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Graphs and Combinatorics (2013) 29:1741–1753
DOI 10.1007/s00373-012-1221-8

ORIGINAL PAPER

On Polygons Excluding Point Sets

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Received: 8 November 2010 / Revised: 19 August 2012 / Published online: 27 September 2012
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Abstract By a polygonization of a finite point set S in the plane we understand a simple polygon having S as the set of its vertices. Let B and R be sets of blue and red points, respectively, in the plane such that $B \cup R$ is in general position, and the convex hull of B contains k interior blue points and l interior red points. Hurtado et al. found sufficient conditions for the existence of a blue polygonization that encloses all red points. We consider the dual question of the existence of a blue polygonization that excludes all red points R. We show that there is a minimal number $K = K(l)$, which is bounded from above by a polynomial in l, such that one can always find a blue polygonization excluding all red points, whenever $k \geq K$. Some other related problems are also considered.

Keywords Polygonization · Convex subdivision · Convex hull · Point set order-type

Radoslav Fulek, Balázs Keszegh and Filip Morić gratefully acknowledge support from Swiss National Science Foundation, Grant No. 200021-125287/1. Partially supported by grant OTKA NK 78439.

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1 Introduction

Let S be a set of points in the plane in *general position*, i.e., such that no three points in S are collinear. A *polygonization* of S is a simple (i.e., closed and non-self-intersecting) polygon P such that its vertex set is S. Polygonizations of point sets have received some attention recently, see e.g. [1,3,6].

We say that a polygon P *encloses* a point set V if all the points of V belong to the interior of P. If all the points of V belong to the exterior of P, then we say that P *excludes* V. Let B and R be disjoint point sets in the plane such that $B \cup R$ is in general position. The elements of B and R will be called *blue* and *red* points, respectively. Also, a polygon whose vertices are blue is a *blue polygon*. A polygonization of B is called a *blue polygonization*. Throughout the paper in the figures we depict a blue point by a black disc, and a red point by a black circle.

Let $conv(X)$ denote the convex hull of a subset $X \subseteq \mathbb{R}^2$. By a *vertex* of $conv(X)$ we understand a 0-dimensional face on its boundary. We assume that all the red points belong to the interior of $conv(B)$, since we can disregard red points lying outside $conv(B)$ for the problems we consider. Let $n \geq 3$ denote the number of vertices of $conv(B)$, $k \geq 1$ the number of blue points in the interior of $conv(B)$, and $l \geq 1$ the number of red points (which all lie in the interior of $conv(B)$ by our assumption).

In [2] and [5] the problem of finding a blue polygonization that encloses the set R was studied, and in Hurtado et al. [5] showed that if the number of vertices of $conv(B)$ is bigger than the number of red points, then there is a blue polygonization enclosing the set R. Moreover, they showed by a simple construction that this result cannot be improved in general.

We propose to study a dual problem, where the goal is to find conditions under which there is a blue polygonization excluding the red points (see Fig. 1a).

Our main result is the following theorem.

Theorem 1 *Let B and R be blue and red point sets in the plane such that $B \cup R$ is in general position and R is contained in the interior of $conv(B)$. Suppose l is the number of red points and k the number of blue points in the interior of $conv(B)$. Then there exists $k_0 = k_0(l) = O(l^4)$, so that whenever $k \geq k_0$, there exists a blue polygonization excluding the set R.*

Note that it is not a priori evident that such k_0 exists. We denote by $K(l)$ the minimum possible value $k_0(l)$ for which the above theorem holds. We also show that k_0 in Theorem 1 must be at least $2l - 1$.

Theorem 2 *For arbitrary $n \geq 3$, $l \geq 1$ and $k \leq 2l - 2$ there is a set of points $B \cup R$ (as before $|B| = n + k$, $|R| = l$ and the set of vertices of the convex hull of $B \cup R$ consists of n blue points) for which there is no polygonization of the blue points that excludes all the red points.*

We consider also a version of the problem where the goal is to use as few inner blue points as possible so as to form a blue polygon excluding the red set (see Fig. 1b). We obtain the following result.

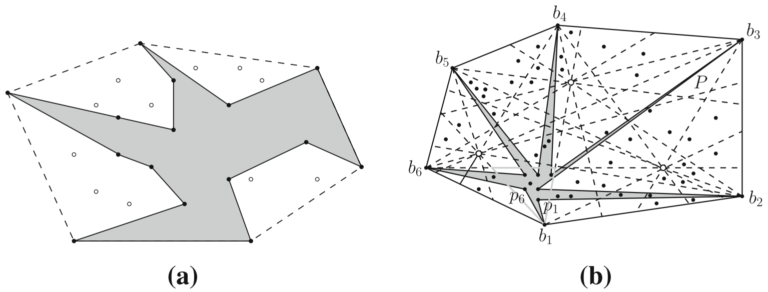


Fig. 1 **a** A blue polygonization excluding all the red points, **b** alternating polygon using few inner blue vertices (color figure online)

Theorem 3 *If $|B| = n+k$, $|R| = l$, $k \geq n^3l^2$ and the convex hull of B contains k blue vertices in its interior, then there exists a simple blue polygonization of a subset of B of size at most $2n$ that contains all the vertices of the convex hull of B, and excludes all the red points.*

Note that the number $2n$ in the previous theorem cannot be improved. Indeed, if we put a red point very close to every side of the convex hull $Conv(B)$, in any blue polygonization we cannot use any side of the convex hull. Thus, at least n inner blue points must be used. The same construction, but in different context, was used in [2].

