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FRACTIONAL BROWNIAN VECTOR FIELDS*

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Abstract. This work puts forward an extended definition of vector fractional Brownian motion (fBm) using a distribution theoretic formulation in the spirit of Gel'fand and Vilenkin's stochastic analysis. We introduce random vector fields that share the statistical invariances of standard vector fBm (self-similarity and rotation invariance) but which, in contrast, have dependent vector components in the general case. These random vector fields result from the transformation of white noise by a special operator whose invariance properties the random field inherits. The said operator combines an inverse fractional Laplacian with a Helmholtz-like decomposition and weighted recombination. Classical fBm's can be obtained by balancing the weights of the Helmholtz components. The introduced random fields exhibit several important properties that are discussed in this paper. In addition, the proposed scheme yields a natural extension of the definition to Hurst exponents greater than one.

Key words. fractional Brownian motion, random vector fields, self-similarity, invariance, Helmholtz decomposition, generalized random processes, Gel'fand–Vilenkin stochastic analysis

AMS subject classifications. 60G18, 60G20, 60G60, 60H40, 60H20, 35S30, 42B20

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1. Introduction.

A one-dimensional fractional Brownian motion (fBm) $B_H(x)$, $x \in \mathbb{R}$, is a nonstationary zero-mean Gaussian random process satisfying $B_H(0) = 0$, with the characteristic property that for any fixed step size $|x - y|$ the increment $B_H(x) - B_H(y)$ is a stationary Gaussian process with a variance proportional to the step size:[1]

$$\mathbf{E}\{|B_H(x) - B_H(y)|^2\} = 2\alpha|x - y|^{2H}.$$

The parameter $H \in (0, 1)$ is known as the Hurst exponent, after Harold Edwin Hurst, a pioneer in the study of long-range statistical dependence [21, 53] (α is an arbitrary positive constant).

The above definition can be extended to the multivariate setting in the style of Lévy's characterization of multiparameter Brownian motion [29, 30], by making the parameter $\boldsymbol{x}$ a vector in $\mathbb{R}^d$ and defining multidimensional fBm as a Gaussian random field with a variogram of the form

$$\mathbf{E}\{|B_H(\boldsymbol{x}) - B_H(\boldsymbol{y})|^2\} = 2\alpha\|\boldsymbol{x} - \boldsymbol{y}\|^{2H}$$

(we shall use bold symbols to denote vector quantities).

FBm's are important examples of stochastic fractals: They are statistically self-similar in the sense that an fBm $B_H(\cdot)$ and its scaled version $\sigma^H B_H(\sigma\cdot)$ have the same statistics. FBm processes have been used to model natural and man-made phenomena in different areas of application including optics, fluid mechanics, seismology,

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[1]The second moment of the increment of a process, defined in the above fashion, is also known as its *variogram* or structure function.

financial mathematics, network traffic analysis, and image processing, among others [13, 14, 15, 24, 27, 33, 39, 40, 48, 54]. Since the notion of invariance that inspires the definition of fractals is also fundamental in physics, it is natural to expect that the scope of applications of such models will further expand with time. For the same reason, investigation of physical invariances and stochastic models characterized by them seems worthwhile.

Our goal in the present paper is to extend the definition of fBm to the vector (multicomponent) field setting. In this undertaking we draw our inspiration from two sources. One is the usual consideration of self-similarity. The second influence comes from physics, as we shall impose on the model a special form of rotation invariance that is compatible with the effect of rotations on coordinates of physical vector fields.

We remark that a trivial vectorial extension of scalar fBm satisfying the conditions of homogeneity and vector rotation invariance can be readily constructed by taking the components of the d-dimensional vector to be independent scalar fBm's. This extension is consistent with the variogram relation [22]

$$\mathbf{E}\{\|\boldsymbol{B}_H(\boldsymbol{x}) - \boldsymbol{B}_H(\boldsymbol{y})\|^2\} = 2\alpha'\|\boldsymbol{x} - \boldsymbol{y}\|^{2H} \tag{1.1}$$

(note that the absolute value has been replaced by a Euclidean norm in the argument of the expectation operator). But it should be emphasized that (1.1) in itself does not specify the cross-correlation structure of the components of $\boldsymbol{B}_H$, and the classical assumption of independent components is not exhaustive. Hence, in this paper we shall consider more general families of fractal vector fields satisfying (1.1) whose vector components can be correlated in ways that lead to a full range of vectorial comportment from fully solenoidal to completely irrotational.

This paper continues the line of reasoning adopted in Tafti, Van De Ville, and Unser [51] (where we considered scalar fBm fields) and more originally in Blu and Unser [4] (where one-dimensional fBm processes were studied). In keeping with these previous works, we shall characterize fBm vector fields as particular solutions of a stochastic fractional differential equation

$$\mathrm{U}\boldsymbol{B}_H = \boldsymbol{W} \tag{1.2}$$

subject to zero boundary conditions at $\boldsymbol{x} = \mathbf{0}$, where $\boldsymbol{W}$ denotes a vector of normalized and independent white noise fields (defined in subsection 3.1). The "whitening" operator U is chosen based on its specific invariance properties that carry over to the random vector field $\boldsymbol{B}_H$. U will turn out to be a generalization of the fractional vector Laplacian $(-\boldsymbol{\Delta})^{\gamma}$, with additional parameters that control the solenoidal versus irrotational tendencies of the solution. Rigorous interpretation and inversion of (1.2) are conducted in the framework of Gel'fand and Vilenkin's theory of generalized random processes and distributional stochastic analysis [18]. Some aspects of this theory that are relevant to our work are summarized in subsection 3.1.

Our characterization by means of a whitening equation gives mathematical meaning to inverse power-law spectra that are traditionally associated with self-similar processes, by providing the mechanics for resolving the singularity of the said spectra at $\boldsymbol{\omega} = \mathbf{0}$ (the noted processes, being nonstationary, do not have power spectra in the classical sense). We should, however, note that in a different approach to the mathematical modeling and simulation of self-similar physical phenomena, the introduction of a cut-off length can provide an alternative way of dealing with the frequency-domain singularity.

Previous related work in the direction of the present paper has appeared, for instance, in Yaglom [62], where a second-order analysis of random vector fields with similar invariance properties was given. Yaglom and others also considered, albeit separately, scalar random processes with stationary nth-order increments [11, 42, 63]. In addition to differences in formulation and approach—for example, in our consideration of singular operators and our focus on characteristic functionals—in the present paper we bring together these separate generalizations (cf. subsection 4.4). Furthermore, our approach is not limited, in essence, to the study of second-order statistics (even though this would have sufficed for the Gaussian fields considered here). This means that by using a similar approach it is possible, without too much difficulty, to construct and completely characterize other—non-Gaussian—models satisfying similar invariance properties, by driving (1.2) with different types of non-Gaussian white noise.

On the applied side, consideration of models in line with (1.1) and their relatives has a long history in fluid dynamics and specifically in the study of turbulence, although the emphasis and methodology are frequently different from ours (see, e.g., Monin and Yaglom [37, Chapter 8], Avellaneda and Majda [2], Carmona [7], Orszag [38], or Klyatskin, Woyczynski, and Gurarie [25]).

In the remainder of this paper we first turn our attention to the search for an operator U satisfying the required invariances (section 2). There, the question of inverting U—which is necessary for solving (1.2)—requires us to consider a particular regularization of singular Fourier integrals. Next, in section 3, we solve (1.2) and give a complete stochastic characterization of generalized vector fBm fields as particular solutions of this equation. A list of the main properties of these random fields is given in section 4. This is followed by computer simulations (section 5) and conclusions (section 6). Proofs of some intermediate results have been deferred until the appendices.

2. Vector operators invariant under rotation and scaling.

2.1. Generalized fractional Laplacians. Let $\boldsymbol{f}(\boldsymbol{u})$, $\boldsymbol{u} \in \mathbb{R}^d$, represent a vector field in terms of the standard coordinates $\boldsymbol{u} = (u_1, \ldots, u_d)$. Consider a second coordinate system $\boldsymbol{x}$ related to $\boldsymbol{u}$ by means of a smooth invertible map $\boldsymbol{\phi} : \mathbb{R}^d \to \mathbb{R}^d$ as per

$$\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{u}).$$

The coordinates of $\boldsymbol{f}$ in the second system are then given by the formula

$$\boldsymbol{f}_{\boldsymbol{\phi}}(\boldsymbol{x}) = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{u}}(\boldsymbol{u})\boldsymbol{f}(\boldsymbol{u})$$

(this can be seen as a consequence of identifying vector fields with differential one-forms and applying the chain rule of differentiation; cf. Rudin [44, paragraphs 10.21, 10.42]).

In particular, for a linear coordinate transformation $\boldsymbol{x} = \mathbf{M}\boldsymbol{u}$, where $\mathbf{M}$ is an invertible $d \times d$ matrix, one has

$$\boldsymbol{f}_{\mathbf{M}}(\boldsymbol{x}) = \mathbf{M}\boldsymbol{f}(\mathbf{M}^{-1}\boldsymbol{x}).$$

It follows that if $\mathbf{M} = \boldsymbol{\Omega}$ is orthogonal (in particular, a rotation matrix), then

$$\boldsymbol{f}_{\boldsymbol{\Omega}}(\boldsymbol{x}) = \boldsymbol{\Omega}\boldsymbol{f}(\boldsymbol{\Omega}^{\mathsf{T}}\boldsymbol{x});$$

and if $\mathbf{M} = \sigma\mathbf{I}$ with $\sigma > 0$ (a scaling), then

$$\boldsymbol{f}_\sigma(\boldsymbol{x}) = \sigma\boldsymbol{f}(\sigma^{-1}\boldsymbol{x}).$$

We shall consider certain *convolution operators* acting on vector fields, as well as their inverses. By the former we mean those operators which can be written in terms of an inverse Fourier integral as per

$$\mathrm{U} : \boldsymbol{f} \mapsto (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\omega}\rangle} \hat{\mathrm{U}}(\boldsymbol{\omega})\hat{f}(\boldsymbol{\omega})\, \mathrm{d}\boldsymbol{\omega}, \tag{2.1}$$

where $\hat{\mathrm{U}}$ is the (matrix-valued) Fourier expression for the operator U and $\hat{\boldsymbol{f}}$ is the Fourier transform of $\boldsymbol{f}$.

Operators of the above type appear in equations of the form

$$\mathrm{U}\boldsymbol{B}_H = \boldsymbol{W}, \tag{2.2}$$

which we shall use to model *statistically self-similar* (homogeneous) and *rotation-invariant* (isotropic) vector fields. These properties are imposed on the solution $\boldsymbol{B}_H$ of the above equation by requiring that the right inverse of U interact in a particular way with rotations and scalings of the coordinate system.[2]

The "invariance" properties the operator U is required to satisfy are the following:

$$\mathrm{U}\boldsymbol{f}_{\boldsymbol{\Omega}} = (\mathrm{U}\boldsymbol{f})_{\boldsymbol{\Omega}} \qquad \text{(rotation invariance);} \tag{2.3}$$

$$\mathrm{U}\boldsymbol{f}_\sigma = \sigma^{2\gamma}(\mathrm{U}\boldsymbol{f})_\sigma \qquad \text{(degree } 2\gamma \text{ homogeneity)} \tag{2.4}$$

(γ relates to one of the main parameters of the family of the random solutions, namely the Hurst exponent, by the relation $H = 2\gamma - d/2$). Note that we shall assume invariance with respect to improper rotations (with $\det \boldsymbol{\Omega} = -1$) as well as proper rotations (with $\det \boldsymbol{\Omega} = 1$).

The above properties translate, respectively, to the following conditions on the Fourier expression of the operator U:

$$\hat{\mathrm{U}}(\boldsymbol{\Omega\omega}) = \boldsymbol{\Omega}\hat{\mathrm{U}}(\boldsymbol{\omega})\boldsymbol{\Omega}^{\mathsf{T}} \qquad \text{(rotation invariance);} \tag{2.5}$$

$$\hat{\mathrm{U}}(\sigma\boldsymbol{\omega}) = \sigma^{2\gamma}\hat{\mathrm{U}}(\boldsymbol{\omega}) \qquad \text{(homogeneity).} \tag{2.6}$$

The following theorem was proved by Arigovindan for $d = 2, 3$ [1]. It can be shown more generally to hold in any number of dimensions.

THEOREM 2.1 (Arigovindan [1]). *A vector convolution operator satisfying properties* (2.3) *and* (2.4) *has a Fourier expression of the form*

$$\hat{\boldsymbol{\Phi}}^{\gamma}_{\boldsymbol{\xi}}(\boldsymbol{\omega}) = \|\boldsymbol{\omega}\|^{2\gamma}\left[\mathrm{e}^{\xi_{\mathrm{irr}}}\frac{\boldsymbol{\omega\omega}^{\mathsf{T}}}{\|\boldsymbol{\omega}\|^2} + \mathrm{e}^{\xi_{\mathrm{sol}}}\left(\mathbf{I} - \frac{\boldsymbol{\omega\omega}^{\mathsf{T}}}{\|\boldsymbol{\omega}\|^2}\right)\right], \tag{2.7}$$

with $\boldsymbol{\xi} = (\xi_{\mathrm{irr}}, \xi_{\mathrm{sol}}) \in \mathbb{C}^2$.

It is easy to verify that the converse of the theorem is also true for arbitrary dimension d. Since we shall be considering real operators, in what follows we shall implicitly assume $\mathrm{e}^{\xi_{\mathrm{irr}}}, \mathrm{e}^{\xi_{\mathrm{sol}}} \in \mathbb{R}$ without further mention.

[2] We shall have to consider a right inverse of U that—unlike U—is not shift invariant and does not correspond to a convolution; hence the solution $\boldsymbol{B}_H$ will not be stationary (more on this later).

