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Localizing gravity on a 't Hooft-Polyakov monopole in seven dimensions

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Abstract

We present regular solutions for a brane world scenario in the form of a 't Hooft-Polyakov monopole living in the three-dimensional spherical symmetric transverse space of a seven-dimensional spacetime. In contrast to the cases of a domain-wall in five dimensions and a string in six dimensions, there exist gravity-localizing solutions for both signs of the bulk cosmological constant. A detailed discussion of the parameter space that leads to localization of gravity is given. A point-like monopole limit is discussed.

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1 Introduction

Recently, there has been renewed interest in brane-models in which our world is represented as a $3+1$-dimensional submanifold (a 3-brane) living in a higher-dimensional space-time [1, 2]. This idea provides an alternative to Kaluza-Klein compactification [3] and gives new insights to a construction of low energy effective theory of the fields of the standard model [1, 4] and gravity [5, 6]. Moreover, it may shed light on gauge hierarchy problem [5, 7] and on cosmological constant problem [8]-[11].

In string theory Standard Model fields are localized on D-branes - [12], whereas from the point of view of field theory brane model could be realized as a topological defect formed by scalar and gauge fields being a solution to the classical equations of motion of the coupled Einstein – Yang-Mills – scalar field equations. In the latter case one should be able to construct a solution leading to a regular geometry and localizing fields of different spins, including gravity.

Quite a number of explicit solutions is already known. In five-dimensions a real scalar field forming a domain wall may serve as a model of 3-brane [13]. Higher dimension topological defects can be qualitatively different from a five-dimensional case from the point of view of localization of different fields on a brane. So, strings in six space-time dimensions were considered in [14]-[19]. In particular, solutions corresponding to a thin local string together with fine-tuning relations (similar to the Randall-Sundrum domain-wall case) were found in [18] (see also [20]). A numerical realization confirming the general results of [18] in a singularity-free geometry and for the case of the Abelian Higgs model has been worked out in [19]. Moving to even higher dimensions may be of interest because of a richer content of fermionic zero modes and because of more complicated structure of transverse space. In the framework of KK compactification monopoles in seven dimensions and instantons in eight dimensions were discussed in [21, 22], and brane-world scenarios in higher dimensions in [16], [23]-[27].

In [23] we considered a general point-like spherically symmetric topological defect as a model of 3-brane and formulated conditions that are necessary for gravity localization on it. A transition from a regular solution to the classical equations of motion to a point-like limit is in fact quite non-trivial for six and higher dimensions (see a detailed discussion for a string case in [19, 20]). The aim of the present paper is to provide an existence proof of a possibility of gravity localization on a regular three-dimensional defect – 't Hooft-Polyakov monopole in seven dimensions, to study the parameter-space of a model that leads to gravity localization and to formulate exactly the meaning of point-like monopole limit. We confirm entirely the previous results, in particular, a possibility of gravity localization on a monopole embedded in a space with both signs of a bulk cosmological constant.

The paper is organized as follows. In section 2 we present the $SO(3)$ invariant Georgi-Glashow model (having 't Hooft-Polyakov monopoles as flat spacetime solutions [28],[29]) coupled to gravity in seven dimensions. The Einstein equations and the field equations are obtained in the case of a generalized 't Hooft-Polyakov ansatz for gauge and scalar fields. Boundary conditions are discussed in section 3, the asymptotic behavior of the solutions at the origin in section 3.1, at infinity in section 3.2. In section 3.3 we give the relations between the brane tension components necessary for warped compactification. Section 4 presents numerical results omitting all technical details. We give explicit sample solutions in section 4.1 and discuss their general dependence on the parameters of the model. We then present the fine-tuning surface (the relation between the independent parameters of the model necessary for gravity localization) in section 4.2. Sections 5 and 6 treat the Prasad-Sommerfield limit and the point-like monopole limit, respectively. While in the former case no gravity localizing solutions exist, in the latter case we demonstrate a possibility of the choice of the model parameters that leads to a fundamental Planck scale in TeV range and small modifications of the Newton's law, while well within the range of applicability of classical gravity. We conclude in section 7. In Appendix A a derivation of the fine-tuning relations is given, whereas in Appendix B a discussion of the numerical details can be found.

2 Field equations

The action for the setup considered in this paper is a straightforward generalization of a gravitating 't Hooft-Polyakov monopole in 4 dimensions (which has been extensively studied in the past [30]-[34]) to the case of seven-dimensional spacetime:

$$S = S_{gravity} + S_{brane}. \tag{1}$$

Here $S_{gravity}$ is the seven-dimensional Einstein-Hilbert action:

$$S_{gravity} = \frac{M_7^5}{2}\int d^7x\sqrt{-g}\left(R - \frac{2\Lambda_7}{M_7^5}\right)\ , \tag{2}$$

g is the determinant of the metric g_{MN} with signature $(-++++++)$. We use the sign conventions for the Riemann tensor of [35]. Upper case latin indices M, N run over $0\ldots6$, lower case latin indices m, n over $4\ldots6$ and greek indices μ, ν over $0\ldots3$. The parameter M_7 denotes the fundamental gravity scale, Λ_7 is the bulk cosmological constant and S_{brane} is the action of Georgi and Glashow [36] containing SU(2) gauge field $W_M^{\tilde a}$ and a scalar triplet $\Phi^{\tilde a}$ (we denote group indices by $\tilde a, \tilde b, \tilde c = 1\ldots3$) :

$$S_{brane} = \int d^7x\sqrt{-g}\,\mathcal{L}_m \quad \text{with} \quad \mathcal{L}_m = -\frac{1}{4}G^{\tilde a}_{MN}G^{\tilde a MN} - \frac{1}{2}\mathcal{D}_M\Phi^{\tilde a}\mathcal{D}^M\Phi^{\tilde a} - \frac{\lambda}{4}\left(\Phi^{\tilde a}\Phi^{\tilde a} - \eta^2\right)^2\ , \tag{3}$$

where η is the vacuum expectation value of the scalar field and $\mathcal{D}_M$ is a covariant derivative,

$$\mathcal{D}_M\Phi^{\tilde a} = \partial_M\Phi^{\tilde a} + e\,\epsilon^{\tilde a\tilde b\tilde c}\,W_M^{\tilde b}\Phi^{\tilde c}. \tag{4}$$

Furthermore one has

$$G^{\tilde a}_{MN} = \partial_M W_N^{\tilde a} - \partial_N W_M^{\tilde a} + e\,\epsilon^{\tilde a\tilde b\tilde c}\,W_M^{\tilde b}W_N^{\tilde c}. \tag{5}$$

The $SO(3)$ symmetry is spontaneously broken down to $U(1)$. The monopole corresponds to the simplest topologically nontrivial field configuration with unit winding number. The Higgs mass is given by $m_H = \eta\sqrt{2\lambda}$. Two of the gauge fields acquire a mass $m_W = e\,\eta$.

The general coupled system of Einsteins equations and the equations of motion for the scalar field and the gauge field following from the above action are

$$\begin{aligned} R_{MN} - \frac{1}{2} g_{MN} R + \frac{\Lambda_7}{M_7^5} g_{MN} &= \frac{1}{M_7^5} T_{MN}\,, & (6)\\ \frac{1}{\sqrt{-g}} \mathcal{D}_M \left(\sqrt{-g}\,\mathcal{D}^M \Phi^{\tilde{a}}\right) &= \lambda\,\Phi^{\tilde{a}} \left(\Phi^{\tilde{b}}\Phi^{\tilde{b}} - \eta^2\right)\,, & (7)\\ \frac{1}{\sqrt{-g}} \mathcal{D}_M \left(\sqrt{-g}\,G^{\tilde{a}MN}\right) &= -e\,\epsilon^{\tilde{a}\tilde{b}\tilde{c}} \left(\mathcal{D}^N \Phi^{\tilde{b}}\right) \Phi^{\tilde{c}}\,, & (8) \end{aligned}$$

where the stress-energy tensor T_{MN} is given by

$$T_{MN} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{brane}}{\delta g^{MN}} = G^{\tilde{a}}_{ML} G^{\tilde{a}\,L}_{N} + \mathcal{D}_M \Phi^{\tilde{a}} \mathcal{D}_N \Phi^{\tilde{a}} + g_{MN}\mathcal{L}_m\,. \tag{9}$$

We are interested in static monopole-like solutions to the set of equations (6)-(8) respecting both, $4D$-Poincaré invariance on the brane and rotational invariance in the transverse space. The fields $\Phi^{\tilde{a}}$ and $W^{\tilde{a}}_M$ (and as a result $G^{\tilde{a}}_{MN}$) should not depend on coordinates on the brane x^μ. The brane is supposed to be located at the center of the magnetic monopole. A general non-factorisable ansatz for the metric satisfying the above conditions is

$$ds^2 = M^2(\rho) g^{(4)}_{\mu\nu} dx^\mu dx^\nu + d\rho^2 + L^2(\rho)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\,, \tag{10}$$

where $g^{(4)}_{\mu\nu}$ is the four-dimensional metric that satisfies the $4D$ Einstein equations with an arbitrary cosmological constant Λ_{phys} [8]. In this paper we will only consider the case of $\Lambda_{\text{phys}} = 0$ and we take $g^{(4)}_{\mu\nu}$ to be the Minkowski metric $\eta_{\mu\nu}$ with signature $(- + + +)$.

