Reordering the Eigenvalues of a Periodic Matrix Pair with Applications in Control

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***Abstract*— Reordering the eigenvalues of a periodic matrix pair is a computational task that arises from various applications related to discrete-time periodic descriptor systems, such as pole placement or linear-quadratic optimal control. However, it is also implicitly present in recently developed robust control methods for linear time-invariant systems. In this contribution, a direct algorithm for performing this task based on the solution of a periodic generalized Sylvester equation is proposed. The new approach is numerically backward stable and it is demonstrated that the resulting deflating subspaces can be much more accurate than those computed by collapsing methods.**

I. Introduction

Let us consider a linear discrete-time descriptor system for which the coefficient matrices change periodically in time:

$$\begin{aligned} E_k x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k + D_k u_k, \end{aligned} \tag{1}$$

with $E_k = E_{k+p} \in \mathbb{R}^{n\times n}$, $A_k = A_{k+p} \in \mathbb{R}^{n\times n}$, $B_k = B_{k+p} \in \mathbb{R}^{n\times m}$, $C_k = C_{k+p} \in \mathbb{R}^{r\times n}$, and $D_k = D_{k+p} \in \mathbb{R}^{r\times m}$ for some period $p \geq 1$. Systems of this form are called *linear discrete-time periodic descriptor systems* and arise naturally from processes that exhibit seasonal or periodic behavior, see, e.g., [5]. In practice, one occasionally encounters systems for which the system matrices E_k and A_k are rectangular and have time-varying dimension. This issue can be resolved by applying the preprocessing techniques presented in [32], [34] and truncating those parts that correspond to spurious characteristic values, which then yields square system matrices of constant dimension.

Provided that all E_k are nonsingular, the *poles* (also called *characteristic values*) of (1) are given by the eigenvalues of the matrix product

$$E_{p-1}^{-1} A_{p-1} E_{p-2}^{-1} A_{p-2} \cdots E_1^{-1} A_1 E_0^{-1} A_0 \tag{2}$$

associated with the *periodic matrix pair* (A_k, E_k). In principle, much of what is said in the following can be directly extended to the case when some of the E_k happen to be singular, see also [16]. For the sake of simplifying the presentation, however, we restrict ourselves to the nonsingular case.

Among the numerically reliable methods for obtaining the eigenvalues of a periodic matrix pair is the periodic QZ algorithm [7], [17]. It computes orthogonal matrices $Q_k, Z_k \in \mathbb{R}^{n\times n}$ with $Z_{k+p} = Z_k$ such that the transformed matrices

$$S_k = Q_k^T A_k Z_k, \quad T_k = Q_k^T E_k Z_{k+1}, \quad k = 0, \ldots, p-1,$$

are all upper triangular, except for S_{p-1} which is in quasi-triangular form. That is, S_{p-1} is block upper triangular with 1×1 and 2×2 blocks on the diagonal. The matrix product

$$T_{p-1}^{-1} S_{p-1} T_{p-2}^{-1} S_{p-2} \cdots T_1^{-1} S_1 T_0^{-1} S_0 \tag{3}$$

is also block upper triangular, has the same eigenvalues as (2), and the periodic matrix pair (S_k, T_k) is said to be in *generalized periodic real Schur form* (GPRSF). The eigenvalues can thus be easily extracted from the 1×1 and 2×2 blocks on the diagonal, although great care has to be exercized to avoid disastrous under- and overflow in this computation [30]. It is usually assumed that the 2×2 blocks correspond to complex conjugate pairs of eigenvalues only.

When applying the periodic QZ algorithm or related methods, no particular order of the eigenvalues on the block diagonal of (3) can be guaranteed. Obtaining a certain order is, however, often a desirable goal. For example, if we have

$$S_k = \begin{bmatrix} S_{11}^{(k)} & S_{12}^{(k)} \\ 0 & S_{22}^{(k)} \end{bmatrix}, \quad T_k = \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix},$$

with $S_{11}^{(k)}$, $T_{11}^{(k)} \in \mathbb{R}^{j\times j}$ such that the upper left part $(S_{11}^{(k)}, T_{11}^{(k)})$ contains all eigenvalues in the open unit disc, then the first j columns of Z_k span stable deflating subspaces. For initial states $x_0 \in \text{span}(Z_0 e_1, \ldots, Z_0 e_j)$ with e_i being the ith unit vector, the states of the open loop system $E_k x_{k+1} = A_k x_k$ satisfy $x_k \in \text{span}(Z_k e_1, \ldots, Z_k e_j)$ and 0 is an asymptotically stable equilibrium.