The operators introduced in Theorem 2.1 generalize vector Laplacians in two senses (fractional orders and reweighting of solenoidal and irrotational components). We shall therefore refer to them as fractional (vector) Laplacians and use the symbol $(-\boldsymbol{\Delta})_{\boldsymbol{\xi}}^{\gamma}$ to denote the operator with Fourier expression $\hat{\boldsymbol{\Phi}}_{\boldsymbol{\xi}}^{\gamma}$. To gain a better understanding of the action of fractional vector Laplacians, it is instructive at this point to recall the Fourier expressions (in standard Cartesian coordinates) of some related vector differential operators:[3]

$$
\begin{array}{ccc}
\mathbf{grad} & \xleftrightarrow{\mathcal{F}} & \mathrm{j}\boldsymbol{\omega}; \\
\mathrm{div} & \xleftrightarrow{\mathcal{F}} & (\mathrm{j}\boldsymbol{\omega})^{\mathsf{T}}; \\
\mathbf{curl} & \xleftrightarrow{\mathcal{F}} & \begin{bmatrix} 0 & -\mathrm{j}\omega_3 & \mathrm{j}\omega_2 \\ \mathrm{j}\omega_3 & 0 & -\mathrm{j}\omega_1 \\ -\mathrm{j}\omega_2 & \mathrm{j}\omega_1 & 0 \end{bmatrix}; \\
\mathbf{grad}\,\mathrm{div} & \xleftrightarrow{\mathcal{F}} & -\boldsymbol{\omega}\boldsymbol{\omega}^{\mathsf{T}}; \\
\mathbf{curl}\,\mathbf{curl} & \xleftrightarrow{\mathcal{F}} & \|\boldsymbol{\omega}\|^2\mathbf{I} - \boldsymbol{\omega}\boldsymbol{\omega}^{\mathsf{T}}; \\
\boldsymbol{\Delta} & \xleftrightarrow{\mathcal{F}} & -\|\boldsymbol{\omega}\|^2\mathbf{I}; \\
\mathrm{E} & \xleftrightarrow{\mathcal{F}} & \frac{\boldsymbol{\omega}\boldsymbol{\omega}^{\mathsf{T}}}{\|\boldsymbol{\omega}\|^2}; \\
(-\boldsymbol{\Delta})_{\boldsymbol{\xi}}^{\gamma} & \xleftrightarrow{\mathcal{F}} & \hat{\boldsymbol{\Phi}}_{\boldsymbol{\xi}}^{\gamma}(\boldsymbol{\omega}) = \|\boldsymbol{\omega}\|^{2\gamma}\left[\mathrm{e}^{\xi_{\mathrm{irr}}}\frac{\boldsymbol{\omega}\boldsymbol{\omega}^{\mathsf{T}}}{\|\boldsymbol{\omega}\|^2} + \mathrm{e}^{\xi_{\mathrm{sol}}}\left(\mathbf{I} - \frac{\boldsymbol{\omega}\boldsymbol{\omega}^{\mathsf{T}}}{\|\boldsymbol{\omega}\|^2}\right)\right].
\end{array}
\tag{2.8}
$$

The penultimate operator (E) and its complement (Id − E) project a vector field onto its curl-free and divergence-free components, respectively. In other words, together they afford a Helmholtz decomposition of the vector field on which they act (these operators appear prominently in fluid dynamics literature [8, 9, 10, 46]). This is because

$$\mathrm{div}\,(\mathrm{Id} - \mathrm{E}) = 0 \quad \text{and} \quad \mathbf{curl}\ \mathrm{E} = 0.$$

In addition, one has

$$\mathrm{E}\ \mathbf{grad} = \mathbf{grad} \quad \text{and} \quad \mathrm{E}\ \mathbf{curl} = 0.$$

(Id − E) is known as the Leray projector in turbulence literature.

Our notation for the fractional vector Laplacian $(-\boldsymbol{\Delta})_{\boldsymbol{\xi}}^{\gamma}$ is motivated by the observation that it can be factorized as

$$(-\boldsymbol{\Delta})_{\boldsymbol{\xi}}^{\gamma} = (-\boldsymbol{\Delta})_{\mathbf{0}}^{\gamma}\left[\mathrm{e}^{\xi_{\mathrm{irr}}}\mathrm{E} + \mathrm{e}^{\xi_{\mathrm{sol}}}\,(\mathrm{Id} - \mathrm{E})\right].$$

In view of the properties of the operator E, this factorization means that the operator $(-\boldsymbol{\Delta})_{\boldsymbol{\xi}}^{\gamma}$ combines a coordinatewise fractional Laplacian with a reweighting of the curl- and divergence-free components of the operand.

2.2. Some properties of $\hat{\boldsymbol{\Phi}}_{\boldsymbol{\xi}}^{\gamma}$. Let us now take a closer look at the family of matrix-valued functions $\hat{\boldsymbol{\Phi}}_{\boldsymbol{\xi}}^{\gamma}$, γ, $\mathrm{e}^{\xi_{\mathrm{irr}}}, \mathrm{e}^{\xi_{\mathrm{sol}}} \in \mathbb{R}$. They, of course, satisfy the required invariances:

$$\hat{\boldsymbol{\Phi}}_{\boldsymbol{\xi}}^{\gamma}(\boldsymbol{\Omega}\boldsymbol{\omega}) = \boldsymbol{\Omega}\hat{\boldsymbol{\Phi}}_{\boldsymbol{\xi}}^{\gamma}(\boldsymbol{\omega})\boldsymbol{\Omega}^{\mathsf{T}};$$
$$\hat{\boldsymbol{\Phi}}_{\boldsymbol{\xi}}^{\gamma}(\sigma\boldsymbol{\omega}) = \sigma^{2\gamma}\hat{\boldsymbol{\Phi}}_{\boldsymbol{\xi}}^{\gamma}(\boldsymbol{\omega}).$$

[3]Note that, while the curl operator is classically defined in three dimensions, the equivalents of **grad** div, **curl curl**, and $\boldsymbol{\Delta} = \mathbf{grad}\,\mathrm{div} - \mathbf{curl}\,\mathbf{curl}$ can be defined in any number of dimensions, for instance by their Fourier symbols. In fact, for arbitrary d, the equivalents of $-\,\mathbf{curl}\,\mathbf{curl}$ and $-\,\mathbf{grad}\,\mathrm{div}$ that appear in the definition of the vector Laplacian correspond, respectively, to the product of d-dimensional curl and divergence with their adjoints.

But they exhibit, in addition, the following properties.

($\hat{\mathbf{\Phi}}$1) *Closedness under multiplication.* We have

$$\hat{\mathbf{\Phi}}_{\boldsymbol{\xi}_1}^{\gamma_1}(\boldsymbol{\omega})\hat{\mathbf{\Phi}}_{\boldsymbol{\xi}_2}^{\gamma_2}(\boldsymbol{\omega}) = \hat{\mathbf{\Phi}}_{\boldsymbol{\xi}_1+\boldsymbol{\xi}_2}^{\gamma_1+\gamma_2}(\boldsymbol{\omega}),$$

which belongs to the same family. To see this, note that the matrix

$$\hat{\mathrm{E}}(\boldsymbol{\omega}) := \frac{\boldsymbol{\omega}\boldsymbol{\omega}^{\mathsf{T}}}{\|\boldsymbol{\omega}\|^2}$$

which appears in the definition of $\hat{\mathbf{\Phi}}_{\boldsymbol{\xi}}^{\gamma}$ is a projection, and therefore

$$\hat{\mathrm{E}}^2 \equiv \hat{\mathrm{E}} \quad \text{and} \quad \hat{\mathrm{E}}(\mathbf{I} - \hat{\mathrm{E}}) \equiv \mathbf{0}.$$

($\hat{\mathbf{\Phi}}$2) *Closedness under matrix inversion.* The inverse of the matrix $\hat{\mathbf{\Phi}}_{\boldsymbol{\xi}}^{\gamma}(\boldsymbol{\omega})$, for $\boldsymbol{\omega} \neq \mathbf{0}$, is equal to $\hat{\mathbf{\Phi}}_{-\boldsymbol{\xi}}^{-\gamma}(\boldsymbol{\omega})$, which is again in the same family. This follows from the previous property and the observation that $\hat{\mathbf{\Phi}}_{\mathbf{0}}^{0}(\boldsymbol{\omega})$ is the identity matrix for $\boldsymbol{\omega} \neq \mathbf{0}$.

($\hat{\mathbf{\Phi}}$3) *Closedness under Fourier transformation.* The family is closed under elementwise Fourier transforms in the particular fashion indicated by the following lemma.

LEMMA 2.2. *Let*

$$\hat{\mathbf{\Psi}}_{\boldsymbol{\xi}}^{\gamma} := \frac{2^{-\gamma}}{\Gamma(\gamma + \frac{d}{2})}\hat{\mathbf{\Phi}}_{\boldsymbol{\xi}}^{\gamma},$$

where Γ *denotes the Gamma function. The elementwise inverse Fourier transform of* $\hat{\mathbf{\Psi}}_{\boldsymbol{\xi}}^{\gamma}$ *(in the sense of generalized functions* [17]*) is given by the formula*

$$\mathcal{F}^{-1}\{\hat{\mathbf{\Psi}}_{\boldsymbol{\xi}}^{\gamma}\} = (2\pi)^{-\frac{d}{2}}\hat{\mathbf{\Psi}}_{\boldsymbol{\zeta}}^{-\gamma-\frac{d}{2}},$$

where $\boldsymbol{\zeta} = (\zeta_{\mathrm{irr}}, \zeta_{\mathrm{sol}})$ *is related to* $\boldsymbol{\xi} = (\xi_{\mathrm{irr}}, \xi_{\mathrm{sol}})$ *by*

$$\mathrm{e}^{\zeta_{\mathrm{irr}}} = \tfrac{2\gamma+d-1}{2\gamma}\mathrm{e}^{\xi_{\mathrm{irr}}} - \tfrac{d-1}{2\gamma}\mathrm{e}^{\xi_{\mathrm{sol}}} \quad \textit{and} \quad \mathrm{e}^{\zeta_{\mathrm{sol}}} = -\tfrac{1}{2\gamma}\mathrm{e}^{\xi_{\mathrm{irr}}} + \tfrac{2\gamma+1}{2\gamma}\mathrm{e}^{\xi_{\mathrm{sol}}}.$$

A proof can be found in Appendix A.

In particular, observe that if $\mathrm{e}^{\xi_{\mathrm{irr}}} = \mathrm{e}^{\xi_{\mathrm{sol}}}$, then $\mathrm{e}^{\zeta_{\mathrm{irr}}} = \mathrm{e}^{\zeta_{\mathrm{sol}}} = \mathrm{e}^{\xi_{\mathrm{irr}}} = \mathrm{e}^{\xi_{\mathrm{sol}}}$.

2.3. Inverse fractional Laplacians. The purpose of inverting the fractional Laplacian operator introduced in the previous subsection is to allow us to solve an equation of the form

$$(-\mathbf{\Delta})_{\boldsymbol{\xi}}^{\gamma}\boldsymbol{g} = \boldsymbol{h}. \tag{2.9}$$

This equation is understood in the sense of the identity

$$\langle(-\mathbf{\Delta})_{\boldsymbol{\xi}}^{\gamma}\boldsymbol{g}, \boldsymbol{f}\rangle = \langle\boldsymbol{h}, \boldsymbol{f}\rangle,$$

which must hold for all test functions $\boldsymbol{f}$ in some appropriate space.[4] In other words, the sides of the former equation are viewed as generalized functions belonging to the dual of the space of test functions $\boldsymbol{f}$.

[4]More precisely, the action of $(-\mathbf{\Delta})_{\boldsymbol{\xi}}^{\gamma}$ on $\boldsymbol{g}$ itself is defined by the duality relation

$$\langle(-\mathbf{\Delta})_{\boldsymbol{\xi}}^{\gamma}\boldsymbol{g}, \boldsymbol{f}\rangle = \langle\boldsymbol{g}, (-\mathbf{\Delta})_{\boldsymbol{\xi}}^{\gamma*}\boldsymbol{f}\rangle,$$

where $(-\mathbf{\Delta})_{\boldsymbol{\xi}}^{\gamma*}$ is the adjoint of the fractional Laplacian. With some abuse of notation, we shall denote $(-\mathbf{\Delta})_{\boldsymbol{\xi}}^{\gamma*}$ by $(-\mathbf{\Delta})_{\bar{\boldsymbol{\xi}}}^{\gamma}$, as the two operators share the same Fourier expression (they are, however, defined on different spaces). Also note that $\boldsymbol{g}$ and $\boldsymbol{h}$ need not belong to the same space, which in turn means that the test functions applied to them may come from different function spaces.

The solution to (2.9) is sought in some generalized function space (e.g., a subspace of $(\mathcal{S}')^d$). Solving (2.9) for general $\boldsymbol{h}$ is made possible by finding a *right inverse* of $(-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}}$ which injects $\boldsymbol{h}$ into the said space of solutions. Let us denote the desired right inverse by $(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$. We shall define its action on $\boldsymbol{h}$ by the action of its adjoint $(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ on test functions:

$$\boldsymbol{g} = (-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{h} \quad \Leftrightarrow \quad \langle \boldsymbol{g}, \boldsymbol{f}\rangle = \langle \boldsymbol{h}, (-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{f}\rangle. \tag{2.10}$$

The adjoint operator is a *left inverse* of the dual Laplacian over the test function space since

$$\langle \boldsymbol{h}, (-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}(-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}}\boldsymbol{f}\rangle = \langle (-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}}(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{h}, \boldsymbol{f}\rangle = \langle \boldsymbol{h}, \boldsymbol{f}\rangle.$$

We hinted previously that in order to use (2.2) with $\mathrm{U} = (-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}}$, $\gamma > 0$, to define self-similar and isotropic random fields we would be seeking a particular right inverse of $(-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}}$ that retains its properties of homogeneity and rotation invariance. Equivalently, the adjoint (i.e., the left inverse) must be homogeneous and rotation invariant. Furthermore, it will be found necessary for our characterization that the range of the left inverse be a subspace of $(\mathrm{L}^2)^d$ (cf. subsection 3.1).

In connection with the fractional Laplacian we make the following observation. Let us first consider test functions belonging to the subspace $\mathcal{S}^d_0$ of $\mathcal{S}^d$ consisting of Schwartz functions with vanishing moments (i.e., zero derivatives of all orders at the origin of the Fourier space). $(-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}}$ is a bijection on this space, and hence also on its dual $(\mathcal{S}^d_0)'$, which can be identified with the quotient space of tempered distributions modulo polynomials, denoted by $(\mathcal{S}^d)'/\Pi$. The left and right inverses of $(-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}}$ therefore coincide on $\mathcal{S}^d_0$ and on $(\mathcal{S}^d_0)'$. On either space, they are both given by the integral

$$(2\pi)^{-d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\omega}\rangle}\hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}}(\boldsymbol{\omega})\boldsymbol{f}(\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega}. \tag{2.11}$$

However, from the identification of $(\mathcal{S}^d_0)'$ with $(\mathcal{S}^d)'/\Pi$ one sees that the extension of the right inverse to $(\mathcal{S}^d)'$ is not unique. This is precisely because $(-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}}$ has a nontrivial null space in $(\mathcal{S}^d)'$ due to the zero of its symbol at $\boldsymbol{\omega} = \boldsymbol{0}$. Correspondingly, the action of the left inverse on an arbitrary test function in $\mathcal{S}^d$ is not a priori well-defined, as its Fourier expression ($\hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}}$) is singular at $\boldsymbol{\omega} = \boldsymbol{0}$.

