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Electromagnetic field created by a macroparticle in an infinitel long and axisymmetric multilayer beam pipe

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Abstract

This paper aims at giving an as complete and detailed as possible derivation of the six electromagnetic field components created by an offset point charge travelling at any speed in an infinitely long circular multilayer beam pipe. Outcomes from this study are a novel and efficient matrix method for the field matching determination of all the constants involved in the field components, and the generalization to any azimuthal mode together with the final summation on all such modes in the impedance formulas. In particular the multimode direct space-charge impedances (both longitudinal and transverse) are given, as well as the wall impedance to any order of precision. New quadrupolar terms for the transverse wall impedance are found, which look negligible in the ultrarelativistic case but might be of significance for low-energy beams. In principle from this analysis the electromagnetic fields created by any particular source, with a finite transverse shape, can then be computed using convolutions.

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Contents

1 Introduction

The impedance (or, equivalently, dispersion relation coefficients) in synchrotrons have been analytically computed for more than forty years [1]. The formalism used, in particular the source terms, changed since then, together with the appoximations made, as machines open up new frequency domains or pipe material evolves in such a way that usual approximations do not apply anymore. A general formalism is therefore very useful, for present and forthcoming impedance calculations. This is particularly true for the LHC, where the wall impedance coming from graphite collimators, combined to the low revolution frequency, makes the classic thick wall formula be wrong in at least part of the frequency domain of interest.
Also, one often limits oneself to the computation of impedance, which is the quantity needed to characterize the effect of the environment on the beam dynamics. However, more generally the six components of the electromagnetic fields are also of interest, because knowing them enables a better understanding of the behaviour of the impedance with frequency and various parameters. There are also relevant from the measurement point of view, since impedance measurements often come from field measurements, especially the magnetic field, and some of the approximations that are then made need to be checked theoretically.
Therefore, we will try here to present a general formalism, introduced first by Bruno Zotter (see Ref. [2] and references therein), with as little approximations as possible, which will enable us to compute analytically the electromagnetic fields and impedance created by a beam in an accelerator. Both longitudinal and transverse cases are studied, for a cylindrical pipe made of any resistive, dielectric or magnetic material, assuming only its linearity, isotropy, homogeneity (in each layer) and the validity of Ohm's law. We derive here the most general and exact (within the assumptions made, see next section) formulae in frequency domain, in the multimode and multilayer case. We restrict ourselves to the single-bunch case[1], and to an infinitely long pipe wall.
The paper is structured as follows. We start by giving the source creating the electromagnetic fields in the pipe in Section 2. Then we introduce Maxwell equations and the various material constants used in Section 3. After that we derive the general expressions for the longitudinal components of the fields in Section 4, from which the transverse components are then computed in Section 5. The various constants are computed in Section 6, using a new matrix method. The electromagnetic forces on a test particle are also derived in Section 7, followed by the impedance calculation in Section 8, with new results concerning the multimode analysis. Our concluding remarks follow finally in Section 9.
Note that the whole paper is expressed in SI (or MKSA) units.

2 Source charges and currents

We consider a macroparticle of charge Q travelling at a speed v along the axis of a cylindrical pipe of inner radius b. In the single-bunch case v is the speed of the beam v_b, but in the multibunch case or for a coasting beam, the source travels on a wave with speed $v \neq v_b$ (see Refs. [2, 3] for more details). The pipe is supposed to be infinitely long (no side effects). In time domain, the charge is supposed to be slightly offset from the center of the pipe by a distance a along the horizontal direction. Neglecting any transient effects, and using the cylindrical coordinates (r, θ, s) (s stands for the coordinate along the axis of the cylinder, which is also assumed to be the azimuthal coordinate along the beam orbit in the accelerator – we therefore neglect all curvature effects which is a good approximation for accelerators of long radius of curvature like the LHC; we refer the reader to Refs. [4–9] for details about such effects), the charge is thus supposed to be at $r = a$, $\theta = 0$ (or $\theta = 2l\pi$ where l is an integer) and $s = vt$ [10, p. 5], so that the charge density[2] is [11]

$$\rho(r, \theta, s; t) = \frac{Q}{a}\delta(r - a)\delta_p(\theta)\delta(s - vt), \tag{2.1}$$

[1] Still, we will sometimes make a difference between the beam wave velocity and the test particle velocity, so that the formalism could be used in the multibunch case as well.

[2] This charge density (and the corresponding current density) is valid when only one bunch is passing through the beam pipe. In circular accelerators we should in principle take into account multiturn effects, in that case a bunch is passing several times at a position s, all separated by an integer number of T_{rev}, the revolution period along the orbit. This means that we have to replace $\delta(s - vt)$ by $\sum_{l=-\infty}^{\infty} \delta(s - v(t - lT_{rev}))$ if at $t = 0$ the bunch is at $s = 0$. For this study only single-turn effects are considered. Maxwell equations being linear we could anyway perform such a multiturn sum on the resulting fields we compute in this paper.

where δ is the Dirac distribution, i.e. such that for any function f, $\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0)$, and δ_p is a $2\pi-$periodic Dirac distribution, i.e. $\delta_p(\theta) = \sum_{l=-\infty}^{\infty} \delta(\theta - 2l\pi)$ (this is to take into account the periodicity of the azimuthal angle θ). As expected we get $\iiint_\Omega \rho(r,\theta,s;t) r\mathrm{d}r\mathrm{d}\theta\mathrm{d}s = Q$ for any volume Ω around $r = a$, $\theta = 0$ and $s = \upsilon t$ (thus the $\frac{1}{a}$ factor in the charge density).
It is convenient to solve Maxwell equations in frequency domain. To do so we write the factor $\delta(s - \upsilon t)$ in terms of its Fourier spectrum [12, 13]

$$\begin{aligned} \delta(s-\upsilon t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jk(s-\upsilon t)}\mathrm{d}k \\ &= \frac{1}{2\pi\upsilon}\int_{-\infty}^{\infty} e^{j\omega\left(t-\frac{s}{\upsilon}\right)}\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{j\omega t}\frac{e^{-jks}}{\upsilon}, \end{aligned} \tag{2.2}$$

where

$$k \equiv \frac{\omega}{\upsilon}, \tag{2.3}$$

is the wave number. We drop the factor $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{j\omega t}$ to proceed to the frequency domain, to get

$$\rho(r,\theta,s;\omega) = \frac{Q}{a\upsilon}\delta(r-a)\delta_p(\theta)e^{-jks}. \tag{2.4}$$

Since the macroparticle is supposed to travel at speed υ along the s axis, the current density is obtained in general by [11]

$$\vec{J} = \rho\upsilon\vec{e_s},$$

$\vec{e_s}$ being the unit vector along the s axis. Therefore we get for our source, in frequency domain

$$\vec{J}(r,\theta,s;\omega) = \frac{Q}{a}\delta(r-a)\delta_p(\theta)e^{-jks}\vec{e_s}. \tag{2.5}$$

We can rewrite Eqs. (2.4) and (2.5) by using the Fourier series expansion on azimuthal modes of the $\delta_p(\theta)$ factor [11, 12]

$$\delta_p(\theta) = \frac{1}{2\pi} + \sum_{m=1}^{\infty}\frac{\cos(m\theta)}{\pi}. \tag{2.6}$$

We then obtain for the charge density [10, p. 5], [14]:

$$\rho(r,\theta,s;\omega) = \sum_{m=0}^{\infty}\frac{P_m\cos(m\theta)}{\pi\upsilon a^{m+1}(1+\delta_{m0})}\delta(r-a)e^{-jks}, \tag{2.7}$$

where $\delta_{m0} = 1$ if $m = 0$, 0 otherwise, and $P_m = Qa^m$ is the m^{th} multipole moment of the source. We can now isolate each azimuthal mode, obtaining finally for the charge density [13, 14]

$$\rho_m(r,\theta,s;\omega) = \frac{P_m\cos(m\theta)}{\pi\upsilon a^{m+1}(1+\delta_{m0})}\delta(r-a)e^{-jks}, \tag{2.8}$$

and for the current density along the s axis

$$J_m(r,\theta,s;\omega) = \frac{P_m\cos(m\theta)}{\pi a^{m+1}(1+\delta_{m0})}\delta(r-a)e^{-jks}. \tag{2.9}$$

For a given azimuthal mode m, Eqs. (2.8) and (2.9) give the charge and current densities in frequency domain that we will use to solve the electromagnetic fields. They are those of a wave of frequency ω propagating along the s axis with a wave number k. Since Maxwell equations are linear in ρ and $\vec{J}$ (see next section), to obtain the fields for the initial sources in $\delta_p(\theta)$ from Eqs. (2.4) and (2.5) we will need to sum the responses from all the azimuthal modes m, as in Eq. (2.7). Then to get back to time domain we should put back the $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{j\omega t}$ factor and integrate our frequency domain solutions.

3 Maxwell equations

The macroscopic Maxwell equations in frequency domain for the electric and magnetic fields $\vec{E}$ and $\vec{H}$ in a general linear and isotropic medium are [2]

$$\begin{aligned} \mathrm{div}\vec{D} &= \rho_m, & (3.1)\\ \vec{\mathrm{curl}}\vec{H} - j\omega\vec{D} &= \vec{J_m}, & (3.2)\\ \vec{\mathrm{curl}}\vec{E} + j\omega\vec{B} &= 0, & (3.3)\\ \mathrm{div}\vec{B} &= 0, & (3.4)\end{aligned}$$

where ρ_m and $\vec{J_m} = J_m\vec{e_s}$ are given in the whole space by Eqs. (2.8) and (2.9). The electric displacement $\vec{D}$ and the magnetic induction $\vec{B}$ are defined using complex permittivities and permeabilities ε_c and μ

$$\begin{aligned} \vec{D} &= \varepsilon_c\vec{E}, & (3.5)\\ \vec{B} &= \mu\vec{H}, & (3.6)\end{aligned}$$

where

$$\begin{aligned} \varepsilon_c &= \varepsilon_0\varepsilon_1 = \varepsilon_0\left(\varepsilon_r' - j\varepsilon_r''\right) = \varepsilon_0\varepsilon_b\left[1 - j\tan\vartheta_E\right] + \frac{\sigma}{j\omega}, & (3.7)\\ \mu &= \mu_0\mu_1 = \mu_0\mu_r\left[1 - j\tan\vartheta_M\right]. & (3.8)\end{aligned}$$

In these expressions, ε_1 (μ_1) is the relative complex permittivity (permeability) of the medium, ε_0 (μ_0) the permittivity (permeability) of vacuum, μ_r is the real part of the relative complex permeability and $\tan\vartheta_M$ is the magnetic loss tangent (that depends on frequency). ε_1 can be written in terms of its real and imaginary parts ε_r' and $-\varepsilon_r''$, or equivalently in terms of the "normal" complex dielectric constant of the medium $\varepsilon_b\left[1 - j\tan\vartheta_E\right]$ (including dielectric losses with the frequency dependent quantity $\tan\vartheta_E$, see Appendix A) and an electric conductivity σ [15, p. 312]. In the general case, σ depends on the angular frequency ω and contributes both to the real and imaginary parts of ε_1. We use in this paper an AC complex conductivity following the Drude model (see Refs. [15, p. 312] and [16, p. 16], with an opposite sign convention for ω in the latter)

$$\sigma = \frac{\sigma_{DC}}{1 + j\omega\tau}, \qquad (3.9)$$

where σ_{DC} is the DC conductivity of the pipe material and τ its relaxation time. It is here important to note that we assume that Ohm's law (in its local sense, i.e. the proportionality between the induced conductive current density and the electric field, at any point) holds for the media involved. Doing so we neglect magnetoresistance effects (see Refs. [16, pp. 11-15 and 234-239] and [17]) and the so-called "anomalous skin effect" [17–24]. Both might appear at low temperature, and very high magnetic fields for the former (several Teslas), or very high frequencies for the latter (see Ref. [25] for some examples of relevant limits).

Equations (3.5) and (3.6) can be derived from the general microsopic Maxwell equations, as shown in Appendix A. The idea is that charges and currents induced in the medium can be taken into account by adding both the polarization term and the conductive term to the electric displacement, instead of adding these charges and currents to the right-hand side of the two inhomogeneous Maxwell equations (3.1) and (3.2)[3].
Note that the minus signs in front of $\tan\vartheta_E$, $\tan\vartheta_M$ and ε_r'' is a convention (see e.g. Ref. [28]) to ensure that the energy dissipation is positive in the medium if $\mu_r\tan(\vartheta_M) \geq 0$ and $\varepsilon_r'' \geq 0$ for positive frequencies (and the

[3]The definitions used for $\vec{D}$ and $\vec{B}$ are consistent with the boundary conditions that will be used later on. In Refs. [2, 14], $\vec{D}$ was instead set to $\varepsilon_0\varepsilon_b\vec{E}$ (with $\tan\vartheta_E$ assumed to be zero) in Eq. (3.1) but not in Eq. (3.2) (the conduction term being treated separately), which has been found to lead to an inconsistency when writing the boundary conditions for the electric displacement radial components at the pipe surface when no surface charges or currents are taken into account (these radial conditions cannot be enforced if the tangential ones are, see Section 6). As only the tangential boundary conditions were used in these references (because the radial ones should have been redundant), it had no consequence on the resulting fields and impedances. The above expression (3.5) for $\vec{D}$ was also found in Refs. [26, 27] (in Ref. [26, p. 33-34], it is likely that there was a misprint in the expression of $\vec{D}$ in Eq. (I.2), since the following wave equations of pages 33 and 34 are consistent with our expression of $\vec{D}$).

opposite condition for negative frequencies[4]). This is shown in Ref. [29, p. 274] (where actually the opposite convention holds since fields are taken to be proportional to $e^{-j\omega t}$ instead of $e^{j\omega t}$).

We will add the superscript (p) to all the quantities related to the properties of the medium where it is not clear from the context, when considering a region of space made of one homogeneous material, i.e. with uniform values of ε_c and μ. The space is thus divided into $N+1$ (with $N \geq 2$) cylindrical layers of radii $b^{(p)}$, as shown in Fig. 1. Finally, when needed we will sometimes assume a positive angular frequency ω. The fields in frequency domain for $\omega < 0$ can be obtained by noting that all the time domain field components should be real, which means that for any field component φ the quantity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{j\omega t}\varphi(\omega) = \frac{1}{2\pi}\int_{0}^{\infty} \mathrm{d}\omega \left[e^{j\omega t}\varphi(\omega) + e^{-j\omega t}\varphi(-\omega)\right] = \frac{1}{2\pi}\int_{0}^{\infty} \mathrm{d}\omega \left[e^{j\omega t}\varphi(\omega) + \left(e^{j\omega t}\varphi(-\omega)^*\right)^*\right], \tag{3.12}$$

is real ($*$ denotes the complex conjugate). This is true if

$$\varphi(-\omega) = \varphi(\omega)^*. \tag{3.13}$$

Therefore we can use the above equation to compute the field components for negative frequencies.

4 Longitudinal components of the electromagnetic f elds

We apply Maxwell equations in a region where ε_c and μ are constant (boundary conditions will be considered in Section 6), so that we will omit the superscript (p).
Applying the $\vec{\mathrm{curl}}$ operator to Maxwell's equation (3.3), we obtain

$$\vec{\mathrm{curl}}\left(\vec{\mathrm{curl}}\vec{E}\right) + j\omega\mu\vec{\mathrm{curl}}\vec{H} = 0.$$

Using the "$\vec{\mathrm{curl}}\ \vec{\mathrm{curl}}$" relation (Eq. (C.1) of Appendix C), injecting Maxwell equations (3.1) and (3.2), and knowing that $\vec{J_m} = \rho_m \upsilon \vec{e_s}$, we then get

$$\nabla^2\vec{E} + \omega^2\varepsilon_c\mu\vec{E} = \frac{1}{\varepsilon_c}\vec{\mathrm{grad}}\rho_m + j\omega\mu\rho_m\upsilon\vec{e_s}. \tag{4.1}$$

Similarly, we can apply the $\vec{\mathrm{curl}}$ operator to Maxwell's equation (3.2) to obtain

$$\vec{\mathrm{curl}}\left(\vec{\mathrm{curl}}\vec{H}\right) - j\omega\varepsilon_c\vec{\mathrm{curl}}\vec{E} = \vec{\mathrm{curl}}\left(\rho_m\upsilon\vec{e_s}\right),$$

which gives, with Eqs. (3.3) and (3.4), using also the expression of the $\vec{\mathrm{curl}}$ operator in cylindrical coordinates from Eq. (B.3) of Appendix B for the right-hand side

$$\nabla^2\vec{H} + \omega^2\varepsilon_c\mu\vec{H} = \upsilon\frac{\partial\rho_m}{\partial r}\vec{e_\theta} - \frac{\upsilon}{r}\frac{\partial\rho_m}{\partial\theta}\vec{e_r}. \tag{4.2}$$

The wave equations (4.1) and (4.2) turn out to be relatively simple for the longitudinal field components. Using the expressions of the gradient and the laplacian in cylindrical coordinates (see Eqs. (B.1), (B.4) and (B.5) of Appendix B), we get the following scalar Helmholtz equations [2, 14]

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial s^2} + \omega^2\varepsilon_c\mu\right]E_s = \frac{1}{\varepsilon_c}\frac{\partial\rho_m}{\partial s} + j\omega\mu\rho_m\upsilon, \tag{4.3}$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial s^2} + \omega^2\varepsilon_c\mu\right]H_s = 0. \tag{4.4}$$

[4]Therefore, if $\tan\vartheta_E$ and $\tan\vartheta_M$ are taken to be independent on frequency, they should still be dependent on the sign of the frequency, and in this case we should write

$$\varepsilon_c = \varepsilon_0\varepsilon_b\left[1 - j\ \mathrm{sign}(\omega)\tan\vartheta_E\right] + \frac{\sigma}{j\omega}, \tag{3.10}$$

$$\mu = \mu_0\mu_r\left[1 - j\ \mathrm{sign}(\omega)\tan\vartheta_M\right]. \tag{3.11}$$

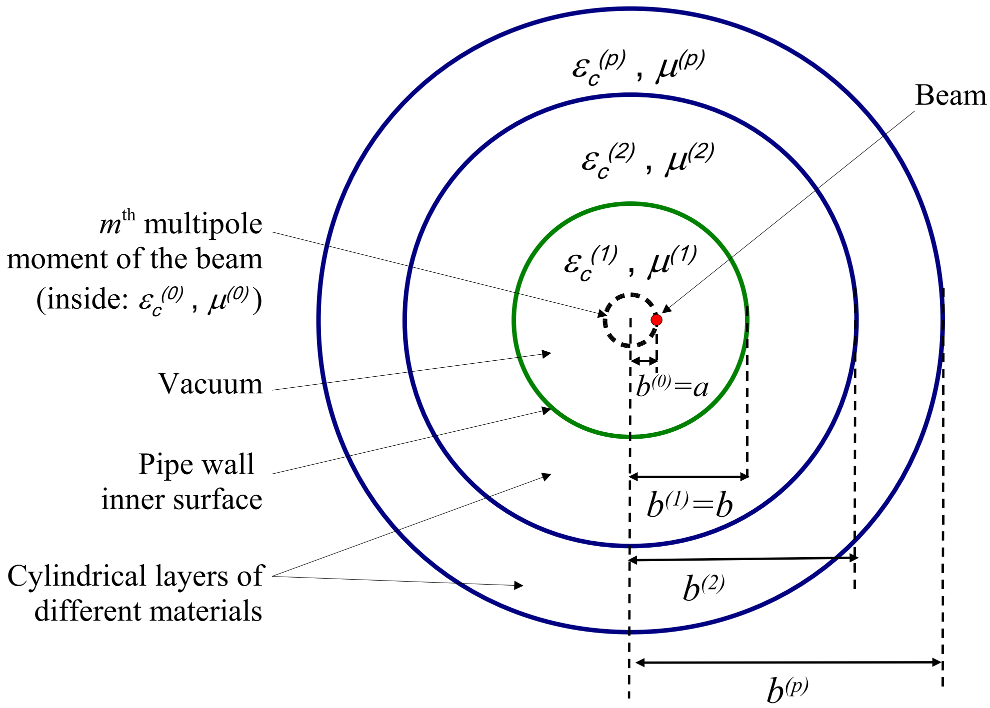

Figure 1: Cross section of the pipe. The region denoted by the superscript (0) is the vacuum inside the m^{th} multipole moment of the beam and is of radius a, and the region denoted by the superscript (1) is the vacuum between the beam and the pipe wall at $r = b$. Subsequent layers can be made of any medium. The last layer (denoted by the superscript (N)) has an infinite radius $b^{(N)} = \infty$. We have also sketched in red the initial beam source, before its azimuthal mode decomposition using Eq. (2.6), which is a point-like charge at $r = a$ and $\theta = 0$.