With the use of spherical coordinates for transverse space

$$\begin{aligned} \vec{e}^{\,\rho} &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,, & (11)\\ \vec{e}^{\,\theta} &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)\,, & (12)\\ \vec{e}^{\,\varphi} &= (-\sin\varphi, \cos\varphi, 0)\,, & (13) \end{aligned}$$

the ’t Hooft-Polyakov ansatz is :

$$\vec{W}_\rho = 0, \quad \vec{W}_\theta = -\frac{1-K(\rho)}{e}\,\vec{e}^{\,\varphi}, \quad \vec{W}_\varphi = \frac{1-K(\rho)}{e}\,\sin\theta\,\vec{e}^{\,\theta}, \quad \vec{W}_\mu = 0 \quad \vec{\Phi} = \frac{H(\rho)}{e\rho}\vec{e}^{\,\rho}\,, \tag{14}$$

with flashes indicating vectors in internal $SO(3)$ space.

Using the ansatz (14) for the fields together with the metric (10) in the coupled system of differential equations (6)-(8) gives

$$
\begin{aligned}
3\,\frac{M''}{M}+2\,\frac{\mathcal{L}''}{\mathcal{L}}+6\,\frac{M'\,\mathcal{L}'}{M\,\mathcal{L}}+3\,\frac{M'^2}{M^2}+\frac{\mathcal{L}'^2}{\mathcal{L}^2}-\frac{1}{\mathcal{L}^2} &= \beta\left(\epsilon_0-\gamma\right)\,, && (15)\\
8\,\frac{M'\,\mathcal{L}'}{M\,\mathcal{L}}+6\,\frac{M'^2}{M^2}+\frac{\mathcal{L}'^2}{\mathcal{L}^2}-\frac{1}{\mathcal{L}^2} &= \beta\left(\epsilon_\rho-\gamma\right)\,, && (16)\\
4\,\frac{M''}{M}+\frac{\mathcal{L}''}{\mathcal{L}}+4\,\frac{M'\,\mathcal{L}'}{M\,\mathcal{L}}+6\,\frac{M'^2}{M^2} &= \beta\left(\epsilon_\theta-\gamma\right)\,, && (17)
\end{aligned}
$$

$$
\begin{aligned}
J''+2\left(2\,\frac{M'}{M}+\frac{\mathcal{L}'}{\mathcal{L}}\right)J'-2\,\frac{K^2J}{\mathcal{L}^2} &= \alpha J\left(J^2-1\right)\,, && (18)\\
K''+\frac{K\left(1-K^2\right)}{\mathcal{L}^2}+4\,\frac{M'}{M}K' &= J^2K\,. && (19)
\end{aligned}
$$

where primes denote derivatives with respect to the transverse radial coordinate r, rescaled by the mass of the gauge boson m_W:

$$r = m_W\rho = \eta\, e\,\rho. \qquad (20)$$

All quantities appearing in the above equations are dimensionless, including α, β and γ:

$$\alpha=\frac{\lambda}{e^2}=\frac{1}{2}\left(\frac{m_H}{m_W}\right)^2\,,\quad \beta=\frac{\eta^2}{M_7^5}\,,\quad \gamma=\frac{\Lambda_7}{e^2\eta^4}\,,\quad \epsilon_i=\frac{f_i}{e^2\eta^4}\,, \qquad (21)$$

with

$$T^\mu_\nu=\delta^\mu_\nu f_0\,,\quad T^\rho_\rho=f_\rho\,,\quad T^\theta_\theta=f_\theta\,,\quad T^\varphi_\varphi=f_\varphi\,,\quad \mathcal{L}=Lm_W\,,\quad J=\frac{H}{\eta\,e\,\rho}. \qquad (22)$$

The dimensionless diagonal elements of the stress-energy tensor are given by

$$
\begin{aligned}
\epsilon_0 &= -\left[\frac{K'^2}{\mathcal{L}^2}+\frac{\left(1-K^2\right)^2}{2\mathcal{L}^4}+\frac{1}{2}J'^2+\frac{J^2K^2}{\mathcal{L}^2}+\frac{\alpha}{4}\left(J^2-1\right)^2\right]\,, && (23)\\
\epsilon_\rho &= \frac{K'^2}{\mathcal{L}^2}-\frac{\left(1-K^2\right)^2}{2\mathcal{L}^4}+\frac{1}{2}J'^2-\frac{J^2K^2}{\mathcal{L}^2}-\frac{\alpha}{4}\left(J^2-1\right)^2\,, && (24)\\
\epsilon_\theta &= \frac{\left(1-K^2\right)^2}{2\mathcal{L}^4}-\frac{1}{2}J'^2-\frac{\alpha}{4}\left(J^2-1\right)^2\,. && (25)
\end{aligned}
$$

The rotational symmetry in transverse space implies that the $(\theta\,\theta)$ and the $(\varphi\varphi)$ components of the Einstein equations are identical (and that $\epsilon_\varphi=\epsilon_\theta$). Equations (15) - (17) are not functionally independent [23]. They are related by the Bianchi identities (or equivalently by conservation of stress-energy $\nabla_M T^M_N=0$). Following the lines of [23] we can define various components of the brane tension per unit length by

$$\mu_i=-\int_0^\infty dr M(r)^4\mathcal{L}(r)^2\epsilon_i(r)\,. \qquad (26)$$

The Ricci scalar and the curvature invariants R, $R_{AB}\,R^{AB}$, $R_{ABCD}\,R^{ABCD}$, $C_{ABCD}\,C^{ABCD}$ with C_{ABCD} being the Weyl tensor are given by

$$
\begin{aligned}
\frac{R}{m_W^2} &= \frac{2}{\mathcal{L}^2} - \frac{2\,\mathcal{L}'^2}{\mathcal{L}^2} - \frac{16\,\mathcal{L}'\,M'}{\mathcal{L}\,M} - \frac{12\,M'^2}{M^2} - \frac{4\,\mathcal{L}''}{\mathcal{L}} - \frac{8\,M''}{M}\,, \qquad (27)\\
\frac{R_{AB}\,R^{AB}}{m_W^4} &= \frac{2}{\mathcal{L}^4} - \frac{4\,\mathcal{L}'^2}{\mathcal{L}^4} + \frac{2\,\mathcal{L}'^4}{\mathcal{L}^4} - \frac{16\,\mathcal{L}'\,M'}{\mathcal{L}^3\,M} + \frac{16\,\mathcal{L}'^3\,M'}{\mathcal{L}^3\,M} + \frac{48\,\mathcal{L}'^2\,M'^2}{\mathcal{L}^2\,M^2} + \\
&\quad \frac{48\,\mathcal{L}'\,M'^3}{\mathcal{L}\,M^3} + \frac{36\,M'^4}{M^4} - \frac{4\,\mathcal{L}''}{\mathcal{L}^3} + \frac{4\,\mathcal{L}'^2\,\mathcal{L}''}{\mathcal{L}^3} + \frac{16\,\mathcal{L}'\,M'\,\mathcal{L}''}{\mathcal{L}^2\,M} + \frac{6\,\mathcal{L}''^2}{\mathcal{L}^2} + \\
&\quad \frac{16\,\mathcal{L}'\,M'\,M''}{\mathcal{L}\,M^2} + \frac{24\,M'^2\,M''}{M^3} + \frac{16\,\mathcal{L}''\,M''}{\mathcal{L}\,M} + \frac{20\,M''^2}{M^2}\,, \qquad (28)\\
\frac{R_{ABCD}\,R^{ABCD}}{m_W^4} &= \frac{4}{\mathcal{L}^4} - \frac{8\,\mathcal{L}'^2}{\mathcal{L}^4} + \frac{4\,\mathcal{L}'^4}{\mathcal{L}^4} + \frac{32\,\mathcal{L}'^2\,M'^2}{\mathcal{L}^2\,M^2} + \frac{24\,M'^4}{M^4} + \frac{8\,\mathcal{L}''^2}{\mathcal{L}^2} + \frac{16\,M''^2}{M^2}\,, \qquad (29)\\
\frac{C_{ABCD}\,C^{ABCD}}{m_W^4} &= \frac{8}{3\,\mathcal{L}^4} - \frac{16\,\mathcal{L}'^2}{3\,\mathcal{L}^4} + \frac{8\,\mathcal{L}'^4}{3\,\mathcal{L}^4} + \frac{128\,\mathcal{L}'\,M'}{15\,\mathcal{L}^3\,M} - \frac{128\,\mathcal{L}'^3\,M'}{15\,\mathcal{L}^3\,M} - \frac{16\,M'^2}{5\,\mathcal{L}^2\,M^2} + \\
&\quad \frac{208\,\mathcal{L}'^2\,M'^2}{15\,\mathcal{L}^2\,M^2} - \frac{64\,\mathcal{L}'\,M'^3}{5\,\mathcal{L}\,M^3} + \frac{24\,M'^4}{5\,M^4} + \frac{32\,\mathcal{L}''}{15\,\mathcal{L}^3} - \frac{32\,\mathcal{L}'^2\,\mathcal{L}''}{15\,\mathcal{L}^3} - \\
&\quad \frac{64\,\mathcal{L}'\,M'\,\mathcal{L}''}{15\,\mathcal{L}^2\,M} + \frac{32\,M'^2\,\mathcal{L}''}{5\,\mathcal{L}\,M^2} + \frac{64\,\mathcal{L}''^2}{15\,\mathcal{L}^2} - \frac{32\,M''}{15\,\mathcal{L}^2\,M} + \frac{32\,\mathcal{L}'^2\,M''}{15\,\mathcal{L}^2\,M} + \\
&\quad \frac{64\,\mathcal{L}'\,M'\,M''}{15\,\mathcal{L}\,M^2} - \frac{32\,M'^2\,M''}{5\,M^3} - \frac{128\,\mathcal{L}''\,M''}{15\,\mathcal{L}\,M} + \frac{64\,M''^2}{15\,M^2}\,. \qquad (30)
\end{aligned}
$$