The problem of finding inverse operators that satisfy the desired properties (invariances and L^2-boundedness of the left inverse) can therefore be reformulated as that of choosing a particular *regularization* of the singular Fourier integral of (2.11) consistent with the said requirements. This will be the subject of the remainder of this subsection.

By a regularization of the singular Fourier integral operator of (2.11) we mean the following. Assume $\boldsymbol{f} \in \mathcal{S}^d_0$ to be a function with vanishing moments; $\boldsymbol{f}$ then satisfies $\partial_{\boldsymbol{k}}\hat{\boldsymbol{f}}(\boldsymbol{0}) = \boldsymbol{0}$ for all nonnegative multi-integers $\boldsymbol{k}$. As was already noted, the above Fourier integral converges for such $\boldsymbol{f}$. Consequently, the restriction of $(-\boldsymbol{\Delta})^{-\gamma}_{-\boldsymbol{\xi}}$ to this subspace of $\mathcal{S}^d$ is well-defined and inverts $(-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}}$ (and, by duality, the adjoint inverse can be applied to the dual of the image of this subspace). A *regularization* of $(-\boldsymbol{\Delta})^{-\gamma}_{-\boldsymbol{\xi}}$ is an extension of it to a larger class of functions, in our case $\mathcal{S}^d$.

There exists a *canonical* regularization of the above singular integral, which is homogeneous and rotation and shift invariant. It can be shown that the canonical regularization, which is the one considered by Gel'fand and Shilov [17] and Hörmander [20], corresponds to a convolution with a homogeneous generalized function. Unfortunately, this regularization fails the third of our requirements, namely L^2-boundedness. We shall therefore have to consider a different regularization of (2.11).

In what follows, we shall always limit our consideration to values of γ such that

$$2\gamma - \tfrac{d}{2} \notin \mathbb{Z}. \tag{2.12}$$

It will be seen later that this condition is equivalent to requiring that the Hurst exponent $H \notin \mathbb{Z}$ in the definition of fBm (see the discussion following Theorem 3.2).

To extend the definition of the (left) inverse from functions with vanishing moments to arbitrary test functions $\boldsymbol{f} \in \mathcal{S}^d$ let us introduce the *regularization operator*

$$\mathrm{R}^\gamma : \boldsymbol{f}(\cdot) \mapsto \boldsymbol{f}(\cdot) - \sum_{|\boldsymbol{k}| \le \lfloor 2\gamma - \frac{d}{2} \rfloor} \mathbf{T}_{\boldsymbol{k}}[\boldsymbol{f}](\cdot)^{\boldsymbol{k}}, \tag{2.13}$$

where $\mathbf{T}_{\boldsymbol{k}}[\boldsymbol{f}]$ denotes the (vector) coefficient of $(\cdot)^{\boldsymbol{k}}$ in the Taylor series expansion of $\boldsymbol{f}(\cdot)$ around $\mathbf{0}$ (we use multiindex notation). Next, consider the operator

$$(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}} : \boldsymbol{f} \mapsto (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\omega} \rangle} \hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}}(\boldsymbol{\omega}) [\mathrm{R}^\gamma \hat{\boldsymbol{f}}](\boldsymbol{\omega}) \, \mathrm{d}\boldsymbol{\omega} \tag{2.14}$$

(defined in the sense of the L^2 Fourier transform). This operator essentially removes sufficiently many terms from the Taylor expansion of $\hat{\boldsymbol{f}}(\boldsymbol{\omega})$ at $\boldsymbol{\omega} = \mathbf{0}$ so as to make the singularity of $\hat{\boldsymbol{\Phi}}_{\boldsymbol{\xi}}^{-\gamma}(\boldsymbol{\omega})$ square integrable. Of key importance is the fact that $(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ maps Schwartz test functions in $\mathcal{S}^d$ to square-integrable functions (assuming, as we already stated, that $2\gamma - \frac{d}{2} \notin \mathbb{Z}$).

PROPOSITION 2.3. *The operator* $(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ *maps* $\mathcal{S}^d$ *into* $(\mathrm{L}^2)^d$ *on the condition that* $2\gamma - d/2 \notin \mathbb{Z}$.

Proof. By Parseval's identity,

$$\begin{aligned}
\|(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}} \boldsymbol{f}\|^2 &= (2\pi)^{-d} \|\hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}} [\mathrm{R}^\gamma \hat{\boldsymbol{f}}](\boldsymbol{\omega})\|^2 \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} [\mathrm{R}^\gamma \hat{\boldsymbol{f}}]^{\mathsf{H}}(\boldsymbol{\omega}) \hat{\boldsymbol{\Phi}}^{-2\gamma}_{-2\mathrm{Re}\,\boldsymbol{\xi}}(\boldsymbol{\omega}) [\mathrm{R}^\gamma \hat{\boldsymbol{f}}](\boldsymbol{\omega}) \, \mathrm{d}\boldsymbol{\omega} \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \sum_{1 \le m, n \le d} \overline{\mathrm{R}^\gamma \hat{f}_m(\boldsymbol{\omega})} \big[\hat{\boldsymbol{\Phi}}^{-2\gamma}_{-2\mathrm{Re}\,\boldsymbol{\xi}}(\boldsymbol{\omega})\big]_{mn} \mathrm{R}^\gamma \hat{f}_n(\boldsymbol{\omega}) \, \mathrm{d}\boldsymbol{\omega}.
\end{aligned}$$

We may consider the behavior of the integrand separately about $\boldsymbol{\omega} = \mathbf{0}$ and at infinity. First, note that $\overline{\mathrm{R}^\gamma \hat{f}_m(\boldsymbol{\omega})} \mathrm{R}^\gamma \hat{f}_n(\boldsymbol{\omega})$ has a zero of order at least $2\lfloor 2\gamma - d/2 \rfloor + 2$ at $\boldsymbol{\omega} = \mathbf{0}$ (cf. the definition of R^γ in (2.13)). Since the singularity of $\big[\hat{\boldsymbol{\Phi}}^{-2\gamma}_{-2\mathrm{Re}\,\boldsymbol{\xi}}(\boldsymbol{\omega})\big]_{mn}$ at $\boldsymbol{\omega} = \mathbf{0}$ is of order -4γ and

$$2\lfloor 2\gamma - d/2 \rfloor + 2 - 4\gamma > -d$$

for $2\gamma - d/2 \notin \mathbb{Z}$, the integral converges about $\boldsymbol{\omega} = \mathbf{0}$.

At infinity, $\overline{\mathrm{R}^\gamma \hat{f}_m(\boldsymbol{\omega})} \mathrm{R}^\gamma \hat{f}_n(\boldsymbol{\omega})$ is dominated by the polynomial term and grows at most like $\|\boldsymbol{\omega}\|^{2\lfloor 2\gamma - d/2 \rfloor}$, while $\big[\hat{\boldsymbol{\Phi}}^{-2\gamma}_{-2\mathrm{Re}\,\boldsymbol{\xi}}(\boldsymbol{\omega})\big]_{mn}$ decays like $\|\boldsymbol{\omega}\|^{-4\gamma}$. We have

$$2\lfloor 2\gamma - d/2 \rfloor - 4\gamma < -d,$$

from where it follows that the integral also converges at infinity. The $(\mathrm{L}^2)^d$ norm of $(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{f}$ is therefore bounded. □

The Hermitian adjoint[5] of $(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ is the operator

$$(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}} : \boldsymbol{f} \mapsto (2\pi)^{-d} \int_{\mathbb{R}^d} \left[\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\omega}\rangle} - \sum_{|\boldsymbol{k}|\leq \lfloor 2\gamma - \frac{d}{2}\rfloor} \frac{\mathrm{j}^{|\boldsymbol{k}|}\boldsymbol{x}^{\boldsymbol{k}}\boldsymbol{\omega}^{\boldsymbol{k}}}{\boldsymbol{k}!} \right] \hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}}(\boldsymbol{\omega})\hat{\boldsymbol{f}}(\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega} \tag{2.15}$$

(see Appendix B).

As was suggested, $(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ and $(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ are named, respectively, the *left* and *right* inverses of $(-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}}$. They satisfy

$$(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}(-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}} = \mathrm{Id} \quad \text{and} \quad (-\boldsymbol{\Delta})^{\gamma}_{\boldsymbol{\xi}}(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}} = \mathrm{Id} \tag{2.16}$$

over $\mathcal{S}^d$. We may further extend the domain of $(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ to a subset of generalized functions (distributions)[6] on $\mathcal{S}^d$, using as definition the duality relation

$$\langle (-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{g}, \boldsymbol{f}\rangle := \langle \boldsymbol{g}, (-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{f}\rangle$$

wherever the right-hand side (r.h.s.) is meaningful and continuous for all $\boldsymbol{f} \in \mathcal{S}^d$.

It is easily verified that $(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ and, by duality, $(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ are rotation invariant and homogeneous. This fact is captured in our next proposition, which we shall prove with the aid of the following lemma.

Lemma 2.4. $\mathrm{R}^{\gamma}[\boldsymbol{f}(\mathbf{M}^{-1}\cdot)](\boldsymbol{x}) = [\mathrm{R}^{\gamma}\boldsymbol{f}(\cdot)](\mathbf{M}^{-1}\boldsymbol{x})$.

Proof. By the uniqueness of the Taylor series expansion,

$$\text{r.h.s.} = \boldsymbol{f}(\mathbf{M}^{-1}\boldsymbol{x}) - \sum_{|\boldsymbol{k}|\leq \lfloor 2\gamma - \frac{d}{2}\rfloor} \mathbf{T}_{\boldsymbol{k}}[\boldsymbol{f}](\mathbf{M}^{-1}\boldsymbol{x})^{\boldsymbol{k}} = \text{l.h.s.} \qquad □$$

Proposition 2.5. *The operators* $(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ *and* $(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ *are rotation invariant and homogeneous in the sense of* (2.3) *and* (2.4).

Proof. For a nonsingular real matrix $\mathbf{M}$,

$$\begin{aligned}(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{f}_{\mathbf{M}}(\boldsymbol{x}) &= (2\pi)^{-d}\int_{\mathbb{R}^d} |\det \mathbf{M}|\, \mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\omega}\rangle}\hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}}(\boldsymbol{\omega})[\mathrm{R}^{\gamma}\mathbf{M}\hat{\boldsymbol{f}}](\mathbf{M}^{\mathsf{T}}\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega}\\ &= (2\pi)^{-d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{j}\langle \mathbf{M}^{-1}\boldsymbol{x},\boldsymbol{\rho}\rangle}\hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}}(\mathbf{M}^{-\mathsf{T}}\boldsymbol{\rho})\mathbf{M}[\mathrm{R}^{\gamma}\hat{\boldsymbol{f}}](\boldsymbol{\rho})\,\mathrm{d}\boldsymbol{\rho}\end{aligned}$$

by Lemma 2.4 and with the change of variables $\boldsymbol{\rho} = \mathbf{M}^{\mathsf{T}}\boldsymbol{\omega}$. Equations (2.5) and (2.6) can now be used to verify the rotation invariance and homogeneity of $(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ (and, by duality, of $(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$). □

---

[5]This adjoint is with respect to the $\mathcal{S}^d$ scalar product

$$\langle \boldsymbol{f}, \boldsymbol{g}\rangle := \int_{\mathbb{R}^d} \boldsymbol{f}^{\mathsf{H}}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \sum_{1\leq i\leq d}\langle f_i, g_i\rangle.$$

[6]By these we mean members of the dual $(\mathcal{S}^d)'$ of $\mathcal{S}^d$. As a matter of fact, $(\mathcal{S}^d)'$ can be identified with $(\mathcal{S}')^d$.

Finally, we note that the scalar counterparts of the vector left and right inverses were defined in our previous paper [51] (as generalizations of the one-dimensional definitions of Blu and Unser [4]) as follows:

$$
\begin{aligned}
(-\grave{\Delta})^{-\gamma} &: f \mapsto (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\omega}\rangle} \|\boldsymbol{\omega}\|^{-2\gamma} \left[\hat{f}(\boldsymbol{\omega}) - \sum_{|\boldsymbol{k}| \le \lfloor 2\gamma - \frac{d}{2} \rfloor} \frac{f^{(\boldsymbol{k})}(\mathbf{0})\boldsymbol{\omega}^{\boldsymbol{k}}}{\boldsymbol{k}!} \right] \mathrm{d}\boldsymbol{\omega}; \\
(-\acute{\Delta})^{-\gamma} &: f \mapsto (2\pi)^{-d} \int_{\mathbb{R}^d} \left[\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\omega}\rangle} - \sum_{|\boldsymbol{k}| \le \lfloor 2\gamma - \frac{d}{2} \rfloor} \frac{\mathrm{j}^{|\boldsymbol{k}|}\boldsymbol{x}^{\boldsymbol{k}}\boldsymbol{\omega}^{\boldsymbol{k}}}{\boldsymbol{k}!} \right] \|\boldsymbol{\omega}\|^{-2\gamma} \hat{f}(\boldsymbol{\omega})\, \mathrm{d}\boldsymbol{\omega}.
\end{aligned}
\tag{2.17}
$$

They share the conjugacy and inversion properties of the vector inverses (cf. (2.10), (2.16)). Notice that the operand, f, is now scalar valued. Also, the vectorial parameters $\xi_{\mathrm{irr}}, \xi_{\mathrm{sol}}$ have no equivalent in the scalar case.