In any region where $r \neq a$, those equations are homogeneous and we can seek solutions by separation of variables, in the general form $R(r)\Theta(\theta)S(s)$. For both E_s and H_s we have

$$\Theta(\theta)S(s)\frac{(rR'(r))'}{r} + \frac{R(r)}{r^2}S(s)\Theta''(\theta) + R(r)\Theta(\theta)S''(s) + \omega^2\varepsilon_c\mu R(r)\Theta(\theta)S(s) = 0, \tag{4.5}$$

where the prime $'$ denotes the derivative with respect to the argument of the function (e.g. $R'(r) = \frac{dR}{dr}$). This gives, when dividing by $R(r)\Theta(\theta)S(s)$

$$\frac{(rR'(r))'}{rR(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} + \frac{S''(s)}{S(s)} + \omega^2\varepsilon_c\mu = 0,$$

or equivalently

$$\begin{cases} \frac{\Theta''(\theta)}{\Theta(\theta)} & = r^2\left(-\omega^2\varepsilon_c\mu - \frac{(rR'(r))'}{rR(r)} - \frac{S''(s)}{S(s)}\right) & = \text{constant} = A, \\ \frac{S''(s)}{S(s)} & = -\omega^2\varepsilon_c\mu - \frac{(rR'(r))'}{rR(r)} - \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} & = \text{constant} = B. \end{cases} \tag{4.6}$$

Therefore both Θ and S are solutions of the harmonic differential equation. θ being the azimuthal coordinate, Θ needs to be $2\pi-$periodic for the field component to be single valued, so that A must be real and negative. The solutions for $\Theta(\theta)$ are then of the form

$$\Theta(\theta) = \kappa_1\cos(\sqrt{-A}\theta) + \kappa_2\sin(\sqrt{-A}\theta),$$

and the periodicity constraint then requires $\sqrt{-A}$ to be an integer, that we will note m_e for the longitudinal component of the electric field, and m_h for that of the magnetic field. We notice also that the whole problem

(including boundary conditions, and source charges and currents from Eqs. (2.8) and (2.9)) have a rotational invariance of angle $\frac{2\pi}{m}$ about the s axis. Consequently, m_e and m_h should be integer multiples of the azimuthal mode number m. Finally, the $\theta = 0 \,(\mathrm{mod}\,\pi)$ plane is a symmetry plane of the problem (corresponding to the invariance with the sign of θ of the charge and current densities as well as boundary conditions), therefore the electric field should also be symmetric with respect to that plane, and in particular the longitudinal component (which is parallel to the symmetry plane). So there is no sine term in Θ_{E_s} for E_s. This is the opposite for the longitudinal component of H_s: we should imagine current loops creating the magnetic field, perpendicular to it; for the longitudinal part of the field, such loops are perpendicular to the $\theta = 0 \,(\mathrm{mod}\,\pi)$ symmetry plane, and their reflected images are current loops where the current flows in the opposite direction, in such a way that the magnetic field created by them is opposite. Therefore $H_s(-\theta) = -H_s(\theta)$, which means that there is no cosine term in Θ_{H_s}.
Similarly to what we saw for the azimuthal dependence of the fields, the whole problem exhibits a translational invariance along the s axis, the translation vector being $\frac{2\pi}{k}\vec{e_s}$. Therefore the resulting fields should exhibit the same invariance, so B must be real and negative, and the longitudinal dependence $S(s)$ can be taken of the form:

$$S(s) = \kappa_3 e^{-j\sqrt{-B}s} + \kappa_4 e^{j\sqrt{-B}s}.$$

From the translational invariance, $\sqrt{-B}$ should then be an integer multiple of k, that we will write lk. If we (temporary) get back to the time domain, applying the inverse Fourier transform $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{j\omega t}$ we see that we get two terms, one whose integrand is proportional to

$$\kappa_3 e^{j\omega\left(t-\frac{ls}{v}\right)},$$

and the other whose integrand is proportional to

$$\kappa_4 e^{j\omega\left(t+\frac{ls}{v}\right)}.$$

Since this must be invariant with respect to the transformation $(t_0, s_0) \longrightarrow (t_0+t, s_0+vt)$ for any (t_0, s_0, t) (which is an invariance property of the time domain initial problem), and recalling that $k \equiv \frac{\omega}{v}$, this clearly means that the second term is zero ($\kappa_4 = 0$) and that $l = 1$.
For the azimuthal and longitudinal parts, we finally get

$$\begin{aligned}
\Theta_{E_s}(\theta) &\propto \cos(m_e\theta), \\
\Theta_{H_s}(\theta) &\propto \sin(m_h\theta), \\
S_{E_s}(s) &\propto e^{-jks}, \\
S_{H_s}(s) &\propto e^{-jks}.
\end{aligned}$$

The radial dependence is then obtained by reinjecting those expressions into Eq.(4.5) and dividing by $\Theta(\theta)S(s)$. For instance for the longitudinal component of $\vec{E}$ we obtain

$$\frac{(rR'(r))'}{r} - \frac{R(r)}{r^2}m_e^2 - R(r)k^2 + \omega^2\varepsilon_c\mu R(r) = 0,$$

or equivalently

$$r^2R''(r) + rR'(r) - \left[m_e^2 + r^2\left(k^2 - \omega^2\varepsilon_c\mu\right)\right]R(r) = 0. \tag{4.7}$$

Now we define the radial propagation constant as in Ref. [14] (using the definitions of Eqs. (3.7) and (3.8), and the identity $\varepsilon_0\mu_0 = \frac{1}{c^2}$ where c is the speed of light in vacuum)

$$\nu^2 = k^2 - \omega^2\varepsilon_c\mu = k^2\left(1 - \beta^2\varepsilon_1\mu_1\right), \tag{4.8}$$

so that

$$\nu = |k|\sqrt{1 - \beta^2\varepsilon_1\mu_1}, \tag{4.9}$$

where the square root of a complex number is defined by $\sqrt{\alpha e^{j\varphi}} = \sqrt{\alpha}e^{j\frac{\varphi}{2}}$ with $-\pi < \varphi \leq \pi$, and $\beta \equiv \frac{v}{c}$ is the relativistic velocity factor. With the change of variable $z = \nu r$ and assuming $\nu \neq 0$ (i.e. $\omega \neq 0$ and $\epsilon_1\mu_1 \neq \frac{1}{\beta^2}$) we get Eq. (D.1) of Appendix D, whose solutions are the modified Bessel functions $I_{m_e}(\nu r)$ and

$K_{m_e}(\nu r)$[5]. The same derivation is applicable to the radial dependence of H_s. Putting all the integration constants together into $R(r)$ and reinserting the superscript (p) for each cylindrical layer (note that at that point, $m_e^{(p)}$ and $m_h^{(p)}$ could be in principle dependent on the layer p), the longitudinal components of the electromagnetic fields can be written [14]

$$E_s^{(p)} = \cos(m_e^{(p)}\theta)e^{-jks}\left[C_{Ie}^{(p)} I_{m_e^{(p)}}\left(\nu^{(p)}r\right) + C_{Ke}^{(p)} K_{m_e^{(p)}}\left(\nu^{(p)}r\right)\right], \tag{4.10}$$

$$H_s^{(p)} = \sin(m_h^{(p)}\theta)e^{-jks}\left[C_{Ih}^{(p)} I_{m_h^{(p)}}\left(\nu^{(p)}r\right) + C_{Kh}^{(p)} K_{m_h^{(p)}}\left(\nu^{(p)}r\right)\right], \tag{4.11}$$

where the subscripts (Ie, Ke, Ih and Kh) of the integration constants are self-explanatory.[6]

5 Transverse components of the electromagnetic f elds

Again, we apply Maxwell equations in a region where ε_c and μ are constant and we omit the superscript (p). Writing the transverse components of Eqs. (3.2) and (3.3) in cylindrical coordinates (see Appendix B) and assuming $r \neq a$, we have the relations

$$\frac{1}{r}\frac{\partial H_s}{\partial \theta} - \frac{\partial H_\theta}{\partial s} = j\omega\varepsilon_c E_r, \tag{5.1}$$

$$\frac{\partial H_r}{\partial s} - \frac{\partial H_s}{\partial r} = j\omega\varepsilon_c E_\theta, \tag{5.2}$$

$$\frac{1}{r}\frac{\partial E_s}{\partial \theta} - \frac{\partial E_\theta}{\partial s} = -j\omega\mu H_r, \tag{5.3}$$

$$\frac{\partial E_r}{\partial s} - \frac{\partial E_s}{\partial r} = -j\omega\mu H_\theta. \tag{5.4}$$

Differentiating with respect to s Eq. (5.4) and combining it to Eq. (5.1), we get, knowing the longitudinal dependence of E_s

$$\frac{\partial^2 E_r}{\partial s^2} + \omega^2\varepsilon_c\mu E_r = -jk\frac{\partial E_s}{\partial r} - j\omega\mu\frac{1}{r}\frac{\partial H_s}{\partial \theta}. \tag{5.5}$$

In the same way, we can differentiate with respect to s Eqs. (5.3), (5.2) and (5.1), then combine them respectively to Eqs. (5.2), (5.3) and (5.4), to get

$$\frac{\partial^2 E_\theta}{\partial s^2} + \omega^2\varepsilon_c\mu E_\theta = -jk\frac{1}{r}\frac{\partial E_s}{\partial \theta} + j\omega\mu\frac{\partial H_s}{\partial r}, \tag{5.6}$$

$$\frac{\partial^2 H_r}{\partial s^2} + \omega^2\varepsilon_c\mu H_r = j\omega\varepsilon_c\frac{1}{r}\frac{\partial E_s}{\partial \theta} - jk\frac{\partial H_s}{\partial r}, \tag{5.7}$$

$$\frac{\partial^2 H_\theta}{\partial s^2} + \omega^2\varepsilon_c\mu H_\theta = -j\omega\varepsilon_c\frac{\partial E_s}{\partial r} - jk\frac{1}{r}\frac{\partial H_s}{\partial \theta}. \tag{5.8}$$

At given r and θ, these four equations can all be written in the form

$$\frac{\partial^2 \psi}{\partial s^2} + \omega^2\varepsilon_c\mu\psi = Ae^{-jks},$$

[5] $I_{m_e}(-\nu r)$ and $K_{m_e}(-\nu r)$ are also solutions of Eq. (4.7) which depends on ν through its square, but these solutions are linearly bound to $I_{m_e}(\nu r)$ and $K_{m_e}(\nu r)$ from Eqs. 9.6.30 and 9.6.31 of Ref. [30], which give in our case

$$I_{m_e}(-\nu r) = (-1)^{m_e} I_{m_e}(\nu r),$$
$$K_{m_e}(-\nu r) = (-1)^{m_e} K_{m_e}(\nu r) - j\pi I_{m_e}(\nu r).$$

[6] Note that with our definition of ν (in particular, its proportionality to the absolute value of k), we have, as will be explained below, $\nu(-\omega) = \nu(\omega)^*$. This comes from the fact that $\nu^2(-\omega) = \left(\nu(\omega)^2\right)^*$ since $\varepsilon_1(-\omega) = \varepsilon_1(\omega)^*$ and a similar relation for μ_1, which can be seen in Eqs. (3.10), (3.11) and (3.9), or more generally in Ref. [15, p. 332]. Then if $\nu^2 = k^2\alpha e^{j\varphi}$, we have $\alpha(-\omega) = \alpha(\omega)$ and $\varphi(-\omega) = -\varphi(\omega)$ which gives $\nu(-\omega) = \nu(\omega)^*$ from $\nu = |k|\sqrt{\alpha}e^{j\frac{\varphi}{2}}$. Equation (D.8) then gives $I_{m_e^{(p)}}\left(\nu(-\omega)^{(p)}r\right) = I_{m_e^{(p)}}\left(\nu(\omega)^{(p)}r\right)^*$ and a similar relation with $K_{m_e^{(p)}}$. The same kind of relations apply to the constants as they are obtained from linear equations whose coefficients depend on $\varepsilon_1^{(p)}$, $\mu_1^{(p)}$, $\nu^{(p)}$ and modified Bessel functions of the arguments $\nu^{(p)}b^{(p)}$ and $\nu^{(p+1)}b^{(p)}$ (see Section 6). Therefore we have $E_s^{(p)}(-\omega) = E_s^{(p)}(\omega)^*$ and the same relation for the magnetic field and the transverse components (from the equations of the next section), which means that Eq. (3.13) is true for the field components without having to apply it "by hand".

ψ being the field component considered, and A a constant with respect to s. The general solution of this equation is

$$\psi(s) = \eta_1 e^{-j\omega\sqrt{\varepsilon_c\mu}s} + \eta_2 e^{j\omega\sqrt{\varepsilon_c\mu}s} + \eta_3 e^{-jks},$$

where η_3 is related to A. The first two terms are obviously not invariant with respect to the translation of vector $\frac{2\pi}{k}\vec{e_s}$ (see Section 4) which means that we have to drop them: $\eta_1 = \eta_2 = 0$. Therefore ψ is proportional to e^{-jks}, the proportionallity constant depending only on r and θ. So the transverse components have the same longitudinal dependence as the longitudinal components (i.e. in e^{-jks}), and we can rewrite Eqs. (5.5) to (5.8) in the form (reinserting the superscript (p) to avoid any confusion) [2, 14]

$$E_r^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\frac{\partial E_s^{(p)}}{\partial r} + \frac{\upsilon\mu^{(p)}}{r}\frac{\partial H_s^{(p)}}{\partial \theta}\right), \tag{5.9}$$

$$E_\theta^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\frac{1}{r}\frac{\partial E_s^{(p)}}{\partial \theta} - \upsilon\mu^{(p)}\frac{\partial H_s^{(p)}}{\partial r}\right), \tag{5.10}$$

$$H_r^{(p)} = \frac{jk}{\nu^{(p)2}}\left(-\frac{\upsilon\varepsilon_c^{(p)}}{r}\frac{\partial E_s^{(p)}}{\partial \theta} + \frac{\partial H_s^{(p)}}{\partial r}\right), \tag{5.11}$$

$$H_\theta^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\upsilon\varepsilon_c^{(p)}\frac{\partial E_s^{(p)}}{\partial r} + \frac{1}{r}\frac{\partial H_s^{(p)}}{\partial \theta}\right). \tag{5.12}$$

6 Field matching

To specify the field components we need to express the boundary conditions between all the cylindrical layers. For simplicity, we will assume from this section onward that the angular frequency ω is positive[7].

6.1 Boundary conditions at $r = a$

Firstly, from Ref. [30] we know that for any $\eta \geq 0$, $I_\eta(0)$ is finite while $K_\eta(z)$ goes to infinity when $|z| \to 0$. Therefore, for the first layer we have

$$C_{Ke}^{(0)} = C_{Kh}^{(0)} = 0. \tag{6.1}$$

We also know (from e.g. Ref. [15, p. 18]) that the electric field component tangential to a boundary between media is always continuous, giving in particular at $r = a$, from Eq. (4.10)

$$\cos(m_e^{(0)}\theta)C_{Ie}^{(0)}I_{m_e^{(0)}}\left(\nu^{(0)}a\right) = \cos(m_e^{(1)}\theta)\left[C_{Ie}^{(1)}I_{m_e^{(1)}}\left(\nu^{(1)}a\right) + C_{Ke}^{(1)}K_{m_e^{(1)}}\left(\nu^{(1)}a\right)\right]. \tag{6.2}$$

Since this is valid for any θ, $m_e^{(0)}$ and $m_e^{(1)}$ are necessarily equal (if the multiplicating factors in front of both $\cos(m_e^{(0)}\theta)$ and $\cos(m_e^{(1)}\theta)$ are zero, it means, since we know from Ref. [30] that $I_{m_e^{(0)}}\left(\nu^{(0)}a\right) \neq 0$, that $E_s = 0$ in the whole region where $r \leq a$ and we can still write $m_e^{(0)} = m_e^{(1)}$ provided $C_{Ie}^{(0)}$ is set to zero).
Equation (4.3) is valid across $r = a$, and following what is done in Ref. [13], we can multiply each side by r and integrate over r between $a - \delta a$ and $a + \delta a$, obtaining

$$\begin{aligned}
&(a+\delta a)\left.\frac{\partial E_s}{\partial r}\right|_{a+\delta a} - (a-\delta a)\left.\frac{\partial E_s}{\partial r}\right|_{a-\delta a} + \int_{a-\delta a}^{a+\delta a}\mathrm{d}r\left(-\frac{m_e^{(0)2}}{r} - rk^2 + r\omega^2\varepsilon_0\mu_0\right)E_s \\
&= \frac{jP_m}{\pi a^{m+1}(1+\delta_{m0})}\cos(m\theta)e^{-jks}\left(\frac{-k}{\varepsilon_0\upsilon} + \omega\mu_0\right)\int_{a-\delta a}^{a+\delta a}\mathrm{d}r\delta(r-a)r \\
&= \frac{-j\omega P_m}{\pi\varepsilon_0\upsilon^2\gamma^2 a^m(1+\delta_{m0})}\cos(m\theta)e^{-jks},
\end{aligned} \tag{6.3}$$

[7] To recover the results at any frequency we would simply need to replace $\frac{k}{\gamma}$ by $\frac{|k|}{\gamma}$ in the expression of the radial propagation constant of vacuum. See also the end of Section 3 and footnote 6.

where we have replaced ε_c and μ by their values in vacuum ε_0 and μ_0, and where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is the relativistic mass factor. When δa goes to zero, the integral term in the left-hand side vanishes since E_s is not infinite at $r = a$. Recalling that in vacuum $\nu^{(0)} = \nu^{(1)} = \frac{k}{\gamma}$, replacing E_s by its expression on each side of the boundary, and dropping the e^{-jks} factor[8], we can rewrite the equation as

$$\frac{ka}{\gamma}\cos(m_e^{(0)}\theta)\left[C_{Ie}^{(1)} I'_{m_e^{(0)}}\left(\frac{ka}{\gamma}\right) + C_{Ke}^{(1)} K'_{m_e^{(0)}}\left(\frac{ka}{\gamma}\right) - C_{Ie}^{(0)} I'_{m_e^{(0)}}\left(\frac{ka}{\gamma}\right)\right] = \frac{-j\omega P_m}{\pi\varepsilon_0 \upsilon^2\gamma^2 a^m(1+\delta_{m0})}\cos(m\theta). \tag{6.4}$$

This relation, valid at any θ, means that we necessarily have

$$m_e^{(0)} = m_e^{(1)} = m, \tag{6.5}$$

and we can divide by $\cos(m\theta)$ both sides of Eq. (6.4). We can then inject Eq. (6.2) taken in the form

$$\left(C_{Ie}^{(0)} - C_{Ie}^{(1)}\right) I_m\left(\frac{ka}{\gamma}\right) = C_{Ke}^{(1)} K_m\left(\frac{ka}{\gamma}\right), \tag{6.6}$$

into Eq. (6.4), to get

$$\frac{ka}{\gamma} C_{Ke}^{(1)}\left[K'_m\left(\frac{ka}{\gamma}\right) I_m\left(\frac{ka}{\gamma}\right) - I'_m\left(\frac{ka}{\gamma}\right) K_m\left(\frac{ka}{\gamma}\right)\right] = \frac{-j\omega P_m}{\pi\varepsilon_0 \upsilon^2\gamma^2 a^m(1+\delta_{m0})} I_m\left(\frac{ka}{\gamma}\right).$$

The term between square brackets is equal to $-\frac{\gamma}{ka}$ from Eq. (D.2) of Appendix D. Finally we obtain

$$C_{Ke}^{(1)} = \frac{j\omega P_m}{\pi\varepsilon_0 \upsilon^2\gamma^2 a^m(1+\delta_{m0})} I_m\left(\frac{ka}{\gamma}\right). \tag{6.7}$$

At $r = a$ there is a surface current density flowing along the s axis. From Ref. [15, p. 18] we know that only the tangential component of the magnetic field H_θ is discontinuous at that point. So H_s is continuous, which gives a relation analogous to Eq. (6.2). Therefore we have $m_h^{(0)} = m_h^{(1)}$ and

$$\left(C_{Ih}^{(0)} - C_{Ih}^{(1)}\right) I_{m_h^{(0)}}\left(\frac{ka}{\gamma}\right) = C_{Kh}^{(1)} K_{m_h^{(0)}}\left(\frac{ka}{\gamma}\right),$$

which we can plug into a similar integration of Eq. (4.4) as what was done above on E_s, but since this time the right-hand side is zero, we get[9]

$$\begin{aligned} C_{Kh}^{(1)} &= 0, \\ C_{Ih}^{(0)} &= C_{Ih}^{(1)}. \end{aligned} \tag{6.8}$$

No further information can be obtained from the boundary condition at $r = a$. Indeed, the discontinuity of H_θ coming from the surface charge density at that point [15, p. 18] is proportional to the discontinuity of $\frac{\partial E_s}{\partial r}$ as can been seen from Eq. (5.12), since both H_s and its derivative with respect to θ are continuous at $r = a$. It will then give the same relation as above (Eq. (6.3)).

6.2 Boundary conditions at the pipe wall inner surface and between each of its layers

We will now consider the boundary conditions for the subsequent layers, i.e. at each $r = b^{(p)}$ for $1 \le p \le N-1$. There are no externally imposed surface charge or currents between each cylindrical layer, which means (see

[8]Note that this equation could have been used in the discussion of Section 4: it prevents any $l \neq 1$ for the e^{-jkls} factor and is another reason to eliminate the e^{jks} term, in the longitudinal dependence of E_s.

[9]This is actually obvious since in Eq. (4.4) there is no source of discontinuity at $r = a$, so no reason for H_s to have a different expression from one side to the other of the ring-shaped source.

