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IMPEDANCES OF TWO DIMENSIONAL MULTILAYER CYLINDRICAL AND FLAT CHAMBERS IN THE NON-ULTRARELATIVISTIC CASE

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Abstract

Two dimensional electromagnetic models (i.e. assuming an infinite length) for the vacuum chamber elements in a synchrotron are often quite useful to give a first estimate of the total beam-coupling impedance. In these models, classical approximations can fail under certain conditions of frequency or material properties. We present here two formalisms for flat and cylindrical geometries, enabling the computation of fields and impedances in the multilayer case without any assumption on the frequency, beam velocity or material properties (except linearity, isotropy and homogeneity).

INTRODUCTION

In this old subject [1], the general formalism of B. Zotter [2] enables the analytical computation of the electromagnetic (EM) fields in frequency domain and the impedance created by a beam in an infinitely long multilayered cylindrical pipe made of any linear materials. Still, improvements of this formalism were possible for better accuracy and computational time, thanks in particular to a matrix formalism for the field matching. Also, it is possible to extend this theory to any azimuthal mode instead of only $m = 0$ and $m = 1$, enabling the computation of nonlinear terms in the EM force.

For multilayer flat chambers, the usual approach is to compute the beam coupling impedances thanks to a formula valid for an axisymmetric geometry multiplied by constant "Yokoya" form factors [3, 4], but this has been shown to fail in the case of non metallic materials such as ferrite [5] which is expected since Yokoya's theory relies on hypotheses that can be wrong for certain materials and/or certain frequencies. Therefore, we show here how we can provide a more general theory of the multilayer flat chamber impedance, going beyond the single-layer case [4] or the double-layer one [6, 7]. We use similar ideas as for a cylindrical geometry and apply them to an infinitely long and large flat chamber.

Details on the derivations below can be found in [8, 9].

ELECTROMAGNETIC FIELDS IN A CYLINDRICAL MULTILAYER CHAMBER

We consider a point-like beam of charge Q travelling at a speed $\upsilon = \beta c$ along the axis of an axisymmetric infinitely long pipe of inner radius b, at the position ($r = a_1, \theta = 0, s = \upsilon t$) in cylindrical coordinates. The source charge density is in frequency domain ($f = \frac{\omega}{2\pi}$), after the usual

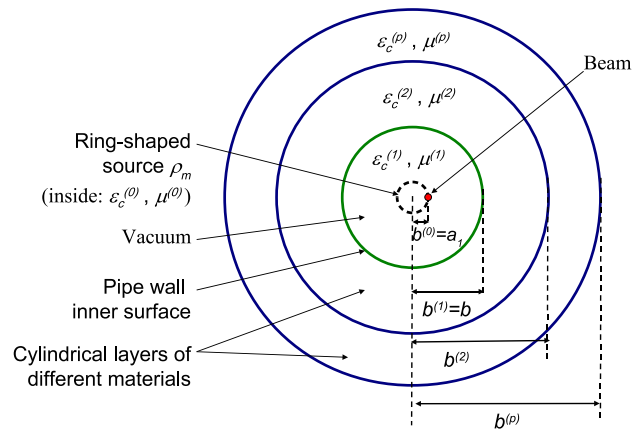

Figure 1: Cross section of the cylindrical chamber.

decomposition on azimuthal modes [8, 10]

$$\rho(r,\theta,s;\omega) = \sum_{m=0}^{\infty}\rho_m = \sum_{m=0}^{\infty}\frac{Q\cos(m\theta)\delta(r-a_1)e^{-jks}}{\pi\upsilon a_1(1+\delta_{m0})}, \quad (1)$$

where $k \equiv \frac{\omega}{\upsilon}$, δ is the delta function, and $\delta_{m0} = 1$ if $m = 0$, 0 otherwise. The space is divided into $N+1$ cylindrical layers of homogeneous, isotropic and linear media (see Fig. 1), each denoted by the superscript (p) $(0 \le p \le N)$. The last layer goes to infinity.

The macroscopic Maxwell equations in frequency domain for the electric and magnetic fields $\vec{E}$ and $\vec{H}$ are written [2]

$$\vec{\mathrm{curl}}\vec{H} - j\omega\vec{D} = \rho_m\upsilon\vec{e_s}, \quad \vec{\mathrm{curl}}\vec{E} + j\omega\vec{B} = 0,$$
$$\mathrm{div}\vec{D} = \rho_m, \quad \mathrm{div}\vec{B} = 0, \quad \vec{D} = \varepsilon_c\vec{E},\ \vec{B} = \mu\vec{H}, \quad (2)$$

where [11]

$$\varepsilon_c = \varepsilon_0\varepsilon_1 = \varepsilon_0\varepsilon_b\left[1 - j\tan\vartheta_E\right] + \frac{\sigma_{DC}}{j\omega(1+j\omega\tau)}, \quad (3)$$
$$\mu = \mu_0\mu_1 = \mu_0\mu_r\left[1 - j\tan\vartheta_M\right]. \quad (4)$$

In these expressions, ε_0 (μ_0) is the permittivity (permeability) of vacuum, ε_b the real dielectric constant, μ_r the real part of the relative complex permeability, $\tan\vartheta_E$ ($\tan\vartheta_M$) the dielectric (magnetic) loss tangent, σ_{DC} the DC conductivity and τ the Drude model relaxation time [12].