DIGRESSION 2.6. *The reader may be wondering why we should bother at all about singular integrals and distinct left and right inverses when we could have—as indicated in the introduction to this subsection—conveniently characterized the solution as an element of the space* $(\mathcal{S}^d)'/\Pi$ *of Schwartz distributions modulo polynomials, on which space the fractional Laplacian is bijective and therefore uniquely invertible. This would indeed be possible, since the space* $\mathcal{S}_0^d$*, being a subspace of a nuclear space, is again nuclear* [41, *Chapter* 5]*; therefore the theorems of Minlos (see* [26, 36]*) which we shall use in subsection* 3.1 *apply to it. By following this approach, one can characterize fractional Brownian vector fields as random elements of* $(\mathcal{S}^d)'/\Pi$ *(i.e., as random equivalence classes of tempered distributions modulo polynomials). One could in fact do even better by considering test functions with a finite number of vanishing moments and their dual spaces (Schwartz distributions modulo polynomials of some finite order), as was done by Dobrushin* [11] *in the scalar setting. (On a related note, the reader might also wish to consult the work of Vedel on the wavelet analysis of the Mumford process* [59]*; see also Bourdaud* [5]*.)*

However, the latter approach—although more straightforward from a technical point of view—does not provide us, at least immediately, with as complete a characterization of the stochastic solutions to (2.2) *as the one we shall see in the following sections.*

As far as the spaces of solutions are concerned, another possibility would be to use fractional Sobolev spaces, as proposed by Ruiz-Medina, Anh, and Angulo [45] *and Kelbert, Leonenko, and Ruiz-Medina* [23]*. It appears that this approach would work especially well when considering Gaussian self-similar vector fields. Working with spaces of generalized functions, on the other hand, allows us to use the method of characteristic functionals* [26, 36, 43]*, which shows its versatility when extending the work to the study of non-Gaussian random models.*

3. Vector fBm. A classical definition of the *scalar* isotropic fractional Brownian motion field with Hurst exponent H (denoted B_H, with $0 < H < 1$) goes as follows: B_H is a zero-mean Gaussian field satisfying $B_H(\mathbf{0}) = 0$, with stationary (Gaussian) increments whose variance depends on the step size as per

$$
\mathbf{E}\{|B_H(\boldsymbol{x}) - B_H(\boldsymbol{y})|^2\} = 2\alpha\|\boldsymbol{x} - \boldsymbol{y}\|^{2H}. \tag{3.1}
$$

This is a generalization of Lévy's characterization of multiparameter Brownian motion [30], to which it reduces for $H = \frac{1}{2}$. The above expectation, as a function of $\boldsymbol{x}$ and $\boldsymbol{y}$, is also known as the *variogram* of the field B_H (denoted here by $\mathrm{Vario}[B_H](\boldsymbol{x}, \boldsymbol{y})$).

A straightforward extension of the above definition to the multicomponent (vector) setting is obtained by requiring the vector-valued field $\boldsymbol{B}_H$ to satisfy

$$\mathbf{E}\{\|\boldsymbol{B}_H(\boldsymbol{x})-\boldsymbol{B}_H(\boldsymbol{y})\|^2\}=2\alpha'\|\boldsymbol{x}-\boldsymbol{y}\|^{2H}, \tag{3.2}$$

where one now considers the Euclidean norm of the increments instead of their absolute values [22]. This definition leaves the cross-correlation structure of the components of $\boldsymbol{B}_H$ unspecified; these are typically assumed to be independent, in which case the components become scalar fBm's of exponent H; i.e., the generalization is trivial. More generally, for a vector-valued random field, one may define a variogram matrix

$$\mathrm{Vario}[\boldsymbol{B}_H](\boldsymbol{x},\boldsymbol{y}):=\mathbf{E}\left\{[\boldsymbol{B}_H(\boldsymbol{x})-\boldsymbol{B}_H(\boldsymbol{y})][\boldsymbol{B}_H(\boldsymbol{x})-\boldsymbol{B}_H(\boldsymbol{y})]^{\mathsf{H}}\right\}.$$

The scalar function given in (3.2) corresponds to the trace of this matrix.

A different approach to defining fBm consists in characterizing it as a linear transformation (essentially a fractional integral) of white noise. In this approach, one starts with a white noise measure on some suitable space and proceeds to derive the probabilistic law (probability measure on a certain space[7]) of fBm from there, showing that it is consistent with the definition given in (3.1) and (3.2). One advantage of this approach is that it is not, in its essence, limited to second-order statistical analysis; this means that one is in principle free to consider non-Gaussian white noises within the same framework.

In the scalar setting, it has been indicated previously in one way or another that the linear transformation of white noise which produces fBm corresponds, in effect, to the right inverse of the scalar fractional Laplacian introduced in (2.17) (see, for instance, Tafti, Van De Ville, and Unser [51], Benassi, Jaffard, and Roux [3], Leonenko [28], and Kelbert, Leonenko, and Ruiz-Medina [23] for the multidimensional case and Samorodnitsky and Taqqu [47] and Blu and Unser [4] for the unidimensional one). One may therefore say that the scalar fractional Brownian field B_H *solves*—we shall elaborate on this—the fractional Poisson equation

$$(-\Delta)^{H/2+d/4}B_H=\epsilon_H W \tag{3.3}$$

subject to boundary conditions imposed by the right inverse (zero at the origin); i.e.,

$$B_H=\epsilon_H(-\acute{\Delta})^{-H/2-d/4}W.$$

In the above formula W denotes a Gaussian white noise field (defined in subsection 3.1) and ϵ_H is a special constant related to α in (3.1) by

$$\epsilon_H=\sqrt{(2\pi)^{\frac{d}{2}}\frac{2^{2H+d/2}\Gamma(H+\frac{d}{2})}{|\Gamma(-H)|}\alpha}.$$

Given that the only essential limitation on H values in the above characterization is the exclusion of integer H (as a consequence of (2.12)), it can also serve as a natural generalization of the definition of fBm to $H>1$ [4, 51].

So far in this section we have identified two approaches towards defining scalar fBm's: first by means of the variogram and then through a transformation of white

[7]The space of tempered distributions is standard [19], although other choices are also possible (cf. the monographs by Vakhania [58] and Talagrand [52] and the papers by Ruiz-Medina, Anh, and Angulo [45] and Kelbert, Leonenko, and Ruiz-Medina [23]).

noise. Early on in this section we also highlighted what may be considered a fundamental property of any reasonable vector generalization of fBm (see (3.2)). We noted that a relatively trivial random vector field with the said property could be constructed by grouping together d independent scalar fBm's.

Next, we shall propose a more general definition of vector fBm consistent with the trace structure of (3.2). We shall not, however, approach the problem by imposing this requirement directly. Instead, following the line of reasoning sketched in the previous paragraph, our characterization relies on solving a stochastic fractional partial differential equation similar to (3.3). From there, we shall then proceed to derive the variogram of the model in section 4 and show that it has the desired trace property.

3.1. The whitening model. As hinted above, we shall take generalized vector fBm to be the solution of the fractional Poisson equation

$$(-\boldsymbol{\Delta})_{\boldsymbol{\xi}}^{H/2+d/4}\boldsymbol{B}_{H,\boldsymbol{\xi}} = \epsilon_H \boldsymbol{W} \tag{3.4}$$

defined using the right inverse as per

$$\boldsymbol{B}_{H,\boldsymbol{\xi}} := \epsilon_H(-\acute{\boldsymbol{\Delta}})_{-\boldsymbol{\xi}}^{-H/2-d/4}\boldsymbol{W}, \tag{3.5}$$

where $\boldsymbol{W}$ is a white noise vector field, to be defined shortly. The first identity is known as a *whitening* equation in signal processing parlance (although there it is applied only to stationary processes). We shall limit our consideration to *real* random fields.

Equations (3.4) and (3.5) may be understood as equivalences in law in a sense we shall now describe. The main reference for the underlying theory of generalized random fields is Gel'fand and Vilenkin [18].

$\boldsymbol{B}_{H,\boldsymbol{\xi}}$ and $\boldsymbol{W}$ are taken to be generalized random fields, i.e., random elements of the continuous duals of certain spaces of test functions. Let us use $\boldsymbol{X}$ to denote one such random element. Under some reasonable consistency conditions, by a generalization of Kolmogorov's extension theorem [36], the stochastic law (infinite-dimensional σ-additive probability measure) of $\boldsymbol{X}$ is fully specified—in the sense of a σ-additive measure on the σ-algebra of Borel cylinder sets—by way of indicating all finite joint distributions of its "scalar products" with test functions. These products are classical random variables denoted as $\langle \boldsymbol{X}, \boldsymbol{f}\rangle$, with $\boldsymbol{f}$ belonging to the desired test function space.

By Minlos's infinite-dimensional generalization of Bochner's theorem [26, 36], it is also possible to uniquely specify the stochastic law of a real random field $\boldsymbol{X}$ by its *characteristic functional*, defined as the expectation

$$L_{\boldsymbol{X}}(\boldsymbol{f}) := \mathbf{E}\{\mathrm{e}^{\mathrm{j}\langle \boldsymbol{X},\boldsymbol{f}\rangle}\},$$

provided the test functions belong to a nuclear space. More precisely, a probability measure on a dual nuclear space gives rise to a positive-definite and continuous characteristic functional, and, conversely, any positive-definite and continuous functional on a nuclear space that evaluates to 1 at $\boldsymbol{f} \equiv \mathbf{0}$ uniquely determines a probability measure on the dual space.

The characteristic functional serves as an infinite-dimensional equivalent of the characteristic function of a random variable. In particular, for any finite number of test functions $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_N$ (N arbitrary), the N-variable function

$$\varphi(\omega_1, \ldots, \omega_N) := L_{\boldsymbol{X}}\left(\sum_{1\le i\le N} \omega_i \boldsymbol{f}_i\right)$$

is the joint characteristic function of the random variables $\langle \boldsymbol{X}, \boldsymbol{f}_i\rangle$, $1 \leq i \leq N$, which is in one-to-one correspondence with their finite-dimensional joint measure by the finite-dimensional version of Bochner's theorem. Characteristic functionals have an entire theory of their own on which we shall not elaborate here, referring instead to Gel'fand and Vilenkin [18] and the survey article by Prohorov [43].

Another useful functional that one may consider is the *correlation form* of $\boldsymbol{X}$, defined as

$$\langle\!\langle \boldsymbol{f}, \boldsymbol{g}\rangle\!\rangle_{\boldsymbol{X}} := \mathbf{E}\left\{\overline{\langle \boldsymbol{X}, \boldsymbol{f}\rangle}\langle \boldsymbol{X}, \boldsymbol{g}\rangle\right\} \quad \text{for} \quad \boldsymbol{f}, \boldsymbol{g} \in \mathcal{S}^d.$$

For real Gaussian fields, it can be shown that the correlation form and characteristic functional are related by

$$L_{\boldsymbol{X}}(\boldsymbol{f}) = \exp[-\tfrac{1}{2}\langle\!\langle \boldsymbol{f}, \boldsymbol{f}\rangle\!\rangle_{\boldsymbol{X}}], \tag{3.6}$$

which is consistent with the understanding that a Gaussian field is completely specified by its second-order statistics.

A reasonable definition of scalar *white noise* can be given as a random field W that has independent values at every point in the sense that for any two test functions f, g with disjoint supports $\langle W, f\rangle$ and $\langle W, g\rangle$ are independent. With the additional assumption that the field has Gaussian statistics, one is led to the standard definition of scalar white Gaussian noise as the field with characteristic functional

$$L_W(f) = \exp[-\tfrac{1}{2}\|f\|_2^2]$$

($\|\cdot\|_2$ denotes the L^2 norm). This random field exists as a random element of $\mathcal{S}'$ (i.e., it corresponds to a unique probability measure on $\mathcal{S}'$), as Minlos [36] has shown. The above characteristic functional also defines a cylinder probability measure on subspaces of L^2.

We shall define the standard white Gaussian noise *vector* $\boldsymbol{W}$ as the field with characteristic functional

$$L_{\boldsymbol{W}}(\boldsymbol{f}) = \exp[-\tfrac{1}{2}\|\boldsymbol{f}\|_2^2] = \exp\left[-\tfrac{1}{2}\sum_{1\leq k\leq d}\|f_k\|_2^2\right].$$

It is clear that $\boldsymbol{W}$ corresponds to a vector of independent scalar white noise fields. Its correlation form is given by the relation

$$\langle\!\langle \boldsymbol{f}, \boldsymbol{g}\rangle\!\rangle_{\boldsymbol{W}} = \langle \boldsymbol{f}, \boldsymbol{g}\rangle = \sum_{1\leq k\leq d}\langle f_k, g_k\rangle. \tag{3.7}$$

DIGRESSION 3.1. *The general form of the characteristic functional of a (not necessarily Gaussian) one-dimensional white noise process can be found in Gel'fand and Vilenkin* [18]. *In the multivariate setting, we note here in particular the characteristic functional of a Poisson white noise field P consisting of Dirac impulses with independent and identically distributed amplitudes with probability measure P_a and a spatial Poisson distribution with parameter λ:*

$$L_P(f) = \exp\left[\lambda\int\int_{\mathbb{R}^d}(\mathrm{e}^{\mathrm{j}af(\boldsymbol{x})} - 1)\,\mathrm{d}\boldsymbol{x}\;P_a(\mathrm{d}a)\right].$$

Poisson white noise can be used to define non-Gaussian stochastic fractals that agree with fractional Brownian models in their second-order statistics [57].

With our framework set as it is, all we now need is a means to derive the law of $\boldsymbol{B}_{H,\boldsymbol{\xi}}$ from that of $\boldsymbol{W}$, i.e., to give probabilistic meaning to (3.5). This we shall do as follows.

By definition, the action of an operator on a generalized function (random or deterministic) is described by the action of its adjoint on test functions. In particular, we have (cf. (2.10))

$$\boldsymbol{B}_{H,\boldsymbol{\xi}} = \epsilon_H(-\acute{\boldsymbol{\Delta}})_{-\boldsymbol{\xi}}^{-H/2-d/4}\boldsymbol{W} \quad \Leftrightarrow \quad \langle \boldsymbol{B}_{H,\boldsymbol{\xi}}, \boldsymbol{f}\rangle = \langle \boldsymbol{W}, \overline{\epsilon_H}(-\grave{\boldsymbol{\Delta}})_{-\overline{\boldsymbol{\xi}}}^{-H/2-d/4}\boldsymbol{f}\rangle \tag{3.8}$$

for all test functions $\boldsymbol{f} \in \mathcal{S}^d$. One may interpret the right-hand equality as an equivalence in joint law for all finite collections of test functions $\boldsymbol{f}$.

