A Sufficient Condition For Computing N-Finger Force-Closure Grasps of 3D Objects

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***Abstract*—We address the problem of computing n-finger force-closure grasps of 3D objects. As $3D$ force-closure grasps involve $6D$ wrench space, we use Pl$\ddot{u}$cker coordinates and Grassmann algebra, to demonstrate that wrenches associated to any three non-aligned contact points of $3D$ objects form a basis of the $6D$ wrench space. Thus, given non-aligned locations of $n-1$ fingers, a $6D$ basis can be extracted form their wrenches. This permits the formulation of a fast and simple sufficient force-closure test. The problem is transformed to searching for a set of locations of the nth finger which wrenches can be uniquely expressed as a strictly negative linear combination of the $6D$ basis. We have implemented the algorithm and confirmed its efficiency by comparing it to the classical convex-hull method [21].**

***Index Terms*—Force-closure grasps, Grassmann algebra.**

I. Introduction

Many researchers are interested in developing humanoid robots to help people in their daily life. Such robots should be autonomous and able to interact with objects around them. Grasping is the central action of object manipulation. A grasp should satisfy several conditions such as stability, collision-avoidance, task compatibility etc. This paper considers generating stable grasps.

The stability of a grasp is characterized by force-closure property [1], under which arbitrary forces and torques exerted on the grasped object can be balanced by the contact forces applied by the fingers. We address the problem of computing n-finger force-closure grasps of 3D objects. We assume hard-finger point contacts with friction.
Salisbury and Roth [2] have proved that a necessary and sufficient condition for force closure is that the primitive contact wrenches resulted by contact forces at the contact points positively span the entire wrench space. This condition is equivalent to that the origin of the wrench space lies strictly inside the convex hull of the primitive contact wrenches [5], [6]. Based on the above necessary and sufficient conditions, various force-closure tests were proposed by Nguyen [12] and Mishra et al. [4].

In the past few years, several force-closure tests were also proposed. By introducing the polyhedral approximation of the non linear friction cone, Y.H. Liu [3] demonstrates that the problem of querying whether the origin lies inside the convex hull is equivalent to a ray shooting problem. Zhu and Wang [20] developed a numerical force closure test based on the concept of the Q-distance. All these methods require considerable computation time.

Heuristic approaches are a way to improve performance. Borst et al. [18] showed that with a strategy to randomly generate grasps and filter them with simple heuristics, the calculation of force-closure grasps can be done very fast. The heuristic of Niparnan and Sudsang [19] relies on a necessary but not sufficient condition of force-closure. It works as a filter that reports a fault positive but not a fault negative force-closure grasps. We propose a sufficient but not necessary method to compute force-closure grasps of 3D objects. Our approach works with general 3D objects and with any number $n \geq 4$ of contacts. The locations, normal directions and friction coefficient of these contacts are known.

The rest of the paper is organized as follows. Section 2 introduces notations and theorems used in the following of the paper. The proposed force-closure test is presented in section 3. Section 4 details the approach to compute n-finger force-closure grasps. Sections 5 shows experimental results and a comparison with the classical convex-hull method. Section 6 concludes.

II. Preliminaries

Our goal is to reduce the overall time to compute possible grasping points of the n-finger force-closure grasps. With a change of mathematical representation, we prove that wrenches, associated to any three non-aligned contact points of 3D objects form a basis of the $6D$ wrench space. This result induce the formulation of a simple sufficient condition of force-closure frictional grasps.
This section presents definitions, theorems and notations necessary for force-closure test elaboration.

A. Grasp Preliminaries

A grasp map or a wrench matrix is crucial to determine if a grasp has force-closure. This paragraph introduces all elements to compute this matrix.

Definition 1: A **grasp** is a set of contacts.

Definition 2: A **contact** is a location where a finger meets the object surface. Information about contact type and local object surface are required. We assume a point contact model with coulomb friction.

Definition 3: A **grasp force** f_i is a force applied by each finger to the object.

To ensure nonslipping at the contact point, the grasp force f_i must satisfy coulomb's law [22], [23]:

$$f_{ix}^2 + f_{iy}^2 \leq \mu^2 f_{iz}^2 \quad (1)$$

where $(f_{ix}^2, f_{iy}^2, f_{iz}^2)$ denotes x, y, z components of the grasp force f_i in the object coordinate frame and μ the friction coefficient.

Definition 4: The non linear constraint in (1) geometrically defines a cone called **friction cone**.