Using the algorithm presented in this contribution, any order of the eigenvalues can be reliably attained by successively swapping adjacent diagonal blocks in the GPRSF. As shown in Section II, such a swapping can be realized via the solution of a periodic generalized Sylvester equation. This approach is in the spirit of various existing swapping procedures [1], [15], [21], [23] for special cases of the GPRSF. In Section III, we discuss several other applications of eigenvalue reordering, such as the solution of periodic Riccati equations. Several other methods have been successfully applied to address these applications; we briefly compare these approaches with our new method in Section IV. Finally, in Section V, some important steps to be taken to turn the presented method into reliable and efficient library software are briefly summarized.

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II. Swapping Diagonal Blocks in the GPRSF

Swapping consists of computing orthogonal matrices U_k, V_k with $V_{k+p} = V_k$ such that

$$\begin{bmatrix} \tilde{S}_{11}^{(k)} & \tilde{S}_{12}^{(k)} \\ 0 & \tilde{S}_{22}^{(k)} \end{bmatrix} = U_k^T \begin{bmatrix} S_{11}^{(k)} & S_{12}^{(k)} \\ 0 & S_{22}^{(k)} \end{bmatrix} V_k, \quad (4)$$

$$\begin{bmatrix} \tilde{T}_{11}^{(k)} & \tilde{T}_{12}^{(k)} \\ 0 & \tilde{T}_{22}^{(k)} \end{bmatrix} = U_k^T \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix} V_{k+1}, \quad (5)$$

for $k = 0, \ldots, p-1$, and

$$\lambda(\tilde{\Pi}_{11}) = \lambda(\Pi_{22}), \quad \lambda(\tilde{\Pi}_{22}) = \lambda(\Pi_{11}), \quad (6)$$

where $\Pi_{ii} = \left[T_{ii}^{(p-1)}\right]^{-1} S_{ii}^{(p-1)} \cdots \left[T_{ii}^{(0)}\right]^{-1} S_{ii}^{(0)}$ ($\tilde{\Pi}_{ii}$ is similarly defined) and $\lambda(\cdot)$ denotes the set of all eigenvalues of a matrix. For our purpose, $S_{ii}^{(k)}, T_{ii}^{(k)} \in \mathbb{R}^{n_i \times n_i}$ are the diagonal blocks of a GPRSF; it can thus be assumed that $n_i \in \{1, 2\}$.

A. Block diagonalization and the PGCSY

If the off-diagonal blocks in (4)–(5) were not present then swapping could be achieved in a very simple manner, by permuting the diagonal blocks. Temporarily admitting nonorthogonal transformations, we can achieve block diagonal form by computing matrices $L_k, R_k \in \mathbb{R}^{n_1 \times n_2}$ with $R_p = R_0$ such that

$$\begin{bmatrix} I & -L_k \\ 0 & I \end{bmatrix} \begin{bmatrix} S_{11}^{(k)} & S_{12}^{(k)} \\ 0 & S_{22}^{(k)} \end{bmatrix} \begin{bmatrix} I & R_k \\ 0 & I \end{bmatrix} = \begin{bmatrix} S_{11}^{(k)} & 0 \\ 0 & S_{22}^{(k)} \end{bmatrix}$$

$$\begin{bmatrix} I & -L_k \\ 0 & I \end{bmatrix} \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix} \begin{bmatrix} I & R_{k+1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} T_{11}^{(k)} & 0 \\ 0 & T_{22}^{(k)} \end{bmatrix}$$

for $k = 0, \ldots, p-1$. This relation can be rewritten as a system of matrix equations:

$$\begin{aligned} S_{11}^{(k)} R_k - L_k S_{22}^{(k)} &= -S_{12}^{(k)}, \\ T_{11}^{(k)} R_{k+1} - L_k T_{22}^{(k)} &= -T_{12}^{(k)}, \end{aligned} \quad (7)$$

which happens to be a so called *periodic generalized (coupled) Sylvester equation*. Following the acronyms used in [20] we refer to it as PGCSY. Equations of this and similar kind play a role in various other applications related to periodic discrete-time systems [35] and their numerical solution is discussed in [10], [28], [31]. In particular, it can be shown that (7) has a unique solution provided that $\lambda(\Pi_{11}) \cap \lambda(\Pi_{22}) = \emptyset$. If this condition is not satisfied then – since $S_{ii}^{(k)}$ and $T_{ii}^{(k)}$ are diagonal blocks of a GPRSF – the eigenvalues of Π_{11} and Π_{22} are actually equal and there is little need for swapping.

