Direct data-driven $\mathcal{H}_2 - \mathcal{H}_\infty$ loop-shaping ⋆

Simone Formentin *, **Alireza Karimi** **, **Sergio M. Savaresi** *

* *Dipartimento di Elettronica e Informazione, Politecnico di Milano, Piazza L. Da Vinci, 32, 20133 Milano (Italy) e-mail: {formentin,savaresi}@elet.polimi.it*

** *Laboratoire d'Automatique, École Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne (Switzerland) e-mail: alireza.karimi@epfl.ch*

Abstract: In this paper, a direct data-driven approach is proposed to tune fixed-order controllers for unknown stable LTI plants in a mixed-sensitivity loop-shaping problem. The method requires a single set of input-output samples and it is based on simple convex optimization techniques; moreover, it guarantees internal stability as the data-length tends to infinity. Compared to a standard model-based approach, the proposed methodology theoretically guarantees the same asymptotical performance in case of correct parameterization, whereas the direct data-driven formulation is less conservative in case of undermodeling. The effectiveness of the method is illustrated via some numerical examples.

Keywords: data-driven control, identification for control, Youla-Kučera parameterization, mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$ loop-shaping, convex optimization.

1. INTRODUCTION

One of the main methods to specify the desired behaviour of a control system is to describe the frequency-domain relations between the different signals in the loop. In robust control theory, $\mathcal{H}_2$-$\mathcal{H}_\infty$ loop-shaping are design techniques that allow one to find a trade-off between different features, *e.g.* settling time and noise rejection, by means of $\mathcal{H}_2$-$\mathcal{H}_\infty$ optimization methods.

The fact that any feedback controller design must reflect a compromise between insensitivity to different disturbances and good stability margins is first identified in Safonov et al. [1981], where the mixed-sensitivity criterion is introduced as a very suitable quality measure of the closed-loop behaviour. Among all different approaches for the solution of such control-design problem, the Youla-Kučera parameterization (see Doyle et al. [1992]) represents one of the most successful. As a matter of fact, by parameterizing the feedback controller with the Youla-Kučera parameter Q, the mixed-sensitivity problem becomes convex in the unknown Q and the final controller is guaranteed to stabilize the closed-loop system. In case of fixed controller order, the loop-shaping problem becomes much more complex, as model-reduction techniques (see Obinata et al. [2001]) must be adopted and closed-loop stability may be compromized.

In the data-driven framework, *i.e.* when a control-oriented model of the plant is difficult to derive from first-principle methods and experimental data are available, a mathematical model is first deduced from data and then the fixed-order controller is designed based on the identified model. In such situations, three optimization problems must be solved to obtain the final controller. The best model with the desired structure is the one that minimizes a certain identification criterion, Q is the transfer function that optimizes the loop-shaping cost and finally the fixed-order controller is the one that, *e.g.*, best fits the frequency response of the optimal controller, if L_2-model reduction techniques are used.

In this paper, a different philosophy is proposed to solve the loop-shaping problem in the data-driven framework. Since the unique aim of model identification is the design of the controller, in the proposed approach this first step is skipped by directly identifying the Youla-Kučera parameter from a single set of data. The final reduced-order controller K is then deduced from the same dataset as the one that minimizes the loop-shaping criterion. The design issue is naturally converted into a convex data-driven optimization problem, if Q and K are linearly parameterized. Furthermore, in both noiseless and noisy environments, the method is "one-shot", *i.e.* it requires only one set of input-output samples, and it allows the designer to avoid all the reasoning about the modeling phase, by still guaranteeing the closed-loop stability. During the presentation of the method, it will be also shown that, in case of correct parameterization, model-based and direct data-driven approaches yield the same performance and that, if undermodeling occurs, the standard technique is more conservative, such that the control system quality may be jeopardized.

Noniterative direct data-driven methodologies for the design of fixed order controllers already exist in the model-reference control framework, *e.g.* the Correlation-based Tuning (CbT, see *e.g.* Karimi et al. [2007] and Van Heusden [2010]) and Virtual Reference Feedback Tuning (VRFT, see *e.g.* Campi et al. [2002] and Formentin et al. [2010]). As far as the authors are aware, this is

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the first work where the direct data-driven philosophy is applied to the mixed-sensitivity loop-shaping problem.

The paper is structured as follows. In Section 2, the mixed-sensitivity loop-shaping problem is formulated in the system-theory analytic framework. Then, Section 3 presents the direct data-driven method in detail. A comparison between the direct data-driven approach and a standard formulation of the model-based approach is developed in Section 4, while some numerical examples are illustrated in Section 5. The paper is ended by some concluding remarks.

2. PRELIMINARIES

Consider the unknown stable LTI SISO plant $G_0(q^{-1})$, where q^{-1} denotes the backward-shift operator, and three weighting functions $W_S(q^{-1})$, $W_T(q^{-1})$ and $W_U(q^{-1})$. The loop-shaping control aim is to design a LTI fixed-order controller $K(q^{-1},\rho)$ so as to minimize

$$J(\rho) = \|W_S S(\rho)\|^2 + \|W_T T(\rho)\|^2 + \|W_U U(\rho)\|^2 \quad , \qquad (1)$$

where:

$$S(\rho) = \frac{1}{1+K(\rho)G_0},$$
$$T(\rho) = \frac{K(\rho)G_0}{1+K(\rho)G_0},$$
$$U(\rho) = \frac{K(\rho)}{1+K(\rho)G_0},$$

and the symbol $\|\cdot\|$ indicates either the $\mathcal{H}_2$- or the $\mathcal{H}_\infty$-norm during the whole paper. Notice that throughout the paper, the arguments t and q^{-1} are sometimes dropped for ease of notation. Note that the criterion (1) is generally non-convex with respect to the parameter vector ρ and that in most cases $J(\rho) = 0$ cannot be achieved.