Ref. [15, p. 18] and Eqs. (3.5) and (3.6)) that the following relations hold across those boundaries (for any θ, s and ω)

$$\begin{aligned}
\varepsilon_c^{(p)} E_r^{(p)}\left(b^{(p)},\theta,s;\omega\right) &= \varepsilon_c^{(p+1)} E_r^{(p+1)}\left(b^{(p)},\theta,s;\omega\right), & (6.9)\\
E_\theta^{(p)}\left(b^{(p)},\theta,s;\omega\right) &= E_\theta^{(p+1)}\left(b^{(p)},\theta,s;\omega\right), & (6.10)\\
E_s^{(p)}\left(b^{(p)},\theta,s;\omega\right) &= E_s^{(p+1)}\left(b^{(p)},\theta,s;\omega\right), & (6.11)\\
\mu^{(p)} H_r^{(p)}\left(b^{(p)},\theta,s;\omega\right) &= \mu^{(p+1)} H_r^{(p+1)}\left(b^{(p)},\theta,s;\omega\right), & (6.12)\\
H_\theta^{(p)}\left(b^{(p)},\theta,s;\omega\right) &= H_\theta^{(p+1)}\left(b^{(p)},\theta,s;\omega\right), & (6.13)\\
H_s^{(p)}\left(b^{(p)},\theta,s;\omega\right) &= H_s^{(p+1)}\left(b^{(p)},\theta,s;\omega\right). & (6.14)
\end{aligned}$$

Equation (6.11) and (6.14) read respectively

$$\begin{aligned}
&\cos(m_e^{(p)}\theta)\left[C_{Ie}^{(p)} I_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right) + C_{Ke}^{(p)} K_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right)\right] = \\
&\qquad \cos(m_e^{(p+1)}\theta)\left[C_{Ie}^{(p+1)} I_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right) + C_{Ke}^{(p+1)} K_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)\right], \quad (6.15)
\end{aligned}$$

$$\begin{aligned}
&\sin(m_h^{(p)}\theta)\left[C_{Ih}^{(p)} I_{m_h^{(p)}}\left(\nu^{(p)}b^{(p)}\right) + C_{Kh}^{(p)} K_{m_h^{(p)}}\left(\nu^{(p)}b^{(p)}\right)\right] = \\
&\qquad \sin(m_h^{(p+1)}\theta)\left[C_{Ih}^{(p+1)} I_{m_h^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right) + C_{Kh}^{(p+1)} K_{m_h^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)\right], \quad (6.16)
\end{aligned}$$

while Eqs. (6.13) and (6.10), using Eqs. (5.12) and (5.10), can be written

$$\begin{aligned}
\frac{1}{{\nu^{(p)}}^2}\left[\upsilon\varepsilon_c^{(p)}\left.\frac{\partial E_s^{(p)}}{\partial r}\right|_{b^{(p)}} + \frac{1}{b^{(p)}}\frac{\partial H_s^{(p)}}{\partial\theta}\left(b^{(p)}\right)\right] &= \frac{1}{{\nu^{(p+1)}}^2}\left[\upsilon\varepsilon_c^{(p+1)}\left.\frac{\partial E_s^{(p+1)}}{\partial r}\right|_{b^{(p)}} + \frac{1}{b^{(p)}}\frac{\partial H_s^{(p+1)}}{\partial\theta}\left(b^{(p)}\right)\right],\\
\frac{1}{{\nu^{(p)}}^2}\left[\frac{1}{b^{(p)}}\frac{\partial E_s^{(p)}}{\partial\theta}\left(b^{(p)}\right) - \upsilon\mu^{(p)}\left.\frac{\partial H_s^{(p)}}{\partial r}\right|_{b^{(p)}}\right] &= \frac{1}{{\nu^{(p+1)}}^2}\left[\frac{1}{b^{(p)}}\frac{\partial E_s^{(p+1)}}{\partial\theta}\left(b^{(p)}\right) - \upsilon\mu^{(p+1)}\left.\frac{\partial H_s^{(p+1)}}{\partial r}\right|_{b^{(p)}}\right],
\end{aligned}$$

which, using Eqs. (4.10) and (4.11), become

$$\begin{aligned}
&\frac{1}{{\nu^{(p)}}^2}\Bigg[\upsilon\varepsilon_c^{(p)}\cos(m_e^{(p)}\theta)\nu^{(p)}\left\{C_{Ie}^{(p)} I'_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right) + C_{Ke}^{(p)} K'_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right)\right\}\\
&\qquad + \frac{m_h^{(p)}}{b^{(p)}}\cos(m_h^{(p)}\theta)\left\{C_{Ih}^{(p)} I_{m_h^{(p)}}\left(\nu^{(p)}b^{(p)}\right) + C_{Kh}^{(p)} K_{m_h^{(p)}}\left(\nu^{(p)}b^{(p)}\right)\right\}\Bigg]\\
&= \frac{1}{{\nu^{(p+1)}}^2}\Bigg[\upsilon\varepsilon_c^{(p+1)}\cos(m_e^{(p+1)}\theta)\nu^{(p+1)}\left\{C_{Ie}^{(p+1)} I'_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right) + C_{Ke}^{(p+1)} K'_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)\right\}\\
&\qquad + \frac{m_h^{(p+1)}}{b^{(p)}}\cos(m_h^{(p+1)}\theta)\left\{C_{Ih}^{(p+1)} I_{m_h^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right) + C_{Kh}^{(p+1)} K_{m_h^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)\right\}\Bigg], \quad (6.17)
\end{aligned}$$

and

$$\frac{1}{{\nu^{(p)}}^2}\left[-\frac{m_e^{(p)}}{b^{(p)}}\sin(m_e^{(p)}\theta)\left\{C_{Ie}^{(p)}I_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right)+C_{Ke}^{(p)}K_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right)\right\}\right.$$
$$\left.-\ \upsilon\mu^{(p)}\sin(m_h^{(p)}\theta)\nu^{(p)}\left\{C_{Ih}^{(p)}I'_{m_h^{(p)}}\left(\nu^{(p)}b^{(p)}\right)+C_{Kh}^{(p)}K'_{m_h^{(p)}}\left(\nu^{(p)}b^{(p)}\right)\right\}\right]$$

$$=\frac{1}{{\nu^{(p+1)}}^2}\left[-\frac{m_e^{(p+1)}}{b^{(p)}}\sin(m_e^{(p+1)}\theta)\left\{C_{Ie}^{(p+1)}I_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)+C_{Ke}^{(p+1)}K_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)\right\}\right.$$
$$\left.-\ \upsilon\mu^{(p+1)}\sin(m_h^{(p+1)}\theta)\nu^{(p+1)}\left\{C_{Ih}^{(p+1)}I'_{m_h^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)+C_{Kh}^{(p+1)}K'_{m_h^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)\right\}\right]. \quad (6.18)$$

Equation (6.15), valid for any θ, tells us that $m_e^{(p)} = m_e^{(p+1)}$ except in the case where

$$C_{Ie}^{(p)}I_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right)+C_{Ke}^{(p)}K_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right)=C_{Ie}^{(p+1)}I_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)+C_{Ke}^{(p+1)}K_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)=0.$$

If this happens, we can express Eq. (6.17) using the derivative with respect to θ of Eq. (6.16) (that is, the continuitity at $b^{(p)}$ of $\frac{\partial H_s}{\partial\theta}$), leading to

$$\frac{1}{{\nu^{(p)}}^2}\upsilon\varepsilon_c^{(p)}\cos(m_e^{(p)}\theta)\nu^{(p)}\left[C_{Ie}^{(p)}I'_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right)+C_{Ke}^{(p)}K'_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right)\right]$$
$$+\left(\frac{1}{{\nu^{(p)}}^2}-\frac{1}{{\nu^{(p+1)}}^2}\right)\frac{m_h^{(p)}}{b^{(p)}}\cos(m_h^{(p)}\theta)\left[C_{Ih}^{(p)}I_{m_h^{(p)}}\left(\nu^{(p)}b^{(p)}\right)+C_{Kh}^{(p)}K_{m_h^{(p)}}\left(\nu^{(p)}b^{(p)}\right)\right]$$

$$=\frac{1}{{\nu^{(p+1)}}^2}\upsilon\varepsilon_c^{(p+1)}\cos(m_e^{(p+1)}\theta)\nu^{(p+1)}\left[C_{Ie}^{(p+1)}I'_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)+C_{Ke}^{(p+1)}K'_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)\right]. \quad (6.19)$$

Then the case $m_e^{(p)} \neq m_e^{(p+1)}$ is possible only if one of the two terms

$$C_{Ie}^{(p)}I'_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right)+C_{Ke}^{(p)}K'_{m_e^{(p)}}\left(\nu^{(p)}b^{(p)}\right)$$

or

$$C_{Ie}^{(p+1)}I'_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)+C_{Ke}^{(p+1)}K'_{m_e^{(p+1)}}\left(\nu^{(p+1)}b^{(p)}\right)$$

is zero, since $m_h^{(p)}$ can be equal to only one of $m_e^{(p)}$ or $m_e^{(p+1)}$ so the the $\cos(m_h^{(p)}\theta)$ term will combine to at most only one of the two other cosine terms. In such a case in either layer p or layer $p+1$ both E_s and $\frac{\partial E_s}{\partial r}$ are zero at $r = b^{(p)}$ which means that the longitudinal component of the electric field is zero in the whole layer (it is determined by two constants only). Should this happend we can still impose $m_e^{(p)} = m_e^{(p+1)}$ by taking the value of the layer that has a non zero E_s, since in the other layer the constants are zeros so the value of the azimuthal mode number does not play any role.
We can repeat this argument at each boundary, and therefore drop the superscript for the quantity $m_e^{(p)}$. Using also Eq. (6.5) we get

$$\forall p \text{ between } 0 \text{ and } N, \quad m_e^{(p)} = m. \quad (6.20)$$

Very similar arguments can be applied to H_s and $m_h^{(p)}$: using first Eq. (6.16), and then, if the radial part of H_s is zero at $r = b^{(p)}$, Eq. (6.18) together with the continuity of $\frac{\partial E_s}{\partial\theta}$, we prove that $m_h^{(p)} = m_h^{(p+1)}$. Applying this at each layer boundary and using the results of Section 6.1 enables us to drop the superscript for this quantity:

$$\forall p \text{ between } 0 \text{ and } N, \quad m_h^{(p)} = m_h. \quad (6.21)$$

We now want to prove that $m = m_h$. To do so we rewrite Eqs. (6.18) and (6.19) using Eqs. (6.15), (6.20) and (6.21), to get

$$\sin(m\theta)\left(\frac{1}{\nu^{(p+1)^2}}-\frac{1}{\nu^{(p)^2}}\right)\frac{m}{b^{(p)}}\left[C_{Ie}^{(p)}I_m\left(\nu^{(p)}b^{(p)}\right)+C_{Ke}^{(p)}K_m\left(\nu^{(p)}b^{(p)}\right)\right]=$$

$$\sin(m_h\theta)\upsilon\left[\frac{\mu^{(p)}}{\nu^{(p)}}\left\{C_{Ih}^{(p)}I'_{m_h}\left(\nu^{(p)}b^{(p)}\right)+C_{Kh}^{(p)}K'_{m_h}\left(\nu^{(p)}b^{(p)}\right)\right\}\right.$$
$$\left.-\frac{\mu^{(p+1)}}{\nu^{(p+1)}}\left\{C_{Ih}^{(p+1)}I'_{m_h}\left(\nu^{(p+1)}b^{(p)}\right)+C_{Kh}^{(p+1)}K'_{m_h}\left(\nu^{(p+1)}b^{(p)}\right)\right\}\right],\quad(6.22)$$

and

$$\cos(m\theta)\upsilon\left[\frac{\varepsilon_c^{(p)}}{\nu^{(p)}}\left\{C_{Ie}^{(p)}I'_m\left(\nu^{(p)}b^{(p)}\right)+C_{Ke}^{(p)}K'_m\left(\nu^{(p)}b^{(p)}\right)\right\}\right.$$
$$\left.-\frac{\varepsilon_c^{(p+1)}}{\nu^{(p+1)}}\left\{C_{Ie}^{(p+1)}I'_m\left(\nu^{(p+1)}b^{(p)}\right)+C_{Ke}^{(p+1)}K'_m\left(\nu^{(p+1)}b^{(p)}\right)\right\}\right]=$$

$$\cos(m_h\theta)\left(\frac{1}{\nu^{(p+1)^2}}-\frac{1}{\nu^{(p)^2}}\right)\frac{m_h}{b^{(p)}}\left[C_{Ih}^{(p)}I_{m_h}\left(\nu^{(p)}b^{(p)}\right)+C_{Kh}^{(p)}K_{m_h}\left(\nu^{(p)}b^{(p)}\right)\right].\quad(6.23)$$

The case when either m or m_h is zero leads obviously to $m = m_h = 0$ since m_h is proportional to m (see Section 4). Now assuming that they are both different from zero, the two above equations (valid for any θ) show that $m = m_h$ unless

$$\begin{aligned}&\left(\frac{1}{\nu^{(p+1)^2}}-\frac{1}{\nu^{(p)^2}}\right)\left[C_{Ie}^{(p)}I_m\left(\nu^{(p)}b^{(p)}\right)+C_{Ke}^{(p)}K_m\left(\nu^{(p)}b^{(p)}\right)\right]\\
&=\frac{\mu^{(p)}}{\nu^{(p)}}\left[C_{Ih}^{(p)}I'_{m_h}\left(\nu^{(p)}b^{(p)}\right)+C_{Kh}^{(p)}K'_{m_h}\left(\nu^{(p)}b^{(p)}\right)\right]\\
&\qquad-\frac{\mu^{(p+1)}}{\nu^{(p+1)}}\left[C_{Ih}^{(p+1)}I'_{m_h}\left(\nu^{(p+1)}b^{(p)}\right)+C_{Kh}^{(p+1)}K'_{m_h}\left(\nu^{(p+1)}b^{(p)}\right)\right]\\
&=\frac{\varepsilon_c^{(p)}}{\nu^{(p)}}\left[C_{Ie}^{(p)}I'_m\left(\nu^{(p)}b^{(p)}\right)+C_{Ke}^{(p)}K'_m\left(\nu^{(p)}b^{(p)}\right)\right]\\
&\qquad-\frac{\varepsilon_c^{(p+1)}}{\nu^{(p+1)}}\left[C_{Ie}^{(p+1)}I'_m\left(\nu^{(p+1)}b^{(p)}\right)+C_{Ke}^{(p+1)}K'_m\left(\nu^{(p+1)}b^{(p)}\right)\right]\\
&=\left(\frac{1}{\nu^{(p+1)^2}}-\frac{1}{\nu^{(p)^2}}\right)\left[C_{Ih}^{(p)}I_{m_h}\left(\nu^{(p)}b^{(p)}\right)+C_{Kh}^{(p)}K_{m_h}\left(\nu^{(p)}b^{(p)}\right)\right]\\
&=0.\end{aligned}\quad(6.24)$$

Should this happen we could express the same boundary conditions on the subsequent layers, leading to the same conclusion ($m = m_h$). In the end the only case that is problematic is when Eq. (6.24) is true for every boundary, i.e. for p between 1 and $N-1$. We can actually show (see Appendix E) that in this case the layers have all the same radial propagation constant $\nu^{(p)}$, that is, that of vacuum, such that $\varepsilon_1^{(p)}\mu_1^{(p)} = 1$ for any p, and that $H_s = 0$ everywhere in space, meaning that we can still write $m_h = m$ provided we set the constants $C_{Ih}^{(p)}$ and $C_{Kh}^{(p)}$ to zero for all the layers. The case where $\varepsilon_1\mu_1 = 1$ is in principle possible for a medium different from vacuum, for instance if it's diamagnetic (i.e. $\mu_r < 1$) and slightly dielectric (i.e. $\varepsilon_b > 1$ with $\varepsilon_b - 1$ small compared to 1), with no loss and an infinite resistivity.

As a consequence, we always have (except in a very specific case, quite unlikely to happen, that we will assume not to occur here, see Appendix E)

$$m = m_h.\quad(6.25)$$

The only unknown coefficients remain the constants in front of the modified Bessel functions in the expression of E_s and H_s of Eqs. (4.10) and (4.11). We have four such constants per layer, so $4(N+1)$ of them on the whole. For the first two layers, we know from Eqs. (6.1), (6.6), (6.7) and (6.8) that only 2 constants ($C_{Ie}^{(1)}$ and $C_{Ih}^{(1)}$) remain to be determined. For the last layer (going to infinity), there can be no function I_m in the expression of the fields, since $I_m(\nu^{(N)}r)$ goes to infinity when r goes to infinity (see e.g. Eq. (9.7.1) in Ref. [30]), such that

$$C_{Ie}^{(N)} = C_{Ih}^{(N)} = 0. \tag{6.26}$$

So $4(N-1)$ constants remain to be determined by the boundary conditions at $r = b^{(p)}$ for $1 \le p \le N-1$, which means that four equations for each boundary are needed. For instance the continuity at those boundaries of $E_\theta^{(p)}$, $E_s^{(p)}$, $H_\theta^{(p)}$ and $H_s^{(p)}$ are sufficient. Continuity of $\varepsilon_c^{(p)} E_r^{(p)}$ and $\mu^{(p)} H_r^{(p)}$ give redundant equations, which can be readily seen from Eqs. (5.1) and (5.3).
Note that the redundancy of D_r continuity would not have occurred if (as done in Refs. [2, 14]) we had used $\vec{D} = \varepsilon_0 \varepsilon_b \vec{E}$ instead of Eq. (3.5) for the definition of the electric displacement field used in Eq. (3.1), while Eqs. (3.2) and (5.1) remain the same. In the absence of surface charge at the boundaries, this would lead to the following boundary condition in replacement of Eq. (6.9)

$$\varepsilon_b^{(p)} E_r^{(p)} \left(b^{(p)}, \theta, s; \omega\right) = \varepsilon_b^{(p+1)} E_r^{(p+1)} \left(b^{(p)}, \theta, s; \omega\right),$$

and consequently to one additional equation per boundary and necessarily to an inconsistency. Still, it had no impact on the final results of Refs. [2, 14] because the continuity of the radial components was never used. Also, consistency is recovered simply by saying that in such a formalism there exists a surface charge density at the layers boundary (physically, those charges are actually induced charges, created by the discontinuity of the complex permittivity – particularly its conductive part – which results in a discontinuity in the current density at that boundary – see also Appendix A).

To solve for all the constants of the problem we will first introduce the free space impedance Z_0 and the field $\vec{G}$ that has the same dimension as the electric field $\vec{E}$

$$\begin{aligned} Z_0 &= \frac{1}{\varepsilon_0 c} = \mu_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}}, & (6.27)\\ \vec{G} &= Z_0 \vec{H}, & (6.28) \end{aligned}$$

and the corresponding constant coefficients for $\vec{G}$

$$\begin{aligned} C_{Ig}^{(p)} &= Z_0 C_{Ih}^{(p)}, \\ C_{Kg}^{(p)} &= Z_0 C_{Kh}^{(p)}. \end{aligned} \tag{6.29}$$

Then, letting

$$\begin{aligned} x^{p+1,p} &= \nu^{(p+1)} b^{(p)}, \\ x^{p,p} &= \nu^{(p)} b^{(p)}, \end{aligned} \tag{6.30}$$

the continuity of E_s and H_s is given by Eqs. (6.15) and (6.16) where we can now drop the cosine and sine factors:

$$\begin{aligned} C_{Ie}^{(p)} I_m(x^{p,p}) + C_{Ke}^{(p)} K_m(x^{p,p}) &= C_{Ie}^{(p+1)} I_m(x^{p+1,p}) + C_{Ke}^{(p+1)} K_m(x^{p+1,p}), & (6.31)\\ C_{Ig}^{(p)} I_m(x^{p,p}) + C_{Kg}^{(p)} K_m(x^{p,p}) &= C_{Ig}^{(p+1)} I_m(x^{p+1,p}) + C_{Kg}^{(p+1)} K_m(x^{p+1,p}). & (6.32) \end{aligned}$$

The continuity of E_θ and H_θ can be written, from Eqs. (6.22) and (6.23) where the cosine and sine factors have been dropped

$$\begin{aligned} &\left(\frac{1}{{\nu^{(p+1)}}^2} - \frac{1}{{\nu^{(p)}}^2} \right) \frac{m}{b^{(p)}} \left[C_{Ie}^{(p)} I_m(x^{p,p}) + C_{Ke}^{(p)} K_m(x^{p,p}) \right] \\ &\qquad - \frac{\beta \mu_1^{(p)}}{\nu^{(p)}} \left[C_{Ig}^{(p)} I'_m(x^{p,p}) + C_{Kg}^{(p)} K'_m(x^{p,p}) \right] \\ &\qquad\qquad = -\frac{\beta \mu_1^{(p+1)}}{\nu^{(p+1)}} \left[C_{Ig}^{(p+1)} I'_m(x^{p+1,p}) + C_{Kg}^{(p+1)} K'_m(x^{p+1,p}) \right], \end{aligned} \tag{6.33}$$

and

$$\frac{\beta\varepsilon_1^{(p)}}{\nu^{(p)}}\left[C_{Ie}^{(p)}I'_m(x^{p,p})+C_{Ke}^{(p)}K'_m(x^{p,p})\right] + \left(\frac{1}{{\nu^{(p)}}^2}-\frac{1}{{\nu^{(p+1)}}^2}\right)\frac{m}{b^{(p)}}\left[C_{Ig}^{(p)}I_m(x^{p,p})+C_{Kg}^{(p)}K_m(x^{p,p})\right] = \frac{\beta\varepsilon_1^{(p+1)}}{\nu^{(p+1)}}\left[C_{Ie}^{(p+1)}I'_m(x^{p+1,p})+C_{Ke}^{(p+1)}K'_m(x^{p+1,p})\right]. \quad (6.34)$$

We can write Eqs. (6.31) and (6.34) in matrix form:

$$\begin{bmatrix} I_m(x^{p+1,p}) & K_m(x^{p+1,p}) \\ \frac{\beta\varepsilon_1^{(p+1)}}{\nu^{(p+1)}}I'_m(x^{p+1,p}) & \frac{\beta\varepsilon_1^{(p+1)}}{\nu^{(p+1)}}K'_m(x^{p+1,p}) \end{bmatrix}\cdot\begin{bmatrix} C_{Ie}^{(p+1)} \\ C_{Ke}^{(p+1)} \end{bmatrix} = \begin{bmatrix} C_{Ie}^{(p)}I_m(x^{p,p})+C_{Ke}^{(p)}K_m(x^{p,p}) \\ \frac{\beta\varepsilon_1^{(p)}}{\nu^{(p)}}\left\{C_{Ie}^{(p)}I'_m(x^{p,p})+C_{Ke}^{(p)}K'_m(x^{p,p})\right\}+ \\ \left(\frac{1}{{\nu^{(p)}}^2}-\frac{1}{{\nu^{(p+1)}}^2}\right)\frac{m}{b^{(p)}}\left\{C_{Ig}^{(p)}I_m(x^{p,p})+C_{Kg}^{(p)}K_m(x^{p,p})\right\} \end{bmatrix}.$$

This can be readily solved for $\begin{bmatrix} C_{Ie}^{(p+1)} \\ C_{Ke}^{(p+1)} \end{bmatrix}$, knowing that the determinant of the first matrix is proportional to the wronskian of the modified Bessel functions, more precisely equal to (see Eq. (D.2)) $-\frac{\beta\varepsilon_1^{(p+1)}}{{\nu^{(p+1)}}^2 b^{(p)}}$. We get, using the inversion formula of a 2×2 matrix (see Appendix F)

$$\begin{bmatrix} C_{Ie}^{(p+1)} \\ C_{Ke}^{(p+1)} \end{bmatrix} = -\frac{{\nu^{(p+1)}}^2 b^{(p)}}{\beta\varepsilon_1^{(p+1)}}\begin{bmatrix} \frac{\beta\varepsilon_1^{(p+1)}}{\nu^{(p+1)}}K'_m(x^{p+1,p}) & -K_m(x^{p+1,p}) \\ -\frac{\beta\varepsilon_1^{(p+1)}}{\nu^{(p+1)}}I'_m(x^{p+1,p}) & I_m(x^{p+1,p}) \end{bmatrix}\cdot \left(\begin{bmatrix} I_m(x^{p,p}) & K_m(x^{p,p}) \\ \frac{\beta\varepsilon_1^{(p)}}{\nu^{(p)}}I'_m(x^{p,p}) & \frac{\beta\varepsilon_1^{(p)}}{\nu^{(p)}}K'_m(x^{p,p}) \end{bmatrix}\cdot\begin{bmatrix} C_{Ie}^{(p)} \\ C_{Ke}^{(p)} \end{bmatrix} + \left\{\frac{1}{{\nu^{(p)}}^2}-\frac{1}{{\nu^{(p+1)}}^2}\right\}\frac{m}{b^{(p)}}\begin{bmatrix} 0 & 0 \\ I_m(x^{p,p}) & K_m(x^{p,p}) \end{bmatrix}\cdot\begin{bmatrix} C_{Ig}^{(p)} \\ C_{Kg}^{(p)} \end{bmatrix}\right). \quad (6.35)$$

Very similarly we can write for $\begin{bmatrix} C_{Ig}^{(p+1)} \\ C_{Kg}^{(p+1)} \end{bmatrix}$, from Eqs. (6.32) and (6.33)

$$\begin{bmatrix} C_{Ig}^{(p+1)} \\ \\ C_{Kg}^{(p+1)} \end{bmatrix} = -\frac{{\nu^{(p+1)}}^2 b^{(p)}}{\beta\mu_1^{(p+1)}} \begin{bmatrix} \frac{\beta\mu_1^{(p+1)}}{\nu^{(p+1)}} K'_m(x^{p+1,p}) & -K_m(x^{p+1,p}) \\ \\ -\frac{\beta\mu_1^{(p+1)}}{\nu^{(p+1)}} I'_m(x^{p+1,p}) & I_m(x^{p+1,p}) \end{bmatrix} \cdot$$

$$\left(\begin{bmatrix} I_m(x^{p,p}) & K_m(x^{p,p}) \\ \\ \frac{\beta\mu_1^{(p)}}{\nu^{(p)}} I'_m(x^{p,p}) & \frac{\beta\mu_1^{(p)}}{\nu^{(p)}} K'_m(x^{p,p}) \end{bmatrix} \cdot \begin{bmatrix} C_{Ig}^{(p)} \\ \\ C_{Kg}^{(p)} \end{bmatrix} + \right.$$

$$\left. \left\{ \frac{1}{{\nu^{(p)}}^2} - \frac{1}{{\nu^{(p+1)}}^2} \right\} \frac{m}{b^{(p)}} \begin{bmatrix} 0 & 0 \\ \\ I_m(x^{p,p}) & K_m(x^{p,p}) \end{bmatrix} \cdot \begin{bmatrix} C_{Ie}^{(p)} \\ \\ C_{Ke}^{(p)} \end{bmatrix} \right). \tag{6.36}$$

