

Can an odd number of fermions be created due to chiral anomaly?

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We describe a possibility of creation of an odd number of fractionally charged fermions in 1+1 dimensional Abelian Higgs model. We point out that for 1+1 dimensions this process does not violate any symmetries of the theory, nor makes it mathematically inconsistent. We construct the proper definition of the fermionic determinant in this model and underline its non-trivial features that are of importance for realistic 3+1 dimensional models with fermion number violation.

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I. INTRODUCTION

It is well known that many gauge theories with non-trivial topological structure allow for violation of fermion number N_F . A familiar example is just the Standard Model. The instanton processes in it lead to non-conservation of N_F by an even number, equal to four times the number (three) of fermionic generations. A model with $SU(2)$ gauge group and just one fermion in fundamental representation would predict, naïvely, the processes that change the vacuum topological number by one which would lead to creation of just one fermion. This type of process contradicts to quite a number of principles of quantum field theory, such as spin-statistics relation, Lorentz invariance, etc. A resolution of the paradox is known: this model turns out to be mathematically inconsistent, because of so called global Witten anomaly [1]. The Witten anomaly is connected with the topological fact that the fourth (four comes from the number of space-time dimensions) homotopy group $\pi^4(SU(2)) = \mathbb{Z}_2$ is non-trivial. This makes it impossible to define a measure in the functional integral over fermion fields in the models with an odd number of fermionic doublets. The anomaly disappears if the number of fermionic doublets is even, but then fermions are always created in pairs.

Clearly, the Witten consistency condition does depend on the dimensionality of space-time and may change if the number of dimensions is not equal to four. For example, in two dimensional Abelian gauge theories, the topological considerations are different. The corresponding homotopy group $\pi^2(U(1)) = 0$ is trivial and the fermionic measure can be defined properly¹. So one may expect existence of processes with one fermion creation in 1+1

dimensions.

This article is devoted to the demonstration that this effect really takes place in 1+1 dimensional models, specifically in an Abelian Higgs model with a chirally charged fermion of half integer charge. It will be shown that the creation of one fermion in 1+1 dimensions does not contradict neither to Lorentz symmetry, nor the calculation of the cross section of such a process leads to some unexpected cancellations.

There are generally two methods with which one can see that the processes with creation or decay of one fermion can take place. We will use both of them in this work. The first one is the analysis of fermion level crossing in the topologically nontrivial background [3, 4, 5, 6]. This picture is straightforward and very intuitive, but it does not allow (at least easily) for calculation of the probability or cross-section of the corresponding process.

The second method uses perturbation theory in instanton background. It was widely used in the calculation of baryon number violating processes [7, 8, 9, 10]. The exponent of the probability is easily obtained in this approach, but the preexponential factor is much harder to calculate. For the theories with chiral fermions it was estimated before only using dimensional considerations for part of the computation. The correct definition of the preexponential factor (or, equivalently, the fermionic determinant) is nontrivial. This was noted for example in [11, 12]. In this article we construct a consistent way to calculate the preexponent in theories with chiral fermions. It is important to note that the same problem also occurs in the usual 4-dimensional electroweak theory, where a similar procedure should be used to obtain the correct prefactor in the instanton transition probability.

The paper is organized as follows. In Section II we analyze the general properties of two-dimensional models, namely Lorentz transformation properties of the Greens' functions and absence of superselection rules and Witten like global anomalies. These properties differ from higher dimensional ones and lead to possibility of one fermion creation. Section III describes the model we study and its vacuum structure. We explain here the creation of one fermion using level crossing approach. Instanton calculation of the cross section is given in the Section IV. Con-

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¹ Strictly speaking Witten like anomaly can occur even in theories with trivial π_{d+1} homotopy group that allow one fermion creation, see [2]. The argument there is inapplicable to 1+1-dimensional case see discussion in the section II C.

clusions are presented in the Section V. In Appendices A–C we describe some technical details of computations.

II. LORENTZ INVARIANCE AND SUPERSELECTION RULES

A. Lorentz invariant one fermion Greens' functions

Usually, processes with an odd number of fermions participating in the reaction are automatically forbidden by Lorentz symmetry. Let us show that in 1+1 dimensions it is not the case, i.e. Lorentz invariant Greens' functions with one fermion can be non-trivial.

Two dimensional spinors transform under a Lorentz boost Λ with rapidity β in the following way,

$$\begin{aligned} \Psi(x) &\rightarrow \Psi'(x) = \Lambda_{\frac{1}{2}} \Psi(\Lambda^{-1}x) \\ &= e^{-\frac{\beta}{2}\gamma^5} \Psi(\Lambda^{-1}x) = \begin{pmatrix} e^{-\frac{\beta}{2}} \Psi_L(\Lambda^{-1}x) \\ e^{\frac{\beta}{2}} \Psi_R(\Lambda^{-1}x) \end{pmatrix}. \end{aligned} \quad (1)$$

Requirement of the Lorentz invariance of the simple Green's function with one fermion has the following form, supposing that the vacuum is Lorentz invariant

$$\begin{aligned} G(x; y) &= \langle 0 | \Psi(x) \phi(y) | 0 \rangle = \langle 0 | U^{-1}(\Lambda) \Psi(x) \phi(y) U(\Lambda) | 0 \rangle \\ &= \langle 0 | \Lambda_{\frac{1}{2}} \Psi(\Lambda^{-1}x) \phi(\Lambda^{-1}y) | 0 \rangle. \end{aligned}$$

Moving y to the coordinate origin, $y = 0$, we get for the left and right components the equations (writing space and time dependence explicitly)

$$\begin{aligned} G_L(x^0, x^1; 0, 0) &= e^{-\frac{\beta}{2}} G_L(x^0 \cosh \beta - x^1 \sinh \beta, \\ &\quad x^1 \cosh \beta - x^0 \sinh \beta; 0, 0), \\ G_R(x^0, x^1; 0, 0) &= e^{\frac{\beta}{2}} G_R(x^0 \cosh \beta - x^1 \sinh \beta, \\ &\quad x^1 \cosh \beta - x^0 \sinh \beta; 0, 0). \end{aligned}$$

These equations allow solution

$$\begin{aligned} G_{L,R}(x^0, x^1; 0, 0) &= \exp \left[\pm \frac{1}{2} \operatorname{atanh} \left(-\frac{x^0}{x^1} \right) \right] f_{L,R}(x_\mu x^\mu) \\ &= \sqrt[4]{\frac{x^0 \mp x^1}{x^0 \pm x^1}} f_{L,R}(x_\mu x^\mu) \end{aligned}$$

with arbitrary functions $f_{L,R}$.

Similar solutions can be found also for more complicated Greens' functions. So, in 1+1 dimensions, thanks to the simple form of Lorentz transformation (1), Greens' functions containing an odd number of fermion fields are not necessarily equal to zero.

B. Absence of superselection rules

We follow here the arguments given in [13]. In 3+1 dimensions a coherent superposition of states with even

$|even\rangle$ and odd $|odd\rangle$ numbers of fermions is incompatible with Lorentz invariance. More precisely, a state with an odd number of fermions is multiplied by (-1) under rotation of 2π of the coordinate system around any axis and under double application of time reversal. Then clearly superpositions of even and odd states would change under previously mentioned transformations which coincide with identity:

$$|even\rangle + |odd\rangle \xrightarrow{2\pi \text{ rotation}} |even\rangle - |odd\rangle.$$

In 1+1 dimensions the Lorentz group consists of a boost only. There is no rotation, and double application of time reversal does not give a factor (-1) . Indeed, time reversal in two dimensions is:

$$T = T_0 K \mathcal{T} = i\gamma^1 K \mathcal{T},$$

where the operator \mathcal{T} changes $t \rightarrow -t$, K performs the complex conjugate and $T_0 = i\gamma^1$ is a matrix in spinor space chosen so that the Dirac equation remains unchanged under time reversal. Note that $i\gamma^1$ is real and symmetric. Then

$$T^2 = i\gamma^1 K i\gamma^1 K = (i\gamma^1)^2 = \mathbb{1}.$$

Parity transformation can also be defined not to give factor (-1) after double application.