Consider now the Youla-Kučera reformulation of (1). The set of all stabilizing controllers for $G_0(q^{-1})$ can be written as (see Doyle et al. [1992])

$$\mathcal{C} = \left\{ C(q^{-1}) = \frac{Q(q^{-1})}{1-Q(q^{-1})G_0(q^{-1})}, Q(q^{-1}) \in \mathcal{H}_\infty \right\}. \quad (2)$$

Then, the three sensitivity functions can be rewritten as $S = (1 - QG_0)$, $T = QG_0$, and $U = Q$. It follows that the criterion (1) is convex in $Q(q^{-1})$ or in the parameters of $Q(q^{-1})$, if it is linearly parameterized. The fixed-order controller is finally found as the model-reduction $K(q^{-1},\rho)$ of the full-order controller $C = \hat{Q}/(1-\hat{Q}G_0)$, where $\hat{Q}(q^{-1}) \in \mathcal{H}_\infty$ is the minimizer of the loop-shaping criterion (1) in the Q-form.

In the Youla-Kučera setting, the reduced-order controller is not guaranteed to internally stabilize the system. In the following theorem, a sufficient condition for guaranteeing internal stability is introduced. This constraint can be included in the final $K(q^{-1},\rho)$ design procedure to insure that also the reduced order controller belongs to $\mathcal{C}$.

Theorem 1. Let $G_0(q^{-1})$ and $\hat{Q}(q^{-1})$ be discrete-time dynamical systems in $\mathcal{H}_\infty$. The controller $K(q^{-1},\rho)$ stabilizes the plant $G_0(q^{-1})$ if

(1) $\Delta(\rho) = G_0 \left[(1-\hat{Q}G_0)K(\rho) - \hat{Q} \right] \in \mathcal{H}_\infty$

(2) the stability radius $\zeta(\rho) = \|\Delta(\rho)\|_\infty$ is less than 1

Proof. Consider the scheme in Figure 1, where $C(q^{-1}) = \hat{Q}(q^{-1})/(1-\hat{Q}(q^{-1})G_0(q^{-1}))$. Since $\hat{Q}(q^{-1})$ belongs to $\mathcal{H}_\infty$, the full-order controller $C(q^{-1})$ internally stabilizes the closed-loop system opened at z. Then, both $S(q^{-1}) = 1 - \hat{Q}(q^{-1})G_0(q^{-1})$ and $T(q^{-1}) = \hat{Q}(q^{-1})G_0(q^{-1})$ are stable. From the Small-Gain Theorem (see Zhou et al. [1996]), a sufficient condition for the closed-loop stability of the interconnected system is that the transfer function between $u(t)$ and $z(t)$ is stable and its infinity norm is less than 1 (requirement 1 and 2).

□

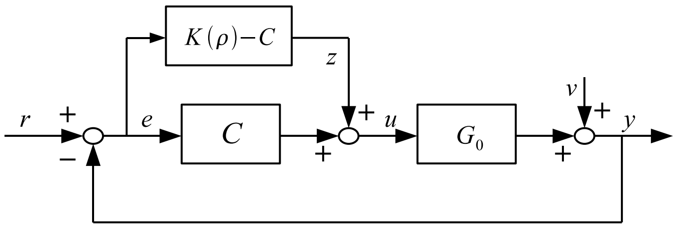

Fig. 1. Closed-loop system with controller $K(\rho)$ and explicit representation of the controller-reduction.

Remark. If the fixed-order controller $K(q^{-1},\rho)$ is stable, requirement 1 is always satisfied since $\hat{Q}(q^{-1}) \in \mathcal{H}_\infty$ by hypotheses and $S(q^{-1}) = 1 - \hat{Q}(q^{-1})G_0(q^{-1})$ is stable. If $K(q^{-1},\rho)$ contains an integrator, it is sufficient to impose that $Q(1) = 1/G_0(1)$ in the Q-design procedure. By doing this way, $S(q^{-1}) = 1-\hat{Q}(q^{-1})G_0(q^{-1})$ has a zero at 1 and $(1-\hat{Q}(q^{-1})G_0(q^{-1}))K(q^{-1},\rho)$ is stable for any ρ.

In practical situation, only an approximation $\hat{G}$ of the real system is known. Therefore, the criterion above could yield to a controller that instabilizes the closed-loop system. A possible reformulation of the stability constraint for stable controllers is presented next.
Let consider only a class of controllers in $\mathcal{H}_\infty$ and suppose that a bound

$$\delta = \sup_{\hat{G}} \left\| \hat{G} - G_0 \right\|_\infty$$

is known. The measure of the stability radius $\hat{\zeta}(G_0,\rho) = \left\| G_0 \left[\left(1 - G_0\hat{Q}\right) K(\rho) - \hat{Q} \right] \right\|_\infty$ for the real plant is such that

$$\begin{aligned} \hat{\zeta}(\rho) \leq \hat{\zeta}'(\rho) = \|G_0\|_\infty \left\| \left(1 - \hat{G}\hat{Q}\right) K(\rho) - \hat{Q} \right\|_\infty + \\ + \|G_0\|_\infty \, \delta \left\| \hat{Q}K(\rho) \right\|_\infty , \end{aligned} \qquad (3)$$

where $\|G_0\|_\infty$ can be either computed from data (see Van Heusden et al. [2007]) or overestimated by means of $\left\|\hat{G}\right\|_\infty + \delta$. It follows that the stability radius for the real plant can be transformed in a (more conservative) convex constraint for ρ, that only depends on $\hat{G}$ and δ. If this new formulation of the stability constraint is used in model-reduction procedure, internal stability can be guaranteed for the real system.

The loop-shaping problem above can be faced both in first-principle and in data-driven modeling settings. In this work, only the data-driven framework will be considered.

