IMPEDANCES OF AN INFINITELY LONG AND AXISYMMETRIC MULTILAYER BEAM PIPE: MATRIX FORMALISM AND MULTIMODE ANALYSIS

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Abstract

Using B. Zotter's formalism, we present here a novel, efficient and exact matrix method for the field matching determination of the electromagnetic field components created by an offset point charge travelling at any speed in an infinitely long circular multilayer beam pipe. This method improves by a factor of more than one hundred the computational time with three layers and allows the computation for more layers than three. We also generalize our analysis to any azimuthal mode and finally perform the summation on all such modes in the impedance formulae. In particular the exact multimode direct space-charge impedances (both longitudinal and transverse) are given, as well as the wall impedances to any order of precision.

INTRODUCTION

In this old subject [1], the general formalism of B. Zotter [2, 3, 4, 5] enables the analytical computation of the electromagnetic (EM) fields in frequency domain and the impedance created by a beam in an infinitely long multi-layered cylindrical pipe made of any linear materials. Still, the implementation of the algorithm to solve the multilayer field matching problem appeared to be a quite subtle matter: numerical errors were a problem until recently when symbolic tools began to be used [6], and then computational time became an issue.

This paper aims at presenting an improvement of Zotter's formalism to overcome these difficulties. Also, we will show that it is possible to extend it to any azimuthal mode instead of only $m = 0$ and $m = 1$. We refer the reader to [7] for more details on the derivations presented below.

ELECTROMAGNETIC CONFIGURATION

We consider a point-like beam of charge Q travelling at a speed $v = \beta c$ along the axis of an axisymmetric infinitely long pipe of inner radius b, at the position $(r = a_1, \theta = 0, s = vt)$ in cylindrical coordinates. The source charge density is in frequency domain ($f = \frac{\omega}{2\pi}$), after the usual decomposition on azimuthal modes [7, 8]

$$\rho(r,\theta,s;\omega) = \sum_{m=0}^{\infty} \rho_m = \sum_{m=0}^{\infty} \frac{Q\cos(m\theta)\delta(r-a_1)e^{-jks}}{\pi v a_1(1+\delta_{m0})}, \tag{1}$$

where $k \equiv \frac{\omega}{v}$, δ is the delta function, and $\delta_{m0} = 1$ if $m = 0$, 0 otherwise. The space is divided into $N + 1$ cylindrical layers of homogeneous, isotropic and linear media (see Fig. 1), each denoted by the superscript (p) $(0 \leq p \leq N)$. The last layer goes to infinity.

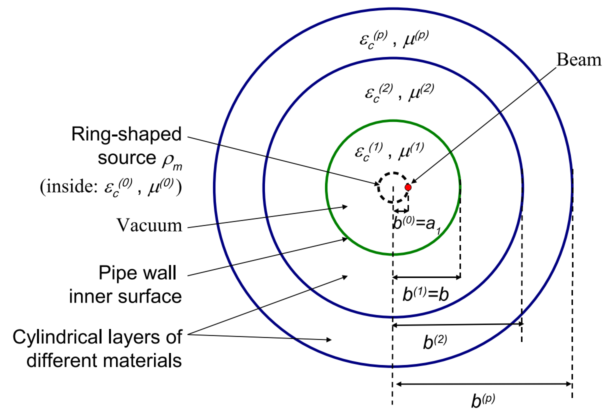

Figure 1: Cross section of the pipe.

The macroscopic Maxwell equations in frequency domain for the electric and magnetic fields $\vec{E}$ and $\vec{H}$ are written [5]

$$\vec{\text{curl}}\vec{H} - j\omega\vec{D} = \rho_m v\vec{e_s}, \quad \vec{\text{curl}}\vec{E} + j\omega\vec{B} = 0,$$
$$\text{div}\vec{D} = \rho_m, \quad \text{div}\vec{B} = 0, \quad \vec{D} = \varepsilon_c\vec{E},\ \vec{B} = \mu\vec{H},$$

where [9]

$$\varepsilon_c = \varepsilon_0\varepsilon_1 = \varepsilon_0\varepsilon_b\left[1 - j\tan\vartheta_E\right] + \frac{\sigma_{DC}}{j\omega\left(1 + j\omega\tau\right)}, \tag{2}$$

$$\mu = \mu_0\mu_1 = \mu_0\mu_r\left[1 - j\tan\vartheta_M\right]. \tag{3}$$

In these expressions, ε_0 (μ_0) is the permittivity (permeability) of vacuum, ε_b the real dielectric constant, μ_r the real part of the relative complex permeability, $\tan\vartheta_E$ ($\tan\vartheta_M$) the dielectric (magnetic) loss tangent, σ_{DC} the DC conductivity and τ the relaxation time. We use the Drude model [10, p. 312] for the AC conductivity, and assume the validity of local Ohm's law.