Finally, we treat the following closely related problem. Given n red and n blue points in general position, we want to draw a polygon separating the two sets, with minimal number of sides. Our result is:

Theorem 4 *Let B and R be sets of n blue and n red points in the plane in general position. Then there exists a simple polygon with at most $3\lceil n/2 \rceil$ sides that separates blue and red points.*

Also, for every n there are sets B and R that cannot be separated by a polygon with less than n sides.

The paper is organized as follows. In Sect. 2 we state some auxiliary lemmas. Section 3 contains the proofs of our main result: Theorem 1 and Theorem 2. The proofs of the other results are deferred to Sect. 4. We finish the paper with the few concluding remarks.

2 Preliminary results

In this section we present several lemmas that we will use throughout the paper. Let us recall that B and R denote sets of blue and red points in the plane. We will assume that they are in general position, i.e., the set $B \cup R$ does not contain three collinear points. We will need the following useful lemma by García and Tejel [4].

Lemma 1 (Partition lemma) *Let P be a set of points in general position in the plane and assume that $p_1, p_2, \ldots, p_n$ are the vertices of the $conv(P)$ and that there are m interior points. Let $m = m_1 + \cdots + m_n$, where the m_i are nonnegative integers. Then* *the convex hull of* P *can be partitioned into* n *convex polygons* $Q_1, \ldots, Q_n$ *such that* Q_i *contains exactly* m_i *interior points (w.r.t.* $conv(P)$*) and* $p_i p_{i+1}$ *is an edge of* Q_i*. (Some interior points can occur on sides of the polygons* $Q_1, \ldots, Q_n$ *and for those points we decide which region they are assigned to.)*

The next corollary will be used as the main ingredient in the proof of Theorem 3.

Corollary 1 *If* $|B| = |R| = n$ *and the blue points are vertices of a convex* n*-gon, while all the red points are in the interior of that* n*-gon, then there exists a simple alternating* $2n$*-gon, i.e., a* $2n$*-gon in which any two consecutive vertices have different colors.*

In the proof of Theorem 1 we will be making a polygon by concatenating several polygonal paths obtained by the following proposition.

Proposition 1 *Let* S *be a set of* n *points in the plane in general position and* p *and* q *two points from* S*. Then one can find a simple polygonal path whose endpoints are* p *and* q *and whose vertices are the* n *given points.*

Proof Let us assume first that both p and q are vertices of $conv(S)$. Consider two non-parallel supporting lines ℓ_1 and ℓ_2 of $conv(S)$ containing points p and q. Let x be the intersection of ℓ_1 and ℓ_2. Denote by $p_1 = p, p_2, \ldots, p_{n-1}, p_n = q$ the points of S according to the order in which they are encountered by rotating the half-line xp around x. Clearly, $pp_2 \ldots p_{n-1}q$ is a simple polygonal path.

It remains to consider the case when one of the points, say p, is in the interior of $conv(S)$. Similarly as before, let $p_1 = q, p_2, \ldots, p_{n-1}$ be the points of $S - \{p\}$ labeled as we encounter them while rotating the half-line pq around p. In this case, $pp_2p_3 \ldots p_{n-1}q$ is a simple polygonal path. □

In order to obtain by our method a bound on $K(l)$ ($|R| = l$) we need to take care of the situation, when the convex hull $conv(B)$ contains too many vertices. For that sake we have the following proposition, which can be established quite easily.

Proposition 2 *There exists a subset* B' *of* B *of size at most* $2l + 1$*, containing only the vertices of* $conv(B)$*, so that all the red points are contained in* $conv(B')$*.*

3 Proof of the main result

The aim of this section is to prove the main result, which is stated in Theorem 1, about sufficient conditions for the existence of a blue polygonization that excludes all the red points.