Assume then that a set of N input-output noisy data $\{u(t), y(t)\}$, $t = 1, .., N$ is available and that these data are generated in open-loop operation according to the system dynamics, *i.e.* $y(t) = G_0(q^{-1})u(t) + v(t)$, where $v(t) = H_0(q^{-1})e(t)$, $H_0(q^{-1})$ is an unknown stable filter and $e(t)$ is a zero mean white noise. Assume also that u is a persistent exciting stationary signal. The standard (model-based) approach for mixed-sensitivity $\mathcal{H}_2$-$\mathcal{H}_\infty$ data-driven design is presented next.

MODEL-BASED ALGORITHM

(1) Choose a class of SISO LTI models
$$\mathcal{G} = \left\{ G(q^{-1}, \theta) \ , \ \theta \in \Theta \subset \mathcal{R}^{dim(\theta)} \right\}. \tag{4}$$
(2) Identify a data-driven model of $G_0(q^{-1})$ as $\hat{G} = G(q^{-1}, \hat{\theta})$, where
$$\hat{\theta} = \arg\min_{\theta} \|G_0 - G(\theta)\|^2 \tag{5}$$
(3) Compute the optimal Youla-Kučera parameter $\hat{Q}$ as the rational transfer function in $\mathcal{H}_\infty$ that minimizes (1). This can be approximately done using a linearly parameterized Q (see Boyd and Barratt [1991])..
(4) Compute the full-order controller in $\mathcal{C}$ that guarantees the optimal sensitivity trade-off as $\hat{C} = \hat{Q}/(1 - \hat{Q}\hat{G})$.
(5) Compute the stable reduced-order controller as $\hat{K} = K(q^{-1}, \hat{\rho})$, where
$$\hat{\rho} = \arg\min_{\rho} J_k(\rho)$$
$$J_k(\rho) = \left\| \hat{C} - K(\rho) \right\|^2 \tag{6}$$
and the condition expressed in Theorem 1 is satisfied with respect to G_0, *i.e.* $\hat{\zeta}'(\rho) < 1$.

In the following section, a suitable way to solve the so-formulated data-driven loop-shaping problem without identifying the plant model is proposed and analyzed. In the proposed method, the stability constraint could also be reformulated with less conservatism.

3. DIRECT DATA-DRIVEN APPROACH

Let the Youla-Kučera parameter be linearly parameterized, *i.e.* $Q(\eta) = \eta^T \beta_Q(q^{-1})$, where $\beta_Q(q^{-1})$ is a vector of orthonormal basis functions with the same dimension of η; analogously, consider a linear parameterization of the controller, *i.e.*
$$K(q^{-1}, \rho) = \rho^T \beta_K(q^{-1}), \tag{7}$$
where $\beta_K(q^{-1})$ is a vector of orthonormal basis functions with the same dimension of ρ.
Consider now the tuning scheme in Figure 2. For each value of the parameter vector, the signals $z_1(t, \eta)$, $z_2(t, \eta)$ and $z_3(t, \eta)$ can be expressed as functions of $u(t)$ and of the output $y(t)$, without including the real plant dynamics:
$$z_1(\eta) = W_S \left(1 - G_0 Q(\eta)\right) u = W_S u - W_S Q(\eta) y,$$

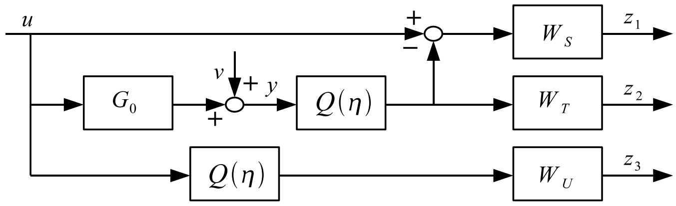

Fig. 2. Tuning scheme for the Youla-Kučera parameter.

$$z_2(\eta) = W_T G_0 Q(\eta) u = W_T Q(\eta) y,$$
$$z_3(\eta) = W_U Q(\eta) u.$$

In a noiseless environment, *i.e.* when $v(t) = 0$, $\forall t$, the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of the generating functions $G_{z_i}(q^{-1}, \theta)$ of such signals, *i.e.* the functions such that $z_i(t, \eta) = G_{z_i}(q^{-1}, \eta)u(t)$, $i = 1, 2, 3$, can be estimated from data. In detail, concerning the $\mathcal{H}_2$-norm, it holds that, for N that tends to infinity,
$$\frac{1}{N} \sum_{t=1}^{N} L(q^{-1}) z_i(t, \eta)^2 \to \|G_{z_i}(\eta)\|_2^2 \ , \ i = 1, 2, 3,$$
where the prefilter $L(q^{-1})$ is such that $\left|L(e^{j\omega})\right|^2 = 1/\hat{\Phi}_{uu}(\omega)$ and $\hat{\Phi}_{uu}(\omega)$ is an estimate of the spectrum of u. An estimate of the $\mathcal{H}_\infty$-norm can be instead asymptotically derived via spectral estimates as suggested in Van Heusden et al. [2007]. Formally, for N that tends to infinity, it holds that
$$\max_{\omega_k} \left| \frac{\hat{\Phi}_{uz_i}(\omega_k, \eta)}{\hat{\Phi}_{uu}(\omega_k)} \right| \to \|G_{z_i}(\eta)\|_\infty \ , \ i = 1, 2, 3,$$
where $\omega_k = 2\pi k/(2l + 1)$, $k = 1, \ldots, l + 1$ and $\hat{\Phi}_{uz_i}(\omega_k, \eta)$ is an estimate of the cross-spectrum between u and z_i. In detail, such spectrum may be computed as
$$\hat{\Phi}_{uz_i}(\omega_k, \eta) = \sum_{\tau=-l}^{l} \hat{R}_{uz_i}(\tau, \eta) e^{-j\tau\omega_k}$$
where $\hat{R}_{uz_i}(\tau, \eta)$ is an estimate of the cross-correlation function between u and z_i
$$\hat{R}_{uz_i}(\tau, \eta) = \frac{1}{N} \sum_{t=1}^{N} u(t - \tau) z_i(t, \eta).$$