From Maxwell equations, one gets for each mode m [13]

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial s^2} + \omega^2\varepsilon_c\mu\right]E_s = \frac{1}{\varepsilon_c}\frac{\partial\rho_m}{\partial s} + j\omega\mu\rho_m\upsilon,$$
$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial s^2} + \omega^2\varepsilon_c\mu\right]H_s = 0.$$

Solutions are sought by separation of variables, in the form $R(r)\Theta(\theta)S(s)$. Θ and S are solutions of the harmonic differential equation. From the symmetries of the problem [8]

$$\Theta_{E_s}(\theta) \propto \cos(m_e\theta), \quad S_{E_s}(s) \propto e^{-jks},$$
$$\Theta_{H_s}(\theta) \propto \sin(m_h\theta), \quad S_{H_s}(s) \propto e^{-jks},$$

where m_e and m_h should be integer multiples of m. R_{E_s} (R_{H_s}) is a combination of modified Bessel functions of order m_e (m_h) and argument νr with $\nu \equiv k\sqrt{1-\beta^2\varepsilon_1\mu_1}$. From the boundary conditions between all the layers it can be first proven [8] that $m_e = m_h = m$. The longitudinal components of the fields are then in each layer (p) [13] (with $\vec{G} = Z_0\vec{H} = \mu_0 c\vec{H}$):

$$E_s^{(p)} = \cos(m\theta)e^{-jks}\left[C_{Ie}^{(p)} I_m\left(\nu^{(p)}r\right) + C_{Ke}^{(p)} K_m\left(\nu^{(p)}r\right)\right],$$
$$G_s^{(p)} = \sin(m\theta)e^{-jks}\left[C_{Ig}^{(p)} I_m\left(\nu^{(p)}r\right) + C_{Kg}^{(p)} K_m\left(\nu^{(p)}r\right)\right],$$

where the constants $C_{Ie}^{(p)}$, $C_{Ke}^{(p)}$, $C_{Ig}^{(p)}$ and $C_{Kg}^{(p)}$ depend on m and ω. The transverse components are found from [13]

$$E_r^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\frac{\partial E_s^{(p)}}{\partial r} + \frac{\beta\mu_1^{(p)}}{r}\frac{\partial G_s^{(p)}}{\partial\theta}\right), \tag{5}$$

$$E_\theta^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\frac{1}{r}\frac{\partial E_s^{(p)}}{\partial\theta} - \beta\mu_1^{(p)}\frac{\partial G_s^{(p)}}{\partial r}\right), \tag{6}$$

$$G_r^{(p)} = \frac{jk}{\nu^{(p)2}}\left(-\frac{\beta\varepsilon_1^{(p)}}{r}\frac{\partial E_s^{(p)}}{\partial\theta} + \frac{\partial G_s^{(p)}}{\partial r}\right), \tag{7}$$

$$G_\theta^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\beta\varepsilon_1^{(p)}\frac{\partial E_s^{(p)}}{\partial r} + \frac{1}{r}\frac{\partial G_s^{(p)}}{\partial\theta}\right). \tag{8}$$

Then, the boundary conditions at $r = a_1$ [10] and the finiteness of the fields at $r = 0$ and $r \to \infty$ give

$$C_{Ke}^{(0)} = C_{Kg}^{(0)} = C_{Kg}^{(1)} = C_{Ie}^{(N)} = C_{Ig}^{(N)} = 0,$$
$$C_{Ke}^{(1)} = \frac{2C}{1+\delta_{m0}} I_m\left(\frac{ka_1}{\gamma}\right),$$
$$C_{Ig}^{(0)} = C_{Ig}^{(1)}, \quad C_{Ie}^{(0)} = C_{Ie}^{(1)} + C_{Ke}^{(1)}\frac{K_m\left(\frac{ka_1}{\gamma}\right)}{I_m\left(\frac{ka_1}{\gamma}\right)}, \tag{9}$$

with $\gamma^{-2} = 1-\beta^2$ and $C = \frac{j\omega\mu_0 Q}{2\pi\beta^2\gamma^2}$. Expressing all the boundary conditions at $b^{(p)}$ for $1 \le p \le N-1$, it can be shown [8] that the constants of one layer are related to those from the adjacent layer through