They must be finite continuous functions for regular geometries we are interested in.

3 Boundary conditions and asymptotics of the solutions

The boundary conditions should lead to a regular solution at the origin. Thus we have to impose

$$
M|_{r=0} = 1\,, \quad M'|_{r=0} = 0\,, \quad \mathcal{L}|_{r=0} = 0\,, \quad \mathcal{L}'|_{r=0} = 1\,, \qquad (31)
$$

for the components of the metric, where the value $+1$ for $M|_{r=0}$ is a convenient choice that can be obtained by rescaling of the brane coordinates.

What concerns the gauge and the scalar fields, the boundary conditions for them are the same as for a monopole solution in the flat space-time, [28]:

$$
J(0) = 0\,, \quad \lim_{r\to\infty} J(r) = 1\,, \qquad (32)
$$

$$
K(0) = 1\,, \quad \lim_{r\to\infty} K(r) = 0. \qquad (33)
$$

Finally, a requirement of gravity localization reads

$$\frac{4\pi M_7^5}{m_W^3}\int_0^\infty M(r)^2\mathcal{L}(r)^2 dr = M_P^2 < \infty \ , \tag{34}$$

what puts a constraint on the behavior of the metric at infinity.

The magnetic charge of the field configuration can either be determined by comparing the stress-energy tensor with the general expression given in [23] or by a direct calculation of the magnetic field strength tensor [28]:

$$\mathcal{G}_{MN} = \frac{\vec{\Phi}\cdot\vec{G}_{MN}}{|\vec{\Phi}|} - \frac{1}{e|\vec{\Phi}|^3}\vec{\Phi}\cdot\left(\mathcal{D}_M\vec{\Phi}\times\mathcal{D}_N\vec{\Phi}\right). \tag{35}$$

The only nonzero component of $\mathcal{G}_{MN}$ is $\mathcal{G}_{\theta\varphi} = -\frac{\sin\theta}{|e|}$. Either way gives $Q = \frac{1}{e}$.

3.1 Behavior at the origin

Once boundary conditions at the center of the defect are imposed for the fields and the metric, the system of equations (15)-(19) can be solved in the vicinity of the origin by developing the fields and the metric into a power series in the (reduced) transverse radial variable r. For the given system this can be done up to any desired order. We give the power series up to third order in r:

$$M(r) = 1 - \frac{1}{60}r^2\,\beta\left(\alpha + 4\,\gamma - 6\,K''(0)^2\right) + \mathcal{O}(r^4)\,, \tag{36}$$

$$\mathcal{L}(r) = r + \frac{1}{360}r^3\,\beta\left(\alpha + 4\,\gamma - 30\,J'(0)^2 - 66\,K''(0)^2\right) + \mathcal{O}(r^5)\,, \tag{37}$$

$$J(r) = r\,J'(0) + r^3\,J'(0)\frac{-9\alpha + \alpha\beta + 4\beta\gamma + 6\beta J'(0)^2 + 18K''(0) + 6\beta K''(0)^2}{90} + \mathcal{O}(r^5)\,, \tag{38}$$

$$K(r) = 1 + \frac{1}{2}r^2\,K''(0) + \mathcal{O}(r^4)\,. \tag{39}$$

It can easily be shown that the power series of $M(r)$ and $K(r)$ only involve even powers of r whereas those of $\mathcal{L}(r)$ and $J(r)$ involve only odd ones. The expressions for $\mathcal{L}(r)$ and $J(r)$ are therefore valid up to 5^{th} order. One observes that the solutions satisfying the boundary conditions at the origin can be parametrized by five parameters $(\alpha, \beta, \gamma, J'(0), K''(0))$. For arbitrary combinations of these parameters the corresponding metric solution will not satisfy the boundary conditions at infinity. Therefore the task is to find those parameter combinations for which (34) is finite. For completeness, we give the zero-th order of the power series solutions for the stress-energy tensor components and the curvature invariants at the origin:

$$\epsilon_0|_{r=0} \;=\; -\frac{1}{4}\left(\alpha + 6\,J'(0)^2 + 6\,K''(0)^2\right)\,, \tag{40}$$

$$\epsilon_\rho|_{r=0} = \epsilon_\theta|_{r=0} \;=\; -\frac{1}{4}\left(\alpha + 2\,J'(0)^2 - 2\,K''(0)^2\right). \tag{41}$$

$$
\begin{aligned}
\frac{R}{m_W^2} &= \frac{\beta}{10}\left(7\,\alpha + 28\,\gamma + 30\,J'(0)^2 + 18\,K''(0)^2\right)\,, &(42)\\
\frac{R_{AB}\,R^{AB}}{m_W^4} &= \frac{\beta^2}{100}\left(7\,\alpha^2 + 56\,\alpha\,\gamma + 112\,\gamma^2 + 60\,\alpha\,J'(0)^2 + 240\,\gamma\,J'(0)^2 + 300\,J'(0)^4\right.\\
&\qquad\left. +12\,\left(3\,\alpha + 12\,\gamma + 70\,J'(0)^2\right)\,K''(0)^2 + 732\,K''(0)^4\right)\,, &(43)\\
\frac{R_{ABCD}\,R^{ABCD}}{m_W^4} &= \frac{\beta^2}{300}\left(17\,\alpha^2 + 136\,\alpha\,\gamma + 272\,\gamma^2 - 60\,\alpha\,J'(0)^2 - 240\,\gamma\,J'(0)^2 + 900\,J'(0)^4\right.\\
&\qquad\left. -36\,\left(9\,\alpha + 36\,\gamma - 110\,J'(0)^2\right)\,K''(0)^2 + 4932\,K''(0)^4\right)\,, &(44)\\
\frac{C_{ABCD}\,C^{ABCD}}{m_W^4} &= \frac{\beta^2}{30}\left(\alpha + 4\,\gamma - 6\,\left(J'(0)^2 + 3\,K''(0)^2\right)\right)^2. &(45)
\end{aligned}
$$

3.2 Behavior at infinity

The asymptotics of the metric functions $M(r)$ and $\mathcal{L}(r)$ far away from the monopole are [23]:

$$
M(r) = M_0\,e^{-\frac{c}{2}r} \quad \text{and} \quad \mathcal{L}(r) = \mathcal{L}_0 = \text{const}\,, \qquad (46)
$$

where only positive values of c lead to gravity localization. This induces the following asymptotics for the stress-energy components and the various curvature invariants:

$$
\begin{aligned}
\lim_{r\to\infty}\epsilon_0(r) = \lim_{r\to\infty}\epsilon_\rho(r) &= -\frac{1}{2{\mathcal{L}_0}^4}\,, &(47)\\
\lim_{r\to\infty}\epsilon_\theta(r) &= \frac{1}{2{\mathcal{L}_0}^4}\,. &(48)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{r\to\infty}\frac{R(r)}{m_W^2} &= -5c^2 + \frac{2}{\mathcal{L}_0^2}\,, &(49)\\
\lim_{r\to\infty}\frac{R_{AB}(r)R^{AB}(r)}{m_W^4} &= 5c^4 + \frac{2}{\mathcal{L}_0^4}\,, &(50)\\
\lim_{r\to\infty}\frac{R_{ABCD}(r)R^{ABCD}(r)}{m_W^4} &= \frac{5}{2}c^4 + \frac{4}{\mathcal{L}_0^4}\,, &(51)\\
\lim_{r\to\infty}\frac{C_{ABCD}(r)C^{ABCD}(r)}{m_W^4} &= \frac{c^4}{6} + \frac{8}{3\,\mathcal{L}_0^4} - \frac{4c^2}{\mathcal{L}_0^2}\,. &(52)
\end{aligned}
$$