By a *wedge* with z as its apex point we mean a convex hull of two non-collinear rays emanating from z. We define an (l-)*zoo* $\mathcal{Z} = (B, R, x, y, z)$ (Fig. 2a) as a set $B = B(\mathcal{Z})$ of blue and $R = R(\mathcal{Z})$, $|R| = l$, red points with two special blue points $x = x(\mathcal{Z}) \in B$, $y = y(\mathcal{Z}) \in B$ and a special point $z = z(\mathcal{Z})$ (not necessarily in B or R) such that:

1. every red point is inside $conv(B)$

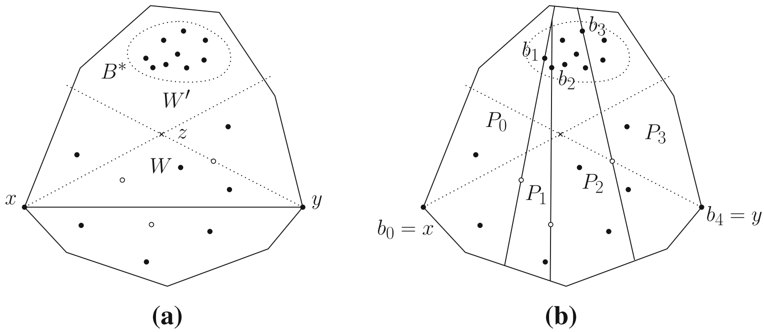


Fig. 2 **a** 3-zoo, **b** Nice partition of 3-zoo into 4 parts

2. x, y are on the boundary of $conv(B)$
3. every red point is contained in the wedge $W = W(\mathcal{Z})$ with apex z and boundary rays zx and zy.

We denote by $B^* = B^*(\mathcal{Z})$ the blue points inside $W' = W'(\mathcal{Z})$, the wedge opposite to $W(\mathcal{Z})$ (i.e., W' is the wedge centrally symmetric to W with respect to its apex). We refer to the points in B^* as to special blue points. We imagine x and y being on the x-axis (with x having smaller x-coordinate than y) and z being above it (see Fig. 2a), and we are assuming that when we talk about objects being below each other in a zoo.

A *nice partition* of an l-zoo is a partition of $conv(B)$ into closed convex parts $P_0, P_1, \ldots, P_m$, for which there exist pairwise distinct special blue points $b_1, \ldots, b_m \in B^*$ (we call $b_0 = x$ and $b_{m+1} = y$) such that for every P_i we have that (see Fig. 2b):

1. no red point is inside P_i, i.e., red points are on the boundaries of the parts
2. P_i has b_i and b_{i+1} on its boundary

A short proof of the next proposition is omitted.

Proposition 3 *Given a zoo $\mathcal{Z}$ with a nice partition, we can draw a polygonal path using all points of $B = B(\mathcal{Z})$ with endpoints $x(\mathcal{Z})$ and $y(\mathcal{Z})$ s.t. all the red points are below the polygonal path.*

The following two lemmas constitute the main part of the proof.

Lemma 2 *Given an l-zoo $\mathcal{Z}$, if $B^* = B^*(\mathcal{Z})$ contains a blue y-monotone convex chain of size $2l - 1$, then it has a nice partition.*

Proof Let $C = \{c_1, c_2, \ldots, c_{2l-1}\}$ denote a y-monotone blue convex chain of size $2l - 1$, so that $y(c_1) < y(c_2) < \ldots < y(c_{2l-1})$. If $l > 1$, without loss of generality, by the y-monotonicity we can assume that the interior of $conv(\{c_i, c_{i+1}, \ldots, c_j\})$ is on the same side of the line $c_i c_j$, for all $1 \le i < j \le 2l - 1$, as an unbounded portion of a positive part of the x-axis.

The special points of the nice partition will be always points of this chain. We start by taking $Q_{-1} = conv(B)$. Then, we recursively define the partition $P_0, P_1, \ldots, P_i, Q_i$

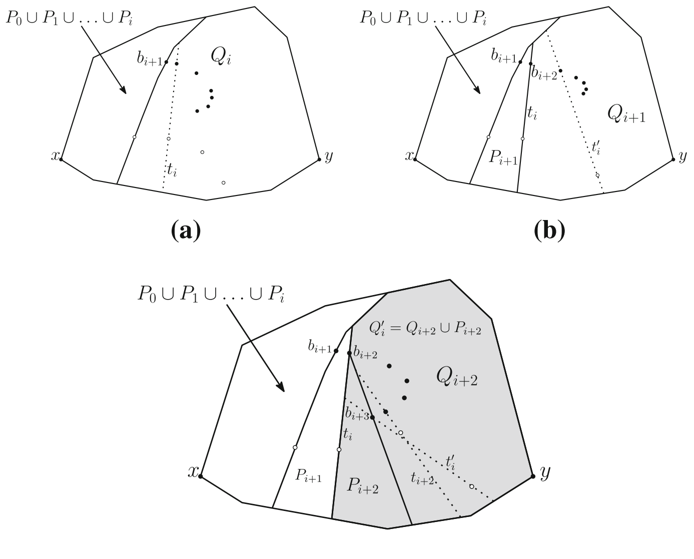


Fig. 3 **a** a general step of the recursion continuing with **b** case (i) or **c** case (ii)

and points $b_1, b_2, \ldots, b_{i+1} \in B^*$ such that for each P_i the two properties needed for a nice partition hold and the remainder Q_i of the zoo is a convex part with b_{i+1} and y on its boundary. We define $R_i = R \cap int(Q_i)$, $C_i = C \cap int(Q_i)$ and either R_i is empty or $|C_i| \geq 2|R_i| - 1$ and then t_i denotes the common tangent of $conv(C_i)$ and $conv(R_i)$, which has the point y and the interior of $conv(C_i)$ and $conv(R_i)$ on the same side (see Fig. 3a for an illustration). We maintain the following:

($\star$) If R_i is nonempty, then $t_i$intersects the boundary of Q_i in a point with highery -coordinate than b_{i+1}.