B. Solution of the PGCSY

One way to solve (7) is to consider a Kronecker product reformulation

$$\begin{bmatrix} \mathcal{S}_{11} & -\mathcal{S}_{22} \\ \mathcal{T}_{11}\mathcal{Z} & -\mathcal{T}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{R} \\ \mathcal{L} \end{bmatrix} = - \begin{bmatrix} \mathcal{S}_{12} \\ \mathcal{T}_{12} \end{bmatrix}, \quad (8)$$

where

$$\mathcal{R} = \begin{bmatrix} \mathrm{vec}(R_0) \\ \vdots \\ \mathrm{vec}(R_{p-1}) \end{bmatrix}, \quad \mathcal{S}_{12} = \begin{bmatrix} \mathrm{vec}(S_{12}^{(0)}) \\ \vdots \\ \mathrm{vec}(S_{12}^{(p-1)}) \end{bmatrix},$$

and $\mathcal{L}$, $\mathcal{T}_{12}$ are defined analogously. Here, the vec operator stacks the columns of a matrix into one long vector [14]. Moreover, $\mathcal{S}_{11}$ and $\mathcal{S}_{22}$ are block diagonal matrices with diagonal entries $I_{n_2} \otimes S_{11}^{(0)}, \ldots, I_{n_2} \otimes S_{11}^{(p-1)}$ and $S_{22}^{(0)T} \otimes I_{n_1}, \ldots, S_{22}^{(p-1)T} \otimes I_{n_1}$, respectively (similarly for $\mathcal{T}_{11}$ and $\mathcal{T}_{22}$). Finally, the block shift matrix $\mathcal{Z}$ is given by

$$\mathcal{Z} = \begin{bmatrix} 0 & I & & \\ & \ddots & \ddots & \\ & & \ddots & I \\ I & & & 0 \end{bmatrix}.$$

Hence, (8) represents a $2pn_1n_2 \times 2pn_1n_2$ linear system of equations. Since we assumed $n_1, n_2 \in \{1, 2\}$, this linear system is of moderate dimension. Moreover, the coefficient matrix has a particular structure, which – after applying an appropriate permutation of the blocks – can be seen to be bordered almost block diagonal (BABD). Efficient and stable algorithms for solving BABD systems are discussed, e.g., in [12].

C. How to obtain orthogonal transformations and guarantee backward stability

For the sake of numerical stability, it is important to avoid explicit block diagonalization and use orthogonal transformations. For this purpose, one computes QR and RQ factorizations

$$\begin{bmatrix} L_k \\ I \end{bmatrix} = U_k \begin{bmatrix} \tilde{L}_k \\ 0 \end{bmatrix}, \quad \begin{bmatrix} I & -R_k \end{bmatrix} = \begin{bmatrix} 0 & \tilde{R}_k \end{bmatrix} V_k^T, \quad (9)$$

for $k = 0, \ldots, p-1$, such that $U_k, V_k \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$ are orthogonal matrices and $\tilde{L}_k \in \mathbb{R}^{n_2 \times n_2}, \tilde{R}_k \in \mathbb{R}^{n_1 \times n_1}$ are upper triangular matrices. Partitioning U_k, V_k conformally with the blocks in (4) and setting $V_p = V_0$, we obtain

$$U_k^T \begin{bmatrix} S_{11}^{(k)} & S_{12}^{(k)} \\ 0 & S_{22}^{(k)} \end{bmatrix} V_k = \begin{bmatrix} \tilde{L}_k S_{22}^{(k)} V_{21}^{(k)} & \star \\ 0 & U_{12}^{(k)T} S_{11}^{(k)} \tilde{R}_k \end{bmatrix},$$

$$U_k^T \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix} V_{k+1} = \begin{bmatrix} \tilde{L}_k T_{22}^{(k)} V_{21}^{(k+1)} & \star \\ 0 & U_{12}^{(k)T} T_{11}^{(k)} \tilde{R}_{k+1} \end{bmatrix},$$

for $k = 0, \ldots, p-1$. From (9), it can be seen that the matrices $U_{12}^{(k)}, V_{21}^{(k)}, \tilde{L}_k, \tilde{R}_k$ are all invertible, which implies – by direct computation – that (6) holds for the diagonal blocks of the transformed matrices. Thus, the matrices U_k and V_k yield the desired swapping (4)–(6).