Let us now define the four following 2×2 matrices, enabling the computation of the values of the constants for the $p+1$ region knowing those of the p region:

$$P^{p+1,p} = -\frac{{\nu^{(p+1)}}^2 b^{(p)}}{\beta\varepsilon_1^{(p+1)}} \begin{bmatrix} \frac{\beta\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} K'_m(x^{p+1,p}) & -K_m(x^{p+1,p}) \\ \\ -\frac{\beta\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} I'_m(x^{p+1,p}) & I_m(x^{p+1,p}) \end{bmatrix} \cdot \begin{bmatrix} I_m(x^{p,p}) & K_m(x^{p,p}) \\ \\ \frac{\beta\varepsilon_1^{(p)}}{\nu^{(p)}} I'_m(x^{p,p}) & \frac{\beta\varepsilon_1^{(p)}}{\nu^{(p)}} K'_m(x^{p,p}) \end{bmatrix},$$

$$Q^{p+1,p} = -\left(\frac{{\nu^{(p+1)}}^2}{{\nu^{(p)}}^2} - 1 \right) \frac{m}{\beta\varepsilon_1^{(p+1)}} \begin{bmatrix} \frac{\beta\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} K'_m(x^{p+1,p}) & -K_m(x^{p+1,p}) \\ \\ -\frac{\beta\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} I'_m(x^{p+1,p}) & I_m(x^{p+1,p}) \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ \\ I_m(x^{p,p}) & K_m(x^{p,p}) \end{bmatrix},$$

$$R^{p+1,p} = -\frac{{\nu^{(p+1)}}^2 b^{(p)}}{\beta\mu_1^{(p+1)}} \begin{bmatrix} \frac{\beta\mu_1^{(p+1)}}{\nu^{(p+1)}} K'_m(x^{p+1,p}) & -K_m(x^{p+1,p}) \\ \\ -\frac{\beta\mu_1^{(p+1)}}{\nu^{(p+1)}} I'_m(x^{p+1,p}) & I_m(x^{p+1,p}) \end{bmatrix} \cdot \begin{bmatrix} I_m(x^{p,p}) & K_m(x^{p,p}) \\ \\ \frac{\beta\mu_1^{(p)}}{\nu^{(p)}} I'_m(x^{p,p}) & \frac{\beta\mu_1^{(p)}}{\nu^{(p)}} K'_m(x^{p,p}) \end{bmatrix},$$

$$S^{p+1,p} = -\left(\frac{{\nu^{(p+1)}}^2}{{\nu^{(p)}}^2} - 1 \right) \frac{m}{\beta\mu_1^{(p+1)}} \begin{bmatrix} \frac{\beta\mu_1^{(p+1)}}{\nu^{(p+1)}} K'_m(x^{p+1,p}) & -K_m(x^{p+1,p}) \\ \\ -\frac{\beta\mu_1^{(p+1)}}{\nu^{(p+1)}} I'_m(x^{p+1,p}) & I_m(x^{p+1,p}) \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ \\ I_m(x^{p,p}) & K_m(x^{p,p}) \end{bmatrix},$$

such that Eqs. (6.35) and (6.36) become[10]

$$\begin{bmatrix} C_{Ie}^{(p+1)} \\ \\ C_{Ke}^{(p+1)} \end{bmatrix} = P^{p+1,p} \cdot \begin{bmatrix} C_{Ie}^{(p)} \\ \\ C_{Ke}^{(p)} \end{bmatrix} + Q^{p+1,p} \cdot \begin{bmatrix} C_{Ig}^{(p)} \\ \\ C_{Kg}^{(p)} \end{bmatrix}, \tag{6.37}$$

$$\begin{bmatrix} C_{Ig}^{(p+1)} \\ \\ C_{Kg}^{(p+1)} \end{bmatrix} = R^{p+1,p} \cdot \begin{bmatrix} C_{Ig}^{(p)} \\ \\ C_{Kg}^{(p)} \end{bmatrix} + S^{p+1,p} \cdot \begin{bmatrix} C_{Ie}^{(p)} \\ \\ C_{Ke}^{(p)} \end{bmatrix}. \tag{6.38}$$

[10]Note that when $m = 0$, $Q^{p+1,p} = S^{p+1,p} = 0$ so the constants for the electric field and those for the magnetic field are uncoupled. This is actually obvious since in this case $H_s = 0$ from Eq. (4.11). A 2×2 matrix is then sufficient to compute the constants of the electric field longitudinal component.

It is crucial to be able to compute accurately these four matrices, which is not straightforward since components can be equal to a difference between very large numbers, especially when $\nu^{(p)}b^{(p)}$ or $\nu^{(p+1)}b^{(p)}$ becomes large: for large arguments I_m and I_m' are exponentially growing while K_m and K_m' are exponentially decaying (see Ref. [30], formulas 9.7.1 to 9.7.4). It is therefore better to write them in the following way

$$
P^{p+1,p} = -\frac{{\nu^{(p+1)}}^2 b^{(p)}}{\varepsilon_1^{(p+1)}} \begin{bmatrix} I_m(x^{p,p})K_m(x^{p+1,p})\left\{\frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}}\frac{K_m'(x^{p+1,p})}{K_m(x^{p+1,p})} - \frac{\varepsilon_1^{(p)}}{\nu^{(p)}}\frac{I_m'(x^{p,p})}{I_m(x^{p,p})}\right\} & K_m(x^{p,p})K_m(x^{p+1,p})\left\{\frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}}\frac{K_m'(x^{p+1,p})}{K_m(x^{p+1,p})} - \frac{\varepsilon_1^{(p)}}{\nu^{(p)}}\frac{K_m'(x^{p,p})}{K_m(x^{p,p})}\right\} \\ I_m(x^{p,p})I_m(x^{p+1,p})\left\{-\frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}}\frac{I_m'(x^{p+1,p})}{I_m(x^{p+1,p})} + \frac{\varepsilon_1^{(p)}}{\nu^{(p)}}\frac{I_m'(x^{p,p})}{I_m(x^{p,p})}\right\} & K_m(x^{p,p})I_m(x^{p+1,p})\left\{-\frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}}\frac{I_m'(x^{p+1,p})}{I_m(x^{p+1,p})} + \frac{\varepsilon_1^{(p)}}{\nu^{(p)}}\frac{K_m'(x^{p,p})}{K_m(x^{p,p})}\right\} \end{bmatrix},
$$

$$
Q^{p+1,p} = -\left(\frac{{\nu^{(p+1)}}^2}{{\nu^{(p)}}^2} - 1\right)\frac{m}{\beta\varepsilon_1^{(p+1)}} \begin{bmatrix} -I_m(x^{p,p})K_m(x^{p+1,p}) & -K_m(x^{p,p})K_m(x^{p+1,p}) \\ I_m(x^{p,p})I_m(x^{p+1,p}) & K_m(x^{p,p})I_m(x^{p+1,p}) \end{bmatrix},
$$

$$
R^{p+1,p} = -\frac{{\nu^{(p+1)}}^2 b^{(p)}}{\mu_1^{(p+1)}} \begin{bmatrix} I_m(x^{p,p})K_m(x^{p+1,p})\left\{\frac{\mu_1^{(p+1)}}{\nu^{(p+1)}}\frac{K_m'(x^{p+1,p})}{K_m(x^{p+1,p})} - \frac{\mu_1^{(p)}}{\nu^{(p)}}\frac{I_m'(x^{p,p})}{I_m(x^{p,p})}\right\} & K_m(x^{p,p})K_m(x^{p+1,p})\left\{\frac{\mu_1^{(p+1)}}{\nu^{(p+1)}}\frac{K_m'(x^{p+1,p})}{K_m(x^{p+1,p})} - \frac{\mu_1^{(p)}}{\nu^{(p)}}\frac{K_m'(x^{p,p})}{K_m(x^{p,p})}\right\} \\ I_m(x^{p,p})I_m(x^{p+1,p})\left\{-\frac{\mu_1^{(p+1)}}{\nu^{(p+1)}}\frac{I_m'(x^{p+1,p})}{I_m(x^{p+1,p})} + \frac{\mu_1^{(p)}}{\nu^{(p)}}\frac{I_m'(x^{p,p})}{I_m(x^{p,p})}\right\} & K_m(x^{p,p})I_m(x^{p+1,p})\left\{-\frac{\mu_1^{(p+1)}}{\nu^{(p+1)}}\frac{I_m'(x^{p+1,p})}{I_m(x^{p+1,p})} + \frac{\mu_1^{(p)}}{\nu^{(p)}}\frac{K_m'(x^{p,p})}{K_m(x^{p,p})}\right\} \end{bmatrix},
$$

$$
S^{p+1,p} = \frac{\varepsilon_1^{(p+1)}}{\mu_1^{(p+1)}} Q^{(p+1,p)},
$$

in which the quotients involving modified Bessel functions and their derivatives can be computed accurately using Eqs. (D.3) and (D.4)

$$
\begin{aligned}
\frac{I_m'(z)}{I_m(z)} &= \frac{I_{m-1}(z)}{I_m(z)} - \frac{m}{z}, \\
\frac{K_m'(z)}{K_m(z)} &= -\frac{K_{m-1}(z)}{K_m(z)} - \frac{m}{z},
\end{aligned} \tag{6.39}
$$

and we can normalize the Bessel functions in the first quotient of these expressions with e^z for I_m and e^{-z} for K_m, which does not change the quotient value (this normalization is available under Matlab®[31], for instance). In order to get more compact expressions, we'd rather rewrite those matrices as the product of three relatively simple matrices

$$P^{p+1,p} = \begin{bmatrix} K_m(x^{p+1,p}) & 0 \\ 0 & -I_m(x^{p+1,p}) \end{bmatrix} \cdot \tilde{P}^{p+1,p} \cdot \begin{bmatrix} I_m(x^{p,p}) & 0 \\ 0 & K_m(x^{p,p}) \end{bmatrix}, \tag{6.40}$$

$$Q^{p+1,p} = \begin{bmatrix} K_m(x^{p+1,p}) & 0 \\ 0 & -I_m(x^{p+1,p}) \end{bmatrix} \cdot \tilde{Q}^{p+1,p} \cdot \begin{bmatrix} I_m(x^{p,p}) & 0 \\ 0 & K_m(x^{p,p}) \end{bmatrix}, \tag{6.41}$$

$$R^{p+1,p} = \begin{bmatrix} K_m(x^{p+1,p}) & 0 \\ 0 & -I_m(x^{p+1,p}) \end{bmatrix} \cdot \tilde{R}^{p+1,p} \cdot \begin{bmatrix} I_m(x^{p,p}) & 0 \\ 0 & K_m(x^{p,p}) \end{bmatrix}, \tag{6.42}$$

$$S^{p+1,p} = \begin{bmatrix} K_m(x^{p+1,p}) & 0 \\ 0 & -I_m(x^{p+1,p}) \end{bmatrix} \cdot \tilde{S}^{p+1,p} \cdot \begin{bmatrix} I_m(x^{p,p}) & 0 \\ 0 & K_m(x^{p,p}) \end{bmatrix}, \tag{6.43}$$

with $\tilde{P}^{p+1,p}$, $\tilde{Q}^{p+1,p}$, $\tilde{R}^{p+1,p}$ and $\tilde{S}^{p+1,p}$ defined by

$$\tilde{P}^{p+1,p} = -\frac{{\nu^{(p+1)}}^2 b^{(p)}}{\varepsilon_1^{(p+1)}} \begin{bmatrix} \frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} \frac{K'_m(x^{p+1,p})}{K_m(x^{p+1,p})} - \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} \frac{I'_m(x^{p,p})}{I_m(x^{p,p})} & \frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} \frac{K'_m(x^{p+1,p})}{K_m(x^{p+1,p})} - \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} \frac{K'_m(x^{p,p})}{K_m(x^{p,p})} \\ \frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} \frac{I'_m(x^{p+1,p})}{I_m(x^{p+1,p})} - \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} \frac{I'_m(x^{p,p})}{I_m(x^{p,p})} & \frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} \frac{I'_m(x^{p+1,p})}{I_m(x^{p+1,p})} - \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} \frac{K'_m(x^{p,p})}{K_m(x^{p,p})} \end{bmatrix}, \tag{6.44}$$

$$\tilde{Q}^{p+1,p} = \left(\frac{{\nu^{(p+1)}}^2}{{\nu^{(p)}}^2} - 1 \right) \frac{m}{\beta \varepsilon_1^{(p+1)}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{6.45}$$

$$\tilde{R}^{p+1,p} = -\frac{{\nu^{(p+1)}}^2 b^{(p)}}{\mu_1^{(p+1)}} \begin{bmatrix} \frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} \frac{K'_m(x^{p+1,p})}{K_m(x^{p+1,p})} - \frac{\mu_1^{(p)}}{\nu^{(p)}} \frac{I'_m(x^{p,p})}{I_m(x^{p,p})} & \frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} \frac{K'_m(x^{p+1,p})}{K_m(x^{p+1,p})} - \frac{\mu_1^{(p)}}{\nu^{(p)}} \frac{K'_m(x^{p,p})}{K_m(x^{p,p})} \\ \frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} \frac{I'_m(x^{p+1,p})}{I_m(x^{p+1,p})} - \frac{\mu_1^{(p)}}{\nu^{(p)}} \frac{I'_m(x^{p,p})}{I_m(x^{p,p})} & \frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} \frac{I'_m(x^{p+1,p})}{I_m(x^{p+1,p})} - \frac{\mu_1^{(p)}}{\nu^{(p)}} \frac{K'_m(x^{p,p})}{K_m(x^{p,p})} \end{bmatrix}, \tag{6.46}$$

$$\tilde{S}^{p+1,p} = \left(\frac{{\nu^{(p+1)}}^2}{{\nu^{(p)}}^2} - 1 \right) \frac{m}{\beta \mu_1^{(p+1)}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{6.47}$$

We can then define the 4×4 matrix $M^{p+1,p}$ by

$$\begin{aligned} M^{p+1,p} &= \begin{bmatrix} P^{p+1,p} & Q^{p+1,p} \\ S^{p+1,p} & R^{p+1,p} \end{bmatrix} \\ &= \begin{bmatrix} K_m(x^{p+1,p}) & 0 & 0 & 0 \\ 0 & -I_m(x^{p+1,p}) & 0 & 0 \\ 0 & 0 & K_m(x^{p+1,p}) & 0 \\ 0 & 0 & 0 & -I_m(x^{p+1,p}) \end{bmatrix} \cdot \tilde{M}^{p+1,p} \cdot \begin{bmatrix} I_m(x^{p,p}) & 0 & 0 & 0 \\ 0 & K_m(x^{p,p}) & 0 & 0 \\ 0 & 0 & I_m(x^{p,p}) & 0 \\ 0 & 0 & 0 & K_m(x^{p,p}) \end{bmatrix}, \end{aligned} \tag{6.48}$$

with

$$\tilde{M}^{p+1,p} = \begin{bmatrix} \tilde{P}^{p+1,p} & \tilde{Q}^{p+1,p} \\ \tilde{S}^{p+1,p} & \tilde{R}^{p+1,p} \end{bmatrix}, \tag{6.49}$$

such that

$$\begin{bmatrix} C_{Ie}^{(p+1)} \\ \\ C_{Ke}^{(p+1)} \\ \\ C_{Ig}^{(p+1)} \\ \\ C_{Kg}^{(p+1)} \end{bmatrix} = M^{p+1,p} \cdot \begin{bmatrix} C_{Ie}^{(p)} \\ \\ C_{Ke}^{(p)} \\ \\ C_{Ig}^{(p)} \\ \\ C_{Kg}^{(p)} \end{bmatrix}. \tag{6.50}$$

When successively applying this relation, we get

$$\begin{bmatrix} C_{Ie}^{(N)} \\ \\ C_{Ke}^{(N)} \\ \\ C_{Ig}^{(N)} \\ \\ C_{Kg}^{(N)} \end{bmatrix} = M^{N,N-1} \cdot M^{N-1,N-2} \cdots M^{2,1} \cdot \begin{bmatrix} C_{Ie}^{(1)} \\ \\ C_{Ke}^{(1)} \\ \\ C_{Ig}^{(1)} \\ \\ C_{Kg}^{(1)} \end{bmatrix}.$$

Recalling, from Eqs. (6.7), (6.8) and (6.26), that

$$\begin{aligned} C_{Ke}^{(1)} &= \frac{j\omega P_m}{\pi\varepsilon_0 \upsilon^2 \gamma^2 a^m (1+\delta_{m0})} I_m\left(\frac{ka}{\gamma}\right), \\ C_{Kg}^{(1)} &= 0, \\ C_{Ie}^{(N)} &= C_{Ig}^{(N)} = 0, \end{aligned} \tag{6.51}$$

and defining

$$\begin{aligned} M &= M^{N,N-1} \cdot M^{N-1,N-2} \cdots M^{2,1} \\ &= D^N \cdot \tilde{M}^{N,N-1} \cdot D^{N-1,N-2} \cdot \tilde{M}^{N-1,N-2} \cdot D^{N-2,N-3} \cdots D^{2,1} \cdot \tilde{M}^{2,1} \cdot D^1 \\ &= D^N \cdot \tilde{M} \cdot D^1, \end{aligned} \tag{6.52}$$

with

$$D^1 = \begin{bmatrix} I_m(x^{1,1}) & 0 & 0 & 0 \\ 0 & K_m(x^{1,1}) & 0 & 0 \\ 0 & 0 & I_m(x^{1,1}) & 0 \\ 0 & 0 & 0 & K_m(x^{1,1}) \end{bmatrix}, \tag{6.53}$$

$$D^{p+1,p} = \begin{bmatrix} I_m(x^{p+1,p+1})K_m(x^{p+1,p}) & 0 & 0 & 0 \\ 0 & -K_m(x^{p+1,p+1})I_m(x^{p+1,p}) & 0 & 0 \\ 0 & 0 & I_m(x^{p+1,p+1})K_m(x^{p+1,p}) & 0 \\ 0 & 0 & 0 & -K_m(x^{p+1,p+1})I_m(x^{p+1,p}) \end{bmatrix}, \tag{6.54}$$

$$D^N = \begin{bmatrix} K_m(x^{N,N-1}) & 0 & 0 & 0 \\ 0 & -I_m(x^{N,N-1}) & 0 & 0 \\ 0 & 0 & K_m(x^{N,N-1}) & 0 \\ 0 & 0 & 0 & -I_m(x^{N,N-1}) \end{bmatrix}, \tag{6.55}$$

$$\tilde{M} = \tilde{M}^{N,N-1} \cdot D^{N-1,N-2} \cdot \tilde{M}^{N-1,N-2} \cdot D^{N-2,N-3} \cdots D^{2,1} \cdot \tilde{M}^{2,1}, \tag{6.56}$$

we can write

$$\begin{bmatrix} 0 \\ C_{Ke}^{(N)} \\ 0 \\ C_{Kg}^{(N)} \end{bmatrix} = M \cdot \begin{bmatrix} C_{Ie}^{(1)} \\ \frac{j\omega P_m}{\pi\varepsilon_0 v^2\gamma^2 a^m(1+\delta_{m0})} I_m\left(\frac{ka}{\gamma}\right) \\ C_{Ig}^{(1)} \\ 0 \end{bmatrix}, \tag{6.57}$$

leading finally to the linear equations

$$\begin{aligned} M_{11}C_{Ie}^{(1)} + M_{13}C_{Ig}^{(1)} &= -M_{12}C_{Ke}^{(1)}, \\ M_{31}C_{Ie}^{(1)} + M_{33}C_{Ig}^{(1)} &= -M_{32}C_{Ke}^{(1)}, \end{aligned}$$

and

$$\begin{aligned} C_{Ke}^{(N)} &= M_{21}C_{Ie}^{(1)} + M_{22}C_{Ke}^{(1)} + M_{23}C_{Ig}^{(1)}, \\ C_{Kg}^{(N)} &= M_{41}C_{Ie}^{(1)} + M_{42}C_{Ke}^{(1)} + M_{43}C_{Ig}^{(1)}. \end{aligned} \tag{6.58}$$

The first two ones can be inverted easily using the inversion formula of a 2×2 matrix (see Appendix F), giving

$$\begin{aligned} C_{Ie}^{(1)} &= -C_{Ke}^{(1)}\frac{M_{12}M_{33} - M_{32}M_{13}}{M_{11}M_{33} - M_{13}M_{31}}, \\ C_{Ig}^{(1)} &= C_{Ke}^{(1)}\frac{M_{12}M_{31} - M_{32}M_{11}}{M_{11}M_{33} - M_{13}M_{31}}. \end{aligned} \tag{6.59}$$

As in Ref. [14] we define α_{TM} and α_{TE} as the proportionality constants respectively between $C_{Ie}^{(1)}$ and $-C_{Ke}^{(1)}$, and between $C_{Ig}^{(1)}$ and $C_{Ke}^{(1)}$:

$$\begin{aligned} \alpha_{\mathrm{TM}} &= \frac{M_{12}M_{33} - M_{32}M_{13}}{M_{11}M_{33} - M_{13}M_{31}}, \\ \alpha_{\mathrm{TE}} &= \frac{M_{12}M_{31} - M_{32}M_{11}}{M_{11}M_{33} - M_{13}M_{31}}. \end{aligned} \tag{6.60}$$

Using Eqs. (6.52), (6.53) and (6.55) we can be further express those constants into

$$\begin{aligned} \alpha_{\mathrm{TM}} &= \frac{K_m(x^{1,1})}{I_m(x^{1,1})}\frac{\tilde{M}_{12}\tilde{M}_{33} - \tilde{M}_{32}\tilde{M}_{13}}{\tilde{M}_{11}\tilde{M}_{33} - \tilde{M}_{13}\tilde{M}_{31}}, \\ \alpha_{\mathrm{TE}} &= \frac{K_m(x^{1,1})}{I_m(x^{1,1})}\frac{\tilde{M}_{12}\tilde{M}_{31} - \tilde{M}_{32}\tilde{M}_{11}}{\tilde{M}_{11}\tilde{M}_{33} - \tilde{M}_{13}\tilde{M}_{31}}. \end{aligned} \tag{6.61}$$

Note that none of these two quantities depend on a, the offset of the source, since the matrices $M^{p+1,p}$ do not depend on a.
Finally, knowing from Eqs. (6.1), (6.6) and (6.8) that

$$\begin{aligned} C_{Ke}^{(0)} &= C_{Kg}^{(0)} = 0, \\ C_{Ig}^{(0)} &= C_{Ig}^{(1)}, \\ C_{Ie}^{(0)} &= C_{Ie}^{(1)} + C_{Ke}^{(1)}\frac{K_m\left(\frac{ka}{\gamma}\right)}{I_m\left(\frac{ka}{\gamma}\right)}, \end{aligned} \tag{6.62}$$

we can compute the constants for all the layers p.
It is worth mentioning that the general multilayer analysis was performed long ago for $m = 0$ and $m = 1$ in Refs. [26, 32], using a different algorithm that was implemented in a computer program called LAWAT, later [33]

converted to Mathematica®[34]. According to Ref. [2], the results seemed to lack accuracy due to numerical errors, so the code was modified to compute the solutions in a symbolic way before performing the numerical evaluation. Still, some problems remained as it was very long to perform the computation (for the $m = 1$ mode) for 3 layers of different materials in the pipe wall, and impossible to perform it for a larger number of layers. Our method, which involves only multiplications of 4×4 matrices and a final simple formula to compute α_{TM} and α_{TE}, overcomes this difficulty. A similar matrix method has also been developped independently in Refs. [35–37].