$$\begin{bmatrix} C_{Ie}^{(p+1)} \\ C_{Ke}^{(p+1)} \\ C_{Ig}^{(p+1)} \\ C_{Kg}^{(p+1)} \end{bmatrix} = M_p^{p+1} \cdot \begin{bmatrix} C_{Ie}^{(p)} \\ C_{Ke}^{(p)} \\ C_{Ig}^{(p)} \\ C_{Kg}^{(p)} \end{bmatrix}, \; M_p^{p+1} = \begin{bmatrix} P_p^{p+1} & Q_p^{p+1} \\ S_p^{p+1} & R_p^{p+1} \end{bmatrix}, \tag{10}$$

where P_p^{p+1}, Q_p^{p+1}, R_p^{p+1} and S_p^{p+1} are 2×2 matrices:

$$P_p^{p+1} = \begin{bmatrix} \frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} I_m^{p,p} K_m'^{\,p+1,p} - \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} K_m^{p+1,p} I_m'^{\,p,p} & \frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} K_m^{p,p} K_m'^{\,p+1,p} - \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} K_m^{p+1,p} K_m'^{\,p,p} \\ -\frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} I_m^{p,p} I_m'^{\,p+1,p} + \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} I_m^{p+1,p} I_m'^{\,p,p} & -\frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} K_m^{p,p} I_m'^{\,p+1,p} + \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} I_m^{p+1,p} K_m'^{\,p,p} \end{bmatrix} \zeta_p^{p+1},$$

$$Q_p^{p+1} = \chi_p^{p+1}\begin{bmatrix} -I_m^{p,p} K_m^{p+1,p} & -K_m^{p,p} K_m^{p+1,p} \\ I_m^{p,p} I_m^{p+1,p} & K_m^{p,p} I_m^{p+1,p} \end{bmatrix},$$

$$R_p^{p+1} = \begin{bmatrix} \frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} I_m^{p,p} K_m'^{\,p+1,p} - \frac{\mu_1^{(p)}}{\nu^{(p)}} K_m^{p+1,p} I_m'^{\,p,p} & \frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} K_m^{p,p} K_m'^{\,p+1,p} - \frac{\mu_1^{(p)}}{\nu^{(p)}} K_m^{p+1,p} K_m'^{\,p,p} \\ -\frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} I_m^{p,p} I_m'^{\,p+1,p} + \frac{\mu_1^{(p)}}{\nu^{(p)}} I_m^{p+1,p} I_m'^{\,p,p} & -\frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} K_m^{p,p} I_m'^{\,p+1,p} + \frac{\mu_1^{(p)}}{\nu^{(p)}} I_m^{p+1,p} K_m'^{\,p,p} \end{bmatrix} \xi_p^{p+1},$$

$$S_p^{p+1} = \frac{\varepsilon_1^{(p+1)}}{\mu_1^{(p+1)}} Q_p^{p+1}, \tag{11}$$

with $\zeta_p^{p+1} = \frac{-\nu^{(p+1)2} b^{(p)}}{\varepsilon_1^{(p+1)}}$, $\chi_p^{p+1} = \frac{m\left(\nu^{(p)2} - \nu^{(p+1)2}\right)}{\nu^{(p)2}\beta\varepsilon_1^{(p+1)}}$, $\xi_p^{p+1} = \frac{-\nu^{(p+1)2} b^{(p)}}{\mu_1^{(p+1)}}$, $I_m^{p+1,p} = I_m\left(\nu^{(p+1)} b^{(p)}\right)$, $I_m^{p,p} = I_m\left(\nu^{(p)} b^{(p)}\right)$ and similar definitions with I_m', K_m and K_m'.

Iteratively applying Eq. (10) and solving leads to

$$C_{Ie}^{(1)} = -C_{Ke}^{(1)}\alpha_{\rm TM} = -C_{Ke}^{(1)}\frac{\mathcal{M}_{12}\mathcal{M}_{33} - \mathcal{M}_{32}\mathcal{M}_{13}}{\mathcal{M}_{11}\mathcal{M}_{33} - \mathcal{M}_{13}\mathcal{M}_{31}},$$
$$C_{Ig}^{(1)} = C_{Ke}^{(1)}\alpha_{\rm TE} = C_{Ke}^{(1)}\frac{\mathcal{M}_{12}\mathcal{M}_{31} - \mathcal{M}_{32}\mathcal{M}_{11}}{\mathcal{M}_{11}\mathcal{M}_{33} - \mathcal{M}_{13}\mathcal{M}_{31}},$$
$$C_{Ke}^{(N)} = \mathcal{M}_{21}C_{Ie}^{(1)} + \mathcal{M}_{22}C_{Ke}^{(1)} + \mathcal{M}_{23}C_{Ig}^{(1)},$$
$$C_{Kg}^{(N)} = \mathcal{M}_{41}C_{Ie}^{(1)} + \mathcal{M}_{42}C_{Ke}^{(1)} + \mathcal{M}_{43}C_{Ig}^{(1)}, \tag{12}$$