We shall now make use of (3.8), (3.7), and (2.14) to find the correlation form of vector fBm:

$$\begin{aligned}
\langle\!\langle \boldsymbol{f}, \boldsymbol{g}\rangle\!\rangle_{\boldsymbol{B}_{H,\boldsymbol{\xi}}} &= \mathbf{E}\left\{\overline{\langle \boldsymbol{B}_{H,\boldsymbol{\xi}}, \boldsymbol{f}\rangle}\langle \boldsymbol{B}_{H,\boldsymbol{\xi}}, \boldsymbol{g}\rangle\right\} \\
&= \mathbf{E}\left\{\overline{\langle \epsilon_H(-\acute{\boldsymbol{\Delta}})_{-\boldsymbol{\xi}}^{-H/2-d/4}\boldsymbol{W}, \boldsymbol{f}\rangle}\langle \epsilon_H(-\acute{\boldsymbol{\Delta}})_{-\boldsymbol{\xi}}^{-H/2-d/4}\boldsymbol{W}, \boldsymbol{g}\rangle\right\} \\
&= |\epsilon_H|^2\mathbf{E}\left\{\overline{\langle \boldsymbol{W}, (-\grave{\boldsymbol{\Delta}})_{-\overline{\boldsymbol{\xi}}}^{-H/2-d/4}\boldsymbol{f}\rangle}\langle \boldsymbol{W}, (-\grave{\boldsymbol{\Delta}})_{-\overline{\boldsymbol{\xi}}}^{-H/2-d/4}\boldsymbol{g}\rangle\right\} \\
&= |\epsilon_H|^2\langle\!\langle (-\grave{\boldsymbol{\Delta}})_{-\overline{\boldsymbol{\xi}}}^{-H/2-d/4}\boldsymbol{f}, (-\grave{\boldsymbol{\Delta}})_{-\overline{\boldsymbol{\xi}}}^{-H/2-d/4}\boldsymbol{g}\rangle\!\rangle_{\boldsymbol{W}} \\
&= |\epsilon_H|^2\langle (-\grave{\boldsymbol{\Delta}})_{-\overline{\boldsymbol{\xi}}}^{-H/2-d/4}\boldsymbol{f}, (-\grave{\boldsymbol{\Delta}})_{-\overline{\boldsymbol{\xi}}}^{-H/2-d/4}\boldsymbol{g}\rangle \\
&= \frac{|\epsilon_H|^2}{(2\pi)^d}\int_{\mathbb{R}^d}[\mathrm{R}^{H/2+d/4}\hat{\boldsymbol{f}}]^{\mathsf{H}}(\boldsymbol{\omega})\hat{\boldsymbol{\Phi}}_{-2\mathrm{Re}\,\boldsymbol{\xi}}^{-H-\frac{d}{2}}(\boldsymbol{\omega})[\mathrm{R}^{H/2+d/4}\hat{\boldsymbol{g}}](\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega}.
\end{aligned}$$

In view of the above identity and (3.6) we have the following theorem.

THEOREM 3.2. *The characteristic functional of the vector fBm $\boldsymbol{B}_{H,\boldsymbol{\xi}}$ is given by*

$$L_{\boldsymbol{B}_{H,\boldsymbol{\xi}}}(\boldsymbol{f}) = \exp\left[-\frac{|\epsilon_H|^2}{2(2\pi)^d}\int_{\mathbb{R}^d}[\mathrm{R}^{\frac{2H+d}{4}}\hat{\boldsymbol{f}}]^{\mathsf{H}}(\boldsymbol{\omega})\hat{\boldsymbol{\Phi}}_{-2\mathrm{Re}\,\boldsymbol{\xi}}^{-H-\frac{d}{2}}(\boldsymbol{\omega})[\mathrm{R}^{\frac{2H+d}{4}}\hat{\boldsymbol{f}}](\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega}\right], \tag{3.9}$$

with $\hat{\boldsymbol{\Phi}}_{-2\mathrm{Re}\,\boldsymbol{\xi}}^{-H-\frac{d}{2}}(\boldsymbol{\omega})$ and the regularization operator $\mathrm{R}^{\frac{2H+d}{4}}$ defined as in (2.7) *and* (2.13), *respectively.*

We remark that the positive-definiteness and continuity of $L_{\boldsymbol{B}_{H,\boldsymbol{\xi}}}$ follow from the positive-definiteness of $L_{\boldsymbol{W}}$ and the continuity of $L_{\boldsymbol{W}}$ and $(-\grave{\boldsymbol{\Delta}})_{-\boldsymbol{\xi}}^{-H/2-d/4}$. These imply the existence of a probability measure corresponding to the given characteristic form, also for $H > 1$, thus extending the definition of fBm outside the usual range of 0 to 1 (however, by (2.12), integer Hurst exponents are once again excluded).

4. Some properties of vector fBm.

In this section we shall establish some of the main properties of the random fields defined in the previous section.

4.1. Self-similarity.

Vector fBm fields are statistically self-similar (fractal) in the sense that the random field $\boldsymbol{B}_{H,\boldsymbol{\xi}}(\sigma\cdot)$ has the same statistical character as the field $\sigma^H \boldsymbol{B}_{H,\boldsymbol{\xi}}$. This can be shown as follows:

$$\begin{aligned}
\langle \boldsymbol{B}_{H,\boldsymbol{\xi}}(\sigma\cdot), \boldsymbol{f}(\cdot)\rangle &= \langle \epsilon_H[(-\acute{\boldsymbol{\Delta}})_{-\boldsymbol{\xi}}^{-\frac{2H+d}{4}}\boldsymbol{W}](\sigma\cdot), \boldsymbol{f}(\cdot)\rangle && \text{by (3.4),}\\
&= \langle \sigma^{H+\frac{d}{2}}\epsilon_H(-\acute{\boldsymbol{\Delta}})_{-\boldsymbol{\xi}}^{-\frac{2H+d}{4}}\{\boldsymbol{W}(\sigma\cdot)\}, \boldsymbol{f}(\cdot)\rangle && \text{by (2.4),}\\
&= \langle \sigma^{H+\frac{d}{2}}\epsilon_H \boldsymbol{W}(\sigma\cdot), (-\grave{\boldsymbol{\Delta}})_{-\bar{\boldsymbol{\xi}}}^{-\frac{2H+d}{4}}\boldsymbol{f}(\cdot)\rangle && \text{by duality,}\\
&= \langle \sigma^{H+\frac{d}{2}}\sigma^{-\frac{d}{2}}\epsilon_H \boldsymbol{W}(\cdot), (-\grave{\boldsymbol{\Delta}})_{-\bar{\boldsymbol{\xi}}}^{-\frac{2H+d}{4}}\boldsymbol{f}(\cdot)\rangle && \text{by the homogeneity of } \boldsymbol{W},\\
&= \langle \sigma^{H}\epsilon_H(-\acute{\boldsymbol{\Delta}})_{-\boldsymbol{\xi}}^{-\frac{2H+d}{4}}\{\boldsymbol{W}(\cdot)\}, \boldsymbol{f}(\cdot)\rangle && \text{by duality,}\\
&= \langle \sigma^{H}\boldsymbol{B}_{H,\boldsymbol{\xi}}(\cdot), \boldsymbol{f}(\cdot)\rangle && \text{by (3.4).}
\end{aligned}$$

4.2. Rotation invariance. For any orthogonal transformation matrix $\boldsymbol{\Omega}$, the random fields $\boldsymbol{B}_{H,\boldsymbol{\xi}}$ and $\boldsymbol{\Omega}\boldsymbol{B}_{H,\boldsymbol{\xi}}(\boldsymbol{\Omega}^{\mathsf{T}}\cdot)$ follow the same stochastic law. The demonstration is similar to the previous one.

4.3. Nonstationarity. Vector fBm is nonstationary. The operator $(-\grave{\boldsymbol{\Delta}})_{-\boldsymbol{\xi}}^{-\frac{2H+d}{4}}$ is not translation invariant, and consequently the random variables

$$\langle \boldsymbol{B}_{H,\boldsymbol{\xi}}, \boldsymbol{f}(\cdot)\rangle = \langle \epsilon_H \boldsymbol{W}, (-\grave{\boldsymbol{\Delta}})_{-\bar{\boldsymbol{\xi}}}^{-\frac{2H+d}{4}}\{\boldsymbol{f}(\cdot)\}\rangle$$

and

$$\langle \boldsymbol{B}_{H,\boldsymbol{\xi}}, \boldsymbol{f}(\cdot+\boldsymbol{h})\rangle = \langle \epsilon_H \boldsymbol{W}, (-\grave{\boldsymbol{\Delta}})_{-\bar{\boldsymbol{\xi}}}^{-\frac{2H+d}{4}}\{\boldsymbol{f}(\cdot+\boldsymbol{h})\}\rangle$$

are not identically distributed in general.

4.4. Stationary $\boldsymbol{n}$th-order increments. We shall now show that the increments of order $\lfloor H\rfloor+1$ of the field $\boldsymbol{B}_{H,\boldsymbol{\xi}}$ are stationary. In particular, for $0<H<1$, $\boldsymbol{B}_{H,\boldsymbol{\xi}}$ has stationary first-order increments, as is the case for standard fBm [53]. To show this, let us first define the nth-order symmetric difference operator $\mathrm{D}_{\boldsymbol{h}_1,\dots,\boldsymbol{h}_n}$ recursively by the relations

$$\begin{gathered}
\mathrm{D}_{\boldsymbol{h}_1}: \boldsymbol{f}(\cdot)\mapsto \boldsymbol{f}(\cdot+\tfrac{\boldsymbol{h}_1}{2})-\boldsymbol{f}(\cdot-\tfrac{\boldsymbol{h}_1}{2}),\\
\mathrm{D}_{\boldsymbol{h}_1,\dots,\boldsymbol{h}_n} := \mathrm{D}_{\boldsymbol{h}_n}\mathrm{D}_{\boldsymbol{h}_1,\dots,\boldsymbol{h}_{n-1}},
\end{gathered}$$

with $\boldsymbol{h}_1,\dots,\boldsymbol{h}_n\in\mathbb{R}^d\backslash\{\boldsymbol{0}\}$. The above operator is represented in the Fourier domain by the expression

$$\prod_{1\le i\le n} 2\sin\tfrac{\langle \boldsymbol{h}_i,\boldsymbol{\omega}\rangle}{2}.$$

We have the following theorem.

THEOREM 4.1. *The vector fBm field $\boldsymbol{B}_{H,\boldsymbol{\xi}}$ has stationary increments of order $\lfloor H\rfloor+1$; that is, the random field*

$$\mathrm{D}_{\boldsymbol{h}_1,\dots,\boldsymbol{h}_n}\boldsymbol{B}_{H,\boldsymbol{\xi}}$$

with $n=\lfloor H\rfloor+1$ is stationary, irrespective of the lengths and directions of the steps $\boldsymbol{h}_i$, $1\le i\le n$.

Proof. We proceed to show that the characteristic functional of the increment field $\boldsymbol{I}_n := \mathrm{D}_{\boldsymbol{h}_1,\dots,\boldsymbol{h}_n}\boldsymbol{B}_{H,\boldsymbol{\xi}}$ is shift invariant, i.e.,

$$L_{\boldsymbol{I}_n}(\boldsymbol{f}(\cdot-\boldsymbol{h})) = L_{\boldsymbol{I}_n}(\boldsymbol{f}),$$

which then directly implies the stationarity of $\boldsymbol{I}_n$.

Indeed, one may write

$$\begin{aligned}
L_{\boldsymbol{I}_n}(\boldsymbol{f}(\cdot-\boldsymbol{h})) &= \mathbf{E}\big\{\exp[\mathrm{j}\langle \boldsymbol{I}_n, \boldsymbol{f}(\cdot-\boldsymbol{h})\rangle]\big\}\\
&= \mathbf{E}\big\{\exp[\mathrm{j}\langle \mathrm{D}_{\boldsymbol{h}_1,\dots,\boldsymbol{h}_n}\boldsymbol{B}_{H,\boldsymbol{\xi}}, \boldsymbol{f}(\cdot-\boldsymbol{h})\rangle]\big\}\\
&= \mathbf{E}\big\{\exp[\mathrm{j}\langle \boldsymbol{B}_{H,\boldsymbol{\xi}}, \mathrm{D}_{-h_n,\dots,-h_1}\boldsymbol{f}(\cdot-\boldsymbol{h})\rangle]\big\}\\
&= L_{\boldsymbol{B}_{H,\boldsymbol{\xi}}}(\mathrm{D}_{-h_n,\dots,-h_1}\{\boldsymbol{f}(\cdot-\boldsymbol{h})\})\\
&= \exp\Bigg[-\frac{|\epsilon_H|^2}{2(2\pi)^d}\int_{\mathbb{R}^d}\Bigg[\mathrm{R}^{\frac{2H+d}{4}}\Bigg\{\hat{\boldsymbol{f}}(\boldsymbol{\omega})\mathrm{e}^{-\mathrm{j}\langle \boldsymbol{h},\boldsymbol{\omega}\rangle}\prod_{1\le i\le\lfloor H\rfloor+1}2\sin\frac{\langle \boldsymbol{h}_i,\boldsymbol{\omega}\rangle}{2}\Bigg\}\Bigg]^{\mathsf{H}}\\
&\qquad\cdot\hat{\boldsymbol{\Phi}}^{-H-\frac{d}{2}}_{-2\mathrm{Re}\,\boldsymbol{\xi}}(\boldsymbol{\omega})\Bigg[\mathrm{R}^{\frac{2H+d}{4}}\Bigg\{\hat{\boldsymbol{f}}(\boldsymbol{\omega})\mathrm{e}^{-\mathrm{j}\langle \boldsymbol{h},\boldsymbol{\omega}\rangle}\prod_{1\le i\le\lfloor H\rfloor+1}2\sin\frac{\langle \boldsymbol{h}_i,\boldsymbol{\omega}\rangle}{2}\Bigg\}\Bigg]\,\mathrm{d}\boldsymbol{\omega}\Bigg].
\end{aligned}$$

Next, note that the partial derivatives at $\boldsymbol{\omega}=\mathbf{0}$ of the function

$$\hat{\boldsymbol{f}}(\boldsymbol{\omega})\mathrm{e}^{-\mathrm{j}\langle \boldsymbol{h},\boldsymbol{\omega}\rangle}\prod_{1\le i\le\lfloor H\rfloor+1}2\sin\frac{\langle \boldsymbol{h}_i,\boldsymbol{\omega}\rangle}{2},\quad \boldsymbol{\omega}\in\mathbb{R}^d,$$

which appears as the argument of the regularization operator $\mathrm{R}^{\frac{2H+d}{4}}$ in the integral, all vanish up to order $\lfloor H\rfloor+1$ at least; this means that the first $\lfloor H\rfloor+1$ terms of its Taylor expansion around the origin are all zero. As a result, the said function is a fixed point of the regularization operator $\mathrm{R}^{\frac{2H+d}{4}}$ (cf. (2.13)).