7 Electromagnetic force inside the pipe

One of the quantity of interest is the Lorentz electromagnetic force $\vec{F}$ on a given test particle inside the vacuum pipe. We assume such a particle has a charge of q and a velocity given by

$$\vec{v_b} = v_b \vec{e_s},$$

where v_b is the velocity of the bunch containing the particle (it can be different from the velocity v of the source terms, in the multibunch and coasting beam cases, where v is not the velocity of the beam but that of the travelling wave [2, 3]). Dropping the superscript (p) for simplicity, the longitudinal component of the force is written

$$F_s = qE_s, \tag{7.1}$$

while the transverse components are (using Eqs. (5.9) to (5.12) and recalling that $\nu = \frac{k}{\gamma}$ in vacuum)

$$F_r = q\left(E_r - v_b \mu_0 H_\theta\right) = \frac{jq\gamma^2}{k}\left[(1-\beta\beta_b)\frac{\partial E_s}{\partial r} + \frac{v - v_b}{r}\mu_0 \frac{\partial H_s}{\partial \theta}\right], \tag{7.2}$$

$$F_\theta = q\left(E_\theta + v_b \mu_0 H_r\right) = \frac{jq\gamma^2}{k}\left[\frac{1-\beta\beta_b}{r}\frac{\partial E_s}{\partial \theta} - (v - v_b)\,\mu_0 \frac{\partial H_s}{\partial r}\right], \tag{7.3}$$

where $\beta_b = \frac{v_b}{c}$ is the relativistic velocity factor of the test particle. In the case when $v = v_b$ (single-bunch case), we get for the transverse forces

$$F_r = \frac{jq}{k}\frac{\partial E_s}{\partial r}, \tag{7.4}$$

$$F_\theta = \frac{jq}{kr}\frac{\partial E_s}{\partial \theta}. \tag{7.5}$$

Therefore in the single-bunch case the force does not depend on the longitudinal component of the magnetic field.

8 Impedance

8.1 Recapitulation: expressions for the f eld components and the electromagnetic force in the single-bunch case

For a given azimuthal mode m, we computed in frequency domain the six components of the electromagnetic fields created by the sources given in Section 2, in each layer (p). The longitudinal components are given by Eqs. (4.10) and (4.11) (recalling that $\vec{G} = Z_0 \vec{H}$ and the equality between all the azimuthal mode numbers):

$$E_s^{(p)} = \cos(m\theta) e^{-jks}\left[C_{Ie}^{(p)} I_m\left(\nu^{(p)} r\right) + C_{Ke}^{(p)} K_m\left(\nu^{(p)} r\right)\right], \tag{8.1}$$

$$G_s^{(p)} = \sin(m\theta) e^{-jks}\left[C_{Ig}^{(p)} I_m\left(\nu^{(p)} r\right) + C_{Kg}^{(p)} K_m\left(\nu^{(p)} r\right)\right], \tag{8.2}$$

where the constants $C_{Ie}^{(p)}$, $C_{Ke}^{(p)}$, $C_{Ig}^{(p)}$ and $C_{Kg}^{(p)}$ (that still depend on m and on frequency) are calculated thanks to the matrices defined in Eqs. (6.44) to (6.49) and (6.52), and the relations (6.50), (6.51), (6.58), (6.59) and (6.62).

The transverse components are then, from Eqs. (5.9) to (5.12)

$$\begin{aligned} E_r^{(p)} &= \frac{jk}{\nu^{(p)2}}\cos(m\theta)e^{-jks}\left[\nu^{(p)}\left\{C_{Ie}^{(p)}I'_m\left(\nu^{(p)}r\right)+C_{Ke}^{(p)}K'_m\left(\nu^{(p)}r\right)\right\}\right. \\ &\left. +\frac{m\beta\mu_1^{(p)}}{r}\left\{C_{Ig}^{(p)}I_m\left(\nu^{(p)}r\right)+C_{Kg}^{(p)}K_m\left(\nu^{(p)}r\right)\right\}\right], \end{aligned} \tag{8.3}$$

$$\begin{aligned} E_\theta^{(p)} &= \frac{jk}{\nu^{(p)2}}\sin(m\theta)e^{-jks}\left[-\frac{m}{r}\left\{C_{Ie}^{(p)}I_m\left(\nu^{(p)}r\right)+C_{Ke}^{(p)}K_m\left(\nu^{(p)}r\right)\right\}\right. \\ &\left. -\beta\mu_1^{(p)}\nu^{(p)}\left\{C_{Ig}^{(p)}I'_m\left(\nu^{(p)}r\right)+C_{Kg}^{(p)}K'_m\left(\nu^{(p)}r\right)\right\}\right], \end{aligned} \tag{8.4}$$

$$\begin{aligned} G_r^{(p)} &= \frac{jk}{\nu^{(p)2}}\sin(m\theta)e^{-jks}\left[\frac{m\beta\varepsilon_1^{(p)}}{r}\left\{C_{Ie}^{(p)}I_m\left(\nu^{(p)}r\right)+C_{Ke}^{(p)}K_m\left(\nu^{(p)}r\right)\right\}\right. \\ &\left. +\nu^{(p)}\left\{C_{Ig}^{(p)}I'_m\left(\nu^{(p)}r\right)+C_{Kg}^{(p)}K'_m\left(\nu^{(p)}r\right)\right\}\right], \end{aligned} \tag{8.5}$$

$$\begin{aligned} G_\theta^{(p)} &= \frac{jk}{\nu^{(p)2}}\cos(m\theta)e^{-jks}\left[\beta\varepsilon_1^{(p)}\nu^{(p)}\left\{C_{Ie}^{(p)}I'_m\left(\nu^{(p)}r\right)+C_{Ke}^{(p)}K'_m\left(\nu^{(p)}r\right)\right\}\right. \\ &\left. +\frac{m}{r}\left\{C_{Ig}^{(p)}I_m\left(\nu^{(p)}r\right)+C_{Kg}^{(p)}K_m\left(\nu^{(p)}r\right)\right\}\right]. \end{aligned} \tag{8.6}$$

The Lorentz electromagnetic force acting on a test particle of charge q located inside the vacuum pipe and outside the ring-shaped beam (in the single-bunch case, i.e. $\upsilon_b = \upsilon$) has a longitudinal component equal to

$$F_s^{(1)} = \frac{jq\omega P_m}{\pi\varepsilon_0\upsilon^2\gamma^2 a^m(1+\delta_{m0})} I_m\left(\frac{ka}{\gamma}\right)\cos(m\theta)e^{-jks}\left[K_m\left(\frac{kr}{\gamma}\right)-\alpha_{\mathrm{TM}}I_m\left(\frac{kr}{\gamma}\right)\right], \tag{8.7}$$

while the transverse components are

$$F_r^{(1)} = -\frac{q\omega P_m}{\pi\varepsilon_0\upsilon^2\gamma^3 a^m(1+\delta_{m0})} I_m\left(\frac{ka}{\gamma}\right)\cos(m\theta)e^{-jks}\left[K'_m\left(\frac{kr}{\gamma}\right)-\alpha_{\mathrm{TM}}I'_m\left(\frac{kr}{\gamma}\right)\right], \tag{8.8}$$

$$F_\theta^{(1)} = \frac{qP_m m}{\pi\varepsilon_0\upsilon\gamma^2 a^m(1+\delta_{m0})r} I_m\left(\frac{ka}{\gamma}\right)\sin(m\theta)e^{-jks}\left[K_m\left(\frac{kr}{\gamma}\right)-\alpha_{\mathrm{TM}}I_m\left(\frac{kr}{\gamma}\right)\right], \tag{8.9}$$

where we used the definitions (6.60) and Eq. (6.51).
We can also compute the forces in the first region, that is, where $r < a$. This gives, using Eqs. (6.62)

$$F_s^{(0)} = \frac{jq\omega P_m}{\pi\varepsilon_0\upsilon^2\gamma^2 a^m(1+\delta_{m0})}\cos(m\theta)e^{-jks}\left[K_m\left(\frac{ka}{\gamma}\right)-\alpha_{\mathrm{TM}}I_m\left(\frac{ka}{\gamma}\right)\right]I_m\left(\frac{kr}{\gamma}\right), \tag{8.10}$$

$$F_r^{(0)} = -\frac{q\omega P_m}{\pi\varepsilon_0\upsilon^2\gamma^3 a^m(1+\delta_{m0})}\cos(m\theta)e^{-jks}\left[K_m\left(\frac{ka}{\gamma}\right)-\alpha_{\mathrm{TM}}I_m\left(\frac{ka}{\gamma}\right)\right]I'_m\left(\frac{kr}{\gamma}\right), \tag{8.11}$$

$$F_\theta^{(0)} = \frac{qP_m m}{\pi\varepsilon_0\upsilon\gamma^2 a^m(1+\delta_{m0})r}\sin(m\theta)e^{-jks}\left[K_m\left(\frac{ka}{\gamma}\right)-\alpha_{\mathrm{TM}}I_m\left(\frac{ka}{\gamma}\right)\right]I_m\left(\frac{kr}{\gamma}\right). \tag{8.12}$$

In principle we should sum the contributions of all the azimuthal modes to calculate the fields created by our initial point source (see Eqs. (2.4) and (2.5)). The resulting expression cannot be evaluated analytically in the general case because all the constants depend on m in a very complicated way. In the next section we will still perform (and then simplify) such a multimode sum on the force in the vacuum region, to get the impedance seen by a test particle.

8.2 Impedance derivation

As explained in the last section, we have to perform a sum on all the azimuthal modes to compute the exact force felt by a test particle in the vacuum region (inside the pipe), which then enables us to compute the impedance seen by the particle at the position in the transverse plane (a_2, θ_2) while the source is at $(a_1, 0)$ (note that from now on we replace a by a_1 in all previous formulas). To the best of our knowledge, such a multimode analysis has not

been done before, as one usually limits oneself to the computation of the modes $m = 0$ ("monopole") and the mode $m = 1$ ("dipole"), respectively associated with the longitudinal and transverse impedance. We will show here that when summing all the azimuthal mode contributions (which is required in principle to get back the fields created by our delta-function point source, so the Green's function of the problem), we get a different expression for the direct space-charge impedance, as well as additional terms in the wall impedance.
Several definitions of the impedance exist in Refs. [13, 38, 39] and [40, p. 74]. We will here write the longitudinal impedance in a general way, inspired by Ref. [39]

$$Z_{\parallel} = -\frac{1}{Q\frac{q}{a_2}} \int \mathrm{d}V E_s J_t^*(a_2, \theta_2), \tag{8.13}$$

where the integration is performed over the volume of the structure considered (usually on a finite length L), and where both E_s and J_t are the sum of all the azimuthal contributions. The $*$ stands for the complex conjugate, E_s is the longitudinal component of the field created by a source at $r = a_1$ and $\theta = 0$ and $\vec{J_t} = J_t \vec{e_s}$ is another current density flowing at the test particle position, whose expression is therefore (see Section 2)

$$J_t(a_2, \theta_2) = \frac{q}{a_2} e^{-jks} \delta(r - a_2) \delta_p(\theta - \theta_2), \tag{8.14}$$

q being the particle charge. This gives for a test particle in the vacuum region outside the source

$$Z_{\parallel}^{(1)} = -\int^L \mathrm{d}s \sum_{m=0}^{\infty} \frac{j\omega}{\pi\varepsilon_0 \upsilon^2 \gamma^2 (1 + \delta_{m0})} I_m\left(\frac{ka_1}{\gamma}\right) \cos(m\theta_2) \left[K_m\left(\frac{ka_2}{\gamma}\right) - \alpha_{\mathrm{TM}}(m, \omega) I_m\left(\frac{ka_2}{\gamma}\right)\right], \tag{8.15}$$

and in the vacuum region where $a_2 < a_1$

$$Z_{\parallel}^{(0)} = -\int^L \mathrm{d}s \sum_{m=0}^{\infty} \frac{j\omega}{\pi\varepsilon_0 \upsilon^2 \gamma^2 (1 + \delta_{m0})} I_m\left(\frac{ka_2}{\gamma}\right) \cos(m\theta_2) \left[K_m\left(\frac{ka_1}{\gamma}\right) - \alpha_{\mathrm{TM}}(m, \omega) I_m\left(\frac{ka_1}{\gamma}\right)\right]. \tag{8.16}$$

Nothing depends on L in the integrands so we can replace both integrals by a multiplcation by L.
We can identify two terms in the impedance: one is a sum over m of terms that do not depend on α_{TM}, i.e. on the pipe wall or even its presence (if there's no pipe wall we obviously have $\alpha_{\mathrm{TM}} = 0$ since there can be no Bessel function I_m in the radial dependence of the field, this function going to infinity with r). This is the so-called direct (and incoherent) space-charge impedance, that we can compute exactly (without making any approximation) using Eq. (D.16). This gives the same result for both $a_2 > a_1$ and $a_2 < a_1$, so an expression valid in the whole vacuum region (we therefore drop the superscript (0) or (1))

$$Z_{\parallel}^{SC,direct} = -\frac{jL\omega}{2\pi\varepsilon_0 \upsilon^2 \gamma^2} K_0\left(\frac{k\sqrt{a_1^2 + a_2^2 - 2a_1 a_2 \cos\theta_2}}{\gamma}\right). \tag{8.17}$$

Note that $\sqrt{a_1^2 + a_2^2 - 2a_1 a_2 \cos\theta_2}$ is, from the law of cosines, the distance (in the transverse plane) between the source and the test particle[11]. This expression differs substantially to what can be found in Ref. [13] for instance, because we have summed all the azimuthal mode contributions in an exact way, which is particularly required for that part of the impedance: even if both $\frac{ka_1}{\gamma}$ and $\frac{ka_2}{\gamma}$ are much smaller than unity, each mode contributes significantly to the final result as it contains the product $I_m\left(\frac{ka_2}{\gamma}\right) K_m\left(\frac{ka_1}{\gamma}\right)$ (for e.g. $a_2 < a_1$) which is of order $\frac{1}{2m}\left(\frac{a_2}{a_1}\right)^m$ (i.e. of order unity) as can be seen from Eqs. (D.13) and (D.14).

[11]This result is actually quite obvious from the observation that the direct space-charge is the part of the impedance that is due to the force created by the beam when no boundaries are present, which is the same force as the one created by a beam at the origin of coordinates on a test particle at a distance r. The longitudinal force can be computed easily when knowing E_s in the mode $m = 0$ by taking the limit $a \to 0$, which gives (there is no boundary so there can be no I_0 function of r in E_s)

$$E_s = \frac{j\omega Q}{2\pi\varepsilon_0 \upsilon^2 \gamma^2} e^{-jks} K_0\left(\frac{kr}{\gamma}\right).$$

Upon integration and normalization according to Eq. (8.13), this gives exactly the direct space-charge longitudinal impedance computed above.

The second term of Eqs. (8.15) and (8.16) (whose radial dependence uses only I_m functions) is the so-called wall impedance, which is not exactly the same as the resistive-wall impedance and has been introduced in Ref. [41]. It contains both the impedance that we would have with a pipe wall made of a perfect conductor (this part is usually called the indirect space-charge impedance) and the part of the impedance coming from the resistivity of the layer(s). Its dependence on the wall properties is contained in $\alpha_{\mathrm{TM}}(m,\omega)$. We can notice that it is the same expression for $a_2 < a_1$ and $a_1 < a_2$ so that we can drop the superscript (0) or (1). To compute the impedance at any order of precision, we need to use the exact formula from Eq. (D.12) for the Bessel function I_m

$$\begin{aligned} Z_{\parallel}^{Wall} &= \frac{jL\omega}{\pi\varepsilon_0 \upsilon^2\gamma^2}\sum_{m=0}^{\infty}\frac{\cos(m\theta_2)}{(1+\delta_{m0})}\alpha_{\mathrm{TM}}(m,\omega)\left(\left(\frac{ka_1}{2\gamma}\right)^m\sum_{n_1=0}^{\infty}\frac{\left(\frac{ka_1}{2\gamma}\right)^{2n_1}}{n_1!(m+n_1)!}\right)\left(\left(\frac{ka_2}{2\gamma}\right)^m\sum_{n_2=0}^{\infty}\frac{\left(\frac{ka_2}{2\gamma}\right)^{2n_2}}{n_2!(m+n_2)!}\right) \\ &= \frac{jL\omega}{\pi\varepsilon_0 \upsilon^2\gamma^2}\sum_{n_1=0}^{\infty}\sum_{\substack{n_2=0 \\ n_2-n_1 \text{ even}}}^{\infty}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2}\left[\sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)}\frac{\cos(m\theta_2)\alpha_{\mathrm{TM}}(m,\omega)}{(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right]. \end{aligned} \tag{8.18}$$

In the linear domain (when both $\frac{ka_1}{\gamma}$ and $\frac{ka_2}{\gamma}$ are much smaller than unity), it is sufficient to compute the first term of the sum only (the next term being of second order). This term comes from the $m=0$ mode and does not depend on the position of the test particle nor on a_1:

$$Z_{\parallel}^{Wall,0,0} = \frac{jL\omega}{2\pi\varepsilon_0\upsilon^2\gamma^2}\alpha_{\mathrm{TM}}(m=0,\omega). \tag{8.19}$$

This result can be inferred from Ref. [14]. More generally we can define a wall impedance of order n_1 in a_1 and n_2 in a_2

$$Z_{\parallel}^{Wall,n_1,n_2} = \begin{cases} 0 & \text{if } n_1-n_2 \text{ is odd,} \\ \frac{jL\omega}{\pi\varepsilon_0\upsilon^2\gamma^2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2}\left[\sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)}\frac{\cos(m\theta_2)\alpha_{\mathrm{TM}}(m,\omega)}{(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right] & \text{if } n_1-n_2 \text{ is even.} \end{cases} \tag{8.20}$$

For instance the next non-zero terms are quadratic:

$$Z_{\parallel}^{Wall,1,1} = \frac{jL\omega}{\pi\varepsilon_0\upsilon^2\gamma^2}\cos(\theta_2)\alpha_{\mathrm{TM}}(m=1,\omega)\left(\frac{ka_1}{2\gamma}\right)\left(\frac{ka_2}{2\gamma}\right), \tag{8.21}$$

$$Z_{\parallel}^{Wall,2,0} = \frac{jL\omega}{2\pi\varepsilon_0\upsilon^2\gamma^2}\alpha_{\mathrm{TM}}(m=0,\omega)\left(\frac{ka_1}{2\gamma}\right)^2, \tag{8.22}$$

$$Z_{\parallel}^{Wall,0,2} = \frac{jL\omega}{2\pi\varepsilon_0\upsilon^2\gamma^2}\alpha_{\mathrm{TM}}(m=0,\omega)\left(\frac{ka_2}{2\gamma}\right)^2. \tag{8.23}$$

Similarly to the longitudinal case we can define the horizontal transverse impedance (see Refs. [13, 38, 39] and [40, p. 77], where we have again chosen a definition inspired from Ref. [39])

$$Z_x = \frac{j}{Q\frac{q}{a_2}}\int \mathrm{d}V\left[\vec{E}+\beta\vec{e_s}\times\vec{G}\right]\cdot\vec{e_x}J_t^*(a_2,\theta_2) = \frac{j}{Q\frac{q}{a_2}}\int \mathrm{d}V\frac{\vec{F}}{q}\cdot\vec{e_x}J_t^*(a_2,\theta_2), \tag{8.24}$$

where x stands for the cartesian horizontal coordinate, $\vec{e_x} = \cos\theta\vec{e_r} - \sin\theta\vec{e_\theta}$, J_t has the same expression as above in Eq. (8.14), and where the fields are the sum of all the azimuthal contributions created by our initial source at $r=a_1$ and $\theta=0$. This gives for $a_2 > a_1$, using Eqs. (8.8) and (8.9)

$$\begin{aligned} Z_x^{(1)} &= -\frac{jL\omega}{\pi\varepsilon_0\upsilon^2\gamma^3}\cos(\theta_2)\sum_{m=0}^{\infty}\frac{1}{1+\delta_{m0}}I_m\left(\frac{ka_1}{\gamma}\right)\cos(m\theta_2)\left[K_m'\left(\frac{ka_2}{\gamma}\right)-\alpha_{\mathrm{TM}}(m,\omega)I_m'\left(\frac{ka_2}{\gamma}\right)\right] \\ &\quad -\frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\sin(\theta_2)\sum_{m=0}^{\infty}\frac{m}{1+\delta_{m0}}I_m\left(\frac{ka_1}{\gamma}\right)\sin(m\theta_2)\left[K_m\left(\frac{ka_2}{\gamma}\right)-\alpha_{\mathrm{TM}}(m,\omega)I_m\left(\frac{ka_2}{\gamma}\right)\right], \end{aligned} \tag{8.25}$$

while for $a_2 < a_1$, using Eqs. (8.11) and (8.12)

$$\begin{aligned} Z_x^{(0)} &= -\frac{jL\omega}{\pi\varepsilon_0 \upsilon^2\gamma^3}\cos(\theta_2)\sum_{m=0}^{\infty}\frac{1}{1+\delta_{m0}}I_m'\left(\frac{ka_2}{\gamma}\right)\cos(m\theta_2)\left[K_m\left(\frac{ka_1}{\gamma}\right)-\alpha_{\mathrm{TM}}(m,\omega)I_m\left(\frac{ka_1}{\gamma}\right)\right] \\ &\quad -\frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\sin(\theta_2)\sum_{m=0}^{\infty}\frac{m}{1+\delta_{m0}}I_m\left(\frac{ka_2}{\gamma}\right)\sin(m\theta_2)\left[K_m\left(\frac{ka_1}{\gamma}\right)-\alpha_{\mathrm{TM}}(m,\omega)I_m\left(\frac{ka_1}{\gamma}\right)\right]. \end{aligned} \tag{8.26}$$

Again, we can identify a direct space-charge term that does not depend on α_{TM}, that we can compute exactly using Eqs. (D.17), (D.18) and (D.19), giving the same expression for both $a_1 < a_2$ and $a_1 > a_2$ so that we can also drop the superscript (0) or (1):

$$Z_x^{SC,direct} = \frac{jL\omega}{2\pi\varepsilon_0\upsilon^2\gamma^3}K_1\left(\frac{k\sqrt{a_1^2+a_2^2-2a_1a_2\cos\theta_2}}{\gamma}\right)\frac{a_2\cos\theta_2-a_1}{\sqrt{a_1^2+a_2^2-2a_1a_2\cos\theta_2}}. \tag{8.27}$$

As in the longitudinal case, this formula is different from the results that can be found in Refs. [13, 14], for the same reasons, that is, the fact that we have summed all the azimuthal modes instead of looking only at the dipolar mode $m = 1$, each of the modes contributing quite significantly to the total result.