To simplify the problem, we linearize the friction cone by a polyhedral convex cone with m sides.

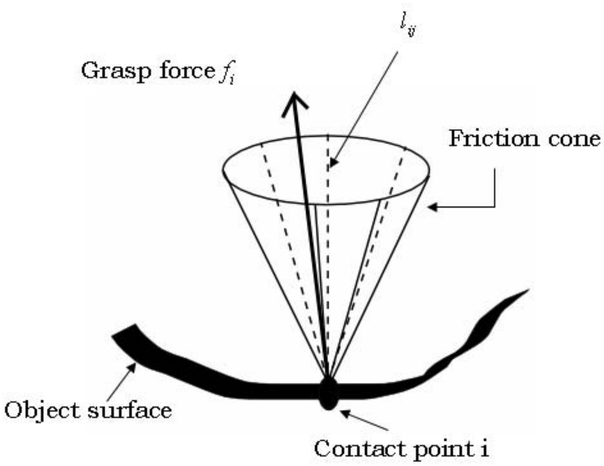

Fig. 1. The grasp force f_i in a linearized friction cone

Under this approximation, the grasp force can be represented as:

$$f_i = \sum_{i=1}^{m} \lambda_{ij} l_{ij}, \quad \lambda_{ij} \geq 0 \quad (2)$$

where l_{ij} represents the j-th edge vector of the polyhedral convex cone. Coefficients λ_{ij} are non negative constants.

Definition 5: A **wrench**, $\underline{w}_i$, is the combination of the force and torque corresponding to the grasp force f_i.

$$\underline{w}_i = \begin{pmatrix} f_i \\ \tau_i \end{pmatrix} = \begin{pmatrix} f_i \\ r_i \times f_i \end{pmatrix} \quad (3)$$

where r_i denotes the position vector of the $i-th$ grasp point in the object coordinate frame origined at the center of mass.

Substituting (2) into (3) provides:

$$\underline{w}_i = \sum_{j=1}^{m} \lambda_{ij} u_{ij} \quad (4)$$

where:

$$u_{ij} = \begin{pmatrix} l_{ij} \\ r_i \times l_{ij} \end{pmatrix} \quad (5)$$

We normalize vectors u_{ij} as follows:

$$w_{ij} = \frac{1}{\|l_{ij}\|} u_{ij} \quad (6)$$

The term $\|l_{ij}\|$ denotes the L_2 norm of vector l_{ij}. Vectors w_{ij} are called **primitive contact wrenches**. Thus, $N = mn$ is the total number of primitve contact wrenches applied at the object by n fingers.

Definition 6: The **wrench matrix**, W, is a $6 \times nm$ matrix where its column vectors are the *primitive contact wrenches*.

$$\mathbf{W} = \begin{pmatrix} l_{11} & \dots & l_{16} & \dots & l_{nm} \\ r_1 \times l_{11} & \dots & r_1 \times l_{16} & \dots & r_n \times l_{nm} \end{pmatrix}$$

B. Force-Closure Preliminaries

The stability of a grasp is characterized by force-closure property. This paragraph presents a definition of this property and an important result (proposition 3). This proposition is used, in section 3, to formulate a frictional force-closure test.

Definition 7: According to the definition of Salisbury [2], a grasp has **force-closure** if and only if any external wrench can be balanced by the wrenches at the fingertips.

Proposition 1: A necessary and sufficient condition for force-closure is that the primitive contact wrenches resulted by contact forces at the contact points positively span the entire 6-dimensional wrench space.
Proof. for a proof, the reader should refer to [2]. ■

Definition 8: A set of vectors , $\{v_i\}$, positively span a vector space if any vector v in this space can be written as a positive linear combination of v_i, namely :

$$v = \sum_{i \in I} \alpha_i v_i, \quad \alpha_i \geq 0 \quad (7)$$

Proposition 2: For any n-dimensional Euclidean space E^n, $n+1$ vectors are necessary to positively span E^n.
Proof. for a proof, the reader should refer to the relative linear algebra results presented by Goldman and Tucker [13]. ■

Lemma 1: Given a set of $n+1$ vectors, $v_1, v_2, ..., v_{n+1}$, in R^n, such that $v_1, v_2, ..., v_n$ are linearly independent and :

$$v_{n+1} = \sum_{i=1}^{n} \alpha_i v_i, \quad \alpha_i < 0 \tag{8}$$

Then each v_i, $i = 1, ..., n+1$, is a unique negative linear combination of the other n vectors [14].
Proof. It is obviously true for v_{n+1}. for any v_i, i=1,...,n, if we solve (8) for v_i, we have:

$$v_i = \frac{1}{\alpha_i} v_{n+1} - \sum_{j=1, j \neq i}^{n+1} \frac{\alpha_j}{\alpha_i} v_j \tag{9}$$