where $\mathcal{M} \equiv M_{N-1}^N \cdot M_{N-2}^{N-1} \cdots M_1^2$ is a 4×4 matrix, and we have defined $\alpha_{\rm TM}$ and $\alpha_{\rm TE}$ as in [13]. When summing all modes m to get the EM response to the initial point-like source in Eq. (1), we get for the total longitudinal electric field in the vacuum region

$$E_{s,tot}^{vac} = Ce^{-jks}\left[K_0\left(\frac{k}{\gamma}\sqrt{a_1^2 + r^2 - 2a_1 r\cos\theta}\right) - 2\sum_{m=0}^{\infty}\frac{\alpha_{\rm TM}(m)\cos(m\theta)}{1+\delta_{m0}} I_m\left(\frac{ka_1}{\gamma}\right) I_m\left(\frac{kr}{\gamma}\right)\right]. \tag{13}$$

Note that similar matrix formalisms in other theoretical frameworks have been developped in [14, 15].

ELECTROMAGNETIC FIELDS IN A FLAT MULTILAYER CHAMBER

Here we also consider a point-like beam of charge Q travelling at a speed $\upsilon = \beta c$ at the position ($x = 0, y = y_1, s = \upsilon t$) in cartesian coordinates, along an infinitely long and large flat chamber of half gap b. We write the source charge density in frequency domain as

$$\rho(x, y, s; \omega) = \frac{Q}{\upsilon}\delta(x)\delta(y - y_1)e^{-jks}, \tag{14}$$

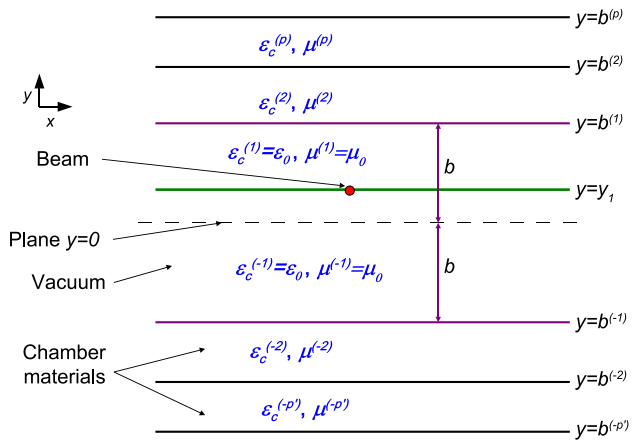

Figure 2: Cross section of the flat chamber.

with the same notations as above. Using the horizontal Fourier transform and dropping the $\int_0^{+\infty} dk_x$ factor, we want first to compute the response to the source

$$\tilde{\rho}(k_x, y, s; \omega) = \frac{Q}{\pi \upsilon} \cos(k_x x)\delta(y - y_1)e^{-jks}, \qquad (15)$$

which corresponds to a surface charge density on the plane $y = y_1$. The space is divided into $N + M$ layers parallel to the $y = 0$ plane (see Fig. 2), denoted by the superscript (p) where $-M \leq p \leq N$, $p \neq 0$, with the same general assumptions as in the axisymmetric case.

The macroscopic Maxwell equations in frequency domain for the electric and magnetic fields $\vec{E}$ and $\vec{H}$ are written as above in Eqs. (2), replacing ρ_m by $\tilde{\rho}$, ε_c and μ being given by Eqs. (3) and (4). One can then get the wave equations

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial s^2} + \omega^2 \varepsilon_c \mu\right] E_s = \frac{1}{\varepsilon_c}\frac{\partial \tilde{\rho}}{\partial s} + j\omega\mu\tilde{\rho}\upsilon,$$
$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial s^2} + \omega^2 \varepsilon_c \mu\right] H_s = 0.$$

Solutions are sought in the form $X(x)Y(y)S(s)$. We get three harmonic differential equations, and from the finiteness of $X(\pm\infty)$ and the symmetries of the problem

$$X_{E_s}(x) \propto \cos(k_{x_e} x), \qquad X_{H_s}(x) \propto \sin(k_{x_h} x),$$
$$S_{E_s}(s) \propto e^{-jks}, \qquad S_{H_s}(s) \propto e^{-jks}.$$