In finite-precisision arithmetic, the above relations are affected by roundoff error, resulting in perturbed transformed matrices

$$\begin{bmatrix} \hat{S}_{11}^{(k)} & \hat{S}_{12}^{(k)} \\ \hat{S}_{21}^{(k)} & \hat{S}_{22}^{(k)} \end{bmatrix}, \quad \begin{bmatrix} \hat{T}_{11}^{(k)} & \hat{T}_{12}^{(k)} \\ \hat{T}_{21}^{(k)} & \hat{T}_{22}^{(k)} \end{bmatrix}. \quad (10)$$

Experimentally it can be observed that the subdiagonal blocks $\hat{S}_{21}^{(k)}$ and $\hat{T}_{21}^{(k)}$ are negligible compared to the rest of the matrix. But it occasionally may happen, in particular if the eigenvalues of Π_{11} and Π_{22} are poorly separated, that $\|\hat{S}_{21}^{(k)}\|_F \gg \mathbf{u}\|S^{(k)}\|_F$ or $\|\hat{T}_{21}^{(k)}\|_F \gg \mathbf{u}\|T^{(k)}\|_F$ for some k, where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix and $\mathbf{u}$ is the unit roundoff. In this case, $\hat{S}_{21}^{(k)}$ and $\hat{T}_{21}^{(k)}$ cannot be set to zero without sacrificing backward stability and the swap must be rejected.

D. The overall algorithm

The considerations made above lead to the following algorithm for swapping an $(n_1+n_2)\times(n_1+n_2)$ periodic matrix pair (S_k, T_k) in GPRSF.

1) *Compute the solutions L_k, R_k of the PGCSY (7) by solving the linear system (8).*
2) *Perform p QR and p RQ factorizations*
$$\begin{bmatrix} L_k \\ I \end{bmatrix} = U_k \begin{bmatrix} \tilde{L}_k \\ 0 \end{bmatrix}, \quad \begin{bmatrix} I & -R_k \end{bmatrix} = \begin{bmatrix} 0 & \tilde{R}_k \end{bmatrix} V_k^T.$$
3) *Compute $\hat{S}_k = U_k^T S_k V_k$ and $\hat{T}_k = U_k^T T_k V_{k+1}$ for $k = 0, \ldots, p-1$.*
4) *If all subdiagonal blocks $\hat{S}_{21}^{(k)}, \hat{T}_{21}^{(k)}$, see (10), are sufficiently small then*
   set $\hat{S}_{21}^{(k)} = \hat{T}_{21}^{(k)} = 0$ and accept swap;
   otherwise
   reject swap.

Properly implemented, this algorithm requires $O(p)$ floating point operations (flops). If it is used to reorder two adjacent diagonal blocks in a larger $n \times n$ periodic matrix pair in GPRSF then the off-diagonal parts must be updated by the transformation matrices U_k and V_k, which additionally requires $O(pn)$ flops.

Several important details regarding the implementation of this algorithm have been omitted. If the matrices S_k and T_k have widly differing norms then a scaling step should be applied in order to avoid artificial ill-conditioning of the linear system (8), see also [15, Ex. 5]. Also, the use of iterative refinement for improving the accuracy of the solution and decreasing the possibility of rejection needs to be investigated. Moreover, the criterion for the decision to be made in Step 4 should be based on a careful error analysis. These and other issues will be addressed in a forthcoming paper [16].

III. Applications

Besides the computation of stable deflating subspaces mentioned in the introduction, eigenvalue reordering is of use in a number of other computational tasks related to periodic discrete-time descriptor systems. In the following, we illustrate two such applications, pole placement and discrete-time optimal control.

A. Pole placement

By applying a linear state feedback law of the form
$$u_k = F_k x_k + v_k$$
with $F_k \in \mathbb{R}^{m\times n}$, $F_{k+p} = F_k$, and the new external input v_k, the open loop system (1) is transformed into the closed loop system
$$\begin{aligned} E_k x_{k+1} &= (A_k + B_k F_k)x_k + B_k v_k \\ y_k &= (C_k + D_k F_k)x_k + D_k v_k, \end{aligned} \tag{11}$$
which is again periodic. The poles of (11) are given by the eigenvalues of the periodic matrix pair $(E_k, A_k + B_k F_k)$. The goal of *pole placement* is to move some or all of these poles to desired locations in the complex plane. For example, a possibly unstable system (1) can be stabilized by moving all poles outside or on the unit circle into the open unit disc.