It is clear by (9) that v_i, i=1,...,n, is a unique negative linear combination of the other n vectors. ■

Proposition 3: A set of $n+1$ vectors $v_1, v_2, ..., v_{n+1}$ in R^n positively span E^n if and only if v_{n+1} is a unique linear combination of v_i, $i = 1, ..., n$ and all coefficients are strictly negative [14].
Proof. for a proof, the reader should refer to [14].■

C. Grassmann Algebra Preliminaries

As $3D$ force-closure grasps involve $6D$ wrench space. With a mere change of mathematical representation, using Grassmann algebra, we prove that wrenches, associated to any three non-aligned contact points of 3D objects, form a basis of the $6D$ wrench space.

Plücker coordinates: Let L be a line in the $3D$ space. Let u be the unit line direction and P a point chosen on L. The direction vector along with its cross product with P are known as Plücker coordinates and are denoted by $(u; P \times u)$. These 6 coordinates represent L in $3D$ space [15], [16]. Consequently a primitive contact wrench, defined as $w_i = (f_i; r_i \times f_i)$ can also be seen as a representation of the line of action L_{fi} of the force f_i applied at the point r_i. The 6 coordinates $(w_{i1}, w_{i2}, ..., w_{i6})$ of w_i are called the Plücker coordinates of the line of action of f.
The Plücker coordinates are homogenous coordinates for a projective space of dimension 5, P^5: the wrenches w_i and λw_i, with $\lambda \neq 0$ both represent the same line L_{fi}. Then every line L_{fi} in the $3D$ space corresponds exactly to one point in P^5. The set of lines form a quadric, called the Grassmannian, defined by $w_1 w_4 + w_2 w_5 + w_3 w_6 = 0$ in this projective space. At this point, we have defined a one-to-one relation between the set of lines in the $3D$ space and points in P^5. The rank of this mapping is 6.

Grassmann algebra : Grassmann studied manifold of lines which rank ranges varies from 0 to 6. The purpose of this study was to find geometric characterization of each variety. We are going to use two main results of this study. For a proof of these results, the reader should refer to [17].

Proposition 4: All lines through one point are of rank 3.

Proposition 5: When all lines meet one special line, they are of rank 5.

III. FOUR FINGER FORCE-CLOSURE GRASPS

At this point, we showed that a $6D$ contact wrench can be represented by the line of action of its corresponding force. We use this mapping to prove that wrenches associated to three non-aligned contact points are of rank 6. This result induces the formulation of a sufficient condition for four finger force-closure grasps.

Proposition 6: Wrenches associated to 3 aligned contact points are at most of rank 5.
Proof. A $6D$ contact wrench can be represented by the line of action of its corresponding force. The lines of action of forces applied at a contact point pass through that point. Thus wrenches associated to 3 aligned contact points meet one line, the one joining the 3 contact points. Consequently, from *proposition 5*, these wrenches are at most of rank 5. ■

Proposition 7: The 6 lines on the sides of a tetrahedron are independent, and thus form a $6D$ basis, (Fig. 2).

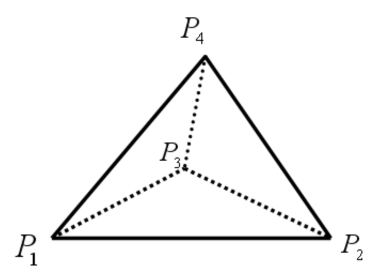

Fig. 2. The 6 lines of a tetrahedron are independent.

Proof. To deal with lines in $3D$-space, we need a 4-dimensional linear space. For a basis of this space we can either take a point, O and 3 vectors e_1, e_2, e_3 or 4 points (p_0, p_1, p_2, p_3). We can relate these by:

$$p_1 = O; \; p_2 = O + e_1; \; p_3 = O + e_2; \; p_4 = O + e_3$$

Any point can be written as a linear combination of these 4 points, for example:

$$P_a = a_1 p_1 + a_2 p_2 + a_3 p_3 + a_4 p_4$$
$$P_b = b_1 p_1 + b_2 p_2 + b_3 p_3 + b_4 p_4$$

where the a_i and b_i are scalars and the sums of the a_i and b_i are unity.