In the beginning when $i = -1$, $|C_i| \geq 2|R_i| - 1$ and ($\star$) holds trivially.

In a general step, $P_0, P_1, \ldots, P_i, Q_i$ being already defined we do the following.

If Q_i does not contain red points inside it, taking $P_{i+1} = Q_i$ and $m = i + 1$ finishes the partitioning. The convex set $P_m = Q_i$ has $b_{m+1} = y$ and $b_m = b_{i+1}$ on its boundary. Hence, the two necessary properties hold for P_m.

Otherwise, let P_{i+1} be the intersection of Q_i with the closed half-plane defined by t_i, which contains x. Trivially, there is no red point inside it. As t_i intersects the boundary of Q_i in a point with higher y-coordinate than b_{i+1}, we have that P_{i+1} has b_{i+1} on its boundary. Let b_{i+2} denote the blue point lying on t_i, trivially b_{i+2} is on the boundary of P_{i+1} too. It is easy to see that the point b_{i+1} has either the lowest or the highest y-coordinate among the points in C_i. We define Q'_i as the closure of $Q_i \setminus P_{i+1}$, $R'_i = R \cap int(Q'_i)$, $C'_i = C \cap int(Q'_i)$, and t'_i denotes the common tangent of $conv(C'_i)$ and $conv(R'_i)$, which has the point y and the interior of $conv(C'_i)$ and

$conv(R'_i)$ on the same side. If t'_i cannot be defined then R'_i is empty and the next step will be the final step, we just take $Q_{i+1} = Q'_i$.

(i) If t'_i intersects the boundary of Q'_i in a point with higher y-coordinate than b_{i+2} then ($\star$) will hold in the next step so we can finish this step by taking $Q_{i+1} = Q'_i$ (see Fig. 3b).

(ii) If t'_i does not intersect the boundary of Q'_i in a point with higher y-coordinate than b_{i+2} then we do the following (see Fig. 3c). Denote by b_{i+3} the blue point on t'_i. Now P_{i+2} is defined as the intersection of Q'_i and the half-plane defined by the line $b_{i+2}b_{i+3}$ and containing x. It is easy to see that P_{i+2} does not contain red points in its interior, and it has both b_{i+2} and b_{i+3} on its boundary. We finish this step by taking Q_{i+2} as the closure of $Q'_i \setminus P_{i+2}$. It remains to prove that in the next step property ($\star$) holds.
First, observe that b_{i+3} has either the lowest or the highest y-coordinate among the points in C_{i+2}. Moreover, it is easy to see that it has to be the lowest one otherwise we would end up in Case (i). Thus, the blue point on the new tangent t_{i+2} is a point of the chain C that is higher than b_{i+3}. Then the intersection of t_{i+2} with the boundary of Q_{i+2} must be a point with higher y-coordinate than b_{i+3} as needed.

The condition $|C_i| \geq 2|R_i| - 1$ holds by induction. Indeed, in each step the number of remaining red points decreases by 1, while the number of remaining blue points decreases at most by 2 except the last step when we never have Case (ii), and thus, the number of remaining blue points decreases also just by 1. □

Remark It is tempting to prove the lemma by divide-and-conquer strategy using the simultaneous partition of the red points and blue points in B^* by a line l into two parts so that the parts on the same side of l have the same size. However, this certainly does not work in a straightforward way, since we need that l passes through a red point and a blue point in B^*.

The next lemma is a variant of the previous one, and it is the key ingredient in the proof of the main theorem in this section.

Lemma 3 *Given an l-zoo $\mathcal{Z}$, if $B^* = B^*(\mathcal{Z})$ contains at least $\Omega(l^2)$ blue points, then it has a nice partition.*

Proof We can suppose that in B^* there is no y-monotone convex chain of size $2l - 1$, because otherwise we can apply Lemma 2 in order to get a desired nice partition.

We start by taking $Q_{-1} = conv(B)$ and $C = C_{-1} = B^*$. As in Lemma 2 we recursively define the partition $P_0, P_1, \ldots, P_i, Q_i$ and points $b_1, b_2, \ldots, b_{i+1}$ such that for each P_i the two properties needed for a nice partition hold and the remainder Q_i of the zoo $\mathcal{Z}$ is a convex part with b_{i+1} and y on its boundary. We define $R_i = R \cap int(Q_i)$, $C_i = C \cap int(Q_i)$.

In a general step, $P_0, P_1, \ldots, P_i, Q_i$ being already defined we do the following.