So there are no superselection rules contradicting with considering configurations with odd number of fermions in 1+1 dimensions.

C. Absence of Witten anomaly

As we already mentioned in the introduction, there is a global Witten anomaly in d -dimensional gauge theories with gauge group G and nontrivial $\pi_d(G)$. This is not the case for our model, because $\pi_2(U(1))$ is zero. But there is a rather simple argument by Goldstone, present in [2], that relates the existence of the global anomaly to the possibility of creation of odd number of fermions in the instanton processes (or to odd number of fermion zero modes in the instanton background). The argument is rather short and nice and we will present it here.

Let us suppose we have a gauge theory with an Yang–Mills instanton. Let us call π the gauge transformation associated with the instanton (which transforms between the vacua that are connected by the instanton), and Λ the corresponding operator acting on the quantum Hilbert space. The Gauss law requires that all gauge or coordinate transformations that can be connected continuously with identity leave the physical states invariant. Λ is not constrained by Gauss law, since π is a topologically nontrivial transformation, and is generally equal to $e^{-i\theta}$, where θ is some phase.

Now, if the instanton is associated with odd number of zero modes, we have $(-1)^F \Lambda (-1)^F = -\lambda$, where $(-1)^F$ counts the fermion number mod 2.

Let us now take an generator J of spatial rotations along some axis, and construct the operator

$$G_s = \pi^{-1} \exp(-isJ) \pi \exp(isJ) .$$

By construction $G_0 = 1$, therefore Gauss law predicts that all physical states $G_s|\text{physical}\rangle$ should be identical. However, $G_{2\pi} = \pi^{-1}(-a)^F \pi(-1)^F = -1$. This means that the Hilbert space does not exist, which is a synonym of a global anomaly [1].

However, in our case this argument fails because of absence of spatial rotations. This means that the two dimensional theories should be free of global anomalies, and this should be the only case free of global anomalies allowing one fermion creation.

III. THE MODEL AND LEVEL CROSSING DESCRIPTION

A. The Model

We are analyzing a chiral Abelian Higgs model in 1+1 dimensions with one fermion of a half charge. The Lagrangian of the model is

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} |D_\mu \phi|^2 + i \bar{\Psi} \gamma^\mu D_\mu \Psi \\ & - \frac{\lambda}{4} (|\phi|^2 - v^2)^2 \\ & + if \left(\bar{\Psi} \frac{1 + \gamma^5}{2} \Psi \phi^* - \bar{\Psi} \frac{1 - \gamma^5}{2} \Psi \phi \right) , \end{aligned} \quad (2)$$

where covariant derivatives are

$$D_\mu \phi = (\partial_\mu - ieA_\mu) \phi , \quad D_\mu \Psi = (\partial_\mu - i\frac{e}{2} \gamma^5 A_\mu) \Psi .$$

We use the two dimensional Dirac matrices representation

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

and Dirac conjugate spinor is $\bar{\Psi} = \Psi^\dagger \gamma^0$.

The charges of the left and right-handed components of the fermion $\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$ differ by a sign, $e_L = -e_R = \frac{e}{2}$. This model has been studied as a toy model for fermionic number non-conservation in electroweak theory in a number of papers, see, e.g. [14, 15, 16, 17, 18, 19, 20].

The particle spectrum consists of a Higgs field with mass $m_H = \sqrt{2}\lambda v$, a vector boson of mass $m_W = ev$, and a Dirac fermion acquiring a mass $F = fv$ via Higgs mechanism. The model is free of gauge anomaly. There is, however, a chiral anomaly leading to non-conservation of fermionic current,

$$J_\mu = J_\mu^L + J_\mu^R = \bar{\Psi} \gamma_\mu \Psi ,$$

with a divergence given by

$$\partial_\mu J_\mu = \partial_\mu J_\mu^L + \partial_\mu J_\mu^R = -q , \quad (3)$$

where $q = \frac{e}{4\pi} \varepsilon_{\mu\nu} F_{\mu\nu}$ is the winding number of the gauge fields configuration. This immediately leads to the conclusion that in topologically nontrivial backgrounds one can get creation of only one fermion.

The simplest description of fermion number violating processes in gauge theories is obtained from the analysis of the fermionic level structure in nontrivial external bosonic fields. First, we have to describe the level structure in different topological vacua, and then analyze the level crossing picture in gauge field background interpolating between vacua with topological numbers different by one.

To clarify the topological structure we will insert the system in a finite box of length L with periodic boundary conditions. At the end, the parameter L can be taken to infinity to recover the infinite space results.

B. Gauge transformations and fermion spectrum

Zero energy configurations of the gauge and Higgs fields are obtained by gauge transformations from the trivial vacuum state

$$\phi^{\text{vac}} = e^{i\alpha(x)} v , \quad A_\mu^{\text{vac}} = \frac{1}{e} \partial_\mu \alpha(x) .$$

These configurations will be called bosonic vacua. In infinite space, or in finite space with periodic boundary conditions for the bosonic fields, the configurations are divided into topological sectors, labeled by the topological number $n = \frac{1}{2\pi} (\alpha(\infty) - \alpha(-\infty))$.

Let us see what happens with fermions when we apply (large) gauge transformations changing the topological number of the vacuum. To leave the Lagrangian (2) invariant fermion fields should transform as

$$\Psi \rightarrow e^{i\alpha(x) \frac{\gamma^5}{2}} \Psi , \quad \bar{\Psi} \rightarrow \bar{\Psi} e^{i\alpha(x) \frac{\gamma^5}{2}} .$$

The fractional fermion charge leads here to some complications. For gauge transformations with odd n the transformation spoils the boundary conditions for the fermion wave function Ψ . So, at least in finite size system, fermion spectra in bosonic vacua with even and odd topological numbers are different. As a result, the energies of the lowest states with odd and even topological numbers are different as well. In other words, the bosonic vacuum states with even n have higher energy than the states with odd n (see Appendix A) and therefore are not the true vacua of the theory² (see Fig. 1).

Let us analyze this feature in more detail.

The fermionic equation of motion is:

$$[i\partial_0 - H_D] \Psi = 0 ,$$

² This difference disappears in the limit of infinite space, see Appendixes A and B.

with Dirac Hamiltonian

$$H_D = \begin{pmatrix} -i\partial_1 - \frac{\epsilon}{2}A_1 & f\phi \\ f\phi^* & i\partial_1 - \frac{\epsilon}{2}A_1 \end{pmatrix}.$$

In trivial background ($A_\mu = 0$, $\phi = v$) in a box of size L with periodic boundary conditions positive and negative energy fermionic solutions have the form

$$\begin{aligned} \Psi_+ &= e^{-iE_l t} \begin{pmatrix} e^{i\frac{2\pi l}{L}x} F \\ e^{i\frac{2\pi l}{L}x} (E_l - k_l) \end{pmatrix}, \\ \Psi_- &= e^{iE_l t} \begin{pmatrix} e^{i\frac{2\pi l}{L}x} (E_l - k_l) \\ -e^{i\frac{2\pi l}{L}x} F \end{pmatrix}, \end{aligned} \quad (4)$$

where momentum and energy are

$$k_l = \frac{2\pi l}{L}, \quad l \in \mathbf{Z}, \quad E_l = \sqrt{F^2 + k_l^2}. \quad (5)$$

Note that for all nonzero momenta there are two degenerate states with equal energy, corresponding to left and right moving particles (and right and left moving antiparticles with negative energy). The state with $k = 0$, $E = F$ is not degenerate.