The parameters c and $\mathcal{L}_0$ are determined by Einsteins equations for large r and are given by [23]:

$$
\begin{aligned}
c^2 &= \frac{5}{32\beta}\left(1 - \frac{16}{5}\gamma\beta^2 \pm \sqrt{1 - \frac{32}{25}\gamma\beta^2}\right)\,, &(53)\\
\frac{1}{\mathcal{L}_0^2} &= \frac{5}{8\beta}\left(1 \pm \sqrt{1 - \frac{32}{25}\gamma\beta^2}\right)\,. &(54)
\end{aligned}
$$

Only the positive signs of the roots lead to solutions with both $c^2 > 0$ and $\frac{1}{\mathcal{L}_0^2} > 0$ [23].

In order to obtain some information about the asymptotic behavior of the fields $J(r)$ and $K(r)$ at infinity we insert relations (46) into (18) and (19). Furthermore we use $K(r) = \delta K(r)$ and $J(r) = 1 - \delta J(r)$ with $\delta K \ll 1$ and $\delta J \ll 1$ for $1 \ll r$ to linearize these equations:

$$\delta J'' - 2\,c\,\delta J' - 2\alpha\delta J \;=\; 0\,, \tag{55}$$

$$\delta K'' - 2\,c\,\delta K' + \left(\frac{1}{\mathcal{L}_0^2} - 1\right)\delta K \;=\; 0\,. \tag{56}$$

By using the ansatz $\delta K = Ae^{-kr}$ and $\delta J = Be^{-jr}$ we find

$$k_{1,2} = -c \pm \sqrt{c^2 - \left(\frac{1}{\mathcal{L}_0^2} - 1\right)}\,, \qquad j_{1,2} = -c \pm \sqrt{c^2 + 2\alpha}\,. \tag{57}$$

To satisfy the boundary conditions we obviously have to impose $k > 0$ and $j > 0$. We distinguish two cases:

1. $\boxed{c > 0}$ Gravity localizing solutions. In this case there is a unique k for $\frac{1}{\mathcal{L}_0^2} < 1$ with the positive sign in (57).

2. $\boxed{c < 0}$ Solutions that do not localize gravity.

   (a) $\boxed{\frac{1}{\mathcal{L}_0^2} > 1}$
      i. $\boxed{c^2 \geq \frac{1}{\mathcal{L}_0^2} - 1}$ Both solutions of (57) are positive.
      ii. $\boxed{c^2 < \frac{1}{\mathcal{L}_0^2} - 1}$ In this case there are no real solutions.

   (b) $\boxed{\frac{1}{\mathcal{L}_0^2} < 1}$ There is a unique solution with the positive sign in (57) .

   (c) $\boxed{\frac{1}{\mathcal{L}_0^2} = 1}$ The important k-value here is $k = -2c$.

It can be easily shown that for large enough r the linear approximation (56) to the equation of motion of the gauge field is always valid. Linearizing the equation for the scalar field however breaks down for $2k < j$ due to the presence of the term $\propto K^2J$ in (18). In that case $J(r)$ approaches 1 as e^{-2kr}. For solutions with $c > 0$ (the case of predominant interest) a detailed discussion of the validity of $2k > j$ gives:

1. $\boxed{\alpha = 0}$ Prasad-Sommerfield limit [39]. One has $2k > j$. The asymptotics of the scalar field is governed by e^{-jr}.

2. $\boxed{0 < \alpha < 2}$ The validity of $2k > j$ depends on different inequalities between α, $\frac{1}{\mathcal{L}_0^2}$ and c^2.

   (a) $\boxed{\alpha + \frac{1}{\mathcal{L}_0^2} \leq 1}$ $\longrightarrow 2k > j$.

(b) $\boxed{\alpha + \frac{1}{\mathcal{L}_0^2} > 1}$ and $\boxed{\frac{\alpha}{2} + \frac{1}{\mathcal{L}_0^2} < 1}$. In this case $2k \geq j$ is equivalent to $c^2 \leq \frac{-\left(1-\frac{\alpha}{2}-\frac{1}{\mathcal{L}_0^2}\right)^2}{1-\alpha-\frac{1}{\mathcal{L}_0^2}}$, where equality in one of the equations implies equality in the other.

(c) $\boxed{\frac{\alpha}{2} + \frac{1}{\mathcal{L}_0^2} \geq 1}$ $\longrightarrow 2k < j$ and the scalar field asymptotics is governed by e^{-2kr}.

3. $\boxed{\alpha > 2}$ $\longrightarrow 2k < j$. Also in this case the gauge field $K(r)$ determines the asymptotics of the scalar field $J(r)$.

3.3 Fine-tuning relations

It is possible to derive analytic relations between the different components of the brane tensions valid for gravity localizing solutions. Integrating linear combinations of Einsteins equations (15)-(17) between 0 and ∞ after multiplication with $M^4(r)\mathcal{L}^2(r)$ gives:

$$\mu_0 - \mu_\theta = \frac{1}{\beta}\int_0^\infty M^4 dr = \int_0^\infty \left(1-K^2\right)\frac{M^4}{\mathcal{L}^2} dr\,, \tag{58}$$

$$\mu_0 - \mu_\rho - 2\mu_\theta = 2\gamma \int_0^\infty M^4 \mathcal{L}^2 dr\,, \tag{59}$$

$$\mu_0 + \mu_\rho + 2\mu_\theta = \alpha \int_0^\infty \left(1-J^2\right) M^4 \mathcal{L}^2 dr\,. \tag{60}$$

To obtain the above relations integration by parts was used where the boundary terms dropped due to the boundary conditions given above. A detailed derivation of these relations is given in the appendix A.

4 Numerical solutions

Details of numerical integrations are given in the Appendix B; in this section we will discuss the results only.

4.1 Examples of numerical solutions

Fig. 1 and Fig. 4 show two different numerical solutions corresponding to positive and negative bulk cosmological constant, respectively. Both solutions localize gravity $c > 0$. The corresponding parameter values are ($\alpha = 1.0000000$, $\beta = 5.50000000$, $\gamma = -0.05434431$) and ($\alpha = 1.0000000$, $\beta = 3.50000000$, $\gamma = 0.02678351$). Figs. 2 and 5 and Figs. 3 and 6 show the corresponding components of the stress-energy tensor and the curvature invariants. For high values of β the metric function $\mathcal{L}(r)$ develops a maximum before attaining its boundary value. Gravity dominates and the volume of the transverse space stays finite. For lower values of β the transverse space has infinite volume and gravity can not be localized. See Figs. 7 and 8 for a solution that does not localize gravity ($c < 0$) and corresponds to ($\alpha = 1.0000000$, $\beta = 1.80000000$, $\gamma = 0.04053600$).