All this means that we have

$$\begin{aligned}
L_{\boldsymbol{I}_n}(\boldsymbol{f}(\cdot-\boldsymbol{h})) &= \exp\Bigg[-\frac{|\epsilon_H|^2}{2(2\pi)^d}\int_{\mathbb{R}^d}\Bigg[\hat{\boldsymbol{f}}(\boldsymbol{\omega})\mathrm{e}^{-\mathrm{j}\langle \boldsymbol{h},\boldsymbol{\omega}\rangle}\prod_{1\le i\le\lfloor H\rfloor+1}2\sin\frac{\langle \boldsymbol{h}_i,\boldsymbol{\omega}\rangle}{2}\Bigg]^{\mathsf{H}}\\
&\qquad\cdot\hat{\boldsymbol{\Phi}}^{-H-\frac{d}{2}}_{-2\mathrm{Re}\,\boldsymbol{\xi}}(\boldsymbol{\omega})\Bigg[\hat{\boldsymbol{f}}(\boldsymbol{\omega})\mathrm{e}^{-\mathrm{j}\langle \boldsymbol{h},\boldsymbol{\omega}\rangle}\prod_{1\le i\le\lfloor H\rfloor+1}2\sin\frac{\langle \boldsymbol{h}_i,\boldsymbol{\omega}\rangle}{2}\Bigg]\,\mathrm{d}\boldsymbol{\omega}\Bigg]\\
&= \exp\Bigg[-\frac{|\epsilon_H|^2}{2(2\pi)^d}\int_{\mathbb{R}^d}[\hat{\boldsymbol{f}}(\boldsymbol{\omega})]^{\mathsf{H}}\hat{\boldsymbol{\Phi}}^{-H-\frac{d}{2}}_{-2\mathrm{Re}\,\boldsymbol{\xi}}(\boldsymbol{\omega})[\hat{\boldsymbol{f}}(\boldsymbol{\omega})]\prod_{1\le i\le\lfloor H\rfloor+1}4\sin^2\frac{\langle \boldsymbol{h}_i,\boldsymbol{\omega}\rangle}{2}\,\mathrm{d}\boldsymbol{\omega}\Bigg] \qquad (4.1)\\
&= L_{\boldsymbol{I}_n}(\boldsymbol{f}),
\end{aligned}$$

which is what we set out to prove. ∎

4.5. The variogram and correlation form of vector fBm. As was seen in the previous paragraph, for $0<H<1$ the random field $\boldsymbol{B}_{H,\boldsymbol{\xi}}$ has stationary first-order increments. In this case we may define its variogram (or second-order structure function) as the correlation matrix of the stationary increment $\boldsymbol{B}_{H,\boldsymbol{\xi}}(\boldsymbol{x})-\boldsymbol{B}_{H,\boldsymbol{\xi}}(\boldsymbol{y}) = \mathrm{D}_{\boldsymbol{x}-\boldsymbol{y}}\boldsymbol{B}_{H,\boldsymbol{\xi}}(\frac{\boldsymbol{x}+\boldsymbol{y}}{2})$. Formally, this is to say

$$\mathrm{Vario}[\boldsymbol{B}_{H,\boldsymbol{\xi}}](\boldsymbol{x},\boldsymbol{y}) := \mathbf{E}\big\{\boldsymbol{I}(\mathbf{0})[\boldsymbol{I}(\mathbf{0})]^{\mathsf{H}}\big\},$$

with $\boldsymbol{I} := \mathrm{D}_{\boldsymbol{x}-\boldsymbol{y}}\boldsymbol{B}_{H,\boldsymbol{\xi}}(\cdot - \frac{\boldsymbol{x}+\boldsymbol{y}}{2})$.

We shall proceed as follows to evaluate the above expression. First, we shall find the cross correlation of the (stationary) scalar random fields $[\boldsymbol{I}]_i$ and $[\boldsymbol{I}]_j$ that constitute the components of the vector $\boldsymbol{I}$. The ijth element of the variogram matrix then corresponds to the value of the said correlation function at $\mathbf{0}$.

We first obtain the correlation form of $\boldsymbol{I}$ from its characteristic functional (derived from (4.1) by setting $n = 1$) by identification (cf. (3.6)):

$$\langle\!\langle \boldsymbol{f}, \boldsymbol{g} \rangle\!\rangle_{\boldsymbol{I}} = \frac{|\epsilon_H|^2}{(2\pi)^d} \int_{\mathbb{R}^d} 4 \sin^2 \langle \tfrac{\boldsymbol{x}-\boldsymbol{y}}{2}, \boldsymbol{\omega} \rangle [\hat{\boldsymbol{f}}(\omega)]^{\mathsf{H}} \hat{\boldsymbol{\Phi}}^{-H-\frac{d}{2}}_{-2\mathrm{Re}\,\boldsymbol{\xi}}(\boldsymbol{\omega}) [\hat{\boldsymbol{g}}(\omega)] \,\mathrm{d}\boldsymbol{\omega}.$$

Next, let $\boldsymbol{f} = \hat{\boldsymbol{e}}_i \phi$ and $\boldsymbol{g} = \hat{\boldsymbol{e}}_j \psi$, where $\hat{\boldsymbol{e}}_i$ and $\hat{\boldsymbol{e}}_j$ denote standard unit vectors in $\mathbb{R}^d$ and ϕ and ψ are scalar test functions. We have

$$\begin{aligned}
\mathbf{E}\{\overline{\langle [\boldsymbol{I}]_i, \phi \rangle} \langle [\boldsymbol{I}]_j, \psi \rangle\} &= \langle\!\langle \boldsymbol{f}, \boldsymbol{g} \rangle\!\rangle_{\boldsymbol{I}} \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} 4 \sin^2 \langle \tfrac{\boldsymbol{x}-\boldsymbol{y}}{2}, \boldsymbol{\omega} \rangle \big[|\epsilon_H|^2 \hat{\boldsymbol{\Phi}}^{-H-\frac{d}{2}}_{-2\mathrm{Re}\,\boldsymbol{\xi}}(\boldsymbol{\omega}) \big]_{ij} \overline{\hat{\phi}(\omega)} \hat{\psi}(\omega)] \,\mathrm{d}\boldsymbol{\omega}.
\end{aligned}$$

By the kernel theorem, this last expression can be written in the spatial domain as

$$(2\pi)^{-d} \int_{\mathbb{R}^d} c(\boldsymbol{t} - \boldsymbol{\tau}) \phi(\boldsymbol{t}) \psi(\boldsymbol{\tau}) \,\mathrm{d}\boldsymbol{t}\mathrm{d}\boldsymbol{\tau},$$

where $c(\boldsymbol{t})$ is the generalized cross-correlation function of the random fields $[\boldsymbol{I}]_i$ and $[\boldsymbol{I}]_j$. $c(\boldsymbol{t})$ is given by the inverse Fourier transform of

$$4 \sin^2 \langle \tfrac{\boldsymbol{x}-\boldsymbol{y}}{2}, \boldsymbol{\omega} \rangle \big[|\epsilon_H|^2 \hat{\boldsymbol{\Phi}}^{-H-\frac{d}{2}}_{-2\mathrm{Re}\,\boldsymbol{\xi}}(\boldsymbol{\omega}) \big]_{ij} = \big(\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}-\boldsymbol{y}, \boldsymbol{\omega} \rangle} - 2 + \mathrm{e}^{\mathrm{j}\langle \boldsymbol{y}-\boldsymbol{x}, \boldsymbol{\omega} \rangle} \big) \big[|\epsilon_H|^2 \hat{\boldsymbol{\Phi}}^{-H-\frac{d}{2}}_{-2\mathrm{Re}\,\boldsymbol{\xi}}(\boldsymbol{\omega}) \big]_{ij},$$

which, by Lemma 2.2, is equal to

$$\alpha \big[\hat{\boldsymbol{\Phi}}^{H}_{\boldsymbol{\eta}}(\boldsymbol{t} + \boldsymbol{x} - \boldsymbol{y}) \big]_{ij} - 2\alpha \big[\hat{\boldsymbol{\Phi}}^{H}_{\boldsymbol{\eta}}(\boldsymbol{t}) \big]_{ij} + \alpha \big[\hat{\boldsymbol{\Phi}}^{H}_{\boldsymbol{\eta}}(\boldsymbol{t} + \boldsymbol{y} - \boldsymbol{x}) \big]_{ij} \tag{4.2}$$

with $\boldsymbol{\eta} = (\eta_{\mathrm{irr}}, \eta_{\mathrm{sol}})$ given by

$$\begin{aligned}
\mathrm{e}^{\eta_{\mathrm{irr}}} &= \tfrac{2H+1}{2H+d} \mathrm{e}^{-2\mathrm{Re}\,\xi_{\mathrm{irr}}} + \tfrac{d-1}{2H+d} \mathrm{e}^{-2\mathrm{Re}\,\xi_{\mathrm{sol}}}; \\
\mathrm{e}^{\eta_{\mathrm{sol}}} &= \tfrac{1}{2H+d} \mathrm{e}^{-2\mathrm{Re}\,\xi_{\mathrm{irr}}} + \tfrac{2H+d-1}{2H+d} \mathrm{e}^{-2\mathrm{Re}\,\xi_{\mathrm{sol}}}.
\end{aligned} \tag{4.3}$$

In particular, we find the ijth element of the variogram matrix by evaluating (4.2) at $\boldsymbol{t} = \mathbf{0}$. This, along with the even symmetry of $\hat{\boldsymbol{\Phi}}^{H}_{\boldsymbol{\eta}}$, yields

$$2\alpha \big[\hat{\boldsymbol{\Phi}}^{H}_{\boldsymbol{\eta}}(\boldsymbol{x} - \boldsymbol{y}) \big]_{ij}$$

as the ijth element of the variogram. We have thus proved the following theorem.

THEOREM 4.2. *The variogram of a normalized vector fBm with parameters* $H \in (0, 1)$ *and* $\boldsymbol{\xi} = (\xi_{\mathrm{irr}}, \xi_{\mathrm{sol}})$ *is*

$$\mathrm{Vario}[\boldsymbol{B}_{H,\boldsymbol{\xi}}](\boldsymbol{x}, \boldsymbol{y}) = 2\alpha \hat{\boldsymbol{\Phi}}^{H}_{(\eta_{\mathrm{irr}}, \eta_{\mathrm{sol}})}(\boldsymbol{x} - \boldsymbol{y}), \tag{4.4}$$

where the dependence of $(\eta_{\mathrm{irr}}, \eta_{\mathrm{sol}})$ *on* $(\xi_{\mathrm{irr}}, \xi_{\mathrm{sol}})$, H, *and* d *is dictated by* (4.3).

COROLLARY 4.3. *For* $0 < H < 1$ *we have*

$$\mathbf{E}\{\|\boldsymbol{B}_{H,\boldsymbol{\xi}}(\boldsymbol{x}) - \boldsymbol{B}_{H,\boldsymbol{\xi}}(\boldsymbol{y})\|^2\} = \alpha'\|\boldsymbol{x} - \boldsymbol{y}\|^{2H}$$

with

$$\alpha' = [\mathrm{e}^{-2\mathrm{Re}\,\xi_{\mathrm{irr}}} + (d-1)\mathrm{e}^{-2\mathrm{Re}\,\xi_{\mathrm{sol}}}]\alpha.$$

The proof is immediate, once it is observed that the above expectation is nothing but the trace of (4.4). We have thus shown that the new definition of fBm is consistent with (3.2).

We further remark that, by (4.3), $\mathrm{e}^{\xi_{\mathrm{irr}}} = \mathrm{e}^{\xi_{\mathrm{sol}}}$ implies $\mathrm{e}^{\eta_{\mathrm{irr}}} = \mathrm{e}^{\eta_{\mathrm{sol}}}$ (and vice versa). Consequently, in the case of classical fBm (where $\xi_{\mathrm{irr}} = \xi_{\mathrm{sol}}$) the variogram matrix is diagonal and the vector components are uncorrelated (and hence independent, due to Gaussianity).

4.6. Wavelet analysis and stationarity. The utility of wavelet analysis in studying fractal processes and turbulent flow has been noted frequently since the early days of wavelet theory, and the stationarizing effect of wavelet transforms on fBm has been widely documented [13, 16, 31, 34, 35, 60]. In this connection, an interesting observation can be made with regard to the scalar products of $\boldsymbol{B}_{H,\boldsymbol{\xi}}$ with test functions that have sufficiently many vanishing moments and, in particular, with respect to the representation of $\boldsymbol{B}_{H,\boldsymbol{\xi}}$ in a biorthogonal wavelet system.[8]

Let $\psi_{n,\boldsymbol{k}} \in \mathrm{L}^2$ and $\tilde{\psi}_{n,\boldsymbol{k}} \in \mathrm{L}^2$ symbolize the primal and dual basis functions of a biorthogonal wavelet system, with n denoting the resolution and $\boldsymbol{k}$ indicating position on a refinable lattice in $\mathbb{R}^d$.