In the case of $n = 1$ bosonic vacuum (with $A_1 = \frac{2\pi}{eL}$, $A_0 = 0$, and $\phi = ve^{i\frac{2\pi x}{L}}$) and periodic boundary conditions³ we get

$$\begin{aligned} \Psi_+ &= e^{-iE_l t} \begin{pmatrix} -e^{i\frac{2\pi l}{L}x} F \\ e^{i\frac{2\pi(l-1)}{L}x} (E_l - k_l) \end{pmatrix}, \\ \Psi_- &= e^{iE_l t} \begin{pmatrix} e^{i\frac{2\pi l}{L}x} (E_l - k_l) \\ e^{i\frac{2\pi(l-1)}{L}x} F \end{pmatrix}, \end{aligned} \quad (6)$$

with momenta and energy

$$k_l = \frac{2\pi(l - \frac{1}{2})}{L}, \quad l \in \mathbf{Z}, \quad E_l = \sqrt{F^2 + k_l^2}. \quad (7)$$

There is no state with $k = 0$ in this case, and all states are doubly degenerate in energy.

We see, that the fermion spectra in bosonic vacua with even and odd topological numbers are indeed different. So, in case of finite space size, a gauge transformation with odd n leads to physical changes in the system. We thus should say that the only allowed gauge transformations (i.e. those that connect physically indistinguishable field configurations) have even $n = \frac{1}{2\pi}(\alpha(L) - \alpha(0))$. Transitions between states with bosonic background being vacuum configurations with $n = 0$ and $n = 1$ are still possible, but they are just tunneling between different (local) minima of the energy of the system (see Fig. 1).

In the limit of infinite space ($L \rightarrow \infty$), however, the difference between energy levels disappears. The total

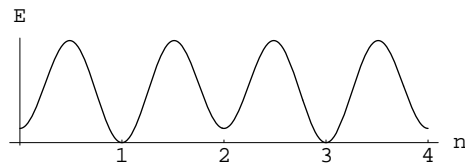


FIG. 1: Picture of the fermionic energy in different bosonic configurations. Bosonic vacua with odd n have a slightly different energy

vacuum energy (or Dirac sea energy) also turns out to be equal in both $n = 0$ and $n = 1$ backgrounds in infinite space limit, see Appendix A. Calculation of the fermion number of the Dirac sea in these backgrounds, performed in Appendix B, gives zero in both backgrounds. In the limit of infinite space transitions from $n = 0$ to $n = 1$ are again vacuum to vacuum transitions, while the vacua are not exactly gauge equivalent, but rather simply degenerate.

C. Level crossing picture

Let us analyze a process in external gauge and Higgs fields interpolating between adjacent bosonic vacua, for example

$$\begin{aligned} \phi^{cl}(x, \tau) &= \frac{v}{\sqrt{2}} e^{-\frac{2\pi i x \tau}{L}} [\cos(\pi\tau) \\ &\quad + i \sin(\pi\tau) \tanh(m_H x \sin(\pi\tau))], \\ A_1^{cl}(x, \tau) &= -\frac{2\pi\tau}{eL}, \end{aligned} \quad (8)$$

with parameter $0 < \tau < 1$. This configuration goes from the vacuum $n = 0$ at $\tau = 0$ to $n = -1$ at $\tau = 1$ minimizing the energy of the intermediate configurations [15]. For each value of the parameter τ we solved numerically the static Dirac equation $H_{D,\tau}\Psi_\tau = E_\tau\Psi_\tau$. Evolution of the energy levels is presented in Figure 2. Exactly one level (level with negative energy with $l = 0$ in (4)) crosses zero. Together with the positive energy level with $l = 0$ they merge into the two degenerate energy states with $l = 0$ and $l = 1$ in $n = -1$ vacua (see (6), or, to be more precise, they go to linear combinations of the $l = 0$ and $l = 1$ states in (6)).

So exactly one fermion should be created in a process with gauge fields interpolating between $n = 0$ and $n = -1$ bosonic vacua.

IV. INSTANTON CALCULATION OF THE CROSS SECTIONS

The level crossing picture described in the previous section does not allow to calculate the probabilities of real processes of one fermion creation (or decay) at low

³ Alternatively one could use the equations in trivial background and impose anti-periodic boundary conditions.

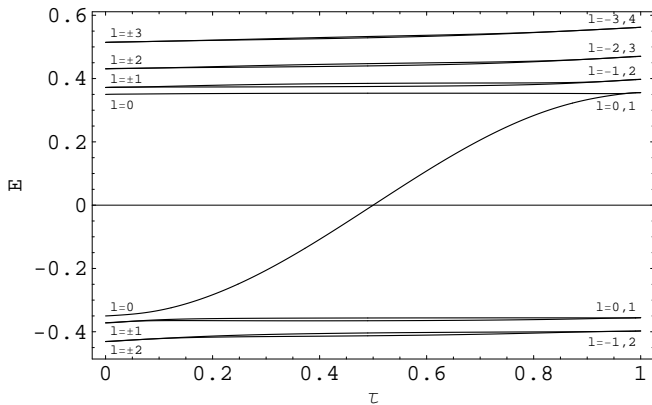


FIG. 2: Fermionic energy levels in the background (8) obtained numerically for finite space of length $L = 50$ and periodic boundary conditions. Fermion mass $F = 0.35$, the charge $e = 1$.

energies. Convenient method for the calculation of such probabilities is given by perturbation theory in the instanton background [7, 8, 9, 10, 21].

The usual prescription is to calculate the Euclidean Greens' functions in instanton background and then apply the LSZ reduction procedure to get matrix elements. The fermionic part of the Green's function contains the fermionic determinant in the instanton background calculated without the zero mode. However, the determinant of the Dirac operator K for a chiral fermion in nontrivial background is hard to define. The operator K itself maps from a Hilbert space to another and its determinant is not defined. The usual trick is to use instead $K^\dagger K$ or KK^\dagger . However in non-trivial background, these two operators do not contain the same number of zero modes. Their determinants, after removing the relevant zero-mode still differ by a constant.

This problem seems to be connected with the fact that usual normalization is performed by division by the vacuum partition function⁴ while the Hilbert spaces for fermionic wave functions are not exactly the same in trivial and one instanton backgrounds [11]. We have to emphasize that this subtlety is not a feature of 1+1-dimensional models but is present in the Standard Model also. In existing calculations of chiral fermion contribution to the instanton transitions, the corresponding normalization was defined using dimensional arguments only [7, 22]. We propose the definition of the required determinant using sort of a valley approximation for the path integral.

In this section we describe the whole procedure in detail. In subsection IV A we describe the instanton solution and the zero modes. Subsection IV B is devoted to

the naïve definition of the Euclidean Greens' functions (and the fermionic determinant) which leads to an inconsistent result. In the subsection IV C we describe a careful definition of the fermionic determinant that resolves the problem. In the last subsection IV D the LSZ reduction formula is used to get matrix elements.