ELECTROMAGNETIC FIELDS

From Maxwell equations, one gets for each mode m [11]

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial s^2} + \omega^2\varepsilon_c\mu\right]E_s = \frac{1}{\varepsilon_c}\frac{\partial\rho_m}{\partial s} + j\omega\mu\rho_m v, \tag{4}$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial s^2} + \omega^2\varepsilon_c\mu\right]H_s = 0. \tag{5}$$

Solutions are sought by separation of variables, in the form $R(r)\Theta(\theta)S(s)$. Θ and S are solutions of the harmonic differential equation. From the symmetries of the problem [7]

$$\Theta_{E_s}(\theta) \propto \cos(m_e\theta), \quad S_{E_s}(s) \propto e^{-jks},$$
$$\Theta_{H_s}(\theta) \propto \sin(m_h\theta), \quad S_{H_s}(s) \propto e^{-jks},$$

where m_e and m_h should be integer multiples of m. R_{E_s} (R_{H_s}) is a combination of modified Bessel functions of order m_e (m_h) and argument νr with $\nu \equiv k\sqrt{1-\beta^2\varepsilon_1\mu_1}$. From the boundary conditions between all the layers it can be first proven [7] that $m_e = m_h = m$. The longitudinal components of the fields are then in each layer (p) [11] (with $\vec{G} = Z_0\vec{H} = \mu_0 c\vec{H}$):

$$E_s^{(p)} = \cos(m\theta)e^{-jks}\left[C_{Ie}^{(p)} I_m\left(\nu^{(p)}r\right) + C_{Ke}^{(p)} K_m\left(\nu^{(p)}r\right)\right], \quad (6)$$

$$G_s^{(p)} = \sin(m\theta)e^{-jks}\left[C_{Ig}^{(p)} I_m\left(\nu^{(p)}r\right) + C_{Kg}^{(p)} K_m\left(\nu^{(p)}r\right)\right], \quad (7)$$

where the constants $C_{Ie}^{(p)}$, $C_{Ke}^{(p)}$, $C_{Ig}^{(p)}$ and $C_{Kg}^{(p)}$ depend on m and ω. The transverse components are found from [11]

$$E_r^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\frac{\partial E_s^{(p)}}{\partial r} + \frac{\beta\mu_1^{(p)}}{r}\frac{\partial G_s^{(p)}}{\partial\theta}\right), \quad (8)$$

$$E_\theta^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\frac{1}{r}\frac{\partial E_s^{(p)}}{\partial\theta} - \beta\mu_1^{(p)}\frac{\partial G_s^{(p)}}{\partial r}\right), \quad (9)$$

$$G_r^{(p)} = \frac{jk}{\nu^{(p)2}}\left(-\frac{\beta\varepsilon_1^{(p)}}{r}\frac{\partial E_s^{(p)}}{\partial\theta} + \frac{\partial G_s^{(p)}}{\partial r}\right), \quad (10)$$

$$G_\theta^{(p)} = \frac{jk}{\nu^{(p)2}}\left(\beta\varepsilon_1^{(p)}\frac{\partial E_s^{(p)}}{\partial r} + \frac{1}{r}\frac{\partial G_s^{(p)}}{\partial\theta}\right). \quad (11)$$

Then, the boundary conditions at $r = a_1$ [8] and the finiteness of the fields at $r = 0$ and $r \to \infty$ give (with $\gamma^{-2} = 1 - \beta^2$)

$$C_{Ke}^{(0)} = C_{Kg}^{(0)} = C_{Kg}^{(1)} = C_{Ie}^{(N)} = C_{Ig}^{(N)} = 0,$$
$$C_{Ke}^{(1)} = \frac{j\omega\mu_0 Q}{\pi\beta^2\gamma^2(1+\delta_{m0})} I_m\left(\frac{ka_1}{\gamma}\right),$$
$$C_{Ig}^{(0)} = C_{Ig}^{(1)}, \quad C_{Ie}^{(0)} = C_{Ie}^{(1)} + C_{Ke}^{(1)}\frac{K_m\left(\frac{ka_1}{\gamma}\right)}{I_m\left(\frac{ka_1}{\gamma}\right)}. \quad (12)$$