Lines are represented in Grassmannian terms by exterior products of points. Hence from these 4 independent basis points we can construct 6 independent lines which intersect to form a tetrahedron :

$$L_1 = p_1 \wedge p_2;\ L_2 = p_1 \wedge p_3;\ L_3 = p_1 \wedge p_4$$
$$L_4 = p_2 \wedge p_3;\ L_5 = p_2 \wedge p_4;\ L_6 = p_3 \wedge p_4$$

Any line is now able to be represented as a linear combination of these 6 basis lines. We can explicitly display this by multiplying out and simplifying the exterior product of two points on a chosen line:

$$L = P_a \wedge P_b = (a_1p_1 + a_2p_2 + a_3p_3 + a_4p_4)\wedge$$
$$(b_1p_1 + b_2p_2 + b_3p_3 + b_4p_4).\ \blacksquare$$

Proposition 8: Wrenches associated to 3 non-aligned contact points are of rank 6.

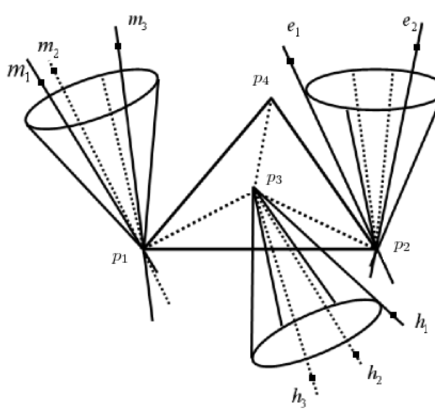

Fig. 3. The wrenches of rank 3 associated to the frictional contact points $p_1,\ p_2$ and p_3.

Proof. Let $p_1,\ p_2\ and\ p_3$ be 3 non-aligned contact points. Consider the friction cone associated to p_1, called CP_1 (Fig. 3). Let $\{m_1,\ m_2,\ m_3\}$ be three points chosen on any 3 non-coplanar lines of this cone. The lines $\{l_1 = p_1 \wedge m_1,\ l_2 = p_1 \wedge m_2,\ l_3 = p_1 \wedge m_3\}$ are of rank 3, (from *proposition 4*). Thus any line that passes through p_1 can be expressed as a linear combination of these 3 lines. Similarly, $\{e_1,\ e_2,\ e_3\}$ and $\{h_1,\ h_2,\ h_3\}$, are associated respectively to the friction cones $CP_2,\ CP_3$ at $p_2,\ p_3$. In the same manner, $\{l_4 = p_2 \wedge e_1,\ l_5 = p_2 \wedge e_2,\ l_6 = p_2 \wedge e_3\}$ and $\{l_7 = p_3 \wedge h_1,\ l_8 = p_3 \wedge h_2,\ l_9 = p_3 \wedge h_3\}$ are either of rank 3. Let p_4 be a point non-coplanar with $p_1,\ p_2,\ p_3$, so these 4 points constitute a tetrahedron.
The lines $(p_1 \wedge p_2)$, $(p_1 \wedge p_3)$ and $(p_1 \wedge p_4)$ can be expressed as a linear combination of $\{p_1 \wedge m_1,\ p_1 \wedge m_2,\ p_1 \wedge m_3\}$ since they all pass through p_1, thus:

$$p_1 \wedge p_2 = \sum_{i=1}^{3} \alpha_i(p_1 \wedge m_i) = \sum_{i=1}^{3} \alpha_i l_i$$
$$p_1 \wedge p_3 = \sum_{i=1}^{3} \beta_i(p_1 \wedge m_i) = \sum_{i=1}^{3} \beta_i l_i$$
$$p_1 \wedge p_4 = \sum_{i=1}^{3} \gamma_i(p_1 \wedge m_i) = \sum_{i=1}^{3} \gamma_i l_i$$

In the same manner, the lines $(p_2 \wedge p_3)$ and $(p_2 \wedge p_4)$ can be expressed as a linear combinations of $\{p_2 \wedge e_1,\ p_2 \wedge e_2,\ p_2 \wedge e_3\}$ since they pass through the contact point p_2. Finally the line $(p_3 \wedge p_4)$ passes through p_3 and thus can be expressed as a linear combination of $\{p_3 \wedge h_1,\ p_3 \wedge h_2,\ p_3 \wedge h_3\}$.