The following algorithm for pole placement is in the spirit of [27]. By applying the periodic QZ algorithm, we may assume without loss of generality that (E_k, A_k) is in GPRSF. Let us furthermore assume that the last subdiagonal entry of A_{p-1} is zero, i.e., we can partition
$$E_k = \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & \eta_k \end{bmatrix}, \; A_k = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & \alpha_k \end{bmatrix}, \; B_k = \begin{bmatrix} B_1^{(k)} \\ \beta_k \end{bmatrix}$$
with $\alpha_k, \eta_k \in \mathbb{R}$ and $\beta_k \in \mathbb{R}^{1\times m}$. Assuming (1) to be completely reachable, it can be guaranteed that there is at least one $j \in [0, p-1]$ with $\beta_j \neq 0$ [22]. Then for a desired closed loop pole λ, we can choose vectors $f_0, \ldots, f_{p-1} \in \mathbb{R}^m$ such that
$$\lambda = \prod_{k=0}^{p-1} \eta_k^{-1}(\alpha_k + \beta_k f_k). \tag{12}$$
Choosing $F_k = f_k e_n^T$, one pole of the closed loop system (11) is moved to λ while the others remain the same. We can place further poles by reordering the matrix pair $(A_k + B_k F_k, E_k)$ so that the nth diagonal entries correspond to an open loop pole, and repeating the described procedure.

There is a lot of freedom in the choice of f_k, even in the single-input case $m = 1$. To keep the scaling of the coefficients A_k balanced, it is advisable to distribute the influence of the state feedback equally over the whole period. For this purpose, we set $\gamma = (\eta_0\eta_1\cdots\eta_{p-1})\lambda$ and choose scalars ζ_k such that $\zeta_0\zeta_1\cdots\zeta_{p-1} = \operatorname{sign}(\gamma)$ and
$$\zeta_k|\gamma|^{1/p} - \alpha_k = c_\star\|\beta_k\|$$
for some constant $c_\star$. To determine this constant one needs to solve
$$(c\|\beta_0\| + \alpha_0)(c\|\beta_1\| + \alpha_1)\cdots(c\|\beta_{p-1}\| + \alpha_{p-1}) = \gamma. \tag{13}$$
This then leads to the state feedback
$$f_k = \begin{cases} \frac{(\zeta_k|\gamma|^{1/p} - \alpha_k)\beta_k^T}{\|\beta_k\|^2} & \text{if } \beta_k \neq 0; \\ 0, & \text{otherwise,} \end{cases}$$
which satisfies (12) by construction and, moreover, $\|f_k\| = |c_\star|$ for all k. Note that (13) needs not be solved exactly to attain near balancing.

The described procedure can be easily extended to complex eigenvalues if one admits periodic complex Schur forms. How to directly work on periodic real Schur forms and avoid complex arithmetic in a numerically reliable manner is not

clear and subject to further investigation. For $m > 1$, the multi-input case, the additional degrees of freedom may be used to attain higher robustness by minimizing the spectral condition numbers of the closed-loop system or the norms of the feedback matrices or a combination of both [33].

B. LQ optimal control

Given a periodic discrete-time descriptor system (1), the aim of LQ optimal control is to find a feedback u_k which stabilizes the system and minimizes

$$\frac{1}{2}\sum_{l=0}^{\infty}(x_l^T Q_l x_l + u_l^T R_l u_l),$$

with $Q_k \in \mathbb{R}^{n\times n}$ symmetric positive semidefinite and $R_k \in \mathbb{R}^{m\times m}$ symmetric positive definite. Moreover, we suppose that the weighting matrices are periodic, i.e., $Q_{k+p} = Q_k$ and $R_{k+p} = R_k$. Under mild assumptions [6], the optimal feedback is linear and unique; it can be expressed as

$$u_k^\star = -(R_k + B_k^T X_{k+1} B_k)^{-1} B_k^T X_{k+1} A_k x_k,$$

where $X_k = X_{k+p}$ is the unique symmetric positive semidefinite solution of the *discrete-time periodic Riccati equation* (DPRE)

