Lemma 6.4, Lemma 6.7 and Proposition 6.8 and closes the discussion for p odd.

Proposition 6.9. *Let p be an odd prime number. If P is a p-group with a cyclic derived subgroup, then $\mathcal{A}_p(P)$ is hCM of dimension $r(P) - 1$.*

We will focus now on the case $p = 2$. So let P be a 2-group with a cyclic Frattini subgroup. By Lemma 3.1, the group P can be written as a direct product $P = Q \times E$ where E is elementary abelian and Q has a cyclic Frattini subgroup, precisely $\Phi(Q) = \Phi(P)$, and a cyclic center. In view of Lemma 6.7 and Proposition 6.8, it is enough to show that $\mathcal{A}_p(Q)$ is hCM.

Without loss of generality, we may thus well assume that P is a 2-group such that $\Omega_1(P) = P$ and both $\Phi(P)$ and $Z(P)$ are cyclic. If $\Phi(P)$ is central, then $\mathcal{A}_p(P)$ is hCM. This can be proved using buildings associated to the quadratic form on $P/Z(P)$. This approach is used for example in Das' paper [5]. As for p odd, we will provide here a proof avoiding buildings.

Proposition 6.10.

1. *If $P = D_8^{*\ell}$ with $\ell \geqslant 1$, then $\mathcal{A}_2(P)$ is hCM of dimension ℓ.*
2. *If $P = D_8^{*\ell} * C_4$ with $\ell \geqslant 0$, then $\mathcal{A}_2(P)$ is hCM of dimension ℓ.*
3. *If $P = D_8^{*\ell} * Q_8$ with $\ell \geqslant 0$, then $\mathcal{A}_2(P)$ is hCM of dimension ℓ.*

Proof. Suppose first that $P = D_8^{*\ell}$. Let $x_1, y_1, \dots, x_\ell, y_\ell$ be symplectic generators of P and let $z = [x_i, y_i]$. We already know from Lemma 4.5, that $\mathcal{A}_2(P)_{>Z(P)} \simeq \mathcal{A}_2(P)_{\geqslant 2}$ has the homotopy type of a wedge of spheres of dimension $\ell - 1 = r(P) - r(Z(P)) - 1$.

Following Lemma 6.6 it remains to show that for any $A \in \mathcal{A}_2(P)_{>Z(P)}$ with $|A| = p$, the poset $\mathcal{A}_2(C_P(A))$ is hCM of dimension ℓ. Such a subgroup A is generated by z and a non-central element g of order 2. Without loss of generality, we may assume $g = x_1$. It is then easy to see that $C_P(A)$ is the subgroup generated by x_1 and the elements x_j, y_j for $j \neq 1$.

Therefore $C_P(A)$ is isomorphic to $C_2 \times D_8^{*(\ell-1)}$. The result for $P = D_8^{*\ell}$ follows now from an induction argument together with the use of Lemma 6.7.

The proof in the two other cases is very similar. When $P = D_8^{*\ell} * C_4$, the centralizers are isomorphic to $C_2 \times D_8^{*(\ell-1)} * C_4$. In the case $P = D_8^{*\ell} * Q_8$ the centralizers are isomorphic to $C_2 \times D_8^{*(\ell-1)} * Q_8$. □

It remains to treat the case when P has a non-central Frattini subgroup, that is $P = D_8^{*\ell} * S$, where S is one of the following groups, all with $m > 1$.

$$D_{2^{m+2}},\ SD_{2^{m+2}},\ Q_{2^{m+2}},\ D^+_{2^{m+3}},\ Q^+_{2^{m+3}},\ D_{2^{m+2}} * C_4,\ SD_{2^{m+2}} * C_4,\ D^+_{2^{m+3}} * C_4.$$

Proposition 6.11. *Let $\ell \geqslant 0$ and $m > 1$. Let $P = D_8^{*\ell} * S$ where S is one of the following groups.*

$$D_{2^{m+2}},\ SD_{2^{m+2}},\ Q_{2^{m+2}},\ D^+_{2^{m+3}},\ Q^+_{2^{m+3}},\ D_{2^{m+2}} * C_4,\ SD_{2^{m+2}} * C_4,\ D^+_{2^{m+3}} * C_4.$$