A. Instanton solution and fermionic zero modes

Let us first review the Euclidean formulation of the model we use. It is described in more detail in Ref. [23]

The Lagrangian (2) may be rewritten in Euclidean space:

$$\mathcal{L}^E = \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}(D_\mu\phi)^\dagger(D_\mu\phi) + V(\phi) + \bar{\Psi}K\Psi, \quad (9)$$

with

$$K = i\gamma_\mu^E D_\mu - if \left(\frac{1 + \gamma^5}{2} \phi^* - \frac{1 - \gamma^5}{2} \phi \right), \quad (10)$$

and $\gamma_0^E = i\gamma^0$, $\gamma_1^E = \gamma^1$. The fields $\bar{\Psi}$ and Ψ are independent variables in Euclidean case, and the gauge transformation reads:

$$\begin{aligned} \Psi &\longrightarrow e^{i\alpha(x)\frac{\tau_3}{2}}\Psi, & \bar{\Psi} &\longrightarrow \bar{\Psi}e^{i\alpha(x)\frac{\tau_3}{2}}, \\ \phi &\longrightarrow e^{i\alpha(x)}\phi. \end{aligned} \quad (11)$$

For comparison, the Lorentz transformation is:

$$\begin{aligned} \Psi(x) &\rightarrow \Psi'(x') = \Lambda_s \Psi(\Lambda^{-1}x'), \\ \bar{\Psi}(x) &\rightarrow \bar{\Psi}'(x') = \bar{\Psi}\Lambda_s^{-1}(\Lambda^{-1}x'), \end{aligned}$$

with $\Lambda_s = \exp(i\gamma^5\frac{\theta}{2})$ being the spinor rotation matrix in two dimensions.

a. Instanton solution. The instanton describing the tunneling between the states $|0\rangle$ and $|n\rangle$ is simply the Nielsen–Olesen vortex with winding number n [24], which is a solution of the Euclidean equations of motion derived from the Lagrangian (9). It is obtained by using the following Ansatz, which is the most general Ansatz consistent with symmetry under spatial rotations accompanied by the corresponding gauge transformations,

$$\phi(r, \theta) = e^{in\theta}\phi(r) \equiv e^{in\theta}vf(r), \quad (12)$$

$$A^i(r, \theta) = \varepsilon^{ij}\hat{r}^j A(r), \quad (13)$$

where $\hat{r} = (\cos\theta, \sin\theta)$ is the unit vector and ε^{ij} is the completely antisymmetric tensor with $\varepsilon^{01} = 1$. The functions A and f have to satisfy the following limits:

$$\begin{aligned} f(r) &\xrightarrow{r\rightarrow 0} cr^{|n|}, & f(r) &\xrightarrow{r\rightarrow\infty} 1, \\ A(r) &\xrightarrow{r\rightarrow 0} 0, & A(r) &\xrightarrow{r\rightarrow\infty} -\frac{n}{er}. \end{aligned}$$

The number ΔN of fermions created in the instanton transition can be computed by integrating (3) over the Euclidean space

$$\Delta N = - \int d^2x \partial_\mu J_\mu = - \int d^2x \frac{e}{4\pi} \varepsilon_{\mu\nu} F_{\mu\nu} = -q,$$

⁴ More precisely by the determinant of the Dirac operator in the trivial background.

where $q = \int d^2x \frac{e}{4\pi} \varepsilon_{\mu\nu} F_{\mu\nu}$ is the winding number of the gauge field configuration. For the instanton configuration (12,13), we have $q = n$.

At large r both approach their asymptotic exponentially

$$f(r) \xrightarrow{r \rightarrow \infty} 1 - f_0 \sqrt{\frac{\pi}{2r}} e^{-mr},$$

$$A(r) \xrightarrow{r \rightarrow \infty} -\frac{n}{er} + \frac{a_0}{e} \sqrt{\frac{\pi}{2r}} e^{-mwr}.$$

Later we will also use this solution in unitary gauge, i.e. gauge where $\phi(r, \theta) \xrightarrow{r \rightarrow \infty} v$ for all directions. The solution in this gauge is singular at the origin, but the singularity is a gauge artefact. Also, for odd n , the fermion zero mode (see next paragraph) is not a single valued function in unitary gauge. However, one may also think of configuration in unitary gauge as a limit of the configuration (12,13) transformed with the gauge function

$$\alpha(\theta) = -n(\theta - 2\pi\Theta^\epsilon(\theta - \pi)),$$

where Θ^ϵ is a function approaching the step function for vanishing ϵ .

b. Fermionic zero modes. According to the index theorem (see for example [25]), the Dirac operator in the background of the instanton satisfies the following relation: $\dim \ker[K] - \dim \ker[K^\dagger] = n$. As the instanton in 1 + 1 dimensions coincides with the vortex, these zero modes may be found by carrying out a similar analysis as in [26]; where the fermionic zero modes on the Nielsen–Olesen string were analyzed for non chiral fermions. In this subsection we present the corresponding equations.

The zero modes are the regular normalizable solutions of the equation $K\Psi = 0$, with A_μ and ϕ given by (12,13). Using spherical mode expansion of the form $\Psi(r, \theta) = \exp\left[-\int_0^r \frac{A(\rho)}{2} d\rho\right] \sum_{m=-\infty}^{\infty} e^{im\theta} \psi^m(r)$ we get

$$Ff(r)\psi_L^m - \left(\frac{\partial}{\partial r} - \frac{m-n-1}{r}\right)\psi_R^{m-n-1} = 0,$$

$$\left(\frac{\partial}{\partial r} + \frac{m}{r}\right)\psi_L^m - Ff(r)\psi_R^{m-n-1} = 0,$$

where $F = fv$ is the fermion mass. We also continue to use indices L and R to denote two components of the spinor, though they are no longer left and right moving in Euclid. In our case, the analysis of [26] shows that for a vortex with topological number $n < 0$ there are exactly $|n|$ fermionic zero modes in the spectrum of K with m in the interval $m \in \{-n+1, \dots, 1, 0\}$ and none in the spectrum of K^\dagger . For $n > 0$ there are no zero modes in the spectrum of K , but n in the spectrum of K^\dagger .

For the case of $n = -1$ studied in [23] the explicit form

of the zero mode is given by

$$\begin{aligned} \Psi_L^0(r) &= -\Psi_R^0(r) \\ &= \text{const} \cdot \exp\left(-\int_0^r \left\{Ff(r') + \frac{e}{2}A(r')\right\} dr'\right) \\ &\xrightarrow{r \rightarrow \infty} U_0 \frac{e^{-Fr}}{\sqrt{r}}. \end{aligned}$$

Note that for massless fermions ($F = 0$), the zero mode decreases as $\frac{1}{\sqrt{r}}$ for large r . It is therefore not normalizable and has a divergent action.

B. Euclidean Greens functions

Let us start from evaluating the generating functional for fermionic Euclidean Green's functions. We will not write here the source terms for bosonic fields explicitly because there is no problem of dealing with the bosonic part here, see, eg. [14]).

$$Z[\bar{\eta}, \eta] = \frac{1}{Z_0} \int \mathcal{D}A_\mu \mathcal{D}\phi e^{-S_{\text{bosonic}} Z_{A,\phi}[\bar{\eta}, \eta]},$$

$$Z_{A,\phi}[\bar{\eta}, \eta] = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{-\int d^2x (\bar{\Psi} K \Psi - \bar{\eta} \Psi - \bar{\Psi} \eta)}. \quad (14)$$

where Z_0 is the same functional integral with zero source terms. At one-loop level the fermionic part of the generating functional can be calculated regarding the bosonic fields A_μ, ϕ as external classical sources, both in the generating functional itself and in the normalization factor Z_0 , which then factorizes in bosonic and fermionic parts.

Let us try to evaluate the fermionic part $Z_{A,\phi}[\bar{\eta}, \eta]/Z_0$. As far as it is just a Gauss integral over Grassman variables we can (at least formally) perform it exactly. To define it we proceed in the spirit of Ref. [8, 9].