By construction, all wavelets at a given resolution n are lattice shifts of one another ($\boldsymbol{k} \in \mathbb{Z}^d$ indexes the refinable lattice $\mathbf{Q}\mathbf{D}^{-n}\mathbb{Z}^d$ with dilation matrix $\mathbf{D} \in \mathbb{Z}^{d\times d}$, $|\det \mathbf{D}| > 1$):

$$\psi_{n,\boldsymbol{k}}(\boldsymbol{x}) = \psi_{n,\boldsymbol{0}}(\boldsymbol{x} - \mathbf{Q}\mathbf{D}^{-n}\boldsymbol{k}).$$

Consider the discrete random field $w_{n,i}$ defined by

$$w_{n,i}[\boldsymbol{k}] := \langle \boldsymbol{B}_{H,\boldsymbol{\xi}}, \hat{\boldsymbol{e}}_i\tilde{\psi}_{\boldsymbol{k}}\rangle.$$

Then

$$w_{n,i}[\boldsymbol{k}] = \langle \epsilon_H(-\acute{\boldsymbol{\Delta}})_{-\boldsymbol{\xi}}^{-\frac{2H+d}{4}}\boldsymbol{W}, \hat{\boldsymbol{e}}_i\tilde{\psi}_{n,\boldsymbol{k}}\rangle = \langle \epsilon_H\boldsymbol{W}, (-\grave{\boldsymbol{\Delta}})_{-\bar{\boldsymbol{\xi}}}^{-\frac{2H+d}{4}}\hat{\boldsymbol{e}}_i\tilde{\psi}_{n,\boldsymbol{k}}\rangle. \tag{4.5}$$

Assuming that $\tilde{\psi}_{n,\boldsymbol{k}}$ has vanishing moments (Fourier-domain zeros at $\boldsymbol{\omega} = 0$) up to degree $\lfloor H \rfloor$ so that $\mathrm{R}^{\frac{2H+d}{4}}\hat{\boldsymbol{e}}_i\tilde{\psi}_{n,\boldsymbol{k}} = \hat{\boldsymbol{e}}_i\tilde{\psi}_{n,\boldsymbol{k}}$, we have

$$(-\grave{\boldsymbol{\Delta}})_{-\bar{\boldsymbol{\xi}}}^{-\frac{2H+d}{4}}\hat{\boldsymbol{e}}_i\tilde{\psi}_{n,\boldsymbol{k}} \;=\; (2\pi)^{-d}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{j}\langle\boldsymbol{x},\boldsymbol{\omega}\rangle}\big[\hat{\boldsymbol{\Phi}}_{-\boldsymbol{\xi}}^{-\frac{2H+d}{4}}(\boldsymbol{\omega})\big]_i^{\mathsf{H}}\hat{\tilde{\psi}}_{n,\boldsymbol{k}}(\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega} \;\in\; \mathrm{L}^2;$$

[8] See Mallat [32] for detailed definitions and properties of wavelet systems. For an account of fractional-order splines and wavelets that are derived from fractional derivative operators see Unser and Blu [55]. A fundamental link between splines and fBm processes was studied in two papers by the same authors [4, 56] and extended to the multiparameter setting by Tafti, Van De Ville, and Unser [51]. The last reference also provides a detailed account of polyharmonic cardinal fractional splines in arbitrary dimensions and their connection with the wavelet analysis of scalar fBm fields.

and it becomes clear that the left inverse is shift invariant over this particular subspace of functions (in contrast to the general case of functions with nonvanishing moments for which it is not, due to the space-dependent operation of $\mathrm{R}^{\frac{2H+d}{4}}$). But then, since $\boldsymbol{W}$ is stationary, by (4.5) we may conclude that in this case the discrete random process $w_{n,i}[\cdot]$ is stationary.

In other words, a wavelet analysis of vector fBm with wavelets whose moments of degrees up to $\lfloor H \rfloor$ vanish yields stationary coefficients at each resolution.

For an overview of how matrix-valued wavelets can be used to estimate the parameters of vector fBm we refer the reader to Tafti et al. [49] and Tafti and Unser [50].

4.7. Link with the Helmholtz decomposition of vector fields. It is possible to study the divergence and curl of vector fBm (the latter for $d = 3$, where it is defined) using adjoint operators.

Taking an arbitrary scalar test function ϕ, for the divergence we have

$$\begin{aligned}\langle \operatorname{div} \boldsymbol{B}_{H,\boldsymbol{\xi}}, \phi\rangle &= -\langle \boldsymbol{B}_{H,\boldsymbol{\xi}}, \mathbf{grad}\,\phi\rangle \\ &= -\langle \boldsymbol{W}, (-\grave{\boldsymbol{\Delta}})_{-\overline{\boldsymbol{\xi}}}^{-\frac{H}{2}-\frac{d}{4}}\, \mathbf{grad}\,\phi\rangle \\ &= -\langle \boldsymbol{W}, \mathrm{e}^{-\overline{\xi_{\mathrm{irr}}}}(-\grave{\boldsymbol{\Delta}})_{\boldsymbol{0}}^{-\frac{H}{2}-\frac{d}{4}}\, \mathbf{grad}\,\phi\rangle \\ &= -\mathrm{e}^{-\overline{\xi_{\mathrm{irr}}}}\langle \boldsymbol{W}, (-\grave{\boldsymbol{\Delta}})_{\boldsymbol{0}}^{-\frac{H}{2}-\frac{d}{4}}\, \mathbf{grad}\,\phi\rangle,\end{aligned} \tag{4.6}$$

where the penultimate step can be verified easily in the Fourier domain as follows:

$$\begin{aligned}(-\grave{\boldsymbol{\Delta}})_{-\overline{\boldsymbol{\xi}}}^{-\frac{H}{2}-\frac{d}{4}}\, \mathbf{grad}\,\phi(\boldsymbol{x}) &= (2\pi)^{-d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\omega}\rangle}\|\boldsymbol{\omega}\|^{-H-\frac{d}{2}} \\ &\qquad \cdot \left[\mathrm{e}^{-\overline{\xi_{\mathrm{irr}}}}\frac{\boldsymbol{\omega}\boldsymbol{\omega}^{\mathsf{T}}}{\|\boldsymbol{\omega}\|^2} + \mathrm{e}^{-\overline{\xi_{\mathrm{sol}}}}\left(\mathbf{I} - \frac{\boldsymbol{\omega}\boldsymbol{\omega}^{\mathsf{T}}}{\|\boldsymbol{\omega}\|^2}\right)\right](\mathrm{j}\boldsymbol{\omega})\hat{\phi}(\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega} \\ &= (2\pi)^{-d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{j}\langle \boldsymbol{x},\boldsymbol{\omega}\rangle}(\mathrm{j}\boldsymbol{\omega})\|\boldsymbol{\omega}\|^{-H-\frac{d}{2}}\left[\mathrm{e}^{-\overline{\xi_{\mathrm{irr}}}} + 0\right]\,\mathrm{d}\boldsymbol{\omega} \\ &= \mathrm{e}^{-\overline{\xi_{\mathrm{irr}}}}(-\grave{\boldsymbol{\Delta}})_{\boldsymbol{0}}^{-\frac{H}{2}-\frac{d}{4}}\, \mathbf{grad}\,\phi.\end{aligned}$$

Similarly, one can take an arbitrary vector test function $\boldsymbol{f}$ and write the following with regard to the curl:

$$\begin{aligned}\langle \mathbf{curl}\, \boldsymbol{B}_{H,\boldsymbol{\xi}}, \boldsymbol{f}\rangle &= \langle \boldsymbol{B}_{H,\boldsymbol{\xi}}, \mathbf{curl}\,\boldsymbol{f}\rangle \\ &= \langle \boldsymbol{W}, (-\grave{\boldsymbol{\Delta}})_{-\overline{\boldsymbol{\xi}}}^{-\frac{H}{2}-\frac{d}{4}}\, \mathbf{curl}\,\boldsymbol{f}\rangle \\ &= \langle \boldsymbol{W}, \mathrm{e}^{-\overline{\xi_{\mathrm{sol}}}}(-\grave{\boldsymbol{\Delta}})_{\boldsymbol{0}}^{-\frac{H}{2}-\frac{d}{4}}\, \mathbf{curl}\,\boldsymbol{f}\rangle \\ &= \mathrm{e}^{-\overline{\xi_{\mathrm{sol}}}}\langle \boldsymbol{W}, (-\grave{\boldsymbol{\Delta}})_{\boldsymbol{0}}^{-\frac{H}{2}-\frac{d}{4}}\, \mathbf{curl}\,\boldsymbol{f}\rangle.\end{aligned} \tag{4.7}$$

The derivation is comparable to that of the previous result, with the difference that one needs to use the Fourier matrix of the curl operator (cf. (2.8)).

We may then deduce from (4.6) that as $|\exp(-\xi_{\mathrm{irr}})| \to 0$, $\boldsymbol{B}_{H,\boldsymbol{\xi}}$ assumes a divergence-free nature. It follows likewise from (4.7) that as $|\exp(-\xi_{\mathrm{sol}})| \to 0$, $\boldsymbol{B}_{H,\boldsymbol{\xi}}$ becomes curl-free.

5. Simulation. The random vector fields that we have described can be simulated on a digital computer in several ways. A simple approach is to take the definition (3.4) in conjunction with (2.15) and apply the operator to simulated white Gaussian noise in the Fourier domain.

A more complex scheme can be set up by considering the scalar products of the vector field with *measurement* test functions and deriving the joint probability distributions of the resulting Gaussian samples, which can then be simulated using standard techniques. For example, a localized test function of the form $\hat{\boldsymbol{e}}_i\psi$ would measure the ith component of the field about a certain location. In implementing this scheme, one may for instance take these measurement functions to be wavelets and simulate the field in keeping with subsection 4.6, taking advantage of the fact that wavelet transform coefficients of vector fBm are stationary (cf. subsection 4.6). (In this connection, see also Elliott and Majda [12]. A Fourier-based technique for simulating processes with power-law spectra was presented in Viecelli and Canfield [61].)

The reader can find examples of simulated two-dimensional vector fBm in Figures 1 and 2. These figures were generated from a single 512×512 pseudorandom noise sequence with different values for the parameters H, ξ_{irr}, and ξ_{sol}; they are available in color only in the online version.

In each instance, we have provided two complementary visualizations. Images on the left were produced by a visualization technique known as line integral convolution (LIC), which consists of local directional smoothing of a noise image in the direction of flow [6] (we used Mathematica's implementation). In these images, more neutral tones indicate larger magnitudes. Arrows are superimposed in white.

In the images on the right, the hue angle encodes local direction, while the local amplitude of the field is indicated by the saturation level (smaller amplitudes are washed out).

The change in smoothness with increasing H is visible in these images, as is the clear effect of the parameters ξ_{irr} and ξ_{sol} on the directional behavior of the field, exhibiting nearly divergence-free and nearly rotation-free extremes as well as the middle ground.

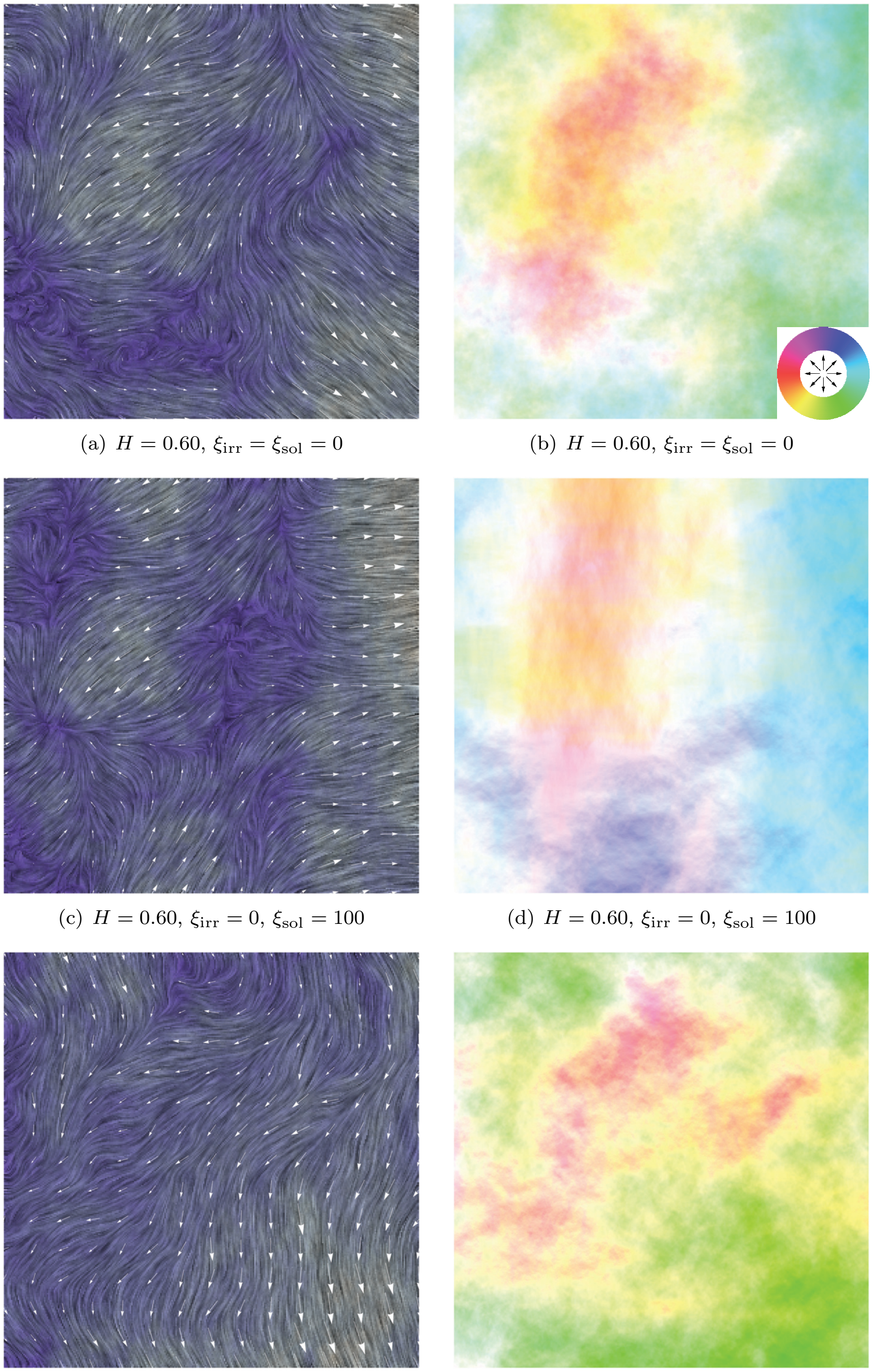
More examples can be found online at http://bigwww.epfl.ch/tafti/gal/vfBm/.

6. Conclusion. In this paper we introduced a family of random vector fields that extend fBm models by providing a means of correlating vector coordinates, which are independent in classical vector fBm models.

The first step in our investigation was to identify vector operators that are invariant under rotations and scalings of the coordinate system and can therefore be used to define random fields that are self-similar (fractal) and rotation invariant. The specific formulation of rotation invariance considered in the present work was inspired by the way physical vector fields transform under changes of coordinates. The operators identified in this step turned out to be generalizations of the vector Laplacian.

Our study of the said operators was aimed at characterizing random vector field models with the desired invariances as solutions of a whitening equation with the said generalized fractional Laplacians acting as whitening operators. To this end, we next addressed the problem of inverting the fractional Laplacian operators. This required us to introduce a new way of regularizing singular integrals, in order to define continuous inverse fractional Laplacian operators that are homogeneous and rotation invariant.