The transverse horizontal wall impedance can be written in a similar way as in the longitudinal case, that is from the second term of Eqs. (8.25) and (8.26). Dropping again the superscript (0) or (1) since it has the same expression for both $a_1 < a_2$ and $a_1 > a_2$, using first Eq. (D.3) and then the exact expansion for I_m and I_{m-1} from Eq. (D.12), we get

$$\begin{aligned} Z_x^{Wall} &= \frac{jL\omega}{\pi\varepsilon_0\upsilon^2\gamma^3}\cos(\theta_2)\sum_{m=0}^{\infty}\frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\cos(m\theta_2)I_m\left(\frac{ka_1}{\gamma}\right)I_{m-1}\left(\frac{ka_2}{\gamma}\right) \\ &\qquad -\frac{jL}{\pi\varepsilon_0\upsilon\gamma^2a_2}\sum_{m=0}^{\infty}\frac{m\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\cos((m+1)\theta_2)I_m\left(\frac{ka_1}{\gamma}\right)I_m\left(\frac{ka_2}{\gamma}\right) \\ &= \frac{jL}{\pi\varepsilon_0\upsilon\gamma^2a_2}\sum_{m=0}^{\infty}\frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\left(\frac{ka_1}{2\gamma}\right)^m\left(\frac{ka_2}{2\gamma}\right)^m\left(\sum_{n_1=0}^{\infty}\frac{\left(\frac{ka_1}{2\gamma}\right)^{2n_1}}{n_1!(m+n_1)!}\right) \\ &\qquad \cdot\left[-m\cos\left((m+1)\theta_2\right)\left(\sum_{n_2=0}^{\infty}\frac{\left(\frac{ka_2}{2\gamma}\right)^{2n_2}}{n_2!(m+n_2)!}\right)\right. \\ &\qquad \left. +2\cos\theta_2\cos(m\theta_2)\left(\sum_{n_2=0}^{\infty}\frac{\left(\frac{ka_2}{2\gamma}\right)^{2n_2}}{n_2!(m+n_2-1)!}\right)\right] \qquad \text{(where } (-1)! = \infty\text{)} \\ &= \frac{jL}{\pi\varepsilon_0\upsilon\gamma^2a_2}\sum_{m=0}^{\infty}\frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\left(\frac{ka_1}{2\gamma}\right)^m\left(\frac{ka_2}{2\gamma}\right)^m\left(\sum_{n_1=0}^{\infty}\frac{\left(\frac{ka_1}{2\gamma}\right)^{2n_1}}{n_1!(m+n_1)!}\right) \\ &\qquad \cdot\left[\sum_{n_2=0}^{\infty}\frac{\left(\frac{ka_2}{2\gamma}\right)^{2n_2}}{n_2!(m+n_2)!}\left(m\cos\left\{(m-1)\theta_2\right)+2n_2\cos\theta_2\cos(m\theta_2)\right\}\right] \\ &= \frac{jL}{\pi\varepsilon_0\upsilon\gamma^2a_2}\sum_{n_1=0}^{\infty}\sum_{\substack{n_2=0 \\ n_2-n_1\ \text{even}}}^{\infty}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2} \\ &\qquad \cdot\left[\sum_{\substack{m=0 \\ n_1-m\ \text{even}}}^{\min(n_1,n_2)}\frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\left(\frac{n_2\cos\theta_2\cos(m\theta_2)+m\sin\theta_2\sin(m\theta_2)}{(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right)\right]. \end{aligned} \tag{8.28}$$

As in the longitudinal case, in the linear domain where $\frac{ka_1}{\gamma} \ll 1$ and $\frac{ka_2}{\gamma} \ll 1$ we can restrict ourselves to the first and principal terms of this sum, that are linear:

$$
\begin{aligned}
Z_x^{Wall,1,1} &= \frac{jLk^2}{4\pi\varepsilon_0 \upsilon\gamma^4}\alpha_{\mathrm{TM}}(m=1,\omega)a_1, \\
Z_x^{Wall,0,2} &= \frac{jLk^2}{4\pi\varepsilon_0 \upsilon\gamma^4}\alpha_{\mathrm{TM}}(m=0,\omega)a_2\cos\theta_2.
\end{aligned} \tag{8.29}
$$

The first term, which does not depend on the position of the test particle but is proportional to the one of the source a_1, is usually called the dipolar term and is often normalized by dividing by a_1 (as in Ref. [14]), thus obtaining an impedance in Ω/m. The second one is proportional to $x_2 = a_2\cos\theta_2$ and is called quadrupolar. This term is new and comes from the fact that we have developped the Bessel function for the $m = 0$ mode instead of considering as usual only the first order term.
More generally we can as in the longitudinal case define a transverse wall impedance of order n_1 in a_1 and n_2 in a_2

$$
Z_x^{Wall,n_1,n_2} = \begin{cases} 0 & \text{if } n_1 - n_2 \text{ is odd,} \\ \frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2} & \\ \quad\cdot\left[\sum_{\substack{m=0\\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\left(\frac{n_2\cos\theta_2\cos(m\theta_2)+m\sin\theta_2\sin(m\theta_2)}{(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right)\right] & \text{if } n_1 - n_2 \text{ is even.} \end{cases} \tag{8.30}
$$

The next non-zero terms are of third order.

Obviously we can also define the vertical transverse impedance by

$$
Z_y = \frac{j}{Q\frac{q}{a_2}}\int \mathrm{d}V\left[\vec{E} + \beta\vec{e_s}\times\vec{G}\right]\cdot\vec{e_y}J_t^*(a_2,\theta_2) = \frac{j}{Q\frac{q}{a_2}}\int \mathrm{d}V\frac{\vec{F}}{q}\cdot\vec{e_y}J_t^*(a_2,\theta_2), \tag{8.31}
$$

where y stands for the cartesian vertical coordinate, $\vec{e_y} = \sin\theta\vec{e_r} + \cos\theta\vec{e_\theta}$, J_t has the same expression as above in Eq. (8.14), and where the fields are the sum of all the azimuthal contributions created by our initial source at $r = a_1$ and $\theta = 0$. This gives for $a_2 > a_1$, using Eqs. (8.8) and (8.9)

$$
\begin{aligned}
Z_y^{(1)} = &-\frac{jL\omega}{\pi\varepsilon_0\upsilon^2\gamma^3}\sin(\theta_2)\sum_{m=0}^{\infty}\frac{1}{1+\delta_{m0}}I_m\left(\frac{ka_1}{\gamma}\right)\cos(m\theta_2)\left[K'_m\left(\frac{ka_2}{\gamma}\right) - \alpha_{\mathrm{TM}}(m,\omega)I'_m\left(\frac{ka_2}{\gamma}\right)\right] \\
&+\frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\cos(\theta_2)\sum_{m=0}^{\infty}\frac{m}{1+\delta_{m0}}I_m\left(\frac{ka_1}{\gamma}\right)\sin(m\theta_2)\left[K_m\left(\frac{ka_2}{\gamma}\right) - \alpha_{\mathrm{TM}}(m,\omega)I_m\left(\frac{ka_2}{\gamma}\right)\right],
\end{aligned}
$$

while for $a_2 < a_1$, using Eqs. (8.11) and (8.12)

$$
\begin{aligned}
Z_y^{(0)} = &-\frac{jL\omega}{\pi\varepsilon_0\upsilon^2\gamma^3}\sin(\theta_2)\sum_{m=0}^{\infty}\frac{1}{1+\delta_{m0}}I'_m\left(\frac{ka_2}{\gamma}\right)\cos(m\theta_2)\left[K_m\left(\frac{ka_1}{\gamma}\right) - \alpha_{\mathrm{TM}}(m,\omega)I_m\left(\frac{ka_1}{\gamma}\right)\right] \\
&+\frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\cos(\theta_2)\sum_{m=0}^{\infty}\frac{m}{1+\delta_{m0}}I_m\left(\frac{ka_2}{\gamma}\right)\sin(m\theta_2)\left[K_m\left(\frac{ka_1}{\gamma}\right) - \alpha_{\mathrm{TM}}(m,\omega)I_m\left(\frac{ka_1}{\gamma}\right)\right].
\end{aligned}
$$

Again, we can identify a direct space-charge term, that we can compute exactly using Eqs. (D.17), (D.18) and (D.19), giving the same expression for both $a_1 < a_2$ and $a_1 > a_2$ so that we can also drop the superscript (0) or (1):

$$
Z_y^{SC,direct} = \frac{jL\omega}{2\pi\varepsilon_0\upsilon^2\gamma^3}K_1\left(\frac{k\sqrt{a_1^2+a_2^2-2a_1a_2\cos\theta_2}}{\gamma}\right)\frac{a_2\sin\theta_2}{\sqrt{a_1^2+a_2^2-2a_1a_2\cos\theta_2}}. \tag{8.32}
$$

The transverse vertical wall impedance can be written in a similar way as in the horizontal one. Dropping again the superscript (0) or (1) since it has the same expression for both $a_1 < a_2$ and $a_1 > a_2$, using first Eq. (D.3) and then the exact expansion for I_m and I_{m-1} from Eq. (D.12), we get

$$\begin{aligned}
Z_y^{Wall} &= \frac{jL\omega}{\pi\varepsilon_0 \upsilon^2\gamma^3}\sin(\theta_2)\sum_{m=0}^{\infty}\frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\cos(m\theta_2) I_m\left(\frac{ka_1}{\gamma}\right) I_{m-1}\left(\frac{ka_2}{\gamma}\right) \\
&\quad -\frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\sum_{m=0}^{\infty}\frac{m\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\sin((m+1)\theta_2) I_m\left(\frac{ka_1}{\gamma}\right) I_m\left(\frac{ka_2}{\gamma}\right) \\
&= \frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\sum_{m=0}^{\infty}\frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\left(\frac{ka_1}{2\gamma}\right)^m\left(\frac{ka_2}{2\gamma}\right)^m\left(\sum_{n_1=0}^{\infty}\frac{\left(\frac{ka_1}{2\gamma}\right)^{2n_1}}{n_1!(m+n_1)!}\right) \\
&\quad \cdot\left[-m\sin\left((m+1)\theta_2\right)\left(\sum_{n_2=0}^{\infty}\frac{\left(\frac{ka_2}{2\gamma}\right)^{2n_2}}{n_2!(m+n_2)!}\right)\right. \\
&\quad \left. +2\sin\theta_2\cos(m\theta_2)\left(\sum_{n_2=0}^{\infty}\frac{\left(\frac{ka_2}{2\gamma}\right)^{2n_2}}{n_2!(m+n_2-1)!}\right)\right] \\
&= \frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\sum_{m=0}^{\infty}\frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\left(\frac{ka_1}{2\gamma}\right)^m\left(\frac{ka_2}{2\gamma}\right)^m\left(\sum_{n_1=0}^{\infty}\frac{\left(\frac{ka_1}{2\gamma}\right)^{2n_1}}{n_1!(m+n_1)!}\right) \\
&\quad \cdot\left[\sum_{n_2=0}^{\infty}\frac{\left(\frac{ka_2}{2\gamma}\right)^{2n_2}}{n_2!(m+n_2)!}\left(-m\sin\left\{(m-1)\theta_2\right)+2n_2\sin\theta_2\cos(m\theta_2)\right\}\right] \\
&= \frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\sum_{n_1=0}^{\infty}\sum_{\substack{n_2=0\\ n_2-n_1\ \text{even}}}^{\infty}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2} \\
&\quad \cdot\left[\sum_{\substack{m=0\\ n_1-m\ \text{even}}}^{\min(n_1,n_2)}\frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\left(\frac{n_2\sin\theta_2\cos(m\theta_2)-m\cos\theta_2\sin(m\theta_2)}{(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right)\right].
\end{aligned} \tag{8.33}$$

As above, in the linear domain where $\frac{ka_1}{\gamma}\ll 1$ and $\frac{ka_2}{\gamma}\ll 1$ we can restrict ourselves to the first and principal term of this sum, that is linear:

$$Z_y^{Wall,0,2}=\frac{jLk^2}{4\pi\varepsilon_0\upsilon\gamma^4}\alpha_{\mathrm{TM}}(m=0,\omega)a_2\sin\theta_2. \tag{8.34}$$

In the transverse vertical impedance we don't have any dipolar term at first order because here $\theta=0$ for the source. On the other hand there is a quadrupolar term, proportional to $y_2=a_2\sin\theta_2$. This term is new and comes from the fact that we have developped the modified Bessel function for the $m=0$ mode instead of considering only the first order term as is usually done. More generally we can as in the horizontal case define a vertical transverse wall impedance of order n_1 in a_1 and n_2 in a_2

$$Z_y^{Wall,n_1,n_2}=\begin{cases} 0 & \text{if } n_1-n_2 \text{ is odd,} \\ \frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2} & \\ \quad\cdot\left[\sum_{\substack{m=0\\ n_1-m\ \text{even}}}^{\min(n_1,n_2)}\frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\left(\frac{n_2\sin\theta_2\cos(m\theta_2)-m\cos\theta_2\sin(m\theta_2)}{(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right)\right] & \text{if } n_1-n_2 \text{ is even.}\end{cases} \tag{8.35}$$

Again, the next non-zero terms are of third order.

8.3 Generalization to a source in $r = a_1$ and $\theta = \theta_1$

We can generalize the impedances computed above in the case of a source at $r = a_1$ and $\theta = \theta_1$ (instead of $\theta = 0$). From the rotational symmetry of the problem about the s axis, the results can be obtained from those in the above analysis by replacing θ_2 with the difference between the two angles $\theta_2 - \theta_1$. This will give directly the longitudinal impedance, while a further step is required for the transverse one, because we would then have $Z_{x'}$ and $Z_{y'}$ for $\vec{e_{x'}}$ along the θ_1 direction, i.e. $\vec{e_r}(\theta_1)$, and $\vec{e_{y'}}$ along the perpendicular of the θ_1 direction, i.e. $\vec{e_\theta}(\theta_1)$. To get back to Z_x and Z_y one then simply needs to compute $Z_x = Z_{x'} \cos\theta_1 - Z_{y'} \sin\theta_1$ and $Z_y = Z_{x'} \sin\theta_1 + Z_{y'} \cos\theta_1$. This gives, for the direct space-charge impedances (defining $x_1 = a_1 \cos\theta_1$ and $y_1 = a_1 \sin\theta_1$)

$$\begin{aligned}
Z_{\parallel}^{SC,direct} &= -\frac{jL\omega}{2\pi\varepsilon_0 \upsilon^2\gamma^2} K_0\left(\frac{k\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{\gamma}\right), \qquad (8.36)\\
Z_{x'}^{SC,direct} &= \frac{jL\omega}{2\pi\varepsilon_0 \upsilon^2\gamma^3} K_1\left(\frac{k\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{\gamma}\right) \frac{x_2\cos\theta_1 + y_2\sin\theta_1 - a_1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}},\\
Z_{y'}^{SC,direct} &= \frac{jL\omega}{2\pi\varepsilon_0 \upsilon^2\gamma^3} K_1\left(\frac{k\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{\gamma}\right) \frac{y_2\cos\theta_1 - x_2\sin\theta_1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}},
\end{aligned}$$

so that

$$Z_{x}^{SC,direct} = \frac{jL\omega}{2\pi\varepsilon_0 \upsilon^2\gamma^3} K_1\left(\frac{k\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{\gamma}\right) \frac{x_2 - x_1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}, \qquad (8.37)$$

$$Z_{y}^{SC,direct} = \frac{jL\omega}{2\pi\varepsilon_0 \upsilon^2\gamma^3} K_1\left(\frac{k\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{\gamma}\right) \frac{y_2 - y_1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}. \qquad (8.38)$$

For the wall part of the impedances we have when $n_1 - n_2$ is even

$$\begin{aligned}
Z_{\parallel}^{Wall,n_1,n_2} &= \frac{jL\omega}{\pi\varepsilon_0\upsilon^2\gamma^2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2}\left[\sum_{\substack{m=0\\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\cos\left(m(\theta_2-\theta_1)\right)\alpha_{\mathrm{TM}}(m,\omega)}{(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right], \qquad (8.39)\\
Z_{x'}^{Wall,n_1,n_2} &= \frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2}\\
&\cdot\left[\sum_{\substack{m=0\\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\left(\frac{n_2\cos(\theta_2-\theta_1)\cos\left(m(\theta_2-\theta_1)\right) + m\sin(\theta_2-\theta_1)\sin\left(m(\theta_2-\theta_1)\right)}{(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right)\right],\\
Z_{y'}^{Wall,n_1,n_2} &= \frac{jL}{\pi\varepsilon_0\upsilon\gamma^2 a_2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2}\\
&\cdot\left[\sum_{\substack{m=0\\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m,\omega)}{1+\delta_{m0}}\left(\frac{n_2\sin(\theta_2-\theta_1)\cos\left(m(\theta_2-\theta_1)\right) - m\cos(\theta_2-\theta_1)\sin\left(m(\theta_2-\theta_1)\right)}{(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right)\right],
\end{aligned}$$

giving in the end

$$
\begin{aligned}
Z_x^{Wall,n_1,n_2} &= \frac{jL}{\pi\varepsilon_0 \upsilon\gamma^2 a_2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2} \\
&\cdot \left[\sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\rm TM}(m,\omega)}{1+\delta_{m0}}\left(\frac{n_2\cos\theta_2\cos\left(m(\theta_2-\theta_1)\right)+m\sin\theta_2\sin\left(m(\theta_2-\theta_1)\right)}{(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right)\right],
\end{aligned}
\tag{8.40}
$$

$$
\begin{aligned}
Z_y^{Wall,n_1,n_2} &= \frac{jL}{\pi\varepsilon_0 \upsilon\gamma^2 a_2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2} \\
&\cdot \left[\sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\rm TM}(m,\omega)}{1+\delta_{m0}}\left(\frac{n_2\sin\theta_2\cos\left(m(\theta_2-\theta_1)\right)-m\cos\theta_2\sin\left(m(\theta_2-\theta_1)\right)}{(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right)\right],
\end{aligned}
\tag{8.41}
$$

while the same terms when $n_1 - n_2$ is odd are all zero.
In the linear domain the first and principal terms of these expressions are written

$$Z_\parallel^{Wall,0,0} = \frac{jL\omega}{2\pi\varepsilon_0\upsilon^2\gamma^2}\alpha_{\rm TM}(m=0,\omega), \tag{8.42}$$

$$Z_x^{Wall,1,1} = \frac{jLk^2}{4\pi\varepsilon_0\upsilon\gamma^4}\alpha_{\rm TM}(m=1,\omega)x_1, \tag{8.43}$$

$$Z_x^{Wall,0,2} = \frac{jLk^2}{4\pi\varepsilon_0\upsilon\gamma^4}\alpha_{\rm TM}(m=0,\omega)x_2, \tag{8.44}$$

$$Z_y^{Wall,1,1} = \frac{jLk^2}{4\pi\varepsilon_0\upsilon\gamma^4}\alpha_{\rm TM}(m=1,\omega)y_1, \tag{8.45}$$

$$Z_y^{Wall,0,2} = \frac{jLk^2}{4\pi\varepsilon_0\upsilon\gamma^4}\alpha_{\rm TM}(m=0,\omega)y_2. \tag{8.46}$$

Note that the new quadrupolar terms we found for the transverse impedance (proportional to x_2 and y_2) will be in the ultrarelativistic case small with respect to the longitudinal impedance, as they are proportional to $\frac{\alpha_{\rm TM}(m=0,\omega)}{\gamma^4}$ whereas $Z_\parallel^{Wall,0,0} \propto \frac{\alpha_{\rm TM}(m=0,\omega)}{\gamma^2}$. Therefore, if $Z_\parallel^{Wall,0,0}$ stays finite when $\gamma \to \infty$ (which should usually be the case), the quadrupolar terms $Z_x^{Wall,0,2}$ and $Z_y^{Wall,0,2}$ go to zero.
For the longitudinal wall impedance, the second order terms could be of interest as well. They are given by

$$Z_\parallel^{Wall,1,1} = \frac{jL\omega k^2}{4\pi\varepsilon_0\upsilon^2\gamma^4}\alpha_{\rm TM}(m=1,\omega)(x_1x_2+y_1y_2), \tag{8.47}$$

$$Z_\parallel^{Wall,2,0} = \frac{jL\omega k^2}{8\pi\varepsilon_0\upsilon^2\gamma^4}\alpha_{\rm TM}(m=0,\omega)(x_1^2+y_1^2), \tag{8.48}$$

$$Z_\parallel^{Wall,0,2} = \frac{jL\omega k^2}{8\pi\varepsilon_0\upsilon^2\gamma^4}\alpha_{\rm TM}(m=0,\omega)(x_2^2+y_2^2). \tag{8.49}$$