Let us start with the trivial background case first. We define the following eigenvalues and eigenvectors

$$K_0^\dagger K_0 \rho_n = \kappa_n^2 \rho_n, \quad K_0 K_0^\dagger \tilde{\rho}_n = \kappa_n^2 \tilde{\rho}_n, \quad (15)$$

where K_0 is the Dirac operator (10) in zero background, and the eigenvectors $\tilde{\rho}$ and ρ are normalized to 1 and connected with the formula

$$\tilde{\rho}_n = \frac{1}{\kappa_n} K_0 \rho_n. \quad (16)$$

Several notes are required here. First, the operators $K_0 K_0^\dagger$ and $K_0^\dagger K_0$ are self conjugate, and thus the sets ρ_n and $\tilde{\rho}_n$ form full orthonormal sets of functions. Second, we are not trying to use operators K (or K^\dagger) to define the eigenfunctions because they map from the space of spinors Ψ to a space with different gauge transformation properties (see (11)). And finally, as far as the background is now just the trivial vacuum, all $\kappa_n \neq 0$, so the relation (16) holds for all n . Also, by convention, we choose all $\kappa_n > 0$.

Now we expand fermionic fields using these eigenmodes

$$\Psi = \sum_n a_n \rho_n, \quad \bar{\Psi} = \sum_n \bar{a}_n \bar{\rho}_n^\dagger$$

and define the functional integral measure as

$$D\Psi D\bar{\Psi} = \prod_n da_n d\bar{a}_n.$$

Then the integration immediately leads to

$$\begin{aligned} Z_0 &= \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp \left[- \int d^2x \bar{\Psi} K_0 \Psi \right] \\ &= \int \prod_n da_n d\bar{a}_n \exp \left[- \sum_n \kappa_n \bar{a}_n a_n \right] = \prod_n \kappa_n. \end{aligned}$$

Analogous procedure we should also apply in the nontrivial background. We find the eigenvalues of the two following equations

$$K^\dagger K \psi_n = \lambda_n^2 \psi_n, \quad K K^\dagger \tilde{\psi}_n = \lambda_n^2 \tilde{\psi}_n, \quad (17)$$

with relation similar to (16) for all $\lambda_n \neq 0$

$$\tilde{\psi}_n = \frac{1}{\lambda_n} K \psi_n. \quad (18)$$

In nontrivial background there may also exist zero eigenvalues, and K is no longer a normal operator⁵, so there may be different number of zero eigenvalues for $K^\dagger K$ and $K K^\dagger$. The index theorem says that $\dim \text{Ker } K^\dagger K - \dim \text{Ker } K K^\dagger = n$, so in one instanton case there should be one more zero mode for $K^\dagger K$ (and it is the only zero mode present). For zero modes there is no relation of the type (18), and we simply define them as

$$K^\dagger K \psi_k^0 = 0, \quad K K^\dagger \tilde{\psi}_l^0 = 0, \quad \text{with } \int |\psi^0|^2 d^2x < \infty.$$

Now we re-expand fermionic fields in terms of the new orthonormal sets $\psi_K = \{\psi_k^0, \psi_n\}$ and $\tilde{\psi}_L = \{\tilde{\psi}_l^0, \tilde{\psi}_n\}$

$$\Psi = \sum_k c_k \psi_k^0 + \sum_n b_n \psi_n, \quad \bar{\Psi} = \sum_l \bar{c}_l \tilde{\psi}_l^{0\dagger} + \sum_n \bar{b}_n \tilde{\psi}_n^\dagger.$$

One should now take care when defining the integration measure, to be consistent with (15)

$$D\Psi D\bar{\Psi} = P \prod_k dc_k \prod_l d\bar{c}_l \prod_n db_n d\bar{b}_n,$$

where P is the Jacobian for the change of variables $\{a_n, \bar{a}_n\} \rightarrow \{c_k, b_n, \bar{c}_l, \bar{b}_n\}$

$$P[A, \phi] = \det[(\rho_n, \psi_k)]^{-1} \det[(\tilde{\psi}_l, \tilde{\rho}_n)]^{-1},$$

⁵ Normal operator is an operator A with the property $A^\dagger A = A A^\dagger$.

where $(\alpha, \beta) = \int dx \bar{\alpha}(x) \beta(x)$ denotes scalar product for spinor functions. Absolute value of P is one, because it corresponds to transition between full orthonormal sets of functions, so it is only a complex phase, which, in general, depends on the background fields A_μ, ϕ . As noted in [8, 9] it is essential to take this phase into account to reconstruct correct perturbative expansion for the theory. In our case, in the leading one-loop approximation this is not important, because there are no instanton orientation to be integrated over—instanton field configurations differ only by translations and gauge transformation. Note that for example in four dimensional nonabelian theory this is not the case.

Performing Gaussian integration over $dc_k d\bar{c}_l db_n d\bar{b}_n$ in (14) we get

$$\begin{aligned} Z_{A, \phi}[\bar{\eta}, \eta] &= P[A, \phi] \times \prod_n \left(\lambda_n + (\bar{\eta}^\dagger, \psi_n) (\tilde{\psi}_n, \eta) \right) \\ &\quad \times \prod_k (\bar{\eta}^\dagger, \psi_k^0) \times \prod_l (\tilde{\psi}_l^0, \eta). \end{aligned} \quad (19)$$

This formula leads to the standard result that nonzero Greens' functions must contain in addition to usual even number of fermionic legs a set of fermionic operators of a special structure, defined by fermionic zero modes. In the instanton case we have only one zero mode, and the simplest nonzero Green's function is given formally by the following expression

$$\begin{aligned} \left[\frac{1}{Z_0} \frac{\delta Z_{A, \phi}[\eta, \bar{\eta}]}{\delta \bar{\eta}} \right] \Big|_{\eta, \bar{\eta}=0} &= \left[\frac{\prod_{n \neq 0} \lambda_n}{\prod_n \kappa_n} \right] \times P[A, \phi] \times \psi_0 \\ &\equiv \sqrt{\det_{\text{ren}}[K_I^\dagger K_I]} \times P[A, \phi] \times \psi_0. \end{aligned} \quad (20)$$

It is easy to see that this quantity is ill defined. The left hand part of the equality has dimension $m^{1/2}$. In the right hand part of the expression ψ_0 has dimension m (as it is normalized to one), P is dimensionless. Thus, the dimension of the infinite product should be $m^{-1/2}$, and not m^{-1} , as could be expected naively.

C. Determinant definition

Let us try to clarify the definition of the determinant. The problem with the description in the previous section is that, strictly speaking, the eigensystems in (15) and in (17) generally belong to different Hilbert spaces—fermions living in trivial and one instanton backgrounds. One may hope that the situation can be cured if one calculates a quantity in a trivial background. A good candidate is the expectation value for two fermion operators in external instanton–antiinstanton background

$$\begin{aligned} \langle 0 | \bar{\Psi}(T) \Psi(-T) | 0 \rangle_{I-A} &= \\ &= \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp \left[- \int d^2x (\bar{\Psi} K_{I-A} \Psi) \right] \bar{\Psi}(T) \Psi(-T), \end{aligned} \quad (21)$$

where index $I - A$ means that everything is calculated in the instanton–antiinstanton background, with instanton and antiinstanton centered at Euclidean time t_0 and $-t_0$ respectively. Just by construction for large t_0 this reproduces the modulus squared of the one fermion expectation value in instanton background

$$\langle 0 | \bar{\Psi}(t_0 + T) \Psi(-t_0 - T) | 0 \rangle_{I-A} \rightarrow | \langle \Psi(-T) \rangle_I |^2 \quad \text{for } t_0 \rightarrow \infty. \quad (22)$$