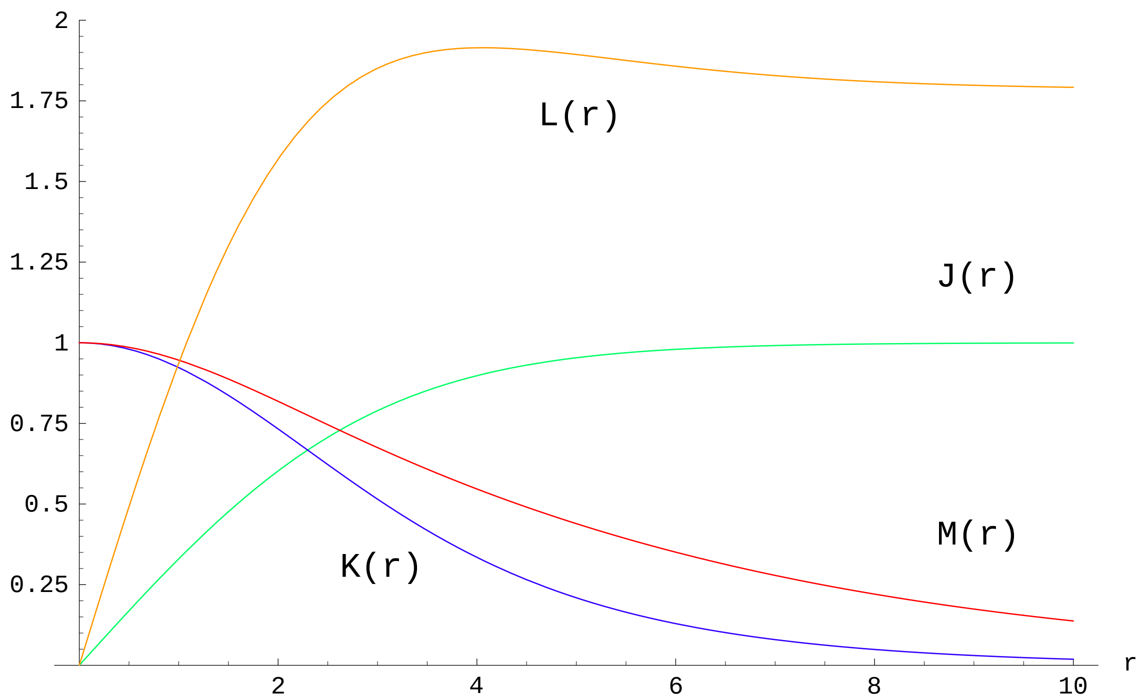

Figure 1: *Gravity-localizing solution with negative bulk cosmological constant corresponding to the parameter values* ($\alpha = 1.00000000$, $\beta = 5.50000000$, $\gamma = -0.05434431$)

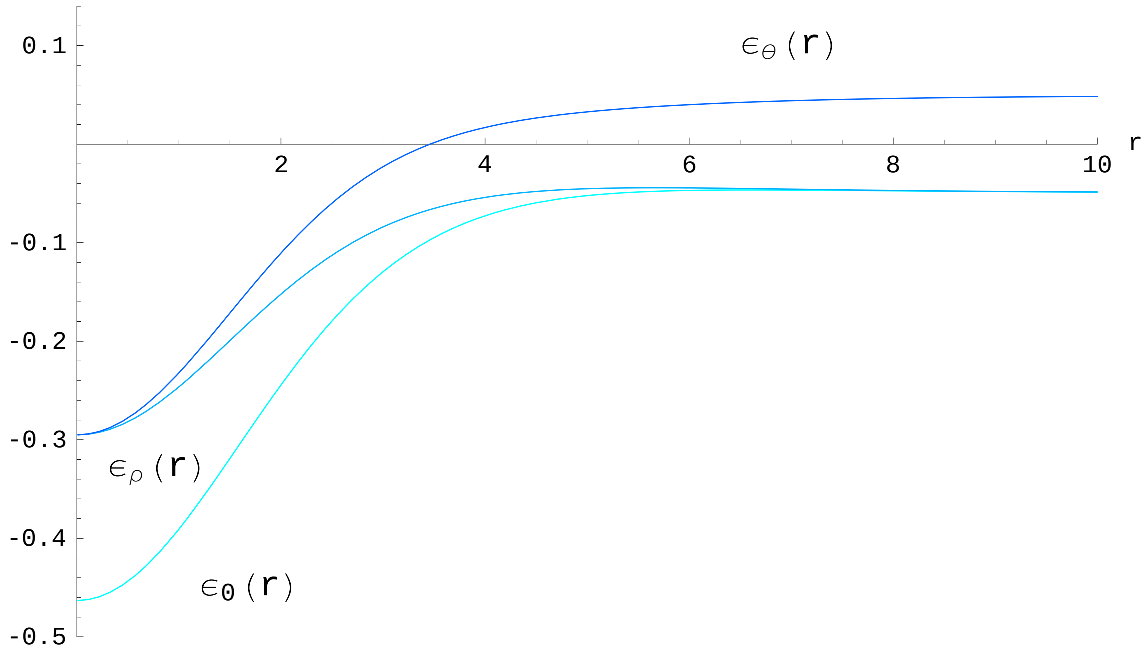

Figure 2: *Stress-energy components for the solution given in Fig. 1.*
($\alpha = 1.00000000$, $\beta = 5.50000000$, $\gamma = -0.05434431$)

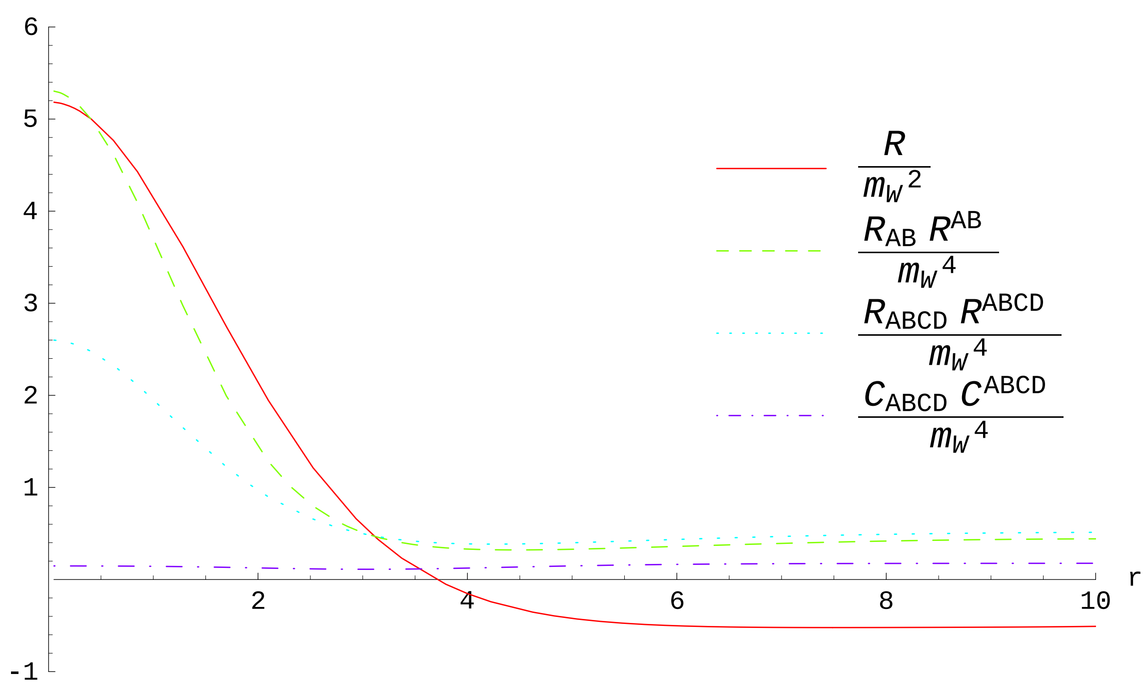

Figure 3: *Curvature invariants for the solution given in Fig. 1.*
($\alpha = 1.00000000$, $\beta = 5.50000000$, $\gamma = -0.05434431$)

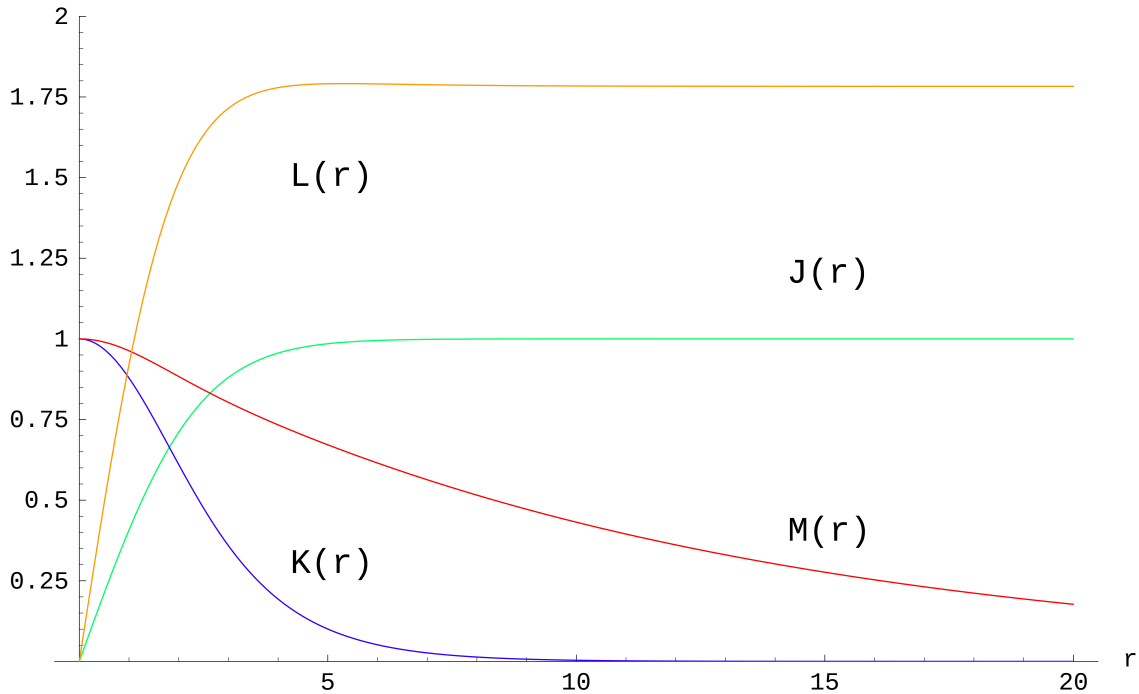

Figure 4: *Gravity-localizing solution with positive bulk cosmological constant corresponding to the parameter values* ($\alpha = 1.00000000$, $\beta = 3.50000000$, $\gamma = 0.02678351$)