From the boundary conditions at $y = b^{(p)}$ and $y = y_1$ it can be shown that $k_{x_e} = k_{x_h} = k_x$ in all the layers [9]. Defining then $\nu^{(p)}$, $\vec{G}$ as above and $k_y^{(p)} = \sqrt{k_x^2 + \nu^{(p)2}}$, we get the fields longitudinal components in layer (p):

$$E_s^{(p)} = \cos(k_x x)e^{-jks}\left[C_{e+}^{(p)} e^{k_y^{(p)} y} + C_{e-}^{(p)} e^{-k_y^{(p)} y}\right],$$
$$G_s^{(p)} = \sin(k_x x)e^{-jks}\left[C_{g+}^{(p)} e^{k_y^{(p)} y} + C_{g-}^{(p)} e^{-k_y^{(p)} y}\right],$$

where the constants $C_{e+}^{(p)}$, $C_{e-}^{(p)}$, $C_{g+}^{(p)}$ and $C_{g-}^{(p)}$ depend on k_x and ω. The transverse components are found from

$$E_x^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\frac{\partial E_s^{(p)}}{\partial x} + \beta\mu_1^{(p)}\frac{\partial G_s^{(p)}}{\partial y}\right), \qquad (16)$$
$$E_y^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\frac{\partial E_s^{(p)}}{\partial y} - \beta\mu_1^{(p)}\frac{\partial G_s^{(p)}}{\partial x}\right), \qquad (17)$$
$$G_x^{(p)} = \frac{jk}{\nu^{(p)2}}\left(-\beta\varepsilon_1^{(p)}\frac{\partial E_s^{(p)}}{\partial y} + \frac{\partial G_s^{(p)}}{\partial x}\right), \qquad (18)$$
$$G_y^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\beta\varepsilon_1^{(p)}\frac{\partial E_s^{(p)}}{\partial x} + \frac{\partial G_s^{(p)}}{\partial y}\right). \qquad (19)$$

Then the boundary conditions at $y = y_1$ give

$$C_{g+}^{(1)} = C_{g+}^{(-1)}, \quad C_{g-}^{(1)} = C_{g-}^{(-1)},$$
$$C_{e+}^{(1)} = C_{e+}^{(-1)} - C\frac{e^{-k_y^{(1)} y_1}}{k_y^{(1)}}, \quad C_{e-}^{(1)} = C_{e-}^{(-1)} + C\frac{e^{k_y^{(1)} y_1}}{k_y^{(1)}},$$

with C defined as in the cylindrical case. We can express all the boundary conditions at $b^{(p)}$ for $1 \leq p \leq N-1$ (upper layers) in a matrix form:

$$\begin{bmatrix} C_{e+}^{(p+1)} \\ C_{e-}^{(p+1)} \\ C_{g+}^{(p+1)} \\ C_{g-}^{(p+1)} \end{bmatrix} = M_p^{p+1} \cdot \begin{bmatrix} C_{e+}^{(p)} \\ C_{e-}^{(p)} \\ C_{g+}^{(p)} \\ C_{g-}^{(p)} \end{bmatrix}, \; M_p^{p+1} = \begin{bmatrix} P_p^{p+1} & Q_p^{p+1} \\ S_p^{p+1} & R_p^{p+1} \end{bmatrix}, \qquad (20)$$

where P_p^{p+1}, Q_p^{p+1}, R_p^{p+1} and S_p^{p+1} are 2×2 matrices:

$$P_p^{p+1} = \begin{bmatrix} \frac{1+\psi_p^{p+1}}{2} e_-^{p,p+1} & \frac{1-\psi_p^{p+1}}{2}\left(e_+^{p,p+1}\right)^{-1} \\ \frac{1-\psi_p^{p+1}}{2} e_+^{p,p+1} & \frac{1+\psi_p^{p+1}}{2}\left(e_-^{p,p+1}\right)^{-1} \end{bmatrix},$$
$$Q_p^{p+1} = \frac{k_x\left(\frac{\nu^{(p+1)2}}{\nu^{(p)2}} - 1\right)}{2\beta k_y^{(p+1)}\varepsilon_1^{(p+1)}} \begin{bmatrix} -e_-^{p,p+1} & -\left(e_+^{p,p+1}\right)^{-1} \\ e_+^{p,p+1} & \left(e_-^{p,p+1}\right)^{-1} \end{bmatrix},$$
$$R_p^{p+1} = \begin{bmatrix} \frac{1+\phi_p^{p+1}}{2} e_-^{p,p+1} & \frac{1-\phi_p^{p+1}}{2}\left(e_+^{p,p+1}\right)^{-1} \\ \frac{1-\phi_p^{p+1}}{2} e_+^{p,p+1} & \frac{1+\phi_p^{p+1}}{2}\left(e_-^{p,p+1}\right)^{-1} \end{bmatrix},$$
$$S_p^{p+1} = \frac{\varepsilon_1^{(p+1)}}{\mu_1^{(p+1)}} Q_p^{p+1}, \qquad (21)$$