Let us now calculate this integral using the method described in Section IV B. We get the eigensystems of the form

$$\begin{aligned} K_{I-A}^\dagger K_{I-A} \Psi_N &= \Lambda_N^2 \Psi_N, \\ K_{I-A} K_{I-A}^\dagger \tilde{\Psi}_N &= \Lambda_N^2 \tilde{\Psi}_N, \end{aligned} \quad (23)$$

where now there are no exact zero modes for both operators, so all eigenfunctions are related by a relation of the form (18). However, we can immediately construct an approximate eigensystem for (23)

$$\begin{aligned} \Lambda_N &= \{ \lambda_n^I; \lambda_n^A; \Lambda_0 \}, \\ \Psi_N &= \{ \psi_n^I(t - t_0); \psi_n^A(t + t_0); \psi_0^I(t - t_0) \}, \\ \tilde{\Psi}_N &= \{ \tilde{\psi}_n^I(t - t_0); \tilde{\psi}_n^A(t + t_0); \tilde{\psi}_0^A(t + t_0) \}, \end{aligned}$$

where Λ_0 is small and goes to zero as $t_0 \rightarrow \infty$. So there are two sets of modes, corresponding to nonzero eigenmodes of the instanton and antiinstanton centered at their locations, and one nearly zero mode Λ_0 , which is constructed out of a zero mode for instanton for Ψ and for antiinstanton for $\tilde{\Psi}$.

It is now trivial to calculate (21) using (19) and differentiating it by $\delta\eta\delta\bar{\eta}$

$$\langle 0 | \bar{\Psi}(T) \Psi(-T) | 0 \rangle_{I-A} = \frac{1}{Z_0} \left(\prod_N \Lambda_N \right) \sum_N \frac{\Psi_N(-T) \bar{\Psi}_N(T)}{\Lambda_N}.$$

The sum is governed by the term with Λ_0 , so we get

$$\langle 0 | \bar{\Psi}(T) \Psi(-T) | 0 \rangle_{I-A} = \frac{(\prod_n \lambda_n^I) (\prod_n \lambda_n^A)}{(\prod_n \kappa_n) (\prod_n \tilde{\kappa}_n)} \Psi_0(-T) \bar{\Psi}_0(T) \quad (24)$$

(no zero mode is present in $\prod_n \lambda_n^I$). It is easy to see, comparing formulas (20), (24) and (22) that

$$\begin{aligned} \langle \Psi(-T) \rangle_I &= \sqrt[4]{\frac{\det'[K_I^\dagger K_I] \det[K_A^\dagger K_A]}{\det[K_0^\dagger K_0] \det[K_0^\dagger K_0]}} \psi_0^I(-T) \\ &\equiv \sqrt{\det_{\text{ren}}[K_I^\dagger K_I]} \times \psi_0, \end{aligned} \quad (25)$$

up to some complex phase, in principle. Calculation and renormalization of the determinant $\det'[K_I^\dagger K_I]$ is described in detail in Ref. [23] and additional subtleties for calculation of the antiinstanton determinant, which has no zero mode, is given in Appendix C. We can then

use (25) as the correct definition of the renormalized determinant in the one instanton background. The dimension of the ratio $\frac{\det'[K_I^\dagger K_I]}{\det[K_0^\dagger K_0]}$ is m^{-2} (zero mode is absent in the numerator), $\frac{\det'[K_A^\dagger K_A]}{\det[K_0^\dagger K_0]}$ has dimension zero (no zero mode here), and ψ_0 is m because of normalization. This whole expression has dimension $m^{1/2}$, which is now correct.

D. Reduction formula

A convenient method to get physical amplitudes from the Greens' functions is provided by LSZ reduction procedure. There is one subtlety in application of the reduction formula in the instanton case, as compared to usually considered topologically trivial situations. The reduction formula is derived using the assumption that field operators are connected with creation-annihilation operators of the physical particles in the same canonical way for all times (both initial and final). For instanton like configurations this is true only in unitary gauge, which is singular at the origin. However, this singularity is of purely gauge type and does not contribute to the poles of the Green's function, so it is safe to use it. At the same time other gauge choices may lead to appearance of nonphysical singularities in the Green's function.

We start from the Euclidean Green's function, calculated in the saddle point approximation

$$\begin{aligned} \langle \Psi(x) h(y_1) \dots h(y_m) \rangle_{\text{inst}} &= \\ &\int d^2 x_0 J(\langle \phi \rangle) \det[K_{\text{scalar}}]^{-1/2} \sqrt{\det_{\text{ren}}[K_I^\dagger K_I]} e^{-S_{\text{inst}}} \\ &\times \psi_0(x - x_0) h_{\text{inst}}(y_1 - x_0) \dots h_{\text{inst}}(y_m - x_0), \end{aligned}$$

where $\det[K_{\text{scalar}}]$ is the determinant of the bosonic field quadratic excitations over the instanton background, see eg. [14], $J(\langle \phi \rangle)$ is the Jacobian appearing from the transition to the integration over the collective coordinate x_0 —instanton center, $\det_{\text{ren}}[K_I^\dagger K_I]$ is the fermionic determinant defined in the previous subsection, ψ_0 is the fermionic zero mode, and $h_{\text{inst}} = \phi_{\text{inst}} - \phi_v$ is the instanton solution for the deviation of the scalar field from vacuum value. In complete analogy it is possible to add gauge fields here. Also pairs of fermion fields can be added, connected with fermion propagator in instanton background.

The meaning of integration over the position of the instanton is clear after going to the momentum representation, where it leads to the energy-momentum conservation

$$\begin{aligned} (2\pi)^2 \delta^2(p + k_1 + \dots + k_m) \tilde{G}(p, \{q\}) &= \\ &\int d^2 x d^2 y_1 \dots d^2 y_m e^{ipx} e^{ik_1 y_1} \dots e^{ik_m y_m} \\ &\times \langle \Psi(x) h(y_1) \dots h(y_m) \rangle_{\text{inst}}. \end{aligned}$$

Using these formulas we get for the Green's function in momentum representation

$$\tilde{G}(p, \{q\}) = J(\langle\phi\rangle) \det[K_{\text{scalar}}]^{-1/2} \left(\det_{\text{ren}}[K_I^\dagger K_I] \right)^{1/2} \times e^{-S_{\text{inst}}} \times \psi_0(p) h_{\text{inst}}(k_1) \dots h_{\text{inst}}(k_m), \quad (26)$$

where $\psi_0(p)$, $h_{\text{inst}}(k)$ are the Fourier transforms of the zero mode and the instanton respectively,

$$\psi_0(p) = \int d^2x e^{ipx} \psi_0(x),$$

etc.

c. Fourier transforms. Let us now calculate Fourier transforms appearing in (26). To get the matrix elements we will be interested only in the pole terms at the physical mass, so we can analyze only infinite contributions from the exponential tails of the solutions.

The instanton solution for the scalar field is (see. [23])

$$h_{\text{inst}}(x) = v(1 - f(r)) \simeq v f_0 K_0(m_H r),$$

where the constant f_0 is determined from the asymptotics of the exact solution $1 - f(r)$ at large r (r is the distance from the instanton origin in Euclid). Thus we get

$$h_{\text{inst}}(k) = \int d^2x e^{ikx} h(x) = -\frac{2\pi f_0 v}{m_H^2 + k^2} + \text{regular terms}.$$

For the fermion zero mode we have

$$\psi_0(x) = \begin{pmatrix} \psi_{0L} \\ \psi_{0R} \end{pmatrix} \rightarrow_{r \rightarrow \infty} \begin{pmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{pmatrix} U_0 \frac{e^{-Fr}}{\sqrt{r}},$$

where the constant U_0 is defined from the exact numerical solution for the zero mode and normalization $\int \psi_0^\dagger \psi_0 d^2x = 1$. The function $\psi_0(x)$ is not well defined in singular gauge, as far as it changes sign when θ changes by 2π . We can say that θ runs from $-\pi$ to π only, i.e. put the cut along the negative x (space coordinate) axis⁶. It is simpler in this case to make calculations after setting explicitly $k_1 = 0$, then we get for the Fourier transform (in Minkowski)

$$\psi_{0R,L}(k_0) = \mp U_0 \sqrt{2\pi} \frac{\sqrt{k_0 \pm k_1}}{F} \left(\frac{e^{\mp i\pi/4}}{F - \sqrt{k_\mu k_\mu}} + \frac{e^{\pm i\pi/4}}{F + \sqrt{k_\mu k_\mu}} \right) + \text{regular terms},$$

where upper and lower signs correspond to ψ_{0R} and ψ_{0L} respectively.