In order to guarantee a consistent approximation of the $\mathcal{H}_\infty$-norm, the choice of l must be such that $l \to \infty$ and $l/N \to 0$ (see Van Heusden et al. [2007]). Notice that, since all signals are linear functions of η with the parameterization of Q selected above, the (approximated) $\mathcal{H}_2$ squared norm is quadratic and the (approximated) $\mathcal{H}_\infty$-norm is convex with respect to the parameter vector. Therefore, in such noiseless setting, the problem of finding $\hat{\eta}$ minimizing (1) is converted in a convex optimization problem, where the addends in the cost function, *i.e* the weighted sensitivities that generate the z_i-s, are directly computed from data. In this way, points 1-2-3 of the standard model-based algorithm (see again Section 2) are reduced to a single identification step.
If $v(t)$ is a generic zero-mean stochastic signal, the problem of minimizing (1) turns out to be a standard errors-in-variables (EIV) problem, where a model has to be identified starting from noisy input-data. In this case, different

solutions are available in the literature to make the procedure insensitive to noise (see *e.g.* Soderstrom [2007]).

The same direct data-driven rationale can be used to condensate points 4-5 of the model-based algorithm in another data-based step, without identifying $G_0(q^{-1})$. As a matter of fact, the linearly parameterized reduced-order controller $K(q^{-1}, \rho)$ can be directly identified from the same data-set used for the computation of the Youla-Kučera parameter. Moreover, it will be shown that the bias due to presence of noise can be easily handled if the problem is formulated using the correlation approach.
Consider Figure 3 where the tracking error $\varepsilon_K(t, \rho)$ is defined as

$$\varepsilon_K(\rho) = (1 - G_0 Q(\hat{\eta}))\, K(\rho) u - Q(\hat{\eta}) u.$$

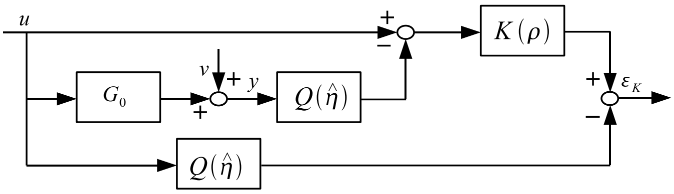

Fig. 3. Tuning scheme for the reduced-order controller.

Introduce now the instrumental variable vector $\zeta(t)$

$$\zeta(t) = [u(t + l_k), \ldots, u(t), \ldots, u(t - l_k)]^T$$

and the decorrelation criterion as

$$J_k^{N,l_k}(\rho) = \left[\frac{1}{N}\sum_{t=1}^{N} \zeta(t)\varepsilon_K(t,\rho)\right]^T \frac{1}{N}\sum_{t=1}^{N} \zeta(t)\varepsilon_K(t,\rho) \quad (8)$$

The following result holds for the formulation above in both noiseless and noisy settings.

Theorem 2. Consider the decorrelation criterion (8), where $\varepsilon_K(t, \rho)$ is generated by the linearly-parameterized controller (7) and filtered with $L_k(q^{-1})$ such that

$$\left|L_k(e^{j\omega})\right| = 1/\hat{\Phi}_{uu}(\omega). \quad (9)$$

Then, as $N, l_k \to \infty$ and $l_k/N \to 0$, the minimizer $\hat{\rho}$ of $J_k^{N,l_k}(\rho)$ is with probability 1 a minimizer of (6), where $C = Q(\hat{\eta})/(1 - Q(\hat{\eta})G_0)$ and $\hat{\eta}$ is the data-driven minimizer of (1).

Proof. Following the same procedure adopted for model-reference criteria in Van Heusden [2010], the criterion can be proved to statistically converge to a continuous function of the cross-correlation indicators $R_{u\epsilon_k}(\tau, \rho) = \mathbb{E}[u(t - \tau)\varepsilon_k(t, \rho)]$, *i.e.*

$$\lim_{N\to\infty} J_k^{N,l_k}(\rho) = \sum_{\tau=-l_k}^{\tau=l_k} R_{u\epsilon_k}(\tau, \rho)^2.$$

Notice then that if $K(q^{-1}, \rho)$ is stable, $(1 - Q(\hat{\eta})G_0)\, K(\rho) - Q(\hat{\eta})$ is stable and that the same holds if $K(q^{-1}, \rho)$ contains an integrator and Q has been constrained such that $Q(1) = 1/G_0(1)$. As a consequence, the squared sum $\sum_{\tau=-l_k}^{l_k} R_{u\epsilon_k}(\tau, \rho)^2$ and its limit $\sum_{\tau=-\infty}^{\infty} R_{u\epsilon_k}(\tau, \rho)^2$ are bounded on the parameter set. Thus, as $N, l_k \to \infty$ and $l_k/N \to 0$, $J_k^{N,l_k}(\rho)$ converges uniformly to $\sum_{\tau=-\infty}^{\infty} R_{u\epsilon_k}(\tau, \rho)^2$ (see Rockafeller [1970]). In frequency-domain, the asymptotical value of $J_k^{N,l_k}(\rho)$ can be rewritten by means of the Parseval theorem as

$$\sum_{\tau=-\infty}^{\infty} R_{u\epsilon_k}(\tau, \rho)^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Phi_{u\varepsilon_k}(\omega)|^2 \, d\omega =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |L|^2 \left|(1 - G_0 Q(\hat{\eta}))\, K(\rho) - Q(\hat{\eta})\right|^2 \Phi_{uu}^2(\omega) d\omega$$

If the data-prefilter is selected according to (9), it then holds that (6) asymptotically tends to (1). It follows that, since the convergence is uniform, the minimizers of the two criteria coincide.

□

The stability constraint can be included in the design problem in three different ways.

(1) *Double-experiment procedure:* a data-driven version of the constraint in Theorem 1 can be formulated with a second open-loop experiment, by feeding the plant with $y(t)_{t=1\ldots N}$ and collecting the output $y'(t)_{t=1\ldots N}$.