Since the lines of the tetrahedron are of rank 6 (from *proposition* 6*)*, they form a basis of R^6. We showed that the lines of the tetrahedron can be expressed as a linear combination of the 9 lines l_i. Thus these 9 lines, associated to the 3 friction cones, are also of rank 6. Consequently, a 6-dimensional basis can be extracted from these 9 lines. We remind the reader that the choice of 3 lines among the m sides of each linearized friction cone is due to the fact that these m lines are of rank 3 (from *proposition* 4*)*.$\blacksquare$

Proposition 9: Assume that the grasp of 3 non-aligned fingers is not force-closure. Suppose that $\{b_i\}$ is the 6-dimensional basis associated to their corresponding contact wrenches. A sufficient condition for a 4-finger force-closure grasp is that there exists a contact wrench γ such that:

- $\gamma\ is\ inside\ the\ linearized\ friction\ cone$
  $of\ the\ 4th\ finger$ (10)
- $\gamma = \sum_{i=1}^{6} \beta_i b_i,\ \beta_i < 0$

$$\Rightarrow\ \gamma = B\beta\quad \Rightarrow\ \beta = B^{-1}\gamma \tag{11}$$

where $B = [b_1, b_2, ..., b_6]$ is a 6×6 matrix and $\beta = [\beta_1, \beta_2, ..., \beta_6]^T$ is a 6×1 strictly negative vector. Thus, a simple multiplication by B^{-1} permits to test if a contact wrench γ, and consequently the location of the $4th$ contact point, ensures a force-closure grasp.

Proof. A necessary and sufficient condition for force-closure is that the primitive contact wrenches resulted by contact forces at the contact points positively span the entire 6-dimensional wrench space, (from *proposition* 1). A set of 7 vectors in R^6 positively span E^6 if and only if the seventh vector is a unique linear combination of the other six vectors and all coefficients are strictly negative, (from *proposition 3*). The seven vectors $\{\gamma, b_i, i = 1, ..6\}$ satisfy these conditions and thus positively span R^6. ■

IV. N-FINGER FORCE-CLOSURE GRASPS SYNTHESIS

We presented, in *proposition 9*, a sufficient condition for four-finger force-closure grasps. This condition can be easily applied to n-finger grasps ($n \geq 4$).
To achieve force-closure, the grasp matrix should positively span the wrench space (*proposition* 1). Our method generates first, randomly, locations of $n-1$ non-aligned fingers. We showed that wrenches associated to 3 non-aligned contact points are of rank 6 (*proposition* 8). Thus, we find then all 6-dimensional basis from the wrenches associated to these $n-1$ contacts. A limited number of basis is selected. A position of the nth finger is located such that an associated contact wrench can be uniquely expressed as a strictly negative linear combination of one of the basis (*proposition* 9).

A. The proposed approach

This paragraph details the different steps of the algorithm computing force closure grasps of a $3D$ object.

Algorithm

```
Input:   - points representing a 3D object
         - linearized friction cone at each point
           and corresponding wrenches
Output: - All possible locations of the n^th finger
BEGIN
    Rand_Fingers (n-1)
    basis = Find_Basis (wrenches)
    rbasis = Rand_Basis (basis)
    for  all object vertices
          if Force_Closure (vertex, rbasis)
                add_vertex_solution (vertexList)

return (vertexList)
END
```

Given a $3D$ representation of an object along with normal directions and a friction coefficient, wrenches associated to each of its vertices are firstly computed. In order to obtain n-finger force-closure grasps, the function *Rand_Fingers* generates randomly, locations of $n-1$ non-aligned fingers on the object surface. All 6-dimensional basis from the wrenches associated to these $n-1$ contacts are determined by *Find_Basis*. A limited number of basis, *rbasis*, is then randomly selected using *Rand_Basis* function. Finally, all object vertices are tested for a n-finger force-closure grasp with *Force_Closure*. A position of the nth finger is located such that an associated contact wrench can be uniquely expressed as a strictly negative linear combination of one of the basis. We choose the wrench associated to the normal force on the nth contact. If the grasp ensures force-closure, *add_vertex_solution* stores the corresponding vertex in a *vertexList*. The latter contains all the possible locations of the nth finger ensuring force-closure grasps.