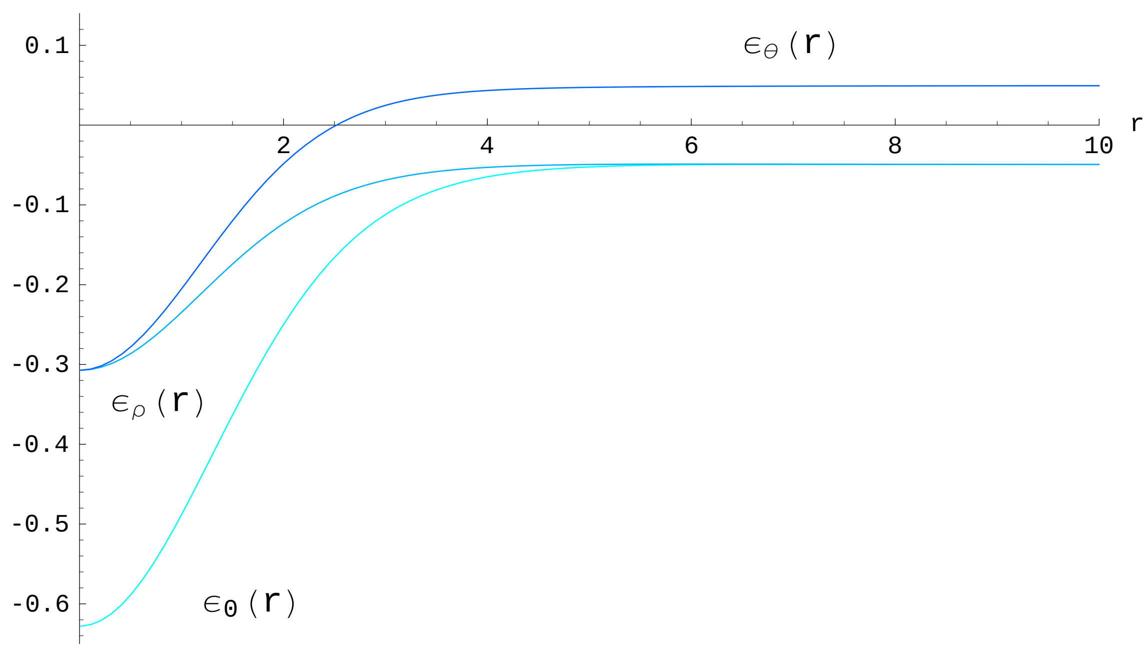

Figure 5: *Stress-energy components for the solution given in Fig. 4.*
($\alpha = 1.00000000$, $\beta = 3.50000000$, $\gamma = 0.02678351$)

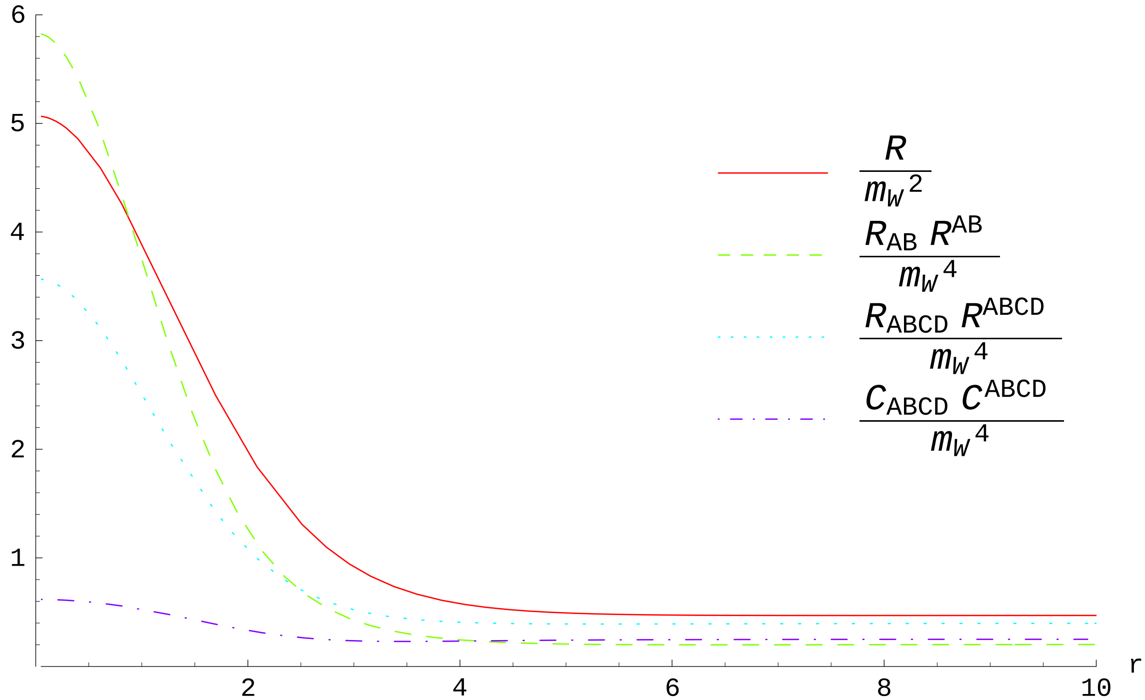

Figure 6: *Curvature invariants for the solution given in Fig. 4.*
($\alpha = 1.00000000$, $\beta = 3.50000000$, $\gamma = 0.02678351$)

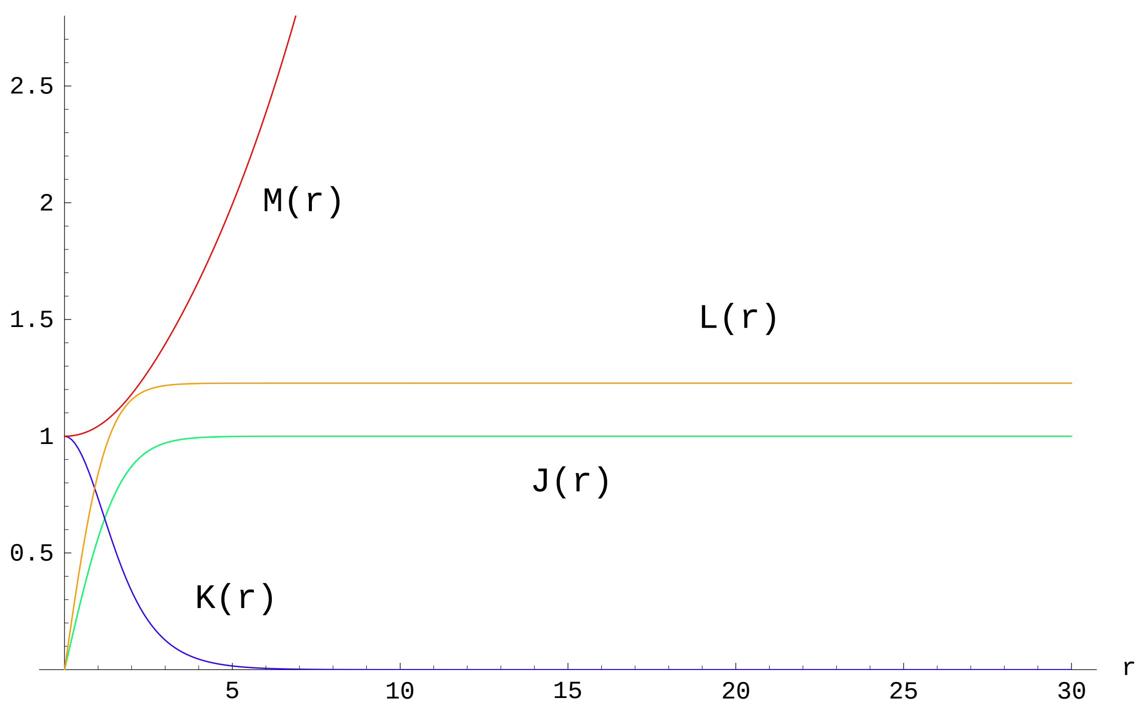

Figure 7: *Example of a solution that does not localize gravity* ($c < 0$) *corresponding to the parameter values* ($\alpha = 1.00000000$, $\beta = 1.80000000$, $\gamma = 0.04053600$)