$$\begin{aligned} 0 = {} & C_k^T Q_k C_k - E_{k-1}^T X_k E_{k-1} + A_k^T X_{k+1} A_k \\ & - A_k^T X_{k+1} B_k (R_k + B_k^T X_{k+1} B_k)^{-1} B_k^T X_{k+1} A_k, \end{aligned} \tag{14}$$

provided that all E_k are invertible. The following $2n \times 2n$ periodic matrix pair is clol associated with (14):

$$(L_k, M_k) = \left(\begin{bmatrix} A_k & 0 \\ -C_k^T Q_k C_k & E_{k-1}^T \end{bmatrix}, \begin{bmatrix} E_{k-1} & B_k R_k^{-1} B_k^T \\ 0 & A_k^T \end{bmatrix} \right).$$

Similarly as for the case $E_k = I_n$ [17], it can be shown that this pair has exactly n eigenvalues inside the unit disk under the reasonable assumption that (1) is d-stabilizable and d-detectable. By reordering the GPRSF of (L_k, M_k) we can compute orthogonal matrices $U_k, V_k \in \mathbb{R}^{2n\times 2n}$ with $V_{k+p} = V_k$ such that

$$U_k^T L_k V_k = \begin{bmatrix} S_{11}^{(k)} & S_{12}^{(k)} \\ 0 & S_{22}^{(k)} \end{bmatrix}, \; U_k^T M_k V_{k+1} = \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix},$$

where the $n \times n$ periodic matrix pair $(S_{11}^{(k)}, T_{11}^{(k)})$ contains all eigenvalues inside the unit disk. If we partition

$$U_k = \begin{bmatrix} U_{11}^{(k)} & U_{12}^{(k)} \\ U_{21}^{(k)} & U_{22}^{(k)} \end{bmatrix}$$

with $U_{ij}^{(k)} \in \mathbb{R}^{n\times n}$, then

$$U_{21}^{(k)} \left[U_{11}^{(k)}\right]^{-1} = X_k E_{k-1},$$

from which X_k can be computed. The proof of this relation is similiar as for the case $p = 1$, see, e.g., [26]. We note that R_k can be ill-conditioned or even singular, e.g., when solving dead-beat control problems [34]. In such cases, it is necesssary to avoid the inversion of R_k and to work instead with $(2n+m)\times(2n+m)$ matrix pairs as described in [26], [29].

C. Other applications

Reordering the eigenvalues of periodic matrix pairs can be used to reliably implement restarting and deflation techniques in numerical methods for solving large-scale product eigenvalue problems, such as the periodic Krylov-Schur and Jacobi-Davidson algorithms [19], [24]. Also, recently developed algorithms [4] for computing deflating subspaces of structured matrix pencils to address linear-quadratic optimal control problems for linear time-invariant systems benefit from our algorithm.

IV. Comparison with other Approaches

In this section, we summarize and compare other existing techniques for reordering eigenvalues and computing deflating subspaces of periodic matrix pairs.

1) Explicit formation of matrix products: For performing the swapping (4) of an $(n_1+n_2)\times(n_1+n_2)$ periodic matrix pair (S_k, T_k), one could explicitly form the matrix product

$$\Pi = T_{p-1}^{-1} S_{p-1} T_{p-2}^{-1} S_{p-2} \cdots T_1^{-1} S_1 T_0^{-1} S_0, \tag{15}$$

and apply standard reordering to obtain the orthogonal transformation matrix V_0. The other transformation matrices can then be generated by propagating V_0 through the triangular factors [8].

Such an approach has been proposed in [17], [25] for the case $n_1 = n_2 = 1$. If $n_1 > 1$ or $n_2 > 1$, triangular matrix-matrix multiplication is not a numerically backward stable operation [18]. Moreover, there are serious numerical difficulties to be expected for long products as the entries of Π become prone to under- and overflow. Nearly singular factors T_k^{-1} pose another source of instability. The following example illustrates some of the drawbacks of working on matrix products.