Expressing all the boundary conditions at $b^{(p)}$ for $1 \le p \le N-1$, it can be shown [7] that the constants of one layer are related to those from the adjacent layer through

$$\begin{bmatrix} C_{Ie}^{(p+1)} \\ C_{Ke}^{(p+1)} \\ C_{Ig}^{(p+1)} \\ C_{Kg}^{(p+1)} \end{bmatrix} = M_p^{p+1} \cdot \begin{bmatrix} C_{Ie}^{(p)} \\ C_{Ke}^{(p)} \\ C_{Ig}^{(p)} \\ C_{Kg}^{(p)} \end{bmatrix}, \quad M_p^{p+1} = \begin{bmatrix} P_p^{p+1} & Q_p^{p+1} \\ S_p^{p+1} & R_p^{p+1} \end{bmatrix}, \quad (13)$$

where P_p^{p+1}, Q_p^{p+1}, S_p^{p+1} and R_p^{p+1} are 2×2 matrices:

$$P_p^{p+1} = \begin{bmatrix} \frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} I_m^{p,p} K_m'^{\,p+1,p} - \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} K_m^{p+1,p} I_m'^{\,p,p} & \frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} K_m^{p,p} K_m'^{\,p+1,p} - \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} K_m^{p+1,p} K_m'^{\,p,p} \\ -\frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} I_m^{p,p} I_m'^{\,p+1,p} + \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} I_m^{p+1,p} I_m'^{\,p,p} & -\frac{\varepsilon_1^{(p+1)}}{\nu^{(p+1)}} K_m^{p,p} I_m'^{\,p+1,p} + \frac{\varepsilon_1^{(p)}}{\nu^{(p)}} I_m^{p+1,p} K_m'^{\,p,p} \end{bmatrix} \zeta_p^{p+1},$$

$$Q_p^{p+1} = \chi_p^{p+1}\begin{bmatrix} -I_m^{p,p} K_m^{p+1,p} & -K_m^{p,p} K_m^{p+1,p} \\ I_m^{p,p} I_m^{p+1,p} & K_m^{p,p} I_m^{p+1,p} \end{bmatrix},$$

$$R_p^{p+1} = \begin{bmatrix} \frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} I_m^{p,p} K_m'^{\,p+1,p} - \frac{\mu_1^{(p)}}{\nu^{(p)}} K_m^{p+1,p} I_m'^{\,p,p} & \frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} K_m^{p,p} K_m'^{\,p+1,p} - \frac{\mu_1^{(p)}}{\nu^{(p)}} K_m^{p+1,p} K_m'^{\,p,p} \\ -\frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} I_m^{p,p} I_m'^{\,p+1,p} + \frac{\mu_1^{(p)}}{\nu^{(p)}} I_m^{p+1,p} I_m'^{\,p,p} & -\frac{\mu_1^{(p+1)}}{\nu^{(p+1)}} K_m^{p,p} I_m'^{\,p+1,p} + \frac{\mu_1^{(p)}}{\nu^{(p)}} I_m^{p+1,p} K_m'^{\,p,p} \end{bmatrix} \xi_p^{p+1},$$

$$S_p^{p+1} = \frac{\varepsilon_1^{(p+1)}}{\mu_1^{(p+1)}} Q_p^{p+1},$$

with $\zeta_p^{p+1} = \frac{-\nu^{(p+1)2} b^{(p)}}{\varepsilon_1^{(p+1)}}$, $\chi_p^{p+1} = \frac{m\left(\nu^{(p)2} - \nu^{(p+1)2}\right)}{\nu^{(p)2}\beta\varepsilon_1^{(p+1)}}$, $\xi_p^{p+1} = \frac{-\nu^{(p+1)2} b^{(p)}}{\mu_1^{(p+1)}}$, $I_m^{p+1,p} = I_m\left(\nu^{(p+1)} b^{(p)}\right)$, $I_m^{p,p} = I_m\left(\nu^{(p)} b^{(p)}\right)$ and similar definitions with I_m', K_m and K_m'.