⁶ The singular gauge can be considered as a limit of gauges obtained by applying smooth gauge transformation with gauge function $\alpha = \theta + 2\pi\Theta^\epsilon(\theta - \pi)$ to the instanton solution, with Θ^ϵ being a smooth function becoming the step function in the limit $\epsilon \rightarrow 0$.

d. Matrix element. As an example let us calculate the matrix element with one fermion and two scalars. It is given by (in Minkowski space-time)

$$iM(p, k_1, k_2) = i\bar{v}(p)(\hat{p} + F)\psi_0(p) \times (-i)(k_1^2 - m_H^2)h_{\text{inst}}(k_1) \times (-i)(k_2^2 - m_H^2)h_{\text{inst}}(k_2) \times J \det[K_{\text{scalar}}]^{-1/2} \sqrt{\det_{\text{ren}}[K_I^\dagger K_I]} e^{-S_{\text{inst}}}.$$

Here $\bar{v}(p)$ is the antifermion spinor normalized like $v(p)\bar{v}(p) = \hat{p} - m$. So, the matrix element is

$$iM(p, k_1, k_2) = i\sqrt{4\pi}U_0(2\pi f_0 v)^2 J \times \det[K_{\text{scalar}}]^{-1/2} \left(\det_{\text{ren}}[K_I^\dagger K_I] \right)^{1/2} e^{-S_{\text{inst}}}. \quad (27)$$

We get a non-zero Lorentz invariant matrix element for a process involving one fermion and two bosons, as announced previously.

The matrix element (27) arise for instance in processes where an antifermion $\bar{\Psi}$ decays into two scalar ϕ if $F > 2m_H$. One may also analyze other Greens' functions. For instance, even simpler Green's function of the form $\langle\Psi h\rangle_{\text{inst}}$ is nonzero in the model, giving boson-fermion mixing.

V. CONCLUSIONS

We have analyzed the Abelian Higgs model in 1+1 dimensions. Half charged chiral fermions with mass generated by Higgs mechanism in this model are created in processes which change the topological number of the vacuum. A peculiar feature of the 1+1 dimensional models makes it possible to create only one fermion in the process where topological vacuum number changes by one. Unlike in similar 3+1 dimensional models, this model does not possess Witten anomaly. Neither this effect contradicts Lorentz symmetry in 1+1 dimensions.

We calculated the probability of such process using perturbation theory in instanton background. Calculation of this probability requires evaluation of the fermionic determinant in one instanton background. We note (see Section IV C) that the fermionic determinant for chiral fermions is very hard to define in topologically nontrivial background, with the main obstacle lying in the correct normalization, which usually requires division by fermion determinant in zero (topologically trivial) background. We want to emphasize, that this problem arises exactly in the same form in 3+1 dimensional theories (separately for each fermionic doublet in case of SU(2) theory). Up to our knowledge the relevant normalization was chosen only on dimensional grounds in literature [7, 22]. We propose a method to deal with the problem in 1+1 dimensions, though direct generalization of it to more dimensions is not trivial.

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APPENDIX A: VACUUM ENERGY

Let us calculate Dirac sea energy in the bosonic vacua with odd and even topological charges.

In sector with $n = 0$ the Dirac sea energy in a box of size L is given by the infinite sum of all negative energy levels in (5)

$$E_0^{\text{vac}} = -F - \frac{4\pi}{L} \sum_{l=1}^{\infty} \sqrt{l^2 + \left(\frac{FL}{2\pi}\right)^2}.$$

A simple method to deal with this sum is to change square roots to powers of $d/2$ and use zeta function regularization (see, eg. [27, 28]) one gets

$$E_0^{\text{vac}} = \frac{F^2 L}{8\pi^{3/2}} \Gamma\left(-\frac{d+1}{2}\right) + \sqrt{\frac{2F}{\pi L}} e^{-FL}, \quad (\text{A1})$$

where d is 1. The first term is just the normal infinite vacuum energy density for massive field, and should be taken care of by normal ordering of the operators in quantization, and the second one is the Casimir force.

Analogous calculation in $n = 1$ using energy levels (7) leads to the sum

$$E_1^{\text{vac}} = -\frac{4\pi}{L} \sum_{l=1}^{\infty} \sqrt{\left(l - \frac{1}{2}\right)^2 + \left(\frac{FL}{2\pi}\right)^2}.$$

This again can be computed in a zeta function regularization style (using eg. [29])

$$E_1^{\text{vac}} = \frac{F^2 L}{8\pi^{3/2}} \Gamma\left(-\frac{d+1}{2}\right) - \sqrt{\frac{2F}{\pi L}} e^{-FL}. \quad (\text{A2})$$

Subtracting (A2) from (A1) we get for the difference of vacuum energies in different gauge vacua

$$\Delta E^{\text{vac}} = E_1^{\text{vac}} - E_0^{\text{vac}} = -2\sqrt{\frac{2F}{\pi L}} e^{-FL}. \quad (\text{A3})$$

We see, that the infinite contribution cancels exactly, and the finite difference goes to zero exponentially with L . Thus, we conclude that in the limit of infinite space there is no energy difference between different vacua, despite of naïvely different fermionic energy levels. As $\Delta E^{\text{vac}} < 0$ for finite system size, the odd bosonic vacua are indeed the real vacua!

Note, that exactly the same result (A3) can be obtained using Pauli-Villars regularization scheme also.

APPENDIX B: FERMION NUMBER OF THE $n = 1$ VACUUM

We calculate here the fermion number in the $n = 1$ vacuum by different means, starting from its definition.

The fermionic Lagrangian is invariant under the following global transformations:

$$\begin{aligned} \Psi &\rightarrow e^{i\theta} \Psi, \\ \Psi^\dagger &\rightarrow e^{-i\theta} \Psi^\dagger. \end{aligned}$$

The conserved Noether current is $j^\mu = \bar{\Psi} \gamma^\mu \Psi$, and the related charge is the fermionic number $N_f = \int j^0 dx = \int \Psi^\dagger \Psi dx$. However, if we quantize the system (Ψ becomes operator and N_f needs normal ordering, $N_f = \frac{1}{2} \int (\Psi^\dagger \Psi - \Psi \Psi^\dagger) dx$) the current is not conserved any more, it suffer from the following anomaly:

$$\partial_\mu j^\mu = \frac{e}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu}.$$

The fermionic number vary in time as

$$\Delta N_f = \int \frac{e}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu} d^2x = \frac{e}{2\pi} \oint A \cdot dl.$$

In the $A_0 = 0$ gauge, if we start with $N_f = 0$ in vacuum $|0\rangle$, then $N_f = 0 + \Delta N_f = \int A_1(x) dx = 1/2$ in the sphaleron configuration and $N_f = 1$ in the vacuum $|1\rangle$. This result is what we expect from the level-crossing picture.