Let $\Delta(\rho)$ be the transfer function between u and a signal z_Δ, *i.e.*

$$z_\Delta(\rho) = (1 - G_0 Q(\hat{\eta})))\, K(\rho) G_0 u - Q(\hat{\eta}) G_0 u.$$

It follows that, in a noiseless environment,

$$z_\Delta(\rho) = (1 - G_0 Q(\hat{\eta})))\, K(\rho) y - Q(\hat{\eta}) y =$$
$$= K(\rho) y - Q(\hat{\eta}) K(\rho) y' - Q(\hat{\eta}) y,$$

that is $z_\Delta(\rho)$ can be computed as a function of known data for each value of ρ. The $\mathcal{H}_\infty$-norm of $\Delta(\rho)$ can then be asymptotically derived as suggested in Van Heusden et al. [2007]. It should be mentioned that if $K(\rho)$ contains an integrator, the equality constraint $Q(1) = 1/G_0(1)$ requires an additional information on the static gain of the process, as an estimate of the plant model is no more available.

(2) *Single-experiment procedure:* if a stabilizing minimum-phase controller C_s is available, it is possible to avoid ad-hoc experiments. Consider again Figure 1, by replacing C with C_s. A different stability condition depending on C_s can be straightforwardly derived by following the same rationale in Theorem 1 and requiring that

$$\Delta_s(\rho) = \frac{G_0\,(K(\rho) - C_s)}{1 + G_0 C_s} \in \mathcal{H}_\infty$$

and

$$\zeta_s(\rho) = \|\Delta_s(\rho)\|_\infty < 1.$$

In such case, $\Delta_s(\rho)$ can be seen as the transfer function between a fictitious reference $r_f(t)$ and z_{Δ_s} (see again the closed-loop scheme in Figure 1), *i.e.*

$$z_{\Delta_s}(\rho) = \frac{G_0\,(K(\rho) - C_s)}{1 + G_0 C_s} r_f,$$

where r_f is given by

$$r_f(t) = C_s^{-1}(q^{-1}) u(t) + y(t).$$

The expression of $z_{\Delta_s}(\rho)$ may be rewritten as

$$z_{\Delta_s}(\rho) = \frac{G_0 K(\rho)}{1+G_0 C_s} r_f - \frac{G_0 C_s}{1+G_0 C_s} r_f =$$
$$= C_s^{-1} K(\rho) \frac{G_0 C_s}{1+G_0 C_s} r_f - y = \left(C_s^{-1} K(\rho) - 1\right) y.$$

Therefore, $z_{\Delta_s}(\rho)$ is completely known from data and the $\mathcal{H}_\infty$-norm of $\Delta(\rho)$ can again be asymptotically derived as suggested in Van Heusden et al. [2007]. Notice that if $K(\rho)$ contains an integrator, also C_s must have it; analogously, C_s must be stable if $K(\rho)$ is stable. However, this is not difficult to achieve in practical situations.

(3) *Single-experiment procedure (trivial):* if a stabilizing minimum-phase controller C_s is not available, the Small-Gain theorem can still be applied to the open-loop transfer function without performing a second experiment. In other words, another sufficient condition of internal stability for the closed-loop system is

$$\Delta_t(\rho) = G_0 K(\rho) \in \mathcal{H}_\infty$$

and

$$\zeta_t(\rho) = \|\Delta_t(\rho)\|_\infty < 1.$$

As for the other criteria, $\Delta_t(\rho)$ can be seen as the transfer function between $u(t)$ and z_{Δ_t}, where

$$z_{\Delta_t}(\rho) = G_0 K(\rho) u = K(\rho) y$$

and the $\mathcal{H}_\infty$-norm of $\Delta_t(\rho)$ can be derived from data. Notice that Δ_t is practically Δ_s for $C_s = 0$; nevertheless, in this approach stability of $K(\rho)$ is required and the closed-loop performance may be significantly limited for the strong conservatism. On the other hand, only one set of experimental data is required and no stabilizing controllers are needed.

The above rationale is derived in a noiseless setting. Several techniques for dealing with noisy data in spectral estimation are available in the literature, see *e.g.* Van Heusden et al. [2007].

Remark. As already said, the stability constraint describes a subset of all stabilizing controllers. In the two-experiment methodology, the centroid of such subset is exactly the optimal full-order controller $C = Q/(1 - G_0 Q)$, whereas in second and third cases, the centroids are respectively C_s and 0. It follows that it is generally better to adopt the double-experiment procedure, since it describes a set that contains controllers close to the optimal one. A critical situation is illustrated in Figure 4. Notwithstanding this limit, the possibility of saving one experiment may be a great advantage in a large variety of practical applications.

The direct data-driven algorithm can be then summarized in the following three points.

DIRECT DATA-DRIVEN ALGORITHM

(1) Choose a class of SISO LTI models

$$\mathcal{Q} = \left\{ Q(q^{-1}, \eta) \ , \ \eta \in \Pi \subset \mathcal{R}^{dim(\eta)} \right\}. \tag{10}$$

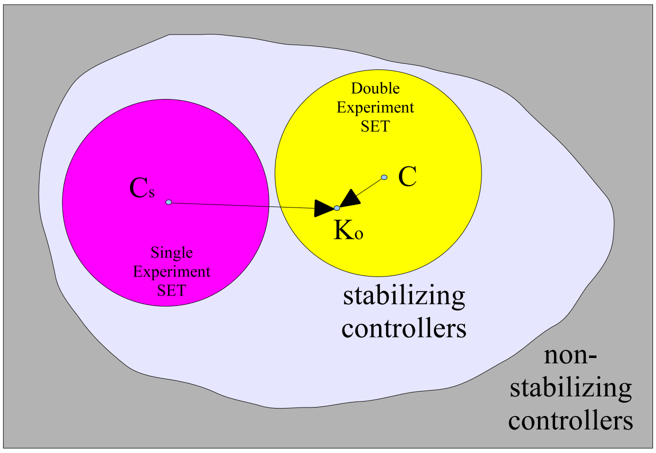

Fig. 4. In the example, if the double-experiment procedure is adopted, the reduced-order optimal controller K_o is included in the set of admissible controllers. Otherwise, K_o is surely substituted by another one, further from the full-order optimal controller C, and thus leading to worse performance.