If Q_i does not contain red points inside it, taking $P_{i+1} = Q_i$ and $m = i + 1$ finishes the partitioning. The convex set $P_m = Q_i$ has $b_{m+1} = y$ and $b_m = b_{i+1}$ on its boundary. Hence, the two necessary properties of a nice partition hold for P_m.

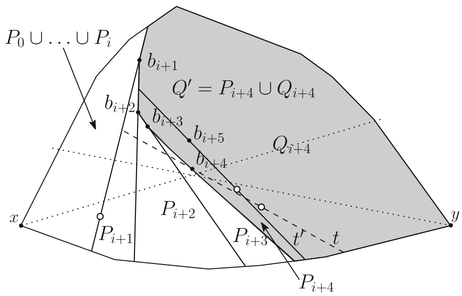


Fig. 4 A general step of the recursion in Lemma 3, $s = 4$

Otherwise, we again define t, the common tangent of $conv(C_i)$ and $conv(R_i)$ which has the point y and the interior of $conv(C_i)$ and $conv(R_i)$ on the same side of t. If t intersects the boundary of Q_i in a point with higher y-coordinate than b_{i+1} then we can finish this step as in Lemma 2 by taking b_{i+2} as the blue point on t, P_{i+1} as the intersection of Q_i with the closed half-plane defined by t and Q_{i+1} as the closure of $Q_i \setminus P_{i+1}$.

If t does not intersect the boundary of Q_i in a point with higher y-coordinate than b_{i+1}, then we define $b_{i+1}, b_{i+2}, \ldots, b_{i+s}, b_{i+s} \in t$, to be the consecutive vertices of $conv(C_i)$, for which the segments with one endpoint x and the other being any of these points, do not cross $conv(C_i)$. As this is a y-monotone convex chain with s vertices, we have that $s < 2l - 1$.

We obtain the regions $P_{i+1}, P_{i+2}, \ldots, P_{i+s-1}$ (see Fig. 4), by cutting Q_i successively with the lines through the pairs $b_{i+1}b_{i+2}, b_{i+2}b_{i+3}, \ldots, b_{i+s-1}b_{i+s}$ (in this order). Evidently, these regions satisfy the property needed for a nice partition. Let Q' stand for the remaining part of Q_i (the gray region in Fig. 4). Furthermore, $R' = R \cap int(Q')$ and $C' = C \cap int(Q')$. We define t' to be the common tangent of $conv(C')$ and $conv(R')$ which has the point y and the interior of $conv(C')$ and $conv(R')$ on the same side. We define b_{i+s+1} to be the blue point on t' and P_{i+s} to be the intersection of Q' with the closed half-plane defined by t' and containing x. Again P_{i+s} satisfies the property needed for a nice partition, as it has b_{i+s+1} and b_{i+s} on its boundary. Indeed, otherwise t' would not intersect the boundary of Q' in a point with higher y-coordinate than b_{i+s}, in which case t' could not be the tangent to $conv(C')$ and $conv(R')$, a contradiction.

Observe that C_{i+s} contains all points of C_i except $b_{i+2}, b_{i+3}, \ldots, b_{i+s+1}$. Because of that, if we proceed in this way recursively, in each step the number of remaining red points decreases by 1, while the number of remaining blue points decreases by $s < 2l - 1$. Thus, if originally, we had $(2l-2)l + 1$ blue points in B^*, we can proceed until the end thereby finding a nice partition of $\mathcal{Z}$. □

Having the previous lemma, we are in the position to prove Theorem 1.

Proof of Theorem 1. First, by Proposition 2 we obtain a subset B', $|B'| = m$, of the vertices of $conv(B)$ of size at most $2l+1$, so that $R \subseteq conv(B')$. Let $b'_0, b'_1, \ldots, b'_{m-1}$ denote the blue points in B' listed according to their cyclic order on the boundary of $conv(B')$. We distinguish two cases.

1°$conv(B')$ *does not contain* $\Omega(l^4)$ *points in its interior*. It follows, that there is a convex region P' containing $\Omega(l^3)$ blue points, which is an intersection of $conv(B)$

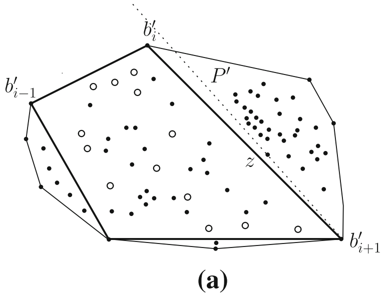


Fig. 5 Partition of $conv(B)$

with a closed half-plane T defined by a line through two consecutive vertices b'_i and b'_{i+1}, for some $0 \leq i < m$ (indices are taken modulo m), on the boundary of $conv(B')$, such that T does not contain the interior of $conv(B')$ (see Fig. 5a). Let B'' denote the set of vertices of $conv(B')$ except b'_i and b'_{i+1}. Observe that we have an l-zoo $\mathcal{Z}$ having $B(\mathcal{Z}) = B \setminus B''$, $R(\mathcal{Z}) = R$, $b_{i'}$ and b'_{i+1} as $x(\mathcal{Z})$ and $y(\mathcal{Z})$, respectively. By the general position of B we can take $z(\mathcal{Z})$ to be a point very close to the line segment $b'_i b'_{i+1}$, so that $B^*(\mathcal{Z})$ contains $\Omega(l^2)$ blue points. Thus, by Lemma 3 we obtain a nice partition of Z. Hence, by Proposition 3 we obtain a blue polygonal path Q having $B \setminus B''$ as a set of vertices. The desired polygonal path is obtained by concatenating the path Q with the convex chain formed by the points in $B'' \cup \{b'_i, b'_{i+1}\}$.