TABLE I
COMPUTING 4-FINGER FORCE-CLOSURE GRASPS RESULTS

	number of Solutions			Time (s)		
	classic	new	ratio	classic	new	ratio
(a)	210	35	16.7%	2.483	0.697	28%
(b)	372	261	70%	3.556	0.719	20%
(c)	566	371	65%	5.1	1.02	20%
(d)	409	223	54.5%	4.78	1.07	22.4%
average			51.5%			22.6%

V. EXPERIMENTAL RESULTS

Our method sacrifices completeness in favor of fast computation. The obvious question is how it competes with a complete method. We choose to use in our experiments the classical complete method based on the construction of a 6D convex hull [21]. The process involves approximating the contact friction cones as a convex sum of a finite number of force vectors around the boundary of the cone, computing the associated object wrench for each force vector, and then finding the convex hull of this set of wrenches. If the origin is contained within this space, the grasp have force-closure. Otherwise, there exists some set of disturbance wrenches that cannot be resisted by the grasp.
We accomplish tests on four 3D object models, shown in (Fig. 4) represented by their vertices and their respective normal directions.

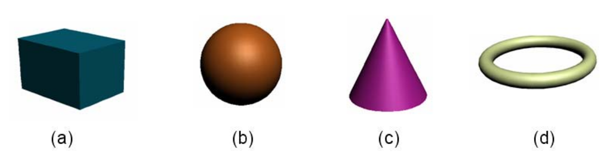

Fig. 4. 4 objects chosen as a testbed for our approach.

The two methods require the cone to be linearized. We use a 6-sided pyramid to represent a linear model of a cone. The test consists of randomly generating non-aligned locations of 3 fingers not in force-closure on each model. All vertices of the model are then tested to see if they ensure a four-finger force-closure grasp. With a 6-sided pyramid, the number of basis computed from the wrenches associated to the 3 fingers is approximately of 18000. We randomly choose 100 basis and test the model vertices for each of these basis. Let n be the number of vertices of a model, thus the time obtained with our method is for $100 \times n$ force-closure tests and the one obtained with the classical method is for n tests. We repeat this procedure 10 times for each model. The result of experiments are shown in table (I). Our method is labelled as "new" and the convex hull method as "classic".
The experiments were run on Pentium Core duo machine with $2GB$ memory and a CPU at 2.13 GHz . The program is implemented in C++.

VI. Discussion

The force-closure test we propose is sufficient but not necessary. In other words, our method reports many fault negative results (the method implies no force-closure when it exists). That is due to two reasons. The first one is the linearization of the friction cone, the second is that a point is not tested for a force-closure with all basis.
The first two columns, in (I), show the number of force-closure grasps found for each model. The fourth and the fifth columns show the corresponding computation time. We should mention that the time of computation varies between the different examples according to the number of their constituting vertices. The box, for example, is constituted of 602 vertices. As we randomly generate 10 times locations of 3 non-aligned fingers, 210 and 35 are respectively the average number of solution vertices found with the classical method and with ours. The computation time are respectively 2.483s and 0.697s. Thus, as we test force-closure for 100 basis, $0.697/(100\times602)$ and $2.483/602$, which are $1.15e-5s$ and $4.1e-3s$, are respectively the force-closure test time of our method and that of the classical method. Similarly, the tore is constituted of 1225 vertices. Thus, $1.07/(1225\times100) =\ 8.73e-006s$ and $4.78/1225 =\ 3.9e-3s$ are the corresponding force-closure test computation time. It is clear that our method is much faster but less complete. It reports many fault negative results. That is why we compute the ratio of the number of force-closure grasps found and the ratio of computation time. According to table (I), our method can find approximately 51.5% of the total number of solutions with a 22.6% time. In other words, it is half complete with the fifth computing time of the complete method.

We believe that the computation time of our method can still be improved by defining a criteria allowing the choice of good basis instead of selecting them randomly.

VII. Conclusions

Computation procedures of force-closure grasps are crucial to solve dextrous manipulation planning problems. Simplifying these procedures is an important contribution for robots to autonomously manipulate objects. Toward this end, we presented an efficient algorithm for computing n-finger force-closure grasps. We used Grassmann algebra to prove that wrenches associated to any three non-aligned contact points of $3D$ objects, form a basis of the 6-dimensional wrench space. Thus, given locations of $n-1$ fingers, not in force-closure, our algorithm is able to find wrench space basis associated to these contact points. It finds then all possible locations of the nth finger. A wrench associated to the nth finger is tested for ensuring n-finger force-closure by a simple inverse basis matrix multiplication.
Our method is general, it can be applied with any number $n \geq 4$ of fingers. The robot can chooses randomly the locations of the $n-1$ fingers on the object surface, with the only condition of the non-linearity of these contact points. Then, it finds the position of the nth finger.

VIII. Acknowledgments

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