These results may also be found by explicit calculations. The sphaleron (kink) case was done eg. in the Chapter 9 of [30]. In short: In the background of the sphaleron we have one zero-mode for Ψ and the other modes come in pairs (particle and anti-particle):

$$\Psi(x, t) = b_0 f_0(x) + \sum_{r=1}^{\infty} b_r e^{-iE_r t} f_r^+(x) + \sum_{r=1}^{\infty} d_r e^{iE_r t} f_r^-(x). \quad (\text{B1})$$

Imposing equal time anticommutating relations

$$\{\Psi_\alpha(x, t), \Psi_\beta^\dagger(y, t)\} = \delta_{\alpha\beta} \delta(x - y)$$

and setting other anticommutators to zero, we get for the operators b, d :

$$\begin{aligned} \{b_r, b_{r'}^\dagger\} &= \{d_r, d_{r'}^\dagger\} = \delta_{rr'} \\ \{b_0, b_0^\dagger\} &= 1 \end{aligned} \quad (\text{B2})$$

and all other anticommutators vanishes. We can calculate the fermion number with (B1) and (B2),

$$\begin{aligned} N_f &= \frac{1}{2} \int (\Psi^\dagger \Psi - \Psi \Psi^\dagger) dx \\ &= b_0^\dagger b_0 - \frac{1}{2} + \sum_{r=1}^{\infty} (b_r^\dagger b_r - d_r^\dagger d_r). \end{aligned} \quad (\text{B3})$$

Application of the operator N_f to the sphaleron configuration with the zero-mode occupied gives $N_f(b_0^\dagger|0\rangle) = 1/2$. Whereas in the case of empty zero energy state: $N_f|0\rangle = -1/2$ (the strange term $-\frac{1}{2}$ in (B3) arise because we have a single state. Such $\frac{1}{2}$ -terms arise for each creation operators, but they cancels between particle b and antiparticle d), In any vacua $|n\rangle$ each states of negative energy (created by d_r , $r = 1, 2, \dots$) correspond to a positive energy state (created by b_r^\dagger , $r = 1, 2, \dots$). The field is

$$\Psi(x, t) = \sum_{r=1}^{\infty} b_r e^{-iE_r t} f_r^+(x) + d_r e^{-iE_r t} f_r^-(x),$$

where the E_r and the f_r depends on the topological number of the vacuum. The fermion number is simply

$$N_f = \sum_{r=1}^{\infty} (b_r^\dagger b_r - d_r^\dagger d_r).$$

In particular $N_f|1\rangle = 0$, $N_f b_1^\dagger|1\rangle = 1$, as in usual vacua.

APPENDIX C: ANTIINSTANTON DETERMINANT

The determinant of the fermionic fluctuations around the anti-instanton $\det'[K^\dagger K_{n=-1}]$ has been computed in Ref. [23]. We need here the same determinant in the background of the instanton ($n = 1$). Noticing that

$K^\dagger K_{n=1} = K K_{n=-1}^\dagger$ allows for better comparison between these two calculations. We may compare the operators $K K_{n=-1}^\dagger$ and $K^\dagger K_{n=-1}$: they have the same spectrum $\{\lambda_n\}_{n \neq 0}$ except that $K^\dagger K$ has a supplementary mode with eigenvalue $\lambda_0 = 0$. The determinant of $\det[K^\dagger K_{n=-1}]$ normalized to vacuum looks like

$$\frac{\det[K^\dagger K_{n=-1}]}{\det[K^\dagger K_{vac}]} = \frac{\lambda_0 \lambda_1 \dots}{\lambda_0^{vac} \lambda_1^{vac} \dots}.$$

Removing the zero mode and inserting the value for the lowest eigenvalue in the vacuum $\lambda_0^{vac} = F^2$ lead to:

$$\frac{\det'[K^\dagger K_{n=-1}]}{\det[K^\dagger K_{vac}]} = \frac{1}{F^2} \frac{\lambda_1 \lambda_2 \dots}{\lambda_1^{vac} \lambda_2^{vac} \dots}.$$

Naively we can guess that in the continuum limit, the eigenvalues in the vacuum are close to each other and

$$\frac{\det'[K^\dagger K_{n=-1}]}{\det[K^\dagger K_{vac}]} \sim \frac{1}{F^2} \frac{\lambda_1 \lambda_2 \dots}{\lambda_0^{vac} \lambda_1^{vac} \dots} = \frac{1}{F^2} \frac{\det[K^\dagger K_{n=1}]}{\det[K^\dagger K_{vac}]}.$$
(C1)

An explicit computation is performed in the following, and shows that this naive expectation is correct in the cases of interests, even if no general proof was found.

The computation of $\det[K^\dagger K_{n=1}]$ differ from the calculation of $\det[K^\dagger K_{n=-1}]$ by the very fact that the radial equations for the $\Psi_{L,R}^m$ are not diagonal⁷ in partial wave space (compare equation (46) of ref. [23]):

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2} - F^2 f^2(r) + \frac{e}{2} (A'(r) + \frac{A(r)}{r}) - \frac{e^2}{4} A^2(r) - m e \frac{A(r)}{r} \right] \Psi_L^m + \left[f \left(f'(r) - \frac{1}{r} f(r) - e A(r) f(r) \right) \right] \Psi_R^{m-2} = 0, \quad (C2)$$

$$\left[f \left(f'(r) - \frac{1}{r} f(r) - e A(r) f(r) \right) \right] \Psi_L^m + \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{(m-2)^2}{r^2} - F^2 f^2(r) + \frac{e}{2} (A'(r) + \frac{A(r)}{r}) - \frac{e^2}{4} A^2(r) + \frac{(m-2)eA(r)}{r} \right] \Psi_R^{m-2} = 0. \quad (C3)$$

Let us rename $\Psi_L^m = \psi_{2m}$ and $\Psi_R^m = \psi_{2m+1}$ and define the operator M_{ij} so that previous equations (C2,C3) are rewritten shortly as $M_{ij} \psi_j = 0$. As in equation (47, 48) of ref. [23], the determinant can be extracted from the

solution of the following differential systems:

$$M_{in} \psi_{nj}(r) = 0, \quad M_{jj}^{vac} \psi_j^{vac}(r) = 0,$$

with boundary conditions

$$\lim_{r \rightarrow 0} \frac{\psi_{ij}(r)}{\psi_i^{vac}(r)} = \delta_{ij}.$$

The determinant is then given by

$$\det \left[\frac{\psi_{ij}(R)}{\psi_i^{vac}(R)} \right].$$

⁷ One is tempted to define a new numbering of the variables to put this matrix in a block diagonal form, however it means that we commute lines at infinity, which is not permitted. Moreover it is not clear how to rearrange the corresponding variables for the vacuum operator.

The non zero elements of the matrix $\frac{\psi_{ij}(R)}{\psi_i^{vac}(R)} = a_{ij}$ are on the diagonal or of the form $a_{2i-3,2i}$, $a_{2i,2i-3}$, for any integer i . Its determinant can be computed with the following formula:

$$\det[a_{ij}] = \prod_{i=-\infty}^{\infty} (a_{2i,2i}a_{2i-3,2i-3} - a_{2i,2i-3}a_{2i-3,2i}) .$$

Note that there is no zero-mode in $K^\dagger K_{n=1}$ and its regularization and renormalization is carried out like in [23].

The results of the numerical computation agree to 10^{-3} accuracy to the formula (C1). An analytical calculation is possible only in very simplified situations. We were able to check formula (C1) for a modified instanton with profile

$$A(r) = \frac{1}{r}\theta(r-a), \quad f(r) = \theta(r-a).$$

The computation is lengthy and will not be given here.

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