(2) Identify a data-driven model of $Q(q^{-1})$ as $\hat{Q} = Q(q^{-1}, \hat{\eta})$, where

$$\hat{\eta} = \arg\min_{\eta} J_N(\eta) \tag{11}$$

and $J_N(\eta)$ can be any user-defined composition of sample-based estimates of $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of the weighted sensitivity functions.

(3) Identify the data-based reduced-order controller as $\hat{K} = K(q^{-1}, \hat{\rho})$, where

$$\hat{\rho} = \arg\min_{\rho} J_k^{N,l_k}(\rho)$$

and one of the two proposed stability constraints is statisfied.

4. COMPARISON

In this section, the new approach and the standard model-based algorithm will be discussed and compared from different points of view, in order to highlight advantages and disadvatages of the two methods.

4.1 Asymptotic results in case of correct parameterization

Define the optimal value for the Youla-Kučera parameter Q_0 and the optimal controller K_0 respectively as

$$Q_0 = \arg \min_{Q \in \mathcal{RH}_\infty} J \ , \ J = J(G_0, Q),$$
$$K_0 = \frac{Q_0}{1 - G_0 Q_0}.$$

Consider then the F.I.R. extension of the selected classes of models, *i.e.* write

$$G(\theta) = \sum_{i=0}^{n_G} \theta_i q^{-i} \ , \ Q(\eta) = \sum_{i=0}^{n_Q} \eta_i q^{-i}$$

By assuming to counteract the effect of noise with Instrumental Variable (IV) techniques (see Ljung and Ljung [1987]), the following asymptotic result holds.

Theorem 3. Assume that $G_0 \in \mathcal{G}$ in model-based procedure and $Q_0 \in \mathcal{Q}$ in direct data-driven case. Then:

(1) the reduced-order controllers guarantee the same asymptotical loop-shaping performance in model-based and in direct data-driven framework.
(2) if K_0 belongs to the class of considered controllers, the minimum of $J(\rho)$ in model-based case and the minimum of $J^N(\rho)$ in the data-driven algorithm coincide, as $N \rightarrow \infty$.

Proof. Let consider the model-based approach first. The F.I.R. estimate of G_0 is $G(\hat{\theta})$, where $\hat{\theta}$ is given by

$$\hat{\theta} = \left[\frac{1}{N} \sum_{t=1}^{N} \psi(t)\psi(t)^T \right]^{-1} \frac{1}{N} \sum_{t=1}^{N} \psi(t)y(t) \tag{12}$$

and $\psi(t) = [u(t), u(t-1), \ldots, u(t-n_G+1)]$. It is well known (see Ljung and Ljung [1987]) that (12) can be written as the sum of three different terms: the real value θ_0, a term due to undermodeling and a third addend depending on noise variance. Since prediction error techniques are used and $G_0 \in \mathcal{G}$, it holds that, asymptotically, $\hat{\theta} \rightarrow \theta_0$. Consequently, $Q \rightarrow Q_0$.
For what concerns the direct data-driven algorithm, the same reasoning can be applied. In few words, since $Q_0 \in \mathcal{Q}$ by hyphoteses, $Q \rightarrow Q_0$ as the number of data grows, if instrumental variables are adopted. This means that the Youla-Kučera parameter minimizing (1) is the same for both the approaches if large data-sets are used. Starting from the same expression for Q, the unique difference between (6) and (8) is the fact that (6) can be computed by means of noiseless simulated data, obtained by feeding $\hat{G}$ with $u(t)$, whereas (8) must be minimized using the set of input-output noisy data. However, the result shown in Theorem 2 assures that, as $N, l_k \rightarrow \infty$ and $l_k/N \rightarrow 0$, the minima of two cost functions coincide (thesis 1). The same result obviously holds if no order-reduction is required (thesis 2).

□

4.2 Discussion about undermodeling

Theorem 3 states that both the approaches are consistent if the right model-order is selected for G and Q. This is not the case in many real-world applications. As obvious, a complete theoretical analysis of the differences between the two approaches would be very complex in this setting, since optimization results are strictly related to the dynamic structure of the plant. Furthermore, in model-based approach, undermodeling of G weighs on Q and then the final value of (1) also depends on how the Youla-Kučera parameter is calculated. However, it may be shown by numerical simulations that sometimes the model-based approach yield worse performance. One of these situations is illustrated and discussed in Section 5.2.

Concerning undermodeling of the controller, a ticklish aspect is the conservativeness of the stability constraint. As explained in Section 2, in order to assure the internal stability for the system with G_0 the modeling error must be taken into account. Moreover, the constraint $\zeta'(\rho) < 1$ may affect closed-loop performance, if the optimal solution is close to the boundary defined by the real constraint. A simple situation where this fact may happen is presented in Section 5.3.
It must be mentioned that the formulation (3) is only one of possible solutions and that different results could be achieved in different situations. In any case, if internal stability for the real system has to be (asymptotically) guaranteed, a fair comparison between the methods must take into account a bound on the modeling error, by introducing more conservativeness in the model-based approach.

5. ILLUSTRATIVE EXAMPLES

Three numerical examples will be used to show, respectively, the effectiveness of the method, a comparison with the model-based approach in case of undermodeling and the different conservativeness degree of the two algorithms. All optimizations are implemented using Yalmip and Sdpt3, available online.