$2°$ *$conv(B')$ contains $\Omega(l^4)$ points in its interior*. Let R_i denote the intersection of R with the triangle $b'_0 b'_i b'_{i+1}$, for all $1 \leq i < m-1$. For each triangle $b'_0 b'_i b'_{i+1}$ we consider the lines through all the pairs r and b, such that $b = b'_0, b'_i$ or b'_{i+1} and $r \in R_i$. For each i, $1 \leq i < m-1$, these lines partition the triangle $b'_0 b'_i b'_{i+1}$ into $O(|R_i|^2)$ 2-dimensional regions. Hence, by doing such a partition in all the triangles $b'_0 b'_i b'_{i+1}$ we partition $conv(B')$ into $O(\sum_{i=1}^{m-2} |R_i|^2) = O(|R|^2)$ regions, each of them fully contained in one of the triangles $b'_0 b'_i b'_{i+1}$. It follows that one of these regions, let us denote it by P', contains at least $\Omega(l^2)$ blue points (see Fig. 5b). Clearly, P' is contained in a triangle $b'_0 b'_i b'_{i+1}$, for some $1 \leq i < m-1$.

For the convenience we rename the points b'_0, b'_i, b'_{i+1} by b_0, b_1, b_2 in clockwise order. We apply Partition Lemma (Lemma 1) on the triangle $b_0 b_1 b_2$, so that we obtain a partition of the triangle $b_0 b_1 b_2$ into three convex polygonal regions P'_0, P'_1, P'_2 (in fact triangles), such that each part contains $\Omega(l^2)$ blue points belonging to $P' \cap P'_j$, for all $0 \leq j \leq 2$, and has $b_j b_{j+1}$ as a boundary segment. We denote by P_0, P_1, P_2 the parts in the partition of $conv(B)$, which is naturally obtained as the extension of the partition of $b_0 b_1 b_2$, so that P_j, $P_j \supseteq P'_j$, has $b_j b_{j+1}$ (indices are taken modulo 3) either as a boundary edge or as a diagonal.

In what follows we show that in each P_j, $0 \leq j \leq 2$, we have an l_j-zoo $\mathcal{Z}_j$, $l_j \leq l$, with b_j as $x(\mathcal{Z}_j)$ and b_{j+1} and $y(\mathcal{Z}_j)$, respectively, and with $\Omega(l^2)$ blue points in $B^*(\mathcal{Z}_j)$.

First, we suppose that there exists a red point in P'_j. We take $z(\mathcal{Z}_j)$ to be the intersection of two tangents t_1 and t_2 from b_j and b_{j+1}, respectively, to $conv(R \cap P'_j)$ that have $conv(R \cap P'_j)$ and $b_j b_{j+1}$ on the same side. Clearly, P' has to be contained in one of

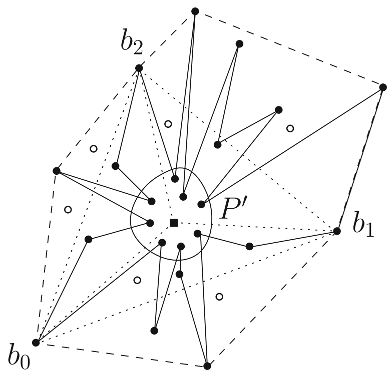


Fig. 6 Forming a polygonization

four wedges defined by t_1 and t_2. However, if P' is not contained in the wedge defined by t_1 and t_2, which has the empty intersection with the line through b_j and b_{j+1}, either P_{j+1} or P_{j-1} cannot have a non-empty intersection with P' (contradiction). Thus, $B^*(\mathcal{Z}_j)$ of $\mathcal{Z}_j$ contains at least $\Omega(l^2)$ blue points.

Hence, we can assume that P'_j does not contain any red point. In this case, by putting z very close to $b_j b_{j+1}$, so that $z \in b_0 b_1 b_2$, we can make sure, that the corresponding wedge above the line $b_j b_{j+1}$ contains all the blue points in P'.