8.4 Checking Panofsky-Wenzel theorem on the derived impedances

We can verify (with Eq. (D.11)) that

$$
\begin{aligned}
\frac{\partial}{\partial x_2}Z_\parallel^{SC,direct} &= -\frac{j\omega L}{2\pi\varepsilon_0\upsilon^2\gamma^2}\frac{k}{\gamma}K_0'\left(\frac{k\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{\gamma}\right)\frac{x_2-x_1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}} \\
&= kZ_x^{SC,direct},
\end{aligned}
\tag{8.50}
$$

$$
\begin{aligned}
\frac{\partial}{\partial y_2}Z_\parallel^{SC,direct} &= -\frac{j\omega L}{2\pi\varepsilon_0\upsilon^2\gamma^2}\frac{k}{\gamma}K_0'\left(\frac{k\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{\gamma}\right)\frac{y_2-y_1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}} \\
&= kZ_y^{SC,direct},
\end{aligned}
\tag{8.51}
$$

which is in agreement with the Panofsky-Wenzel theorem as stated in Ref. [40, p. 90]. Also, knowing that

$$\left(\frac{a_2}{a_1}\right)^m \{\cos(m(\theta_2-\theta_1)) + j\sin(m(\theta_2-\theta_1))\} = \frac{(x_2+jy_2)^m}{(x_1+jy_1)^m},$$

which means

$$\cos(m(\theta_2-\theta_1)) = \frac{1}{2}\left(\frac{a_1}{a_2}\right)^m \left[\left(\frac{x_2+jy_2}{x_1+jy_1}\right)^m + \left(\frac{x_2-jy_2}{x_1-jy_1}\right)^m\right],$$

and

$$j\sin(m(\theta_2-\theta_1)) = \frac{1}{2}\left(\frac{a_1}{a_2}\right)^m \left[\left(\frac{x_2+jy_2}{x_1+jy_1}\right)^m - \left(\frac{x_2-jy_2}{x_1-jy_1}\right)^m\right],$$

we get (when n_1-n_2 is even, the case when it's odd leading obviously to the same relations)

$$\begin{aligned}
\frac{\partial}{\partial x_2} Z_\parallel^{Wall,n_1,n_2} &= \frac{jL\omega}{\pi\varepsilon_0 v^2\gamma^2}\left(\frac{k}{2\gamma}\right)^{n_1+n_2} \sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m,\omega)}{2(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!} \\
&\quad \cdot (x_1^2+y_1^2)^{\frac{n_1+m}{2}} \frac{\partial}{\partial x_2}\left[(x_2^2+y_2^2)^{\frac{n_2-m}{2}} \left\{\left(\frac{x_2+jy_2}{x_1+jy_1}\right)^m + \left(\frac{x_2-jy_2}{x_1-jy_1}\right)^m\right\}\right] \\
&= \frac{jL\omega}{\pi\varepsilon_0 v^2\gamma^2}\left(\frac{k}{2\gamma}\right)^{n_1+n_2} \sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m,\omega)(x_1^2+y_1^2)^{\frac{n_1+m}{2}}}{2(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!} \\
&\quad \cdot \left[(n_2-m)x_2(x_2^2+y_2^2)^{\frac{n_2-m-2}{2}} \left\{\left(\frac{x_2+jy_2}{x_1+jy_1}\right)^m + \left(\frac{x_2-jy_2}{x_1-jy_1}\right)^m\right\}\right. \\
&\quad \left. +(x_2^2+y_2^2)^{\frac{n_2-m}{2}} \left\{\frac{m}{x_1+jy_1}\left(\frac{x_2+jy_2}{x_1+jy_1}\right)^{m-1} + \frac{m}{x_1-jy_1}\left(\frac{x_2-jy_2}{x_1-jy_1}\right)^{m-1}\right\}\right] \\
&= \frac{jL\omega}{\pi\varepsilon_0 v^2\gamma^2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2} \sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m,\omega)}{(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!} \\
&\quad \cdot \left[\frac{n_2-m}{a_2}\cos\theta_2\cos(m(\theta_2-\theta_1)) + \frac{m}{a_2^2}\{x_2\cos(m(\theta_2-\theta_1))\right. \\
&\qquad\qquad \left. +jy_2(-j)\sin(m(\theta_2-\theta_1))\}\right] \\
&= \frac{jL\omega}{\pi\varepsilon_0 v^2\gamma^2 a_2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2} \sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m,\omega)}{(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!} \\
&\quad \cdot \left[n_2\cos\theta_2\cos(m(\theta_2-\theta_1)) + m\sin\theta_2\sin(m(\theta_2-\theta_1))\right], \\
&= kZ_x^{Wall,n_1,n_2}, \qquad (8.52)
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial}{\partial y_2} Z_{\parallel}^{Wall,n_1,n_2} &= \frac{jL\omega}{\pi\varepsilon_0 \upsilon^2\gamma^2}\left(\frac{k}{2\gamma}\right)^{n_1+n_2} \sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m,\omega)}{2(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!} \\
&\quad \cdot (x_1^2+y_1^2)^{\frac{n_1+m}{2}} \frac{\partial}{\partial y_2}\left[(x_2^2+y_2^2)^{\frac{n_2-m}{2}}\left\{\left(\frac{x_2+jy_2}{x_1+jy_1}\right)^m + \left(\frac{x_2-jy_2}{x_1-jy_1}\right)^m\right\}\right] \\
&= \frac{jL\omega}{\pi\varepsilon_0 \upsilon^2\gamma^2}\left(\frac{k}{2\gamma}\right)^{n_1+n_2} \sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m,\omega)(x_1^2+y_1^2)^{\frac{n_1+m}{2}}}{2(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!} \\
&\quad \cdot \Bigg[(n_2-m)y_2(x_2^2+y_2^2)^{\frac{n_2-m-2}{2}}\left\{\left(\frac{x_2+jy_2}{x_1+jy_1}\right)^m + \left(\frac{x_2-jy_2}{x_1-jy_1}\right)^m\right\} \\
&\quad + (x_2^2+y_2^2)^{\frac{n_2-m}{2}}\left\{\frac{jm}{x_1+jy_1}\left(\frac{x_2+jy_2}{x_1+jy_1}\right)^{m-1} - \frac{jm}{x_1-jy_1}\left(\frac{x_2-jy_2}{x_1-jy_1}\right)^{m-1}\right\}\Bigg] \\
&= \frac{jL\omega}{\pi\varepsilon_0 \upsilon^2\gamma^2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2} \sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m,\omega)}{(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!} \\
&\quad \cdot \Bigg[\frac{n_2-m}{a_2}\sin\theta_2\cos\left(m(\theta_2-\theta_1)\right) + \frac{m}{a_2^2}\left\{jx_2 j\sin\left(m(\theta_2-\theta_1)\right)\right. \\
&\qquad\qquad \left.+ y_2\cos\left(m(\theta_2-\theta_1)\right)\right\}\Bigg] \\
&= \frac{jL\omega}{\pi\varepsilon_0 \upsilon^2\gamma^2 a_2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2} \sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m,\omega)}{(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!} \\
&\quad \cdot \left[n_2\sin\theta_2\cos\left(m(\theta_2-\theta_1)\right) - m\cos\theta_2\sin\left(m(\theta_2-\theta_1)\right)\right], \\
&= kZ_y^{Wall,n_1,n_2}, \qquad (8.53)
\end{aligned}
$$

which are also in agreement with the Panofsky-Wenzel theorem.

9 Conclusion

This paper aimed at giving an as complete and detailed as possible derivation of the electromagnetic fields created by an offset point charge travelling at any speed in a multilayer infinitely long circular beam pipe. One must keep in mind that the basic assumptions of the derivation are the geometry of the pipe (uniform, infinitely long, cylindrical, and with no curvature along its axis) and the linearity of the pipe materials together with the validity of local Ohm's law (thus neglecting magnetoresitance and the anomalous skin effect).

Some of the results found here, in particular the general form for the longitudinal and transverse field components, were found long ago, but some detailed explanations on how they were derived were still missing in the references cited. In particular, we tried to explain thoroughly the equality between the azimuthal mode numbers of the electric field and magnetic field longitudinal components of all the subsequent layers.

A new procedure has also been devised for the field matching determination of all the constants, involving a matrix method, which is not completely new as similar methods were also devised in Refs. [35–37] but not in the same general formalism. A numerical implementation follows directly from the application of the equations in Section 6, having in mind that it is better to use first a symbolic code to obtain the whole multilayer formula of a particular problem, before plugging in numerical values. In a future paper the comparison between previously adopted approaches and our matrix method will be shown.

Quite surprisingly appear some new results from this study, due to the generalization to any azimuthal mode together with the final summation on all the modes m, to get the Green's function, that is, the solution of the initial problem involving delta-function sources. In principle from this analysis the electromagnetic fields created by any particular source, with a finite transverse shape, can then be computed using convolutions. The main new outcomes are the formulas given in Section 8.3, in particular the multimode direct space-charge impedance in Eqs. (8.36), (8.37) and (8.38). As the direct space-charge is usually not computed from the resistive-wall computation we did, but directly, using the absence of wall boundary and the particular shape of the real source, those “space-charge Green’s functions” should not give new information on the current state of comprehension and modelisation of space-charge. On the other hand, the new quadrupolar terms for the transverse wall impedance in Eqs. (8.44) and (8.46), which look negligible in the ultrarelativistic case but maybe not for low-energy beams, might be of relevance. Future numerical and experimental studies are still needed to quantify the impact of these analytically derived quantities in reality.

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A Appendix A: Macroscopic Maxwell equations in frequency domain for a general linear medium

In a medium, time domain macroscopic Maxwell equations are derived from the exact microscopic ones that hold for the electromagnetic fields $\vec{e}$ and $\vec{b}$ in vacuum, given the **total** microscopic charge density ρ and current density $\vec{j}$ [29, p. 1-2], [29, p. 105], [15, p. 248]

$$\begin{aligned} \mathrm{div}\vec{e} &= \frac{\rho}{\varepsilon_0}, \\ \vec{\mathrm{curl}}\vec{b} - \frac{\partial \vec{e}}{c^2 \partial t} &= \mu_0 \vec{j}, \\ \vec{\mathrm{curl}}\vec{e} + \frac{\partial \vec{b}}{\partial t} &= 0, \\ \mathrm{div}\vec{b} &= 0, \end{aligned}$$

where c is the speed of light in vacuum, ε_0 the permittivity of vacuum and μ_0 its permeability. When averaging these equations over elements of volume "physically infinitesimal", we get similar equations for the macroscopic averaged quantites $\vec{E} = \overline{\vec{e}}$, $\vec{B} = \overline{\vec{b}}$, $\overline{\rho}$ and $\vec{J} = \overline{\vec{j}}$ [15, p. 250-251]

$$\mathrm{div}\vec{E} = \frac{\overline{\rho}}{\varepsilon_0}, \tag{A.1}$$

$$\vec{\mathrm{curl}}\vec{B} - \frac{\partial \vec{E}}{c^2 \partial t} = \mu_0 \vec{J}, \tag{A.2}$$

$$\vec{\mathrm{curl}}\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \tag{A.3}$$

$$\mathrm{div}\vec{B} = 0. \tag{A.4}$$

The total mean charge $\overline{\rho}$ and current $\vec{J}$ are the sum of several terms:

- "External" charges and currents, i.e. independent of the electromagnetic fields and imposed by an external source. For instance, in our study the external charges are those of Eq. (2.1) and the external current density is related to them thanks to $\vec{J}_{ext} = \rho_{ext} v \vec{e_s}$,
- "Induced" charges and currents, i.e. that come from the action of the fields themselves on the medium. The induced charges and currents obviously vanish in vacuum (neglecting quantum effects).

We will now express the induced charges and currents, first in the case of a linear medium without dielectric or magnetic losses, and then in a general linear medium that can exhibit such losses.

A.1 Linear medium without dielectric or magnetic losses

In such a medium, which can be either a dielectric or a conductor and has a certain magnetic permeability, we can indentify three terms in the induced charges and currents, as we will see below. From the linear superposition principle we can consider separately the induced charges and currents coming from the dielectric aspect of the material to those coming from its conductive aspect[12]:

- For the dielectric aspect of the material we have [29, p. 34], [15, p. 153-156]

  $$\rho^{diel}_{induced} = -\mathrm{div}\vec{P},$$

  $\vec{P}$ being the polarization, that is, in a linear medium without loss:

  $$\vec{P} = \varepsilon_0 \chi_e \vec{E}, \tag{A.5}$$

[12] At a given frequency only one of these two aspects will be relevant, the material being either a conductor or a dielectric. Putting both aspects (dielectric and conductor) in the same material can be thought of being somehow artificial, but it enables to write a general formalism that will be suited for both cases, which is necessary to get formulas valid at any frequency since at high frequencies a conductor behaves like a dielectric (see also Ref. [16, p. 777]).

with χ_e the electric susceptibility, which is also equal to $\varepsilon_b - 1$ by definition of the dielectric constant ε_b (using the notations of Section 3). So we obtain for the dielectric induced charge density

$$\rho^{diel}_{induced} = -\varepsilon_0 \mathrm{div}\left(\chi_e \vec{E}\right). \tag{A.6}$$

The current density is then obtained from the continuity equation [15, p. 3][13]

$$\frac{\partial \rho^{diel}_{induced}}{\partial t} + \mathrm{div}\vec{J}_{induced} = 0, \tag{A.7}$$

so that

$$\mathrm{div}\vec{J}_{induced} = \mathrm{div}\frac{\partial \vec{P}}{\partial t},$$

which from Eq. (C.2) gives a current density as the sum of two terms

$$\vec{J}_{induced} = \varepsilon_0 \frac{\partial\left(\chi_e \vec{E}\right)}{\partial t} + \vec{\mathrm{curl}}\vec{M}. \tag{A.8}$$

$\vec{M}$ is a vector field called magnetization, which has nothing to do with the dielectric or conductive aspect of the medium but is related to $\vec{B}$ thanks to the magnetic susceptibility $\chi_m = \mu_r - 1$ (using the notations of Section 3) [29, p. 105], [15, p. 192]

$$\vec{M} = \frac{\chi_m}{\mu_0(1+\chi_m)}\vec{B}. \tag{A.9}$$

Therefore we have for the dielectric part of the induced current density

$$\vec{J}^{diel}_{induced} = \varepsilon_0 \frac{\partial\left(\chi_e \vec{E}\right)}{\partial t}, \tag{A.10}$$

and for the induced magnetic part

$$\vec{J}^{mag}_{induced} = \vec{\mathrm{curl}}\left(\frac{\chi_m}{\mu_0(1+\chi_m)}\vec{B}\right). \tag{A.11}$$

- For the conductive aspect of the medium, we have from Ohm's law

$$\vec{J}^{cond}_{induced} = \sigma \vec{E}, \tag{A.12}$$

where σ is the conductivity (see Section 3) of the material. From the continuity equation we then get

$$\frac{\partial \rho^{cond}_{induced}}{\partial t} = -\mathrm{div}\left(\sigma \vec{E}\right),$$

so that

$$\rho^{cond}_{induced} = -\int^t \mathrm{d}t' \mathrm{div}\left(\sigma \vec{E}\right). \tag{A.13}$$

[13] The continuity equation is not an additional equation to the problem but a consequence of Maxwell equations: taking the divergence of Eq. (A.2), and using Eqs. (C.2) and (A.1) we get

$$\mu_0 \mathrm{div}\vec{J} = -\frac{\partial \mathrm{div}\vec{E}}{c^2 \partial t} = -\frac{1}{\varepsilon_0 c^2}\frac{\partial \rho}{\partial t}$$

so that

$$\mathrm{div}\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

from $\varepsilon_0 \mu_0 c^2 = 1$. This continuity equation can be applied separately to each part of the induced charges and currents since it should be true for a dielectric that has no conductivity, or a conductor that has no dielectric susceptibility, or more generally assuming that we can always consider seperately the dielectric charges and currents from the conductive ones.

Finally, substituting the induced charges and currents from Eqs. (A.6), (A.10), (A.11), (A.12) and (A.13) into the right hand side of Maxwell's inhomogeneous equations (A.1) and (A.2) and adding the external charges and currents ρ_{ext} and $\vec{J}_{ext}$, we get

$$\begin{aligned}
\varepsilon_0 \mathrm{div}\vec{E} &= \rho_{ext} - \varepsilon_0 \mathrm{div}\left(\chi_e \vec{E}\right) - \int^t \mathrm{d}t' \mathrm{div}\left(\sigma \vec{E}\right), \\
\frac{1}{\mu_0}\vec{\mathrm{curl}}\vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} &= \vec{J}_{ext} + \varepsilon_0 \frac{\partial\left(\chi_e \vec{E}\right)}{\partial t} + \vec{\mathrm{curl}}\left(\frac{\chi_m}{\mu_0(1+\chi_m)}\vec{B}\right) + \sigma \vec{E},
\end{aligned}$$

which leads to

$$\mathrm{div}\left(\varepsilon_0 \varepsilon_b \vec{E} + \int^t \mathrm{d}t' \sigma \vec{E}\right) = \rho_{ext}, \tag{A.14}$$

$$\vec{\mathrm{curl}}\left(\frac{1}{\mu_0 \mu_r}\vec{B}\right) - \frac{\partial}{\partial t}\left(\varepsilon_0 \varepsilon_b \vec{E} + \int^t \mathrm{d}t' \sigma \vec{E}\right) = \vec{J}_{ext}. \tag{A.15}$$

A.2 Linear medium with dielectric and magnetic losses

In a general linear medium, there can be losses (i.e. dissipation of energy) that are due to some delay between the cause (an electromagnetic field) and its effect on the medium (polarization and/or magnetization). We can introduce them by replacing Eqs. (A.5) and (A.9) by the following integrals over all previous instants [29, p. 266]:

$$\vec{P} = \varepsilon_0 \int_0^\infty \mathrm{d}\tau f(\tau)\vec{E}(t-\tau), \tag{A.16}$$

$$\vec{M} = \frac{1}{\mu_0}\int_0^\infty \mathrm{d}\tau g(\tau)\vec{B}(t-\tau), \tag{A.17}$$

where f and g are two functions (usually decaying with time). When substituting these equations into the induced charge and current densities (dielectric and magnetic part, as the conductive part remains the same), we get instead of Eqs. (A.14) and (A.15)

$$\mathrm{div}\left(\varepsilon_0 \hat{\varepsilon}_b \vec{E} + \int^t \mathrm{d}t' \sigma \vec{E}\right) = \rho_{ext}, \tag{A.18}$$

$$\vec{\mathrm{curl}}\left(\frac{1}{\mu_0}\widehat{\frac{1}{\mu_r}}\vec{B}\right) - \frac{\partial}{\partial t}\left(\varepsilon_0 \hat{\varepsilon}_b \vec{E} + \int^t \mathrm{d}t' \sigma \vec{E}\right) = \vec{J}_{ext}, \tag{A.19}$$

where the linear operators $\hat{\varepsilon}_b$ and $\widehat{\frac{1}{\mu_r}}$ are defined by

$$\begin{aligned}
\hat{\varepsilon}_b \vec{E} &= \vec{E}(t) + \int_0^\infty \mathrm{d}\tau f(\tau)\vec{E}(t-\tau), \\
\widehat{\frac{1}{\mu_r}}\vec{B} &= \vec{B}(t) - \int_0^\infty \mathrm{d}\tau g(\tau)\vec{B}(t-\tau).
\end{aligned}$$

In frequency domain (that is, after applying a Fourier transform as explained in Section 2), we can substitute all occurences of $\frac{\partial}{\partial t}$ by the multiplicative factor $j\omega$ and all occurences of $\int^t \mathrm{d}t'$ by $\frac{1}{j\omega}$. Also, we can replace the convolution product in the linear operator $\hat{\varepsilon}_b$ (respectively, $\widehat{\frac{1}{\mu_r}}$) by a multiplication between the Fourier transform $\tilde{f}$ of f (resp. $\tilde{g}$ for g) and that of $\vec{E}$ (resp. $\vec{B}$). Now we can always write them in the form

$$\begin{aligned}
1 + \tilde{f} &= \varepsilon_b(\omega)\left[1 - j\tan\vartheta_E(\omega)\right], \\
1 - \tilde{g} &= \frac{1}{\mu_r(\omega)\left[1 - j\tan\vartheta_M(\omega)\right]},
\end{aligned}$$

where ε_b, ϑ_E, μ_r and ϑ_M (defined in Section 3) are real and can depend on frequency. At first order, we can reasonably assume that ε_b and μ_r are constants (as can also be inferred from the theory on linear medium without losses, see previous section). For the imaginary part, this can never be exactly the case since we should always have, as specified in Ref. [15, p. 262], $\mathrm{Im}\left(\tilde{f}(-\omega)\right) = -\mathrm{Im}\left(\tilde{f}(\omega)\right)$ and a similar relation for $\tilde{g}$. Finally, we get for the two first (inhomogeneous) Maxwell equations in frequency domain[14]

$$\begin{aligned} \mathrm{div}\left[\left(\varepsilon_0\varepsilon_b(1-j\tan\vartheta_E)+\frac{\sigma}{j\omega}\right)\vec{E}\right] &= \rho_{ext}, & \text{(A.20)}\\ \vec{\mathrm{curl}}\left[\frac{1}{\mu_0\mu_r(1-j\tan\vartheta_M)}\vec{B}\right]-j\omega\left[\left(\varepsilon_0\varepsilon_b(1-j\tan\vartheta_E)+\frac{\sigma}{j\omega}\right)\vec{E}\right] &= \vec{J}_{ext}. & \text{(A.21)}\end{aligned}$$

Since the two homogeneous Maxwell's equation (A.3) and (A.4) do not depend on induced charges and currents, we get the equations of Section 3 by defining the electric displacement $\vec{D}$ and magnetic field $\vec{H}$ (while $\vec{B}$ is called the magnetic induction) in the following way[15]

$$\begin{aligned} \vec{D} &= \left[\varepsilon_0\varepsilon_b(1-j\tan\vartheta_E)+\frac{\sigma}{j\omega}\right]\vec{E}, & \text{(A.22)}\\ \vec{H} &= \frac{1}{\mu_0\mu_r(1-j\tan\vartheta_M)}\vec{B}, & \text{(A.23)}\end{aligned}$$

therefore obtaining the four general frequency domain macroscopic Maxwell equations in a linear medium with losses

$$\begin{aligned} \mathrm{div}\vec{D} &= \rho_{ext}, & \text{(A.24)}\\ \vec{\mathrm{curl}}\vec{H}-j\omega\vec{D} &= \vec{J}_{ext}, & \text{(A.25)}\\ \vec{\mathrm{curl}}\vec{E}+j\omega\vec{B} &= 0, & \text{(A.26)}\\ \mathrm{div}\vec{B} &= 0. & \text{(A.27)}\end{aligned}$$

We should stress again that these equations suppose linearity of the medium. This is in particular not true for ferromagnetic materials (where the relation between the magnetization and the magnetic field is hysteretic and strongly non-linear), except for small fields in an untreated material (that is, not previously magnetized). Also, isotropy, homogeneity and the validity of Ohm's law have been assumed.

B Appendix B: Vector operations in cylindrical coordinates

The following formulas can be found in many textbooks of mechanics or electrodynamics, and in particular in Ref. [29, p. 452-453].

B.1 Gradient

For any scalar field f, the gradient in cylindrical coordinates (r,θ,s) (the basis unit vector being $\vec{e_r}$, $\vec{e_\theta}$ and $\vec{e_s}$) is given by

$$\vec{\mathrm{grad}}f = \frac{\partial f}{\partial r}\vec{e_r}+\frac{1}{r}\frac{\partial f}{\partial\theta}\vec{e_\theta}+\frac{\partial f}{\partial s}\vec{e_s}. \qquad \text{(B.1)}$$

B.2 Divergence

For any vector field $\vec{A}$, the divergence in cylindrical coordinates (r,θ,s) is given by

$$\begin{aligned} \mathrm{div}\vec{A} &= \frac{\partial A_r}{\partial r}+\frac{A_r}{r}+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{\partial A_s}{\partial s},\\ &= \frac{1}{r}\frac{\partial(rA_r)}{\partial r}+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{\partial A_s}{\partial s}. & \text{(B.2)}\end{aligned}$$

[14] Alternatively, we could have introduced the losses directly in the frequency domain equations, as in Ref. [15, p. 262], by defining the complex permittivity and permeability.