5.1 Example 1.

Let the mixed-sensitivity $\mathcal{H}_2$-loop-shaping problem be

$$\min_{\rho} J(\rho) = \min_{\rho} \|W_S S(\rho)\|_2^2 + \|W_T T(\rho)\|_2^2 + \|W_U U(\rho)\|_2^2$$

where the plant and the weighting functions are

$$G_0(q^{-1}) = \frac{0.9592q^{-1}(1 - 1.79q^{-1} + 0.8822q^{-2})}{(1 - 0.8546q^{-1})(1 - 1.867q^{-1} + 0.9274q^{-2})},$$

$$W_S = \frac{q^{-1} - 0.55q^{-2}}{1 - 0.95q^{-1}} \ , \ W_T = \frac{10q^{-1} - 9.5q^{-2}}{1 - 0.55q^{-1}} \ , \ W_U = 1.$$

Such system dynamics are very typical in speed-control problems of servomechanisms with elastic load. The most critical aspect for control design is represented by the couple resonance/anti-resonance, that in this case are low-damped.
A PI controller is used to control the plant, *i.e.* $\rho = [\rho_0, \ \rho_1]^T$, whereas Q is parameterized as a 50^{th}-order F.I.R. filter. The set of input-output data is collected by exciting the system with a maximum length PRBS. A white noise disturbance has been added to the output in order to get a Signal-to-Noise ratio (SNR) equal to 10. All signals are sampled at $2KHz$. Mixed-sensitivity performance obtained by direct controller identification is illustrated in Figure 5. Notice that performance is very good in terms of closed-loop bandwidth and attenuation of the control action; nevertheless, a resonant mode still affects the sensitivity functions. In order to reduce this effect, as in model-based approach, one possibility is to increase the controller order. Otherwise, the weighting functions can be suitably modified, *e.g.*:

$$W_S = \frac{q^{-1} - 0.6q^{-2}}{1 - 0.95q^{-1}} \ , \ W_T = \frac{10q^{-1} - 9.5q^{-2}}{1 - 0.6q^{-1}} \ , \ W_U = 1.$$

This selection obviously yields slower closed-loop responses, but it allows the designer to leave the resonant modes of the plant at higher frequencies. A comparison between step responses in the two cases is illustrated in Figure 6.

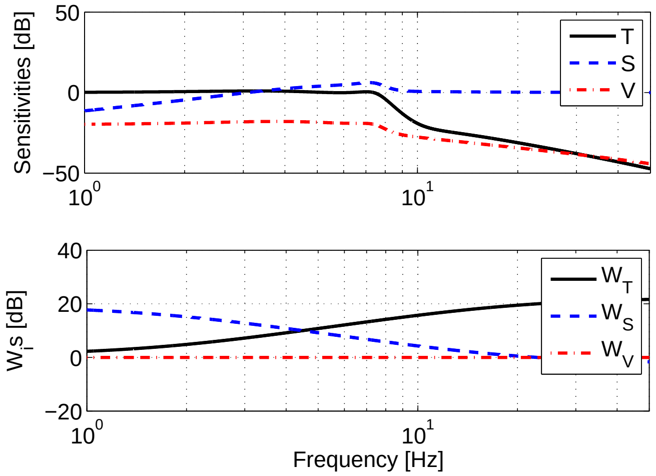

Fig. 5. Closed-loop sensitivity functions after direct data-driven synthesis (above) and magnitude diagrams of the adopted weighting functions (below).

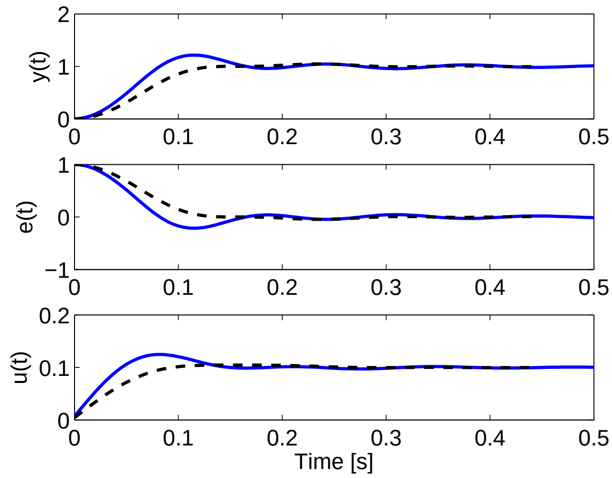

Fig. 6. Time-responses to step excitation of the reference signal: first choice (solid) and second choice (dashed) of the weighting functions.

5.2 Example 2.

Let consider the oscillator system

$$G_0(q^{-1}) = \frac{q^{-1}}{1 - 0.8911q^{-1}0.4705q^{-2}}$$

and the $\mathcal{H}_2$-loop-shaping problem defined in Example 1 where

$$W_S = \frac{q^{-1} - 0.2q^{-2}}{1 - 0.95q^{-1}}\ ,\ W_T = \frac{q^{-1} - 0.95q^{-2}}{1 - 0.2q^{-1}}\ ,\ W_U = 1.$$

If the same experiment as Example 1 is performed, the model-based and the direct data-driven procedures can be applied, by simply selecting the order of models G and Q and the controller structure. Figure 7 shows the normalized value of the cost function $J(\rho)$ for different orders of the model plant and of the Youla-Kučera parameter. In both the methods, it has been assumed that no controller reduction occurs. The better behaviour of the direct data-driven approach may be explained by noticing that the real plant is characterized by a pure autoregressive part. It follows that, inside the cost function, an F.I.R. structure for Q is sufficient to completely cancel all system dynamics, if the number of parameters is at least equal to the order of the system (recall that in sensitivity and complementary sensitivity expressions, G_0 is multiplied by the Youla-Kučera parameter). On the contrary, an F.I.R. approximation of the process is not accurate enough to correctly fit the frequency response of G_0. This error in the approximation of the process also influences the design of Q, that is consequently no more optimal for the $\mathcal{H}_2$-loop-shaping criterion.