with

$$\psi_p^{p+1} = \frac{\nu^{(p+1)2} k_y^{(p)} \varepsilon_1^{(p)}}{\nu^{(p)2} k_y^{(p+1)} \varepsilon_1^{(p+1)}}, \; \phi_p^{p+1} = \frac{\nu^{(p+1)2} k_y^{(p)} \mu_1^{(p)}}{\nu^{(p)2} k_y^{(p+1)} \mu_1^{(p+1)}},$$
$$e_+^{p,p+1} = e^{\left(k_y^{(p)} + k_y^{(p+1)}\right) b^{(p)}}, \; e_-^{p,p+1} = e^{\left(k_y^{(p)} - k_y^{(p+1)}\right) b^{(p)}}.$$

We can write the same relations as in Eqs. (20) to (21) for the lower layers $-M \leq p \leq -1$ simply by replacing $p+1$ with $p-1$. Defining then $\mathcal{M} \equiv M_{N-1}^{N} \cdot M_{N-2}^{N-1} \cdots M_1^2$, $\mathcal{M}' = M_{-M+1}^{-M} \cdot M_{-M+2}^{-M+1} \cdots M_{-1}^{-2}$, and the 4×4 matrix $\mathcal{P}$ with the lines 1 and 3 of $\mathcal{M}$ and 2 and 4 of $\mathcal{M}'$ in this order, using

the finiteness of $Y(\pm\infty)$, we solve and find [9]

$$C_{e+}^{(1)} = -\frac{C}{k_y^{(1)}}\left[\chi_1(k_x)e^{k_y^{(1)}y_1} + \eta_1(k_x)e^{-k_y^{(1)}y_1}\right],$$
$$C_{e-}^{(-1)} = -\frac{C}{k_y^{(1)}}\left[\chi_2(k_x)e^{k_y^{(1)}y_1} + \eta_2(k_x)e^{-k_y^{(1)}y_1}\right], \quad (22)$$

with ($i = 1$ or 2)

$$\chi_i(k_x) = \left(\mathcal{P}^{-1}\right)_{i1}\mathcal{M}_{12} + \left(\mathcal{P}^{-1}\right)_{i2}\mathcal{M}_{32},$$
$$\eta_i(k_x) = \left(\mathcal{P}^{-1}\right)_{i3}\mathcal{M}'_{21} + \left(\mathcal{P}^{-1}\right)_{i4}\mathcal{M}'_{41}. \quad (23)$$

Note that χ_i and η_i depend on k_x but not on y_1. The total fields due to our initial point-like source are obtained by integration over k_x. Using the polar coordinates (r, θ) in the (x, y) plane, in the vacuum region we can cast E_s into [9]:

$$E_{s,tot}^{vac} = Ce^{-jks}\left[K_0\left(\frac{k}{\gamma}\sqrt{x^2 + (y - y_1)^2}\right)\right.$$
$$\left. -4\sum_{m,n=0}^{+\infty}\frac{\alpha_{mn}\cos\left(n\theta - \frac{n\pi}{2}\right)}{(1+\delta_{m0})(1+\delta_{n0})}I_m\left(\frac{ky_1}{\gamma}\right)I_n\left(\frac{kr}{\gamma}\right)\right], \quad (24)$$

where α_{mn} are obtained by integrals that can be computed numerically:

$$\alpha_{mn} = \int_0^\infty du\cosh(mu)\cosh(nu)\left[\chi_1\left(\frac{k}{\gamma}\sinh u\right)\right.$$
$$+(-1)^m\eta_1\left(\frac{k}{\gamma}\sinh u\right) + (-1)^n\chi_2\left(\frac{k}{\gamma}\sinh u\right)$$
$$\left. +(-1)^{m+n}\eta_2\left(\frac{k}{\gamma}\sinh u\right)\right]. \quad (25)$$

The first term in $E_{s,tot}^{vac}$ is the direct space-charge part, independent on the chamber. The other term is the "wall" part of the fields, as α_{mn} depend only on the chamber properties and on ω. Those coefficients are the analoguous of $\alpha_{\mathrm{TM}}(m)$ in the cylindrical case.