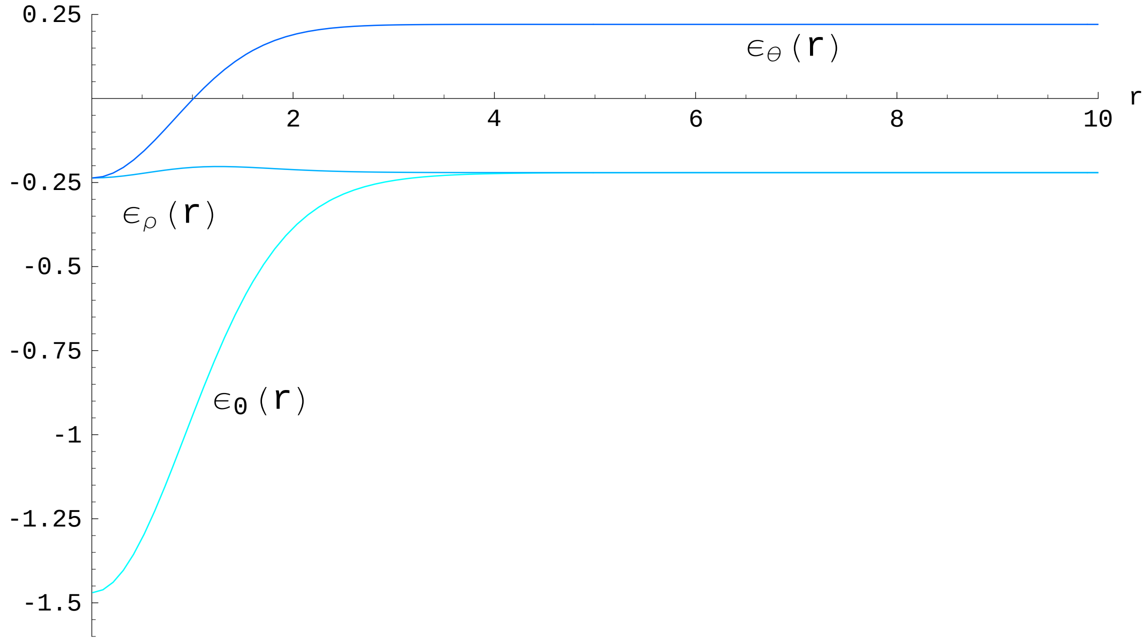

Figure 8: *Stress-energy components for the solution given in Fig. 7.*
($\alpha = 1.00000000$, $\beta = 1.80000000$, $\gamma = 0.04053600$)

4.2 The fine-tuning surface

Fig. 9 shows the fine-tuning surface in parameter-space. A point on this surface (α_0, β_0, γ_0) corresponds to a particular solution with the metric asymptotics (46) for both values of the sign of c. The bold line separates gravity localizing solutions $c > 0$ from solutions that do not localize gravity $c < 0$. The parameter space has been thoroughly exploited within the rectangles $\Delta\alpha \times \Delta\beta = [0,\, 10] \times [0.5,\, 4.0]$ and $\Delta\alpha \times \Delta\beta = [0,\, 1] \times [3.5,\, 10]$ in the (α, β)-plane. The series of solutions presented in section 4.1 can be used to illustrate their dependence on the parameter β (strength of gravity) for a for a fixed value of $\alpha = 1$.

It can be seen from Fig. 9 that for every fixed α there is a particular value of β such that c equals zero, which is the case for all points on the solid line shown in Fig. 9. By looking at eq. (53) we immediately see that $c = 0$ is equivalent to $\beta^2\gamma = \frac{\Lambda_7}{e^2 M_7^{10}} = \frac{1}{2}$. We will discuss the limit $c \ll 1$ in more detail in section 6. It will turn out to be the most physical case where the monopole can be considered to be point-like since the fields attain their vacuum values much earlier than the metric goes to zero outside the core. A solution corresponding to that case is given in Fig. 17, where gravity (parametrized by β) is just strong enough to provide a finite volume for transverse space. If for fixed value of α we increase β (starting from $c = 0$), c becomes more and more positive, the Planck mass becomes smaller and the monopole size increases, see Figs. 1 and 4. If on the other hand β is decreased (from $c = 0$ on), c becomes more and more negative and the metric $M(r)$ blows up exponentially, as in Fig. 7. Gravity is no longer strong enough to provide for a finite Planck mass.

For α=0 (the Prasad-Sommerfield limit, to be discussed in section 5) it can be read off from

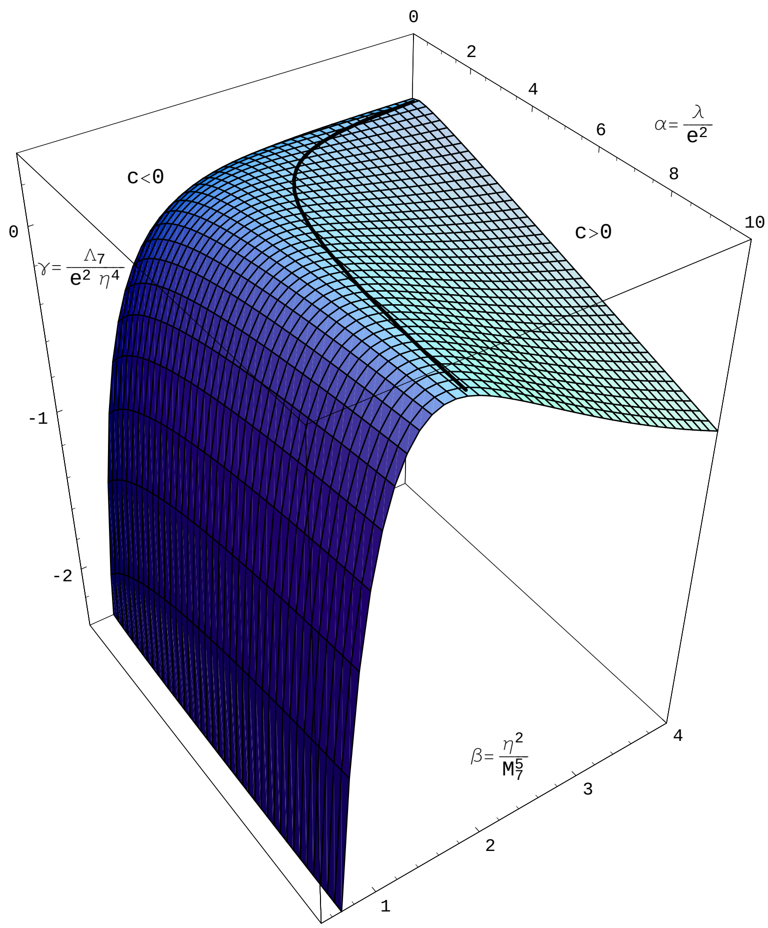

Figure 9: *Fine-tuning surface for solutions with the metric asymptotics (46). The bold line separates solutions that localize gravity* ($c > 0$) *from those that do not* ($c < 0$)*. Numerically obtained values of* $\gamma = \frac{\Lambda_7}{e^2\eta^4}$ *are plotted as a function of* $\alpha = \frac{\lambda}{e^2}$ *and* $\beta = \frac{\eta^2}{M_7^5}$.

Fig. 9 that there are no solutions that localize gravity. All $\alpha = 0$ solutions lie in the $c < 0$ part of the surface.

5 Prasad-Sommerfield limit ($\alpha = 0$)

The Prasad-Sommerfield limit ($\alpha = 0$) was exploited numerically for β-values ranging from 0.4 to about 70. The corresponding intersection of the fine-tuning surface Fig. 9 and the plane $\alpha = 0$ is given in Fig. 10. There exist no gravity localizing solutions as the separating line in Fig. 9 indicates. The point in Fig. 10 corresponds to the solution shown in Figs. 11 and 12. One sees that γ tends to zero for β going to infinity and that γ tends to $-\infty$ for β going to zero.