Iteratively applying Eq. (13) and solving leads to

$$C_{Ie}^{(1)} = -C_{Ke}^{(1)}\alpha_{\mathrm{TM}} = -C_{Ke}^{(1)}\frac{\mathcal{M}_{12}\mathcal{M}_{33} - \mathcal{M}_{32}\mathcal{M}_{13}}{\mathcal{M}_{11}\mathcal{M}_{33} - \mathcal{M}_{13}\mathcal{M}_{31}},$$
$$C_{Ig}^{(1)} = C_{Ke}^{(1)}\alpha_{\mathrm{TE}} = C_{Ke}^{(1)}\frac{\mathcal{M}_{12}\mathcal{M}_{31} - \mathcal{M}_{32}\mathcal{M}_{11}}{\mathcal{M}_{11}\mathcal{M}_{33} - \mathcal{M}_{13}\mathcal{M}_{31}},$$
$$C_{Ke}^{(N)} = \mathcal{M}_{21}C_{Ie}^{(1)} + \mathcal{M}_{22}C_{Ke}^{(1)} + \mathcal{M}_{23}C_{Ig}^{(1)},$$
$$C_{Kg}^{(N)} = \mathcal{M}_{41}C_{Ie}^{(1)} + \mathcal{M}_{42}C_{Ke}^{(1)} + \mathcal{M}_{43}C_{Ig}^{(1)}, \quad (14)$$

where $\mathcal{M} \equiv M_{N-1}^{N} \cdot M_{N-2}^{N-1} \cdots M_1^2$.

As an example we have plotted in Fig. 2 the fields in a two-layer structure. For this plot we have taken away the e^{-jks} factor, and the direct space charge part of the fields in the vacuum region by setting $C_{Ke}^{(1)} = 0$ and $C_{Ie}^{(0)} = C_{Ie}^{(1)}$.

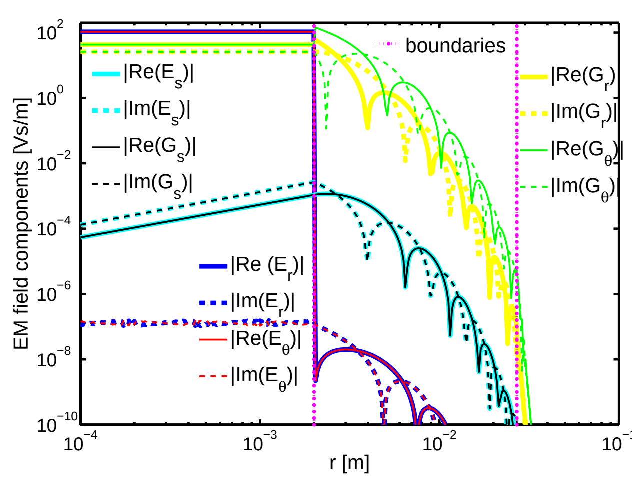

Figure 2: EM fields along the direction $\theta = \frac{\pi}{4}$ for the $m = 1$ mode at 1MHz in a graphite tube (b = 2mm, thickness=25mm, $\sigma_{DC,gra} = 10^5$S/m, $\tau_{gra} = 0.8$ ps), surrounded by stainless steel ($\sigma_{DC,ss} = 10^6$S/m, $\tau_{ss} = 0$). In all layers $\varepsilon_b = \mu_r = 1$ and $\vartheta_E = \vartheta_M = 0$. Beam parameters are $\gamma = 479.7$, $Q = 1$ C and $a_1 = 10\mu$m.

MULTIMODE IMPEDANCE

From the electromagnetic fields we can compute the impedances (longitudinal and transverse) for a test particle located at $r = a_2$ and $\theta = \theta_2$, using the definitions [11]

$$Z_{\parallel} = -\frac{1}{Q}\int \mathrm{d}s E_s(a_2, \theta_2, s; \omega) e^{jks}, \tag{15}$$

$$Z_x = \frac{j}{Q}\int \mathrm{d}s \left[E_x(a_2, \theta_2, s; \omega) - \beta G_y(a_2, \theta_2, s; \omega)\right] e^{jks}. \tag{16}$$

In these we can plug the total fields created by the point-like beam, i.e. the sum on all the azimuthal modes m, instead of only the modes $m = 0$ or 1. When doing so on the direct space-charge part of the fields (obtained when $\alpha_{\mathrm{TM}} = \alpha_{\mathrm{TE}} = 0$), we get the exact multimode direct space-charge impedances [7]:

$$Z_{\parallel}^{SC,dir.} = -\frac{jL\mu_0\omega}{2\pi\beta^2\gamma^2} K_0\left(\frac{kd_{1,2}}{\gamma}\right), \tag{17}$$

$$Z_{x}^{SC,dir.} = \frac{jL\mu_0\omega}{2\pi\beta^2\gamma^3} K_1\left(\frac{kd_{1,2}}{\gamma}\right)\frac{a_2\cos\theta_2 - a_1}{d_{1,2}}, \tag{18}$$