IMPEDANCES

For both cases, we can now proceed to the impedances (longitudinal and transverse) for a test particle located at $(x_2 = r_2\cos\theta_2, y_2 = r_2\sin\theta_2)$, and generalizing (thanks to the symmetries of both geometries) the source position at $(x_1 = r_1\cos\theta_1, y_1 = r_1\sin\theta_1)$. Using the definitions from [13] and Eqs. (16) to (19) in vacuum (valid for both the cylindrical and flat chamber cases):

$$Z_\parallel = -\frac{1}{Q}\int^L ds E_{s,tot}^{vac}(x_2, y_2, s; \omega)e^{jks},$$
$$Z_x = -\frac{1}{kQ}\int^L ds\frac{\partial E_{s,tot}^{vac}}{\partial x}(x_2, y_2, s; \omega)e^{jks},$$
$$Z_y = -\frac{1}{kQ}\int^L ds\frac{\partial E_{s,tot}^{vac}}{\partial y}(x_2, y_2, s; \omega)e^{jks}, \quad (26)$$

the integration going over the length L of the element.

Direct Space-charge Impedance

From the direct space-charge part of $E_{s,tot}^{vac}$, in both the axisymmetric and flat chamber cases we get the multimode direct space-charge impedances [8]:

$$Z_\parallel^{SC,direct} = -\frac{jL\mu_0\omega}{2\pi\beta^2\gamma^2}K_0\left(\frac{kd_{1,2}}{\gamma}\right),$$
$$Z_x^{SC,direct} = \frac{jL\mu_0\omega}{2\pi\beta^2\gamma^3}K_1\left(\frac{kd_{1,2}}{\gamma}\right)\frac{x_2 - x_1}{d_{1,2}},$$
$$Z_y^{SC,direct} = \frac{jL\mu_0\omega}{2\pi\beta^2\gamma^3}K_1\left(\frac{kd_{1,2}}{\gamma}\right)\frac{y_2 - y_1}{d_{1,2}}, \quad (27)$$

with $d_{1,2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ the distance between the source and the test particles.

Wall Impedance in the Cylindrical Case

We obtain the "wall" impedance [11] (i.e. the impedance due to the chamber itself, including the indirect space-charge or perfect conductor part of the fields, but excluding the direct space-charge isolated above) when writing Eqs. (26) with Eq. (13) without the direct space-charge. Up to first order in the source and test positions we get

$$Z_\parallel^{Wall} = \frac{jL\mu_0\omega}{2\pi\beta^2\gamma^2}\alpha_{\mathrm{TM}}(0),$$
$$Z_x^{Wall} = \frac{jLZ_0k^2}{4\pi\beta\gamma^4}\left[\alpha_{\mathrm{TM}}(1)x_1 + \alpha_{\mathrm{TM}}(0)x_2\right],$$
$$Z_y^{Wall} = \frac{jLZ_0k^2}{4\pi\beta\gamma^4}\left[\alpha_{\mathrm{TM}}(1)y_1 + \alpha_{\mathrm{TM}}(0)y_2\right]. \quad (28)$$

In transverse, in addition to the usual dipolar impedance (coefficient in front of x_1 and y_1 in Z_x^{Wall} and Z_y^{Wall}), we find a term proportional to x_2 or y_2, which is a transverse quadrupolar impedance [16, 17]. In most classical theories this term is thought to be 0 in axisymmetric structures; we find here that in principle it is not the case.

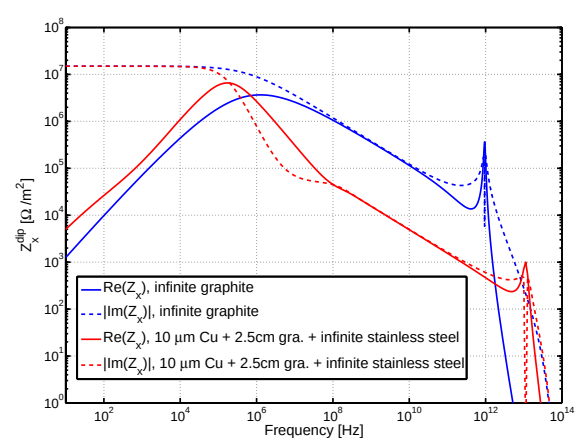

Figure 3: Dipolar wall impedance/L (i.e. per unit length) for a round collimator of one or three layers ($\gamma = 479.6$, $b = 2$mm, $\sigma_{DC,Cu} = 5.9 \cdot 10^7$S/m, $\tau_{Cu} = 27$ fs, $\sigma_{DC,gra} = 10^5$S/m, $\tau_{gra} = 0.8$ ps, $\sigma_{DC,ss} = 10^6$S/m, $\tau_{ss} = 0$, and in all layers $\varepsilon_b = \mu_r = 1$ and $\vartheta_E = \vartheta_M = 0$).

In Fig. 3 we show the dipolar wall impedance in the case of a graphite round collimator, with one layer or three layers. The difference between the two is mainly due to the copper coating in the three layers case, and this difference decreases at low frequencies because the fields penetrate deep inside the collimator wall.