1. *If $S = SD_{2^{m+2}}$ and $\ell \geqslant 1$, then $\mathcal{A}_2(P)$ is not hCM.*
2. *If $S = D^+_{2^{m+3}}$, then $\mathcal{A}_2(P)$ is not hCM.*
3. *In all other cases, $\mathcal{A}_2(P)$ is hCM of dimension d, where*
   (a) *$d = 0$, if $P = S = SD_{2^{m+2}}$;*
   (b) *$d = \ell$, if $S = Q_{2^{m+2}}$;*
   (c) *$d = \ell + 1$, if S is either $D_{2^{m+2}}$, $D_{2^{m+2}} * C_4$, $SD_{2^{m+2}} * C_4$ or $Q^+_{2^{m+3}}$;*
   (d) *$d = \ell + 2$, if $S = D^+_{2^{m+3}} * C_4$.*

Proof. 1. By Lemma 4.5 and Proposition 6.5, a necessary condition for $\mathcal{A}_2(P)$ to be hCM is that $\mathcal{A}_2(P)_{\geqslant 2}$ must be a wedge of spheres all of the same dimension. In view of Proposition 4.10, we obtain immediately that $\mathcal{A}_2(P)$ is not hCM when $P = D_8^{*\ell} * SD_{2^{m+2}}$ with $\ell \geqslant 1$.

2. By Lemma 6.3, it is enough to exhibit two subgroups that are maximal in $\mathcal{A}_2(P)$ but that do not have the same rank. Let z be a generator of $Z(P)$ and let $x_1, y_1, \dots, x_\ell, y_\ell$ be symplectic generators of $D_8^{*\ell}$ such that $[x_i, y_i] = z$. Let a, b, u be generators of $D^+_{2^{m+3}}$ with a and b of order 2, u of order 2^{m+1}, $aua^{-1} = u^{1+2^m}$ and $bub^{-1} = u^{-1}$. On the one hand, the subgroup $\langle z, x_1, \dots, x_\ell, a, b\rangle$ is maximal in $\mathcal{A}_2(P)$ and has rank $\ell + 3$. On the other hand, the subgroup $\langle z, x_1, \dots, x_\ell, ub\rangle$ is also maximal in $\mathcal{A}_2(P)$, but has rank $\ell + 2$.

3. The proof in the remaining cases is very similar to the proof of Proposition 6.10, so that we will not give full details. Indeed, we will only give an outline of the proof that $\mathcal{A}_2(P)$ is hCM when $P = D_8^{*\ell} * SD_{2^{m+2}} * C_4$. Our motivation for doing so is that it contrasts with the fact that $\mathcal{A}_2(P)$ is actually not hCM when $P = D_8^{*\ell} * SD_{2^{m+2}}$.

So let $P = D_8^{*\ell} * SD_{2^{m+2}} * C_4$. Let c be a generator of the subgroup C_4 and let u, a be generators of the subgroup $SD_{2^{m+2}}$ with u of order 2^{m+1}, a of order 2 and $aua^{-1} = u^{-1+2^m}$. Let $w = u^{2^{m-1}}$ and $z = w^2 = c^2$. Let $x_1, y_1, \dots, x_\ell, y_\ell$ be symplectic generators of the subgroup $D_8^{*\ell}$ such that $[x_i, y_i] = z$. Recall from Lemma 6.6 that we have to show that, for any minimal subgroup A in $\mathcal{A}_2(P)_{>Z(P)}$, the poset $\mathcal{A}_2(C_P(A))$ is hCM of dimension $r(P) - 1 = \ell + 1$. Such a subgroup A is generated by z and an element $g = xs$ of order 2 with x in the subgroup $D_8^{*\ell}$ and s in the subgroup $SD_{2^{m+2}} * C_4$. All cases can be reduced to one of the following. To ease verifications, we remark that $[a, w] = z$ and $(wc)^2 = 1$.

If $g = a$, then $C_P(A)$ is generated by the subgroup $D_8^{*\ell}$ and the two elements c and a.

If $g = auc$, then $C_P(A)$ is generated by the subgroup $D_8^{*\ell}$ and the two elements c and auc.

If $g = x_1 y_1 au$, then $C_P(A)$ is generated by $x_1 y_1 au$, $x_1 wc$, $y_1 wc$, c and the elements x_j, y_j for $j \neq 1$. It is useful to remark here that the subgroup generated by $x_1 wc$ and $y_1 wc$ is isomorphic to D_8. Note also that this is, in some sense, the crucial case as the next remark will show.

If $g = x_1 y_1 ac$, then $C_P(A)$ is generated by $x_1 y_1 ac$, $x_1 wc$, $y_1 wc$, c and the elements x_j, y_j for $j \neq 1$.

In these four cases, $C_P(A)$ is isomorphic to $(D_8^{*\ell} * C_4) \times C_2$.