Thus, in every P_j, $0 \le j \le 2$, we have $\mathcal{Z}_j$ with b_j and b_{j+1} as $x(\mathcal{Z}_j)$ and $y(\mathcal{Z}_j)$, respectively, the set of blue points in P_j as $B(\mathcal{Z}_j)$, and the set of red points in P_j as $R(\mathcal{Z}_j)$. By using Proposition 3 on a nice partition of $\mathcal{Z}_j$ obtained by Lemma 3 we obtain a polygonal path using all the blue points in P_j which joins b_j and b_{j+1}, and which has all the red points in P_j on the "good" side. Finally, the required polygonization is obtained by concatenating the paths obtained by Lemma 3 (see Fig. 6). □

We finish this section with the proof of the complementary lower bound.

Proof of Theorem 2. For fixed n and $l \ge 1$ and $k = 2l - 2$ we define the set B as follows (see Fig. 7). We put two blue points x and y on the x-axis, x being left from y. In the upper half-plane we put $n - 2$ blue points $Z = \{z_1, z_2, \ldots, z_{n-2}\}$ close to each other such that $Z' = \{x, y\} \cup Z$ are in convex position. Let us call a vertex in Z a z-vertex. Furthermore, we put $l - 1$ blue points (not necessarily in convex position) to the interior of $conv(Z')$ close to the z-vertices, we call them b-vertices. Next, we put l red points in the interior of $conv(Z')$, all below the lines xz_{n-2} and yz_1 such that together with x and y they form a convex chain $xr_1r_2 \ldots r_l y$. Finally, for each segment $r_i r_{i+1}$, we put a blue point l_i a bit below its midpoint. We call these l-vertices (lower blue vertices). This way we added $l - 1$ more blue points. Suppose that there exists a polygon P through all the blue points excluding all the red points. Starting with a b-vertex we take the vertices of the polygon one by one until we reach an l-vertex, say l_i. The vertex preceding l_i on the polygon cannot be x, as in this case r_1 would be in the interior of P, and similarly it cannot be y as then r_l would be in the interior of P. If it is a z-vertex then r_i or r_{i+1} is inside P. Thus, it can be only a b-vertex. Now, the vertex following l_i on the polygon cannot be neither x, y nor an l-vertex as

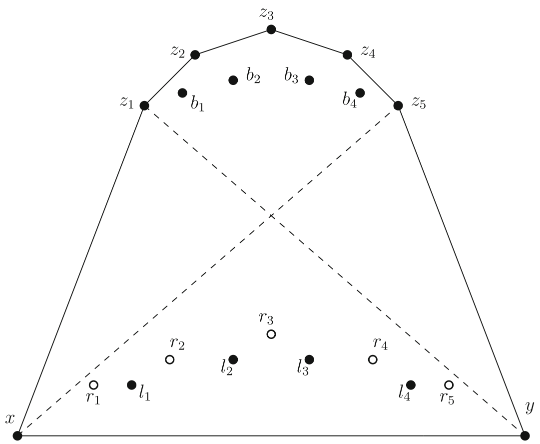


Fig. 7 Lower bound construction

in all of these cases r_i or r_{i+1} would be inside P. For the same reason it cannot be a z-vertex. Hence, it must be a b-vertex. Now, we find the next l-vertex on the polygon. Again, the vertex before and after it must be a b-vertex. Proceeding this way we see that every l-vertex is preceded and followed by a b-vertex. As we have other vertices on the polygon too, it means that the number of b-vertices is at least one more than the number of l-vertices, a contradiction. □

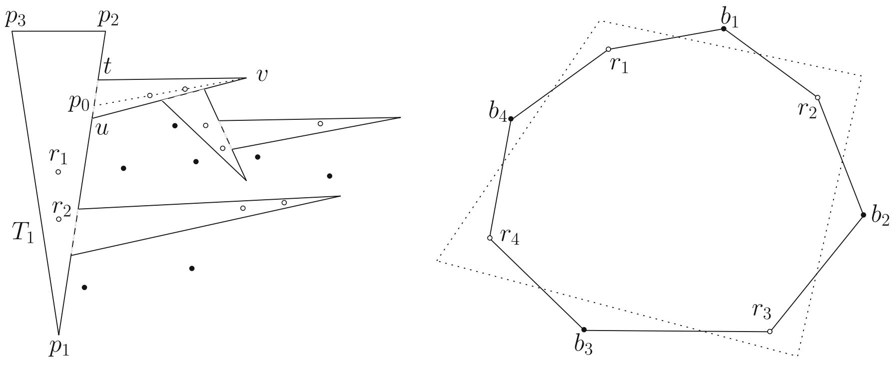
4 Variants on the problem

We present the proofs of two results (Theorems 3 and 4) on the problems which are of similar flavor as that of our central problem treated in the previous section. In comparison with our main results both proofs are quite simple.

Proof of Theorem 3. Let $b_1, \dots, b_n$ be the vertices of the convex hull. Consider all the lines determined by one blue point from the convex hull and one red point. It is easy to see that by drawing these nl lines the interior of $conv(B)$ is divided into no more than $(nl)^2$ 2-dimensional regions. Since we have at least $K'(l, n)$ interior blue points, it follows that there is a region that contains at least n blue points (see Fig. 1b).