Example 1: Let $D = \mathrm{diag}(1, 10^{-1}, 10^{-2})$ and consider the periodic matrix pair $(E_k, A_k) = (I_3, Q_{k+1}^T D Q_k)$ where Q_k are random orthogonal matrices with $Q_{k+p} = Q_k$. For various periods p, we computed the eigenvector of the matrix product Π belonging to the eigenvalue 10^{-p} using two different approaches. In the first approach, denoted by RSF below, we formed Π explicitly and computed the reordered Schur form of Π such that 10^{-p} appears in the top left corner. Then, at least in exact arithmetic, the first column of the orthogonal transformation matrix is the desired eigenvector. In the second approach, denoted by GPRSF, we used the reordered generalized periodic Schur form instead, yielding the desired eigenvector in the first column of the "outer" transformation matrix Z_0. The following table contains the angles between the computed and exact eigenvectors.

	$p = 10$	$p = 15$	$p = 20$
RSF	2×10^{-13}	9×10^{-05}	2×10^{-01}
GPRSF	3×10^{-16}	4×10^{-16}	3×10^{-16}

It can be observed that, in contrast to GPRSF, the accuracy of the eigenvectors computed by RSF drops rapidly with increasing p, being completely polluted by roundoff error for $p \geq 20$.

2) Collapsing matrix pairs: Collapsing the periodic matrix pair into a single matrix pair [2], [3], [11] offers a more robust alternative than forming Π, especially when some of the factors E_k are nearly singular. This approach, which was shown to be numerically backward stable for $p = 2$, is based on the following lemma.

Lemma 2 ([2]): Let $U^T E_0 X = C$ and $V^T A_1 X = S$ be the generalized singular value decomposition [14] of $E_0, A_1 \in \mathbb{R}^{n\times n}$, i.e., $U, V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, $X \in \mathbb{R}^{n\times n}$ is nonsingular, and $C, S \in \mathbb{R}^{n\times n}$ are nonnegative, diagonal matrices. Define $\tilde{E}_1 = CV^T$ and $\tilde{A}_0 = SU^T$. Then

$$A_1 E_0^{-1} = \tilde{E}_1^{-1} \tilde{A}_0.$$

Lemma 2 turns a product of the form $E_1^{-1} A_1 E_0^{-1} A_0$ into a product of the form $(\tilde{E}_1 E_1)^{-1} (\tilde{A}_0 A_0)$. The successive application to a periodic matrix pair (A_k, E_k) yields $\Pi = \tilde{E}^{-1}\tilde{A}$ for some matrices $\tilde{E}, \tilde{A} \in \mathbb{R}^{n\times n}$. Now, the reordered generalized real Schur form (GRSF) can be applied to compute the outer deflating subspaces of the periodic matrix pair. The inner deflating subspaces can be found by a simple substitution procedure, see [2].

Repeating the numerical experiments from Example 1, we found that collapsing led to the following angles between the exact and computed eigenvectors belonging to the eigenvalue 10^{-p}.

	$p = 10$	$p = 15$	$p = 20$
GRSF	3×10^{-15}	7×10^{-07}	1×10^{-02}

Although such a statement does not hold in general, this example demonstrate that GRSF can, when compared to RSF, result in higher accuracy even for the case $E_k = I$. Nevertheless, the rapid loss of accuracy for increasing values of p is not cured by using GRSF and GPRSF remains the method of choice.

3) Block cyclic embedding: Lifting [13] is a popular technique to turn a discrete-time periodic (descriptor) system into an equivalent time-invariant (descriptor) system. These techniques can also be used to extract the deflating subspaces of an $n\times n$ periodic matrix pair of period p from the deflating subspaces of an embedded $pn \times pn$ block diagonal/block cyclic matrix pair, see, e.g., [24] for more details. Besides being more costly, it was shown in [9] that such an approach may also sometimes lead to serious loss of accuracy due the fact that the condition number of the deflating subspaces can be considerably increased by lifting.

V. Conclusions

A new method for reordering the eigenvalues of a periodic matrix pair in generalized periodic real Schur form was presented. Unlike other approaches, this method is guaranteed to be numerically backward stable, an important property to attain high accuracy in the resulting deflating subspaces, as confirmed by the numerical experiments.

Ongoing work, to be reported in [16], is directed towards making the presented approach more robust and efficient. For example, solving the arising periodic Sylvester equations by more accurate methods in combination with a careful error analysis may help avoid unnecessary rejections in the swapping algorithm. It is planned to include the reordering algorithm into a broader, publicly available software library for solving periodic eigenvalue problems, which can then be used to address the applications described in Section III. These developments include recursive blocked methods and software (PRECSY) for periodic Sylvester equations, building on RECSY [20].

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