If $g = x_1 a$, then $C_P(A)$ is generated by $x_1 a$, $y_1 wc$, c and the elements x_j, y_j for $j \neq 1$.
If $g = x_1 wc$, then $C_P(A)$ is generated by $x_1 wc$, $y_1 a$, c and the elements x_j, y_j for $j \neq 1$.
If $g = x_1 auc$, then $C_P(A)$ is generated by $x_1 auc$, $y_1 wc$, c and the elements x_j, y_j for $j \neq 1$.
In these three cases, $C_P(A)$ is isomorphic to $(D_8^{*\ell-1} * C_4) \times C_2 \times C_2$.

If $g = wc$, then $C_P(A)$ is generated by the subgroup $D_8^{*\ell}$ and the two elements u and wc, so that in this case $C_P(A)$ is isomorphic to $(D_8^{*\ell} * C_{2^{m+1}}) \times C_2$.

In all these cases, it follows from previous results, namely Lemma 6.7, Lemma 6.4 and Proposition 6.10, that $\mathcal{A}_2(C_P(A))$ is hCM of dimension $\ell + 1$.

If $g = x_1$, then $C_P(A)$ is generated by x_1, the elements x_j, y_j for $j \neq 1$ and the subgroup $SD_{2^{m+2}} * C_4$.
If $g = x_1 y_1 w$, then $C_P(A)$ is generated by $x_1 y_1 w$, $y_1 a$, u, c and the elements x_j, y_j for $j \neq 1$.
If $g = x_1 y_1 c$, then $C_P(A)$ is generated by $x_1 y_1 c$, u, a, c and the elements x_j, y_j for $j \neq 1$.

In all these three cases, the group $C_P(A)$ is then isomorphic to the group $(D_8^{*(\ell-1)} * SD_{2^{m+2}} * C_4) \times C_2$. It follows then from previous results (Lemma 6.7 and Lemma 6.4) and an induction argument on ℓ (the case $\ell = 0$ being trivial) that $\mathcal{A}_2(C_P(A))$ is hCM of dimension $\ell + 1$. $\square$

Remark 6.12. The crucial case in the preceding proof is when $g = x_1 y_1 au$. In this situation, we obtained $C_P(A) = (D_8^{*(\ell-1)} * D_8 * C_4) \times C_2$. The subgroup $D_8 * C_4$ is generated by $y_1 w, x_1 wc$ and c.

There is an analogous situation for the group $Q = D_8^{*\ell} * SD_{2^{m+2}}$, but for this group, $C_Q(A)$ would be isomorphic to $D_8^{*\ell-1} * Q_8$, which has the homotopy type of a wedge of spheres but not of the required dimension. This is roughly why $\mathcal{A}_2(Q)$ is not hCM. The subgroup Q_8 would be here generated by the two elements $x_1 w$ and $y_1 w$, of order 4.

Because of the presence of the central element c of order 4, when P is the group $D_8^{*\ell} * SD_{2^{m+2}} * C_4$, the two elements $x_1 w$ and $y_1 w$, of order 4, can be modified by c in order to change their order. This is the well-known isomorphism $Q_8 * C_4 \cong D_8 * C_4$. In this situation, the centralizer has the homotopy type of a wedge of spheres of the required dimension allowing $\mathcal{A}_2(P)$ to be hCM.

Proposition 6.13. *Let P be a 2-group with a cyclic derived subgroup.*

1. *If $\Omega_1(P)$ is isomorphic to $D_8^{*\ell} * SD_{2^{m+2}}$ or $D_8^{*\ell-1} * D^+_{2^{m+3}}$ for some $\ell \geqslant 1$ and $m > 1$, then $\mathcal{A}_2(P)$ is not hCM.*
2. *In all other cases, $\mathcal{A}_2(P)$ is hCM.*

Remark 6.14. The question whether $\mathcal{A}_p(P)$ is hCM when P is a p-group with a cyclic derived subgroup has already been considered by Matucci [10], but with strong restrictions in the case $p = 2$. Using more topological arguments, such as gluing lemmas, he was able to handle the two cases $P = E \times (D_8^{*\ell} * D_{2^{m+2}})$ and $P = E \times (D_8^{*\ell} * SD_{2^{m+2}})$, where E is elementary abelian.

In this section, we have restricted our attention to p-groups, but as the next result shows the hCM property can be transferred to p-nilpotent groups. Following Matucci's arguments given in Section 5 of [10], the following corollary should remain true if G is taken to be a solvable group containing a Sylow p-subgroup with a cyclic derived subgroup.

Proposition 6.15. *(See Corollary 11.4 in [12].) Let G be a p-nilpotent group with $P = G/O_{p'}(G)$. If $\mathcal{A}_p(P)$ is hCM of dimension d, then $\mathcal{A}_p(G)$ is hCM of dimension d.*

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