Once these inverse operators were identified, we were able to set the problem of characterizing the random models in the framework of Gel'fand and Vilenkin's theory of stochastic analysis. Specifically, we used the method of characteristic functionals to



(a) $H = 0.60$, $\xi_{\mathrm{irr}} = \xi_{\mathrm{sol}} = 0$

(b) $H = 0.60$, $\xi_{\mathrm{irr}} = \xi_{\mathrm{sol}} = 0$

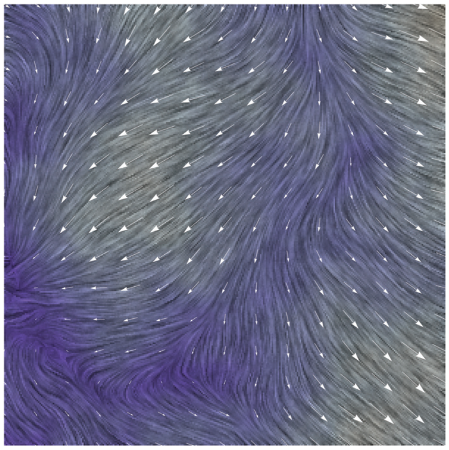
(c) $H = 0.60$, $\xi_{\mathrm{irr}} = 0$, $\xi_{\mathrm{sol}} = 100$

(d) $H = 0.60$, $\xi_{\mathrm{irr}} = 0$, $\xi_{\mathrm{sol}} = 100$

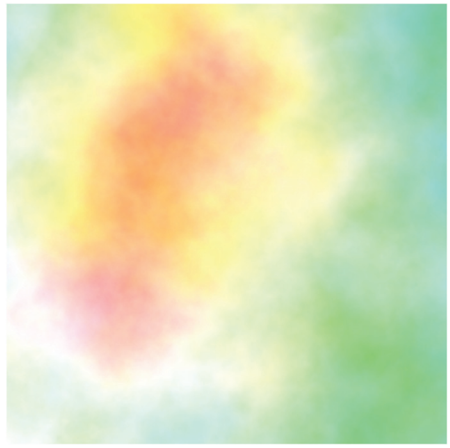
(e) $H = 0.60$, $\xi_{\mathrm{irr}} = 100$, $\xi_{\mathrm{sol}} = 0$

(f) $H = 0.60$, $\xi_{\mathrm{irr}} = 100$, $\xi_{\mathrm{sol}} = 0$

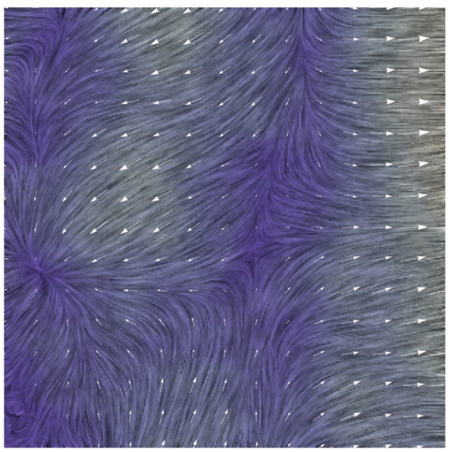
FIG. 1. *Simulated vector fBm with* $H = 0.6$ *and varying* ξ_{irr} *and* ξ_{sol}. *Left column: LIC visualization with arrows superimposed in white. Right column: directional behavior with local direction coded by the hue angle (see inset) and local amplitude represented by color saturation level (smaller amplitudes are bleached out).*



(a) $H = 0.90$, $\xi_{\mathrm{irr}} = \xi_{\mathrm{sol}} = 0$

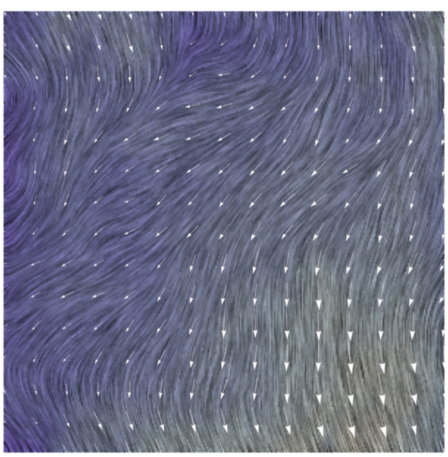


(b) $H = 0.90$, $\xi_{\mathrm{irr}} = \xi_{\mathrm{sol}} = 0$

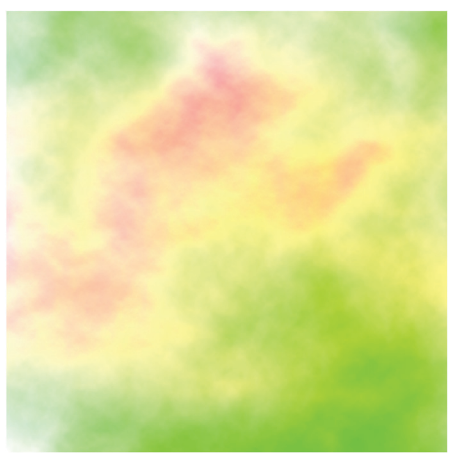


(c) $H = 0.90$, $\xi_{\mathrm{irr}} = 0$, $\xi_{\mathrm{sol}} = 100$

(d) $H = 0.90$, $\xi_{\mathrm{irr}} = 0$, $\xi_{\mathrm{sol}} = 100$



(e) $H = 0.90$, $\xi_{\mathrm{irr}} = 100$, $\xi_{\mathrm{sol}} = 0$



(f) $H = 0.90$, $\xi_{\mathrm{irr}} = 100$, $\xi_{\mathrm{sol}} = 0$

FIG. 2. *Simulated vector fBm with* $H = 0.9$ *and varying* ξ_{irr} *and* ξ_{sol}. *See the caption of Figure* 1 *for a description of the different visualizations.*

provide a complete probabilistic characterization of the new random vector fields. Using this methodology, we were also able to extend the definition of fractional Brownian fields to Hurst exponents beyond the usual range of $0 < H < 1$.

Similar to classical fBm, the fractional Brownian vector fields introduced in the present work are nonstationary but have stationary nth-order increments and can also be stationarized by means of wavelet analysis. In addition, in accordance with classical fBm models, these random fields exhibit statistical self-similarity (fractality) and rotation invariance, which are in fact properties they inherit from inverse fractional Laplacian operators. On the other hand, the directional properties of these new models have no scalar counterpart. Significantly, these models can exhibit a range of vectorial behavior, from completely irrotational (curl-free) to fully solenoidal (divergence-free).

Considering the versatility of these stochastic vector field models, potential stochastic modeling applications can exist in different disciplines such as fluid mechanics and turbulence physics, field theory, and image processing.

Appendix A. Proof of Lemma 2.2. Let

$$f_\lambda(\boldsymbol{\omega}) := 2^{-\frac{\lambda}{2}} \frac{\|\boldsymbol{\omega}\|^\lambda}{\Gamma(\frac{\lambda+d}{2})}.$$

We note the following facts concerning the above function:

$$\mathcal{F}^{-1}\{f_\lambda(\boldsymbol{\omega})\} = (2\pi)^{-\frac{d}{2}} f_{-\lambda-d}(\boldsymbol{x});$$

$$\frac{f_\lambda(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|^2} = \tfrac{1}{\lambda+d-2} f_{\lambda-2}(\boldsymbol{\omega});$$

$$\partial_i\partial_j f_\lambda(\boldsymbol{\omega}) = \tfrac{\lambda}{\lambda+d-2} f_{\lambda-2}(\boldsymbol{\omega}) \left[\delta_{ij} + (\lambda-2)\frac{\omega_i\omega_j}{\|\boldsymbol{\omega}\|^2}\right],$$

where δ_{ij} is Kronecker's delta.

The ijth element of $\hat{\boldsymbol{\Psi}}^\gamma_{\boldsymbol{\xi}}$ is

$$\left[\hat{\boldsymbol{\Psi}}^\gamma_{\boldsymbol{\xi}}(\boldsymbol{\omega})\right]_{ij} = f_{2\gamma}(\boldsymbol{\omega}) \left[\mathrm{e}^{\xi_{\mathrm{sol}}}\delta_{ij} - (\mathrm{e}^{\xi_{\mathrm{sol}}} - \mathrm{e}^{\xi_{\mathrm{irr}}})\frac{\omega_i\omega_j}{\|\boldsymbol{\omega}\|^2}\right].$$

Using the cited properties of f_λ we can write

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\left[\hat{\boldsymbol{\Psi}}^\gamma_{\boldsymbol{\xi}}(\boldsymbol{\omega})\right]_{ij}\right\} &= (2\pi)^{-\frac{d}{2}}\left[\mathrm{e}^{\xi_{\mathrm{sol}}}\delta_{ij} f_{-2\gamma-d}(\boldsymbol{x}) + \tfrac{\mathrm{e}^{\xi_{\mathrm{sol}}}-\mathrm{e}^{\xi_{\mathrm{irr}}}}{2\gamma+d-2}\partial_i\partial_j f_{-2\gamma-d+2}(\boldsymbol{x})\right]\\
&= (2\pi)^{-\frac{d}{2}} f_{-2\gamma-d}(\boldsymbol{x})\left[\mathrm{e}^{\xi_{\mathrm{sol}}}\delta_{ij} + \tfrac{\mathrm{e}^{\xi_{\mathrm{sol}}}-\mathrm{e}^{\xi_{\mathrm{irr}}}}{2\gamma}\left(\delta_{ij} - (2\gamma+d)\frac{x_i x_j}{\|\boldsymbol{x}\|^2}\right)\right]\\
&= (2\pi)^{-\frac{d}{2}} f_{-2\gamma-d}(\boldsymbol{x})\left[\mathrm{e}^{\zeta_{\mathrm{irr}}}\frac{x_i x_j}{\|\boldsymbol{x}\|^2} + \mathrm{e}^{\zeta_{\mathrm{sol}}}\left(\delta_{ij} - \frac{x_i x_j}{\|\boldsymbol{x}\|^2}\right)\right]
\end{aligned}$$

with

$$\mathrm{e}^{\zeta_{\mathrm{irr}}} = \tfrac{2\gamma+d-1}{2\gamma}\mathrm{e}^{\xi_{\mathrm{irr}}} - \tfrac{d-1}{2\gamma}\mathrm{e}^{\xi_{\mathrm{sol}}} \quad \text{and} \quad \mathrm{e}^{\zeta_{\mathrm{sol}}} = -\tfrac{1}{2\gamma}\mathrm{e}^{\xi_{\mathrm{irr}}} + \tfrac{2\gamma+1}{2\gamma}\mathrm{e}^{\xi_{\mathrm{sol}}}. \qquad \square$$

Appendix B. Conjugacy of $(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ and $(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$. We proceed to show that for all test functions $\boldsymbol{f}$ and $\boldsymbol{g} \in \mathcal{S}^d$,

$$\langle(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{f}, \boldsymbol{g}\rangle = \langle\boldsymbol{f}, (-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{g}\rangle.$$

Using Parseval's identity and the definition of $(-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}$ in (2.14) we can write

$$\begin{aligned}\langle \boldsymbol{f}, (-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{g}\rangle &= (2\pi)^{-d}\int_{\mathbb{R}^d}[\hat{\boldsymbol{f}}(\boldsymbol{\omega})]^{\mathsf{H}}\hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}}(\boldsymbol{\omega})[\mathrm{R}^\gamma\hat{\boldsymbol{g}}](\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega}\\ &= (2\pi)^{-d}\int_{\mathbb{R}^d}[\hat{\boldsymbol{f}}(\boldsymbol{\omega})]^{\mathsf{H}}\hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}}(\boldsymbol{\omega})\left[\hat{\boldsymbol{g}}(\boldsymbol{\omega}) - \sum_{|\boldsymbol{k}|\leq\lfloor 2\gamma-\frac{d}{2}\rfloor}\mathbf{T}_{\boldsymbol{k}}[\hat{\boldsymbol{g}}]\boldsymbol{\omega}^{\boldsymbol{k}}\right]\mathrm{d}\boldsymbol{\omega}. \qquad \text{(B.1)}\end{aligned}$$

Moreover,

$$\mathbf{T}_{\boldsymbol{k}}[\hat{\boldsymbol{g}}] = \frac{\hat{\boldsymbol{g}}^{(\boldsymbol{k})}(\mathbf{0})}{\boldsymbol{k}!} = \int_{\mathbb{R}^d}\frac{(-\mathrm{j})^{\boldsymbol{k}}\boldsymbol{x}^{\boldsymbol{k}}}{\boldsymbol{k}!}\boldsymbol{g}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

By combining this and (B.1) we get

$$\begin{aligned}\langle \boldsymbol{f}, (-\grave{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{g}\rangle &= (2\pi)^{-d}\int_{\mathbb{R}^d}[\hat{\boldsymbol{f}}(\boldsymbol{\omega})]^{\mathsf{H}}\hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}}(\boldsymbol{\omega})\\ &\qquad\cdot\left[\int_{\mathbb{R}^d}\left(\mathrm{e}^{-\mathrm{j}\langle\boldsymbol{x},\boldsymbol{\omega}\rangle} - \sum_{|\boldsymbol{k}|\leq\lfloor 2\gamma-\frac{d}{2}\rfloor}\frac{(-\mathrm{j})^{\boldsymbol{k}}\boldsymbol{x}^{\boldsymbol{k}}\boldsymbol{\omega}^{\boldsymbol{k}}}{\boldsymbol{k}!}\right)\boldsymbol{g}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right]\mathrm{d}\boldsymbol{\omega}\\ &= \int_{\mathbb{R}^d}\left[(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{f}(\boldsymbol{x})\right]^{\mathsf{H}}\boldsymbol{g}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},\end{aligned}$$

where the last step follows from exchanging the order of integration and using the definition of the right inverse given in (2.15), together with the identity $\left[\hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}}(\boldsymbol{\omega})\right]^{\mathsf{H}} = \hat{\boldsymbol{\Phi}}^{-\gamma}_{-\boldsymbol{\xi}}(\boldsymbol{\omega})$. The last integral is equal to the scalar product $\langle(-\acute{\boldsymbol{\Delta}})^{-\gamma}_{-\boldsymbol{\xi}}\boldsymbol{f}, \boldsymbol{g}\rangle$ by definition. ☐

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