with L the length of the pipe element considered and $d_{1,2} = \sqrt{a_1^2 + a_2^2 - 2a_1a_2\cos\theta_2}$ the distance between the source and the test. In the same way, we obtain the wall impedances [9] to any order n_1 in a_1 and n_2 in a_2 [7]:

$$Z_{\parallel}^{W,n_1,n_2} = \frac{jL\mu_0\omega}{\pi\beta^2\gamma^2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2}\left[\sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\cos(m\theta_2)\alpha_{\mathrm{TM}}(m)}{(1+\delta_{m0})(\frac{n_1-m}{2})!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right] f_{n_1}^{n_2},$$

$$Z_{x}^{W,n_1,n_2} = \frac{jZ_0L}{\pi\beta\gamma^2a_2}\left(\frac{ka_1}{2\gamma}\right)^{n_1}\left(\frac{ka_2}{2\gamma}\right)^{n_2}\left[\sum_{\substack{m=0 \\ n_1-m \text{ even}}}^{\min(n_1,n_2)} \frac{\alpha_{\mathrm{TM}}(m)\left(n_2\cos\theta_2\cos(m\theta_2) + m\sin\theta_2\sin(m\theta_2)\right)}{(1+\delta_{m0})\left(\frac{n_1-m}{2}\right)!(\frac{n_1+m}{2})!(\frac{n_2-m}{2})!(\frac{n_2+m}{2})!}\right] f_{n_1}^{n_2},$$

where $f_{n_1}^{n_2} = 0$ if $n_1 - n_2$ is odd, 1 otherwise. We show below the first order terms, that are linear:

$$Z_{\parallel}^{W,0,0} = \frac{jL\mu_0\omega}{2\pi\beta^2\gamma^2}\alpha_{\mathrm{TM}}(m=0), \tag{19}$$

$$Z_{x}^{W,1,1} = \frac{jLZ_0k^2}{4\pi\beta\gamma^4}\alpha_{\mathrm{TM}}(m=1)a_1, \tag{20}$$

$$Z_{x}^{W,0,2} = \frac{k}{2\gamma^2}Z_{\parallel}^{W,0,0}a_2\cos\theta_2. \tag{21}$$

The first two terms above are the usual longitudinal and transverse dipolar impedances. The third term is new: it is a transverse quadrupolar impedance (proportional to $x_2 = a_2\cos(\theta_2)$) that is usually thought to be 0 in axisymmetric structures.

The impedance computation using the matrix formalism presented above is now much more efficient than previously; we can for instance compute the transverse dipolar impedance from Eq. (20) for a pipe made of two or five layers, as shown in Fig. 3. In this particular case, the two and five layers calculations differ only in $\mathrm{Re}(Z_x)$ and below ~ 100Hz.

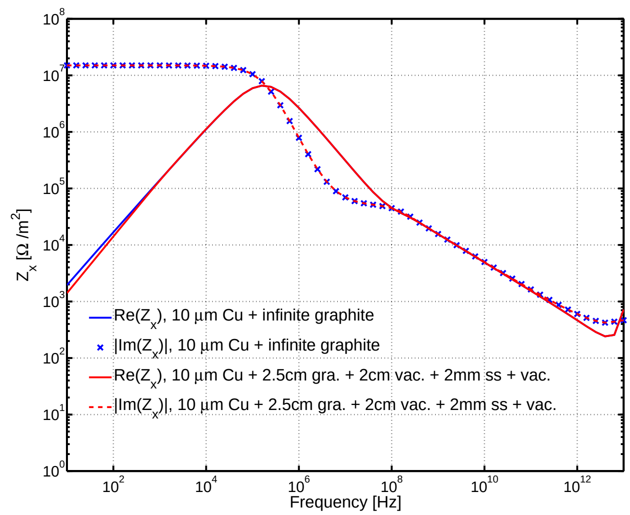

Figure 3: Horizontal dipolar impedance (divided by a_1 and L) for a multilayer pipe (see Fig. 2 for parameters; for copper $\sigma_{DC,Cu} = 5.9 \cdot 10^7$S/m and $\tau_{Cu} = 27$ fs).

CONCLUSION

Our new matrix method, which involves only multiplications of 4×4 matrices and a final simple formula, overcomes the computational difficulties of Zotter's formalism. Note that similar matrix formalisms in other theoretical frameworks have been developped in [12, 13]. Moreover, we have derived a multimode extension of the formalism, enabling the computation of all the nonlinear terms in the electromagnetic fields and impedances.

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