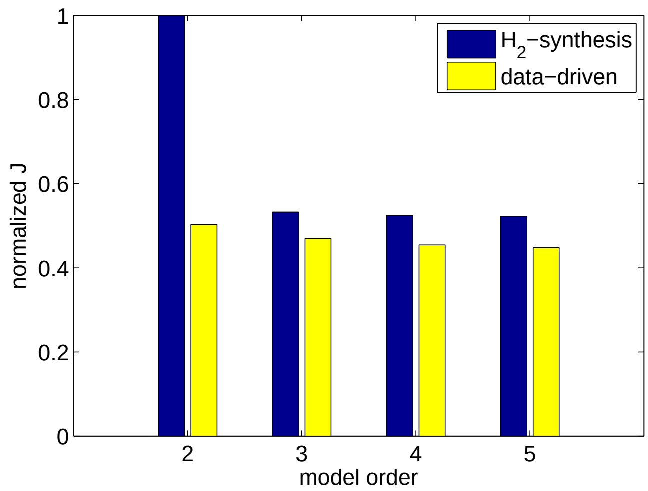

Fig. 7. Mixed-sensitivity cost for decreasing undermodeling degree.

5.3 Example 3.

The aim of the following example is to show that different conservativeness degrees of model-based and direct data-driven approach can strongly affect optimization performance even in very simple applications.
Let the system and the loop-shaping cost criterion be

$$G_0(q^{-1}) = \frac{1.2(1 - 0.87q^{-1})}{1 - 0.4q^{-1}},$$
$$J(\rho) = \|W_S S(\rho)\|_2^2 + \|W_U U(\rho)\|_2^2$$

where

$$W_S = \frac{1 - 0.4q^{-1}}{1 - 0.9q^{-1}}\ ,\ W_U = 0.5.$$

The controller $K(\rho) = \rho$ is proportional and both the adopted model structures for G and Q are characterized by only one parameter, *i.e.* $G(q^{-1}, \theta) = \theta q^{-1}$ and $Q(\eta) = \eta$. Assume that a data-set obtained via an open-loop experiment analogous to the one described in Exercise 1 is available.
The so-formulated problem is obviously affected by undermodeling in any approach. As a matter of fact, $G_0 \notin \mathcal{G}$ and the proportional Q is too simple to cancel the process dynamics, as it would happen in spectral-factorization-based computation of the Youla-Kučera parameter.
In model-based approach, the conservative stability criterion in (3) is adopted and, specifically, the numerical value

$\delta = 1.536$ of the modeling error is assumed to be known. In Figure 8, the $\mathcal{H}_2$-cost as function of ρ is shown. In

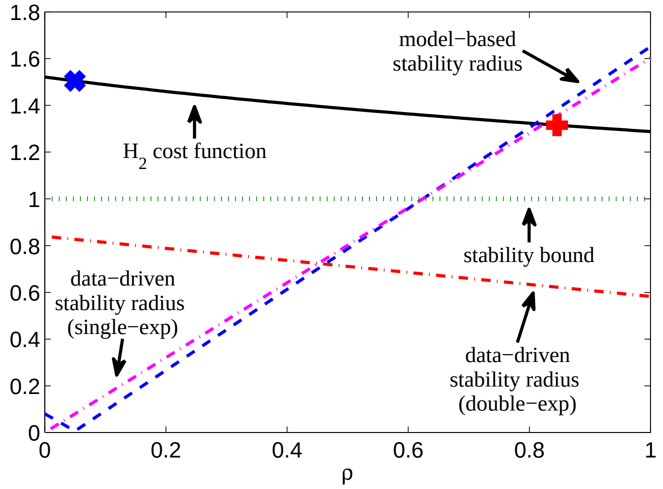

Fig. 8. Loop-shaping cost function and stability radii as functions of the proportional control action. The symbol (+) indicates the result of the direct data-driven synthesis with the double-experiment constraint, whereas the symbol (X) denotes the result of the model-based design.

the same diagram, the stability radii for both the design approaches are illustrated: the model-based radius (3) (dashed line), the direct data-driven constraint obtained with two experiments (light dash-dotted line) and the direct data-driven constraint deriving from a single experiment without knowledge of stabilizing controllers (dark dash-dotted line). Notice that the direct data-driven constraint obtained by means of the trivial stabilizing controller is more conservative than the one obtained with two different experiments.

In the data-driven approach, it holds that the minimizer $\hat{\rho} = 0.8457$ does not correspond with the stationary point of $J(\rho)$. This is due to the fact that the optimization procedure passes through the identification of a reduced-order Q. Notwithstanding this, $\hat{\rho}$ corresponds to $J_{min} = 1.315$ and such value is better than any possible result of the model-based procedure, in which ρ must belong to $[0, 0.63]$ and then J_{min} must be always greater than 1.357. It can be concluded that the model-based approach may yield worse performance for the different design procedures (see again Example 2), but also for the higher conservativeness degree due to the stability constraint.

6. CONCLUSIONS

A data-driven approach for controller design in mixed-sensitivity $\mathcal{H}_2$-$\mathcal{H}_\infty$ loop-shaping framework has been proposed. The method is based on convex optimization techniques and it is limited to stable plants. The main idea is to directly derive the Youla-Kučera parameter from a set of input-output samples and to perform a second identification step to identify a fixed-order controller from the same data-set. The algorithm does not require to identify the plant dynamics, but it still guarantees the same asymptotic performance of a standard model-based approach in case of correct parameterization. Internal stability of the closed-loop system with the resulting controller is asymptotically achieved by means of a convex $\mathcal{H}_\infty$-constraint. Furthermore, such stability constraints for the proposed technique are generally less conservative than the analogous one for the standard model-based approach. Some numerical examples show the effectiveness of the method and its potential in case of undermodeling. Nonetheless, no theoretical results are available for comparison in case of $G_0 \notin \mathcal{G}$ or $Q_0 \notin \mathcal{Q}$. Properties of the direct data-driven algorithm in these situations need to be investigated, together with the extension of the methodology to unstable plants.

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