Let $p_1, \dots, p_n$ be blue points that lie inside one region. By Corollary 1 it follows that there exists a simple $2n$-polygon P whose vertices are taken alternatingly from the sets $\{b_1, \dots, b_n\}$ and $\{p_1, \dots, p_n\}$. It is easy to see from the proof of Corollary 1 (based on Lemma 1) that this $2n$-gon satisfies the following property: *for each point x from the interior of the $2n$-gon there is a blue point b_i such that the segment $b_i x$ is entirely contained in the $2n$-gon.*

By relabeling the points if necessary we can assume that $P = b_1 p_1 \dots b_n p_n$. We claim that P does not contain any red point in its interior. Suppose the contrary, i.e., there exists a red point r in the interior of P. Then there exists a blue vertex b_i such that the segment $b_i r$ lies in the interior of P. Hence, the line l through b_i and r intersects the line segment $p_{i-1} p_i$ (where $p_0 = p_n$), which cannot be true because all the points



(a) **(b)**

Fig. 8 **a** Construction of the red-blue separation, **b** the lower bound construction for red-blue separation

$p_1, \ldots, p_n$ lie in the same closed half-plane defined by l. This contradiction finishes the proof. □

Proof of Theorem 4. Let $R = \{r_1, \ldots, r_n\}$, where $x(r_1) \leq x(r_2) \leq \ldots \leq x(r_n)$. By choosing the coordinate system appropriately we can assume that $x(r_1) = x(r_2) = 0$. Due to the general position we can find numbers $a, b > 0$ large enough so that for certain $c > 0$ the triangle T_1 (see Fig. 8a) with vertices $p_1 = (0, -a)$, $p_2 = (c, b)$, $p_3 = (-c, b)$ has the following properties:

- T_1 contains r_1 and r_2 and does not contain any other red or blue points
- all the lines $r_{2i-1}r_{2i}$ ($i = 2, 3, \ldots$) intersect the boundary of T_1

We will proceed by enlarging the polygon T_1 adding to it in each step three new vertices so that the new polygon contains the next pair of red points and no blue points. Since the line r_3r_4 intersects the boundary of T_1 at some point p_0 we can find two points t and u on the boundary of T_1 close enough to p_0 and a point v on the line r_3r_4 close to one of the points r_3, r_4, so that the triangle tuv can be joined with T_1 thereby creating a new polygon T_2 that contains the points r_1, r_2, r_3, r_4, and does not contain any other red or blue point. Notice that the condition requiring, that any line determined by two consecutive red points intersects the boundary of T_2, is still satisfied, since it was already true for T_1. Observe that T_2 has six vertices.

We can continue in this way by adding the pairs r_i, r_{i+1} for $i = 5, 7, \ldots, 2\lfloor n/2 \rfloor - 1$ one by one. In the end we get a polygon $T_{\lfloor n/2 \rfloor}$, that contains all the red points, except r_n in case of odd n, has $3\lfloor n/2 \rfloor$ vertices, and does not contain any blue point. If n is even, we are done. Otherwise we can add in the same manner three new vertices to $T_{\lfloor n/2 \rfloor}$ in order to include r_n as well.

Finally, let us show that we cannot always find a separating polygon with less than n sides. Let $r_1, b_1, r_2, b_2, \ldots, r_n, b_n$ be the vertices of a convex $2n$-gon appearing in that order on the circumference and set $R = \{r_1, \ldots, r_n\}$ and $B = \{b_1, \ldots, b_n\}$ (see Fig. 8b). Let P be any polygon that separates the two sets. Obviously, each of the $2n$ segments $r_1b_1, b_1r_2, \ldots, r_nb_n, b_nr_1$ must be intersected by a side of P. Since one side of P can intersect simultaneously at most two of these segments, it follows that P must have at least n sides. □

5 Concluding remarks

Theorem 1 in Sect. 3 proves the existence of a total blue polygonization excluding red points if we have enough inner blue points. We showed an upper bound on $K(l)$, the needed number of inner blue points, that is polynomial, but likely not tight. We conjecture that the upper bound is $2l - 1$, which meets the lower bound in Theorem 2. If $l \leq 2$ then a non-trivial case-analysis shows that the conjecture holds. If finding the right values of $K(l)$ for all l turns out to be out of reach, it is natural to ask the following.

Question 1 What is the right order of magnitude of $K(l)$?

One could obtain a better upper bound on $K(l)$, e.g., by proving Lemma 3 with a weaker requirement on the number of blue points in $W(\mathcal{Z})$, which we suspect is possible.

Question 2 Does Lemma 3 still hold, if we require only to have $\Omega(l)$ points in $W(\mathcal{Z})$, instead of $\Omega(l^2)$?

Finally, the bounds we have on the minimal number of sides for the red-blue separating polygon do not meet.

Problem 1 *Improve the bounds n or/and* $3\lceil n/2 \rceil$ *in Theorem 4.*

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