Wall Impedance in the Flat Chamber Case

Plugging now Eq. (24) into Eqs. (26) we get for the wall impedances up to first order in the source and test positions

$$\begin{aligned} Z_{\|}^{Wall} &= \frac{jL\mu_0\omega}{2\pi\beta^2\gamma^2}\alpha_{00}, \\ Z_x^{Wall} &= \frac{jLZ_0k^2}{4\pi\beta\gamma^4}(\alpha_{02}-\alpha_{00})(x_1-x_2), \\ Z_y^{Wall} &= \frac{jLZ_0k^2}{4\pi\beta\gamma^4}\left[\frac{2\gamma}{k}\alpha_{01}+2\alpha_{11}y_1+(\alpha_{00}+\alpha_{02})y_2\right]. \end{aligned} \quad (29)$$

Due to the absence of top-bottom symmetry, there is a constant term in the vertical impedance. Also, contrary to usual ultrarelativistc results, $Z_x^{Wall,quad} \neq -Z_y^{Wall,quad}$.

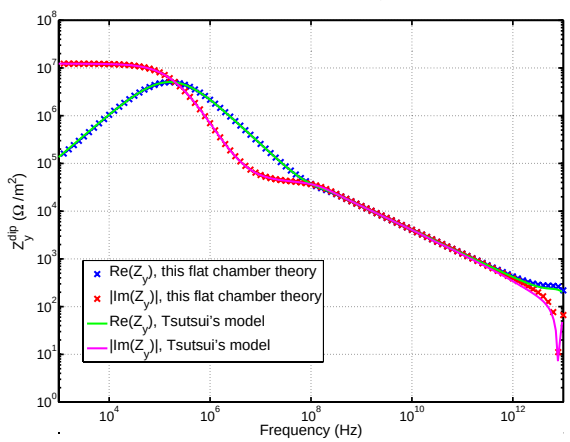

Figure 4: Vertical dipolar impedance/L for a three-layer copper coated graphite flat collimator (parameters in Fig. 3). In Tsutsui's model the third layer is replaced by a perfect conductor and the plates perpendicular to the large flat jaws are 25cm apart.

In Fig. 4 we have plotted the vertical dipolar impedance of a copper coated graphite flat collimator, comparing our results to Tsutsui's model [18] on a rectangular geometry. The agreement between the two approaches is very good.

Form Factors Between the Two Geometries

The ratio of the flat chamber impedances to the cylindrical ones (longitudinal for $F_{\|}$, dipolar term only for the others) give us form factors that are a frequency and material dependent generalization of the Yokoya factors:

$$\begin{aligned} F_{\|} &= \frac{\alpha_{00}}{\alpha_{\rm TM}(0)},\ F_x^{dip} = \frac{\alpha_{02}-\alpha_{00}}{\alpha_{\rm TM}(1)},\ F_y^{dip} = \frac{2\alpha_{11}}{\alpha_{\rm TM}(1)}, \\ F_x^{quad} &= \frac{\alpha_{00}-\alpha_{02}}{\alpha_{\rm TM}(1)},\ F_y^{quad} = \frac{\alpha_{00}+\alpha_{02}}{\alpha_{\rm TM}(1)}. \end{aligned} \quad (30)$$

In Fig. 5 we have plotted those form factors for the case of

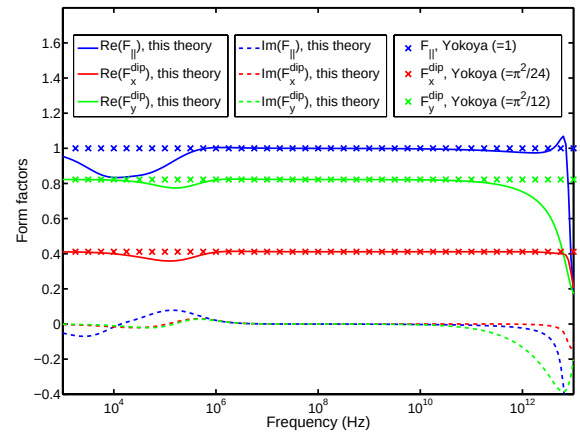

Figure 5: Form factors in the case of a three-layer copper coated collimator (see Figs. 3 and 4).

the copper coated collimator already investigated in Figs. 3 and 4. Deviations from the usual Yokoya factors are significant mainly at high frequencies but can also be seen below 1 MHz, in particular for the longitudinal impedance.

CONCLUSION

Two dimensional models giving the EM fields and impedance in respectively cylindrical and flat multilayer chambers have been presented. They rely only on basic assumptions on the materials (linearity, isotropy and homogeneity) such that they are valid in principle at any frequency and for any beam velocity. Thanks to the matrix formalism used, the number of layers in the structure is no longer an issue.

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