[15] Note that the dielectric loss tangent cannot be easily separated from the conductive part, which also generate an imaginary part in the total dispersion $\varepsilon_b(\omega)\left[1-j\tan\vartheta_E(\omega)\right]+\frac{\sigma}{j\omega}$. So we must be very careful when applying numerical values from tables in these formulas: for instance, the loss tangent could already "contain" the conductivity.

B.3 Curl

For any vector field $\vec{A}$, the curl in cylindrical coordinates (r, θ, s) is given by

$$\begin{aligned}
\left(\vec{\text{curl}}\vec{A}\right)_r &= \frac{1}{r}\frac{\partial A_s}{\partial \theta} - \frac{\partial A_\theta}{\partial s}, \\
\left(\vec{\text{curl}}\vec{A}\right)_\theta &= \frac{\partial A_r}{\partial s} - \frac{\partial A_s}{\partial r}, \\
\left(\vec{\text{curl}}\vec{A}\right)_s &= \frac{\partial A_\theta}{\partial r} + \frac{A_\theta}{r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}, \\
&= \frac{1}{r}\left[\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right].
\end{aligned} \tag{B.3}$$

B.4 Scalar laplacian

For any scalar field f, the laplacian in cylindrical coordinates (r, θ, s) is given by

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial s^2}. \tag{B.4}$$

B.5 Vector laplacian

For any vector field $\vec{A}$, the vector laplacian in cylindrical coordinates (r, θ, s) is given by

$$\begin{aligned}
\left(\nabla^2\vec{A}\right)_r &= \nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta}, \\
\left(\nabla^2\vec{A}\right)_\theta &= \nabla^2 A_\theta - \frac{A_\theta}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta}, \\
\left(\nabla^2\vec{A}\right)_s &= \nabla^2 A_s.
\end{aligned} \tag{B.5}$$

C Appendix C: Various relations between vector operations

From for instance Ref. [15, cover page], we know that

$$\vec{\text{curl}}\left(\vec{\text{curl}}\right) = \vec{\text{grad}}\,(\text{div}) - \nabla^2, \tag{C.1}$$

$$\text{div}\left(\vec{\text{curl}}\right) = 0. \tag{C.2}$$

D Appendix D: Various properties of the modif ed Bessel functions

In all the following η and z are complex numbers while m is a positive integer.
Modified Bessel functions $I_\eta(z)$ and $K_\eta(z)$ are independent solutions of the differential equation [30]

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} - \left(z^2 + \eta^2\right)y = 0. \tag{D.1}$$

From Ref. [30], we have the following relations between the modified Bessel functions (z^* stands for the complex conjugate of z)

$$
\begin{aligned}
I'_\eta(z)K_\eta(z) - K'_\eta(z)I_\eta(z) &= \frac{1}{z}, && \text{(D.2)}\\
I'_\eta(z) &= I_{\eta-1}(z) - \frac{\eta}{z}I_\eta(z), && \text{(D.3)}\\
K'_\eta(z) &= -K_{\eta-1}(z) - \frac{\eta}{z}K_\eta(z), && \text{(D.4)}\\
K'_\eta(z) &= -K_{\eta+1}(z) + \frac{\eta}{z}K_\eta(z), && \text{(D.5)}\\
I_{-m}(z) &= I_m(z), && \text{(D.6)}\\
K_{-m}(z) &= K_m(z), && \text{(D.7)}\\
I_m(z^*) &= I_m(z)^*, && \text{(D.8)}\\
K_m(z^*) &= K_m(z)^*, && \text{(D.9)}\\
I'_0(z) &= I_1(z), && \text{(D.10)}\\
K'_0(z) &= -K_1(z). && \text{(D.11)}
\end{aligned}
$$

The same reference gives also expansions for small arguments

$$
\begin{aligned}
I_m(z) &= \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k!(m+k)!} \quad \text{(also valid for } m=-1 \text{ with the convention } (-1)!=\infty\text{)}, && \text{(D.12)}\\
I_m(z) &\sim_{|z|\to 0} \frac{\left(\frac{1}{2}z\right)^m}{m!}, && \text{(D.13)}\\
K_m(z) &\sim_{|z|\to 0} \frac{1}{2}(m-1)!\left(\frac{1}{2}z\right)^{-m} \qquad (m \text{ strictly positive integer}), && \text{(D.14)}
\end{aligned}
$$

where $o_{|z|\to 0}\left(f(|z|)\right)$ is a function such that $\frac{o_{|z|\to 0}(f(|z|))}{f(|z|)} \to 0$ when $|z| \to 0$.
Finally, from Ref. [42] the following formula holds, for any complex numbers ϕ, η, z_1 and z_2 such that $|z_1 e^{\pm j\phi}| < |z_2|$

$$
\sum_{m=-\infty}^{\infty} I_m(z_1)K_{\eta+m}(z_2)e^{jm\phi} = K_\eta\left(\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi}\right)\left(\frac{z_2 - z_1e^{-j\phi}}{\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi}}\right)^\eta. \qquad \text{(D.15)}
$$

This in particular gives for $\eta = 0$, $0 < z_1 < z_2$ and ϕ real numbers, taking only the real part of the formula, and recalling Eqs. (D.6) and (D.7)

$$
\sum_{m=0}^{\infty} \frac{1}{1+\delta_{m0}} I_m(z_1)K_m(z_2)\cos(m\phi) = \frac{1}{2}K_0\left(\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi}\right), \qquad \text{(D.16)}
$$

where $\delta_{m0} = 1$ if $m = 0$, 0 otherwise. We can differentiate this equation with respect to z_1, z_2 and ϕ, giving (using Eq. (D.11))

$$
\begin{aligned}
\sum_{m=0}^{\infty} \frac{1}{1+\delta_{m0}} I'_m(z_1)K_m(z_2)\cos(m\phi) &= -K_1\left(\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi}\right)\frac{z_1 - z_2\cos\phi}{2\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi}}, && \text{(D.17)}\\
\sum_{m=0}^{\infty} \frac{1}{1+\delta_{m0}} I_m(z_1)K'_m(z_2)\cos(m\phi) &= -K_1\left(\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi}\right)\frac{z_2 - z_1\cos\phi}{2\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi}}, && \text{(D.18)}\\
\sum_{m=0}^{\infty} \frac{m}{1+\delta_{m0}} I_m(z_1)K_m(z_2)\sin(m\phi) &= K_1\left(\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi}\right)\frac{z_1z_2\sin\phi}{2\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi}}. && \text{(D.19)}
\end{aligned}
$$

E Appendix E: Consequences of Eq. (6.24) for all the pipe wall boundaries

In Section 6.2, we could not prove that the azimuthal mode numbers m and m_h are equal if for all the pipe wall boundaries Eq. (6.24) is true. We will show here the consequences of such a situation.
From this equation together with the continuity of E_s and H_s as stated in Eqs. (6.15) and (6.16) (dropping the sine and cosine factors thanks to Eqs. (6.20) and (6.21)), we have at each boundary $r = b^{(p)}$ with $1 \leq p \leq N-1$ (using the notations of Section 6.2, i.e. $x^{p+1,p} = \nu^{(p+1)} b^{(p)}$ and $x^{p,p} = \nu^{(p)} b^{(p)}$)

$$
\begin{aligned}
C_{Ie}^{(p)} I_m(x^{p,p}) + C_{Ke}^{(p)} K_m(x^{p,p}) &= C_{Ie}^{(p+1)} I_m(x^{p+1,p}) + C_{Ke}^{(p+1)} K_m(x^{p+1,p}), && \text{(E.1)} \\
\frac{\varepsilon_c^{(p)}}{\nu^{(p)}} \left[C_{Ie}^{(p)} I'_m(x^{p,p}) + C_{Ke}^{(p)} K'_m(x^{p,p}) \right] &= \frac{\varepsilon_c^{(p+1)}}{\nu^{(p+1)}} \left[C_{Ie}^{(p+1)} I'_m(x^{p+1,p}) + C_{Ke}^{(p+1)} K'_m(x^{p+1,p}) \right], && \text{(E.2)} \\
C_{Ih}^{(p)} I_{m_h}(x^{p,p}) + C_{Kh}^{(p)} K_{m_h}(x^{p,p}) &= C_{Ih}^{(p+1)} I_{m_h}(x^{p+1,p}) + C_{Kh}^{(p+1)} K_{m_h}(x^{p+1,p}), && \text{(E.3)} \\
\frac{\mu^{(p)}}{\nu^{(p)}} \left[C_{Ih}^{(p)} I'_{m_h}(x^{p,p}) + C_{Kh}^{(p)} K'_{m_h}(x^{p,p}) \right] &= \frac{\mu^{(p+1)}}{\nu^{(p+1)}} \left[C_{Ih}^{(p+1)} I'_{m_h}(x^{p+1,p}) + C_{Kh}^{(p+1)} K'_{m_h}(x^{p+1,p}) \right], && \text{(E.4)}
\end{aligned}
$$

to which we can add the relations (from Eqs. (6.1), (6.6), (6.7), (6.8) and (6.26))

$$
\begin{aligned}
C_{Ke}^{(1)} &= \frac{j\omega P_m}{\pi \varepsilon_0 \upsilon^2 \gamma^2 a^m (1+\delta_{m0})} I_m\left(\frac{ka}{\gamma}\right), \\
\left(C_{Ie}^{(0)} - C_{Ie}^{(1)} \right) I_m\left(\frac{ka}{\gamma}\right) &= C_{Ke}^{(1)} K_m\left(\frac{ka}{\gamma}\right), \\
C_{Ke}^{(0)} &= C_{Kh}^{(0)} = C_{Kh}^{(1)} = 0, \\
C_{Ih}^{(0)} &= C_{Ih}^{(1)}, \\
C_{Ie}^{(N)} &= C_{Ih}^{(N)} = 0. && \text{(E.5)}
\end{aligned}
$$

Putting together first Eqs. (E.1) and (E.2), then Eqs. (E.3) and (E.4), and writing them in matrix form, we have

$$
\begin{bmatrix} I_m(x^{p+1,p}) & K_m(x^{p+1,p}) \\ \frac{\varepsilon_c^{(p+1)}}{\nu^{(p+1)}} I'_m(x^{p+1,p}) & \frac{\varepsilon_c^{(p+1)}}{\nu^{(p+1)}} K'_m(x^{p+1,p}) \end{bmatrix} \cdot \begin{bmatrix} C_{Ie}^{(p+1)} \\ C_{Ke}^{(p+1)} \end{bmatrix} = \begin{bmatrix} C_{Ie}^{(p)} I_m(x^{p,p}) + C_{Ke}^{(p)} K_m(x^{p,p}) \\ \frac{\varepsilon_c^{(p)}}{\nu^{(p)}} \left\{ C_{Ie}^{(p)} I'_m(x^{p,p}) + C_{Ke}^{(p)} K'_m(x^{p,p}) \right\} \end{bmatrix}, \quad \text{(E.6)}
$$

and

$$
\begin{bmatrix} I_{m_h}(x^{p+1,p}) & K_{m_h}(x^{p+1,p}) \\ \frac{\mu^{(p+1)}}{\nu^{(p+1)}} I'_{m_h}(x^{p+1,p}) & \frac{\mu^{(p+1)}}{\nu^{(p+1)}} K'_{m_h}(x^{p+1,p}) \end{bmatrix} \cdot \begin{bmatrix} C_{Ih}^{(p+1)} \\ C_{Kh}^{(p+1)} \end{bmatrix} = \begin{bmatrix} C_{Ih}^{(p)} I_{m_h}(x^{p,p}) + C_{Kh}^{(p)} K_{m_h}(x^{p,p}) \\ \frac{\mu^{(p)}}{\nu^{(p)}} \left\{ C_{Ih}^{(p)} I'_{m_h}(x^{p,p}) + C_{Kh}^{(p)} K'_{m_h}(x^{p,p}) \right\} \end{bmatrix}. \quad \text{(E.7)}
$$

These can be solved readily using the inversion formula of a 2×2 matrix (see Appendix F), noticing that the determinant of the first matrix is proportional to the wronskian of the modified Bessel functions, more precisely equal to (see Eq. (D.2)) $-\frac{\varepsilon_c^{(p+1)}}{\nu^{(p+1)^2} b^{(p)}}$ for the first equation and $-\frac{\mu^{(p+1)}}{\nu^{(p+1)^2} b^{(p)}}$ for the second one. We get

$$
\begin{bmatrix} C_{Ie}^{(p+1)} \\ C_{Ke}^{(p+1)} \end{bmatrix} = -\frac{\nu^{(p+1)^2} b^{(p)}}{\varepsilon_c^{(p+1)}} \begin{bmatrix} \frac{\varepsilon_c^{(p+1)}}{\nu^{(p+1)}} K'_m(x^{p+1,p}) & -K_m(x^{p+1,p}) \\ -\frac{\varepsilon_c^{(p+1)}}{\nu^{(p+1)}} I'_m(x^{p+1,p}) & I_m(x^{p+1,p}) \end{bmatrix} \cdot \begin{bmatrix} I_m(x^{p,p}) & K_m(x^{p,p}) \\ \frac{\varepsilon_c^{(p)}}{\nu^{(p)}} I'_m(x^{p,p}) & \frac{\varepsilon_c^{(p)}}{\nu^{(p)}} K'_m(x^{p,p}) \end{bmatrix} \cdot \begin{bmatrix} C_{Ie}^{(p)} \\ C_{Ke}^{(p)} \end{bmatrix}, \quad \text{(E.8)}
$$

and

$$\begin{bmatrix} C_{Ih}^{(p+1)} \\ \\ C_{Kh}^{(p+1)} \end{bmatrix} = -\frac{\nu^{(p+1)^2} b^{(p)}}{\mu^{(p+1)}} \begin{bmatrix} \frac{\mu^{(p+1)}}{\nu^{(p+1)}} K'_{m_h}(x^{p+1,p}) & -K_{m_h}(x^{p+1,p}) \\ -\frac{\mu^{(p+1)}}{\nu^{(p+1)}} I'_{m_h}(x^{p+1,p}) & I_{m_h}(x^{p+1,p}) \end{bmatrix} \cdot \begin{bmatrix} I_{m_h}(x^{p,p}) & K_{m_h}(x^{p,p}) \\ \frac{\mu^{(p)}}{\nu^{(p)}} I'_{m_h}(x^{p,p}) & \frac{\mu^{(p)}}{\nu^{(p)}} K'_{m_h}(x^{p,p}) \end{bmatrix} \cdot \begin{bmatrix} C_{Ih}^{(p)} \\ \\ C_{Kh}^{(p)} \end{bmatrix}. \quad \text{(E.9)}$$

We call $M_e^{p+1,p}$ and $M_h^{p+1,p}$ the matrices relating respectively $\begin{bmatrix} C_{Ie}^{(p+1)} \\ C_{Ke}^{(p+1)} \end{bmatrix}$ to $\begin{bmatrix} C_{Ie}^{(p)} \\ C_{Ke}^{(p)} \end{bmatrix}$ and $\begin{bmatrix} C_{Ih}^{(p+1)} \\ C_{Kh}^{(p+1)} \end{bmatrix}$ to $\begin{bmatrix} C_{Ih}^{(p)} \\ C_{Kh}^{(p)} \end{bmatrix}$, i.e. the product of the two 2×2 matrices in Eqs. (E.8) and (E.9), multiplied respectively by $-\frac{\nu^{(p+1)^2} b^{(p)}}{\varepsilon_c^{(p+1)}}$ and $-\frac{\nu^{(p+1)^2} b^{(p)}}{\mu^{(p+1)}}$. When successively applying the relations (E.8) and (E.9) for each boundary, we get

$$\begin{bmatrix} C_{Ie}^{(N)} \\ \\ C_{Ke}^{(N)} \end{bmatrix} = M_e^{N,N-1} \cdot M_e^{N-1,N-2} \cdots M_e^{2,1} \cdot \begin{bmatrix} C_{Ie}^{(1)} \\ \\ C_{Ke}^{(1)} \end{bmatrix},$$

$$\begin{bmatrix} C_{Ih}^{(N)} \\ \\ C_{Kh}^{(N)} \end{bmatrix} = M_h^{N,N-1} \cdot M_h^{N-1,N-2} \cdots M_h^{2,1} \cdot \begin{bmatrix} C_{Ih}^{(1)} \\ \\ C_{Kh}^{(1)} \end{bmatrix},$$

which we can rewrite, recalling Eqs. (E.5) and defining $M(h) = M_h^{N,N-1} \cdot M_h^{N-1,N-2} \cdots M_h^{2,1}$ and $M(e) = M_e^{N,N-1} \cdot M_e^{N-1,N-2} \cdots M_e^{2,1}$

$$\begin{bmatrix} 0 \\ C_{Ke}^{(N)} \end{bmatrix} = M(e) \cdot \begin{bmatrix} C_{Ie}^{(1)} \\ C_{Ke}^{(1)} \end{bmatrix}, \quad \text{(E.10)}$$

and

$$\begin{bmatrix} 0 \\ C_{Kh}^{(N)} \end{bmatrix} = M(h) \cdot \begin{bmatrix} C_{Ih}^{(1)} \\ 0 \end{bmatrix}. \quad \text{(E.11)}$$

From the first equation we have $C_{Ie}^{(1)} = -\frac{M(e)_{12}}{M(e)_{11}} C_{Ke}^{(1)}$, and all the other constants for the longitudinal component of the electric field are determined by applying successively the matrices $M_e^{p+1,p}$. Note that $M(e)_{11} \neq 0$, otherwise we would have $M(e)_{12} = 0$ since $C_{Ke}^{(1)} \neq 0$ from Eq. (E.5), meaning that the determinant of $M(e)$ is zero which is not possible from the expression of $M_e^{p+1,p}$, whose determinant is proportional to a product of wronskians of modified Bessel functions.

Also we necessarily have that for all the layers $\nu^{(p+1)} = \nu^{(p)}$: if this was not the case for one layer then Eq. (6.24) gives for that layer the additional relation

$$C_{Ie}^{(p)} I_m(x^{p,p}) + C_{Ke}^{(p)} K_m(x^{p,p}) = 0,$$

which gives another relation between $C_{Ie}^{(p)}$ and $C_{Ke}^{(p)}$, whereas those constants are already known. The rather exceptional case when the computed constants already exactly verify this relation won't be analysed here. Therefore one of the consequences of Eq. (6.24) is that all the layers have the same radial propagation constant, namely that of vacuum $\nu = \frac{k}{\gamma}$ (γ being the relativistic mass factor), which also means that $\varepsilon_1 \mu_1 = 1$ for all the pipe wall materials.

From Eq. (E.11) we have $M(h)_{11} C_{Ih}^{(1)} = 0$. The case $M(h)_{11} = 0$ means a particular relation holds between ν and all the $\mu^{(p)}$ and $b^{(p)}$ of the layers. The other equation being $C_{Kh}^{(N)} = M(h)_{21} C_{Ih}^{(1)}$, with $M(h)_{21} \neq 0$ (otherwise the determinant of $M(h)$ would be zero, which is not the case from the expression of $M_h^{p+1,p}$), this would lead to the fact that H_s is not fully determined, despite that all the boundary conditions have been used: the coefficient $C_{Ih}^{(1)}$ can have any value, as well as the integer quotient $\frac{m_h}{m}$. We won't analyse fully this (again) exceptional case here [16].

[16] This is in principle possible but quite unlikely to happen. For instance if we have one layer going to infinity in the pipe, such that $N = 2$, $M(h) = M_h^{2,1}$, we have (with $b^{(1)} = b$, $\mu^{(1)} = \mu_0$, $\mu^{(2)} = \mu$, $\nu = \frac{k}{\gamma}$ and using Eq. (D.2))

$$M(h)_{11} = -\frac{kb}{\gamma\mu}\left[\mu K'_{m_h}\left(\frac{kb}{\gamma}\right) I_{m_h}\left(\frac{kb}{\gamma}\right) - \mu_0 K_{m_h}\left(\frac{kb}{\gamma}\right) I'_{m_h}\left(\frac{kb}{\gamma}\right)\right] = -\frac{kb}{\gamma\mu}\left[(\mu-\mu_0) K'_{m_h}\left(\frac{kb}{\gamma}\right) I_{m_h}\left(\frac{kb}{\gamma}\right) - \frac{\mu_0\gamma}{kb}\right], \quad \text{(E.12)}$$

If $M(h)_{11} \neq 0$, then $C_{Ih}^{(1)} = 0$ and by successively applying $M_h^{p+1,p}$ we get $C_{Ih}^{(p)} = C_{Kh}^{(p)} = 0$ for any p. Therefore, in the general case Eq. (6.24) has for consequences that $\varepsilon_1\mu_1 = 1$ for all the layers and $H_s = 0$ everywhere.

F Appendix F: Inversion of a 2×2 matrix

Given a 2×2 matrix of the form

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{F.1}$$

whose determinant $ad - bc$ is non zero, its inverse is given by

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{F.2}$$

This can be checked simply by multiplying the two matrices.

---

so that when

$$\mu_1 \equiv \frac{\mu}{\mu_0} = 1 + \frac{\gamma}{kbK'_{m_h}\left(\frac{kb}{\gamma}\right) I_{m_h}\left(\frac{kb}{\gamma}\right)}, \tag{E.13}$$

we get $M(h)_{11} = 0$. μ_1 is often taken as real and constant with frequency, so Eq. (E.13) means that there might be particular frequencies (related to different $\frac{m_h}{m}$ integer ratio) at which the behaviour of H_s becomes not fully determined. In the case of sufficiently small frequencies such that $\frac{kb}{\gamma} \ll 1$, using Eqs. (D.5), (D.13) and (D.14), we have

$$\frac{kb}{\gamma}K'_{m_h}\left(\frac{kb}{\gamma}\right) I_{m_h}\left(\frac{kb}{\gamma}\right) = -\frac{kb}{\gamma}K_{m_h+1}\left(\frac{kb}{\gamma}\right) I_{m_h}\left(\frac{kb}{\gamma}\right) + m_h K_{m_h}\left(\frac{kb}{\gamma}\right) I_{m_h}\left(\frac{kb}{\gamma}\right) \approx -1 + \frac{m_h}{2}\frac{(m_h-1)!}{m_h!} = -\frac{1}{2} \text{ (for } m_h \neq 0\text{)},$$

so that the condition above reads $\mu_1 = -1$, which, together with $\varepsilon_1 = -1$ from $\varepsilon_1\mu_1 = 1$, is quite unlikely to happen.

List of notations

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