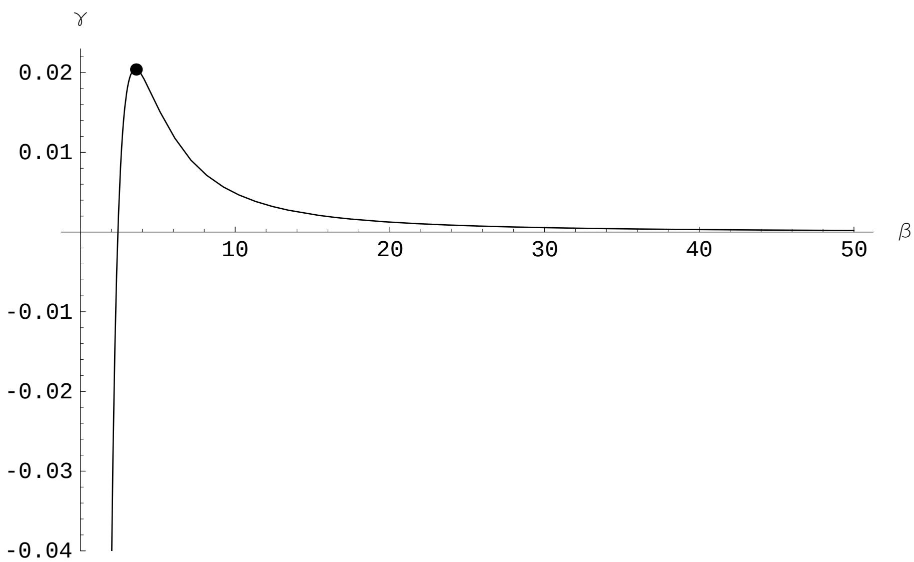

Figure 10: *Section of the fine-tuning surface Fig. 9 corresponding to the Prasad-Sommerfield limit $\alpha = 0$. None of the shown combinations of β and γ-values correspond to solutions that lead to warped compactification. The point indicates the sample solution given below.*

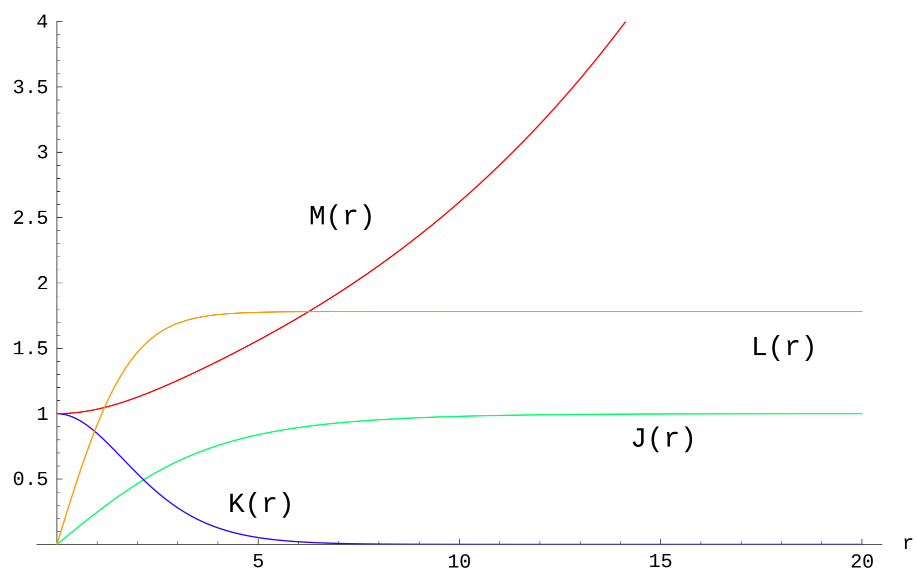

Figure 11: *Sample solution for the Prasad-Sommerfield limit corresponding to the parameter values* ($\alpha = 0.00000000$, $\beta = 3.60000000$, $\gamma = 0.02040333$)

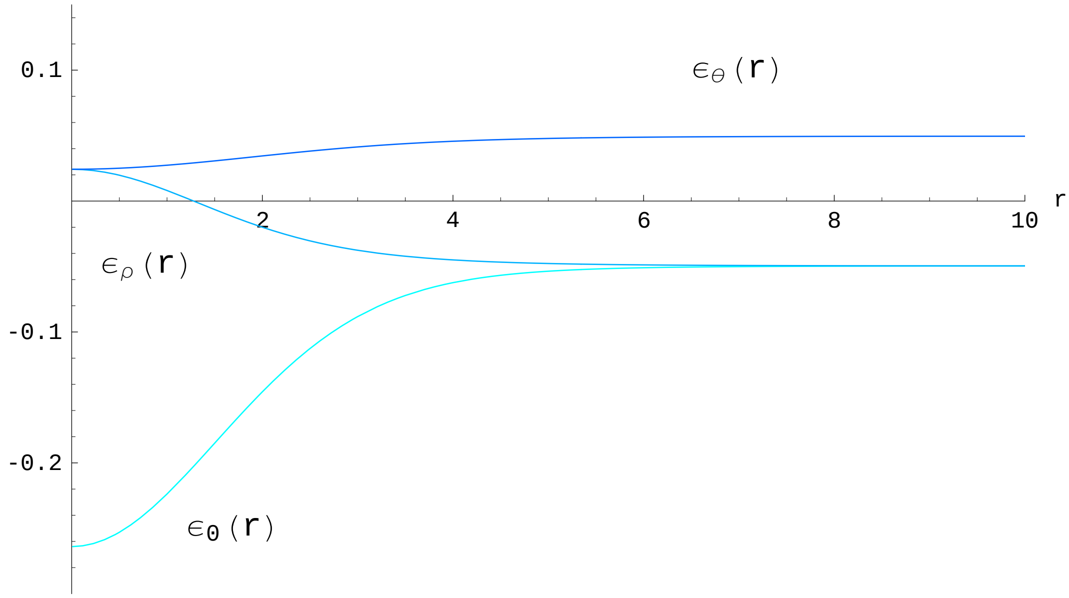

Figure 12: *Stress-energy components for the solution given in Fig. 11.*
($\alpha = 0.00000000$, $\beta = 3.60000000$, $\gamma = 0.02040333$)

6 A point-like monopole limit - physical requirements on solutions

The fine monopole can be characterized by $c \ll 1$. As anticipated in section 4.2, $c = 0$ in eq. (53) immediately leads to $\beta^2\gamma = \frac{\Lambda_7}{e^2 M_7^{10}} = \frac{1}{2}$. Hence we deduce that the fine-monopole limit can not be realized for a negative bulk cosmological constant. In addition, from (54) it follows that $\mathcal{L}_0 = \sqrt{\beta}$. The $c \ll 1$ limit is qualitatively different from its analogue in the $6D$-string case [19]. The solutions do not correspond to strictly local defects. The Einstein equations never decouple from the field equations. Due to the particular metric asymptotics (46), the stress-energy tensor components tend to constants at infinity in transverse space. In the $6D$-string case the fine-string limit was realized as a strictly local defect having stress energy vanishing exponentially outside the string core. Despite this difference the discussions of the physical requirements are very similar. In the following we show that in the fine monopole-limit the dimension-full parameters of the system Λ_7, M_7, m_W, λ, e can be chosen in such a way that all of the following physical requirements are simultaneously satisfied:

1. M_P^2 equals $(1.22 \cdot 10^{19}\text{GeV})^2$.
2. The corrections to Newtons law do not contradict latest measurements.
3. Classical gravity is applicable in the bulk.
4. Classical gravity is applicable in the monopole core ($r = 0$).

To find solutions with the above mentioned properties it is possible to restrict oneself to a particular value of α, e.g. $\alpha = \frac{1}{2}$. This choice corresponds to equal vector and Higgs masses $m_W = m_H$. Even though extra dimensions are infinite, the fact that $M(r)$ decreases exponentially permits the definition of an effective "size" r_0 of the extra dimensions:

$$M = M_0\, e^{-\frac{c}{2}r} = M_0\, e^{-\frac{c\, m_W}{2}\frac{r}{m_W}} \quad \Rightarrow \quad r_0 \equiv \frac{2}{c\, m_W}\,. \tag{61}$$

In order to solve the hierarchy problem in similar lines to [5] we parametrize the fundamental gravity scale as follows:

$$M_7 = \kappa\, 10^3\, \mathrm{GeV}\,. \tag{62}$$

$\kappa = 1$ then sets the fundamental scale equal to the electroweak scale.

1. The expression for the square of the Planck mass M_P^2 can be approximated in the fine-monopole limit by using the asymptotics (46) for the metric in the integral (34) rather than the exact (numerical) solutions. This gives

$$M_P^2 \approx \frac{4\pi M_7^5}{m_W^3}\, M_0^2\, \mathcal{L}_0^2\, \frac{1}{c}\,. \tag{63}$$

By using one of Einstein's equations at infinity

$$\mathcal{L}_0^2 = \frac{1}{4c^2 + 2\beta\gamma}\,, \tag{64}$$

and by developing to lowest order in c one finds

$$M_P^2 \approx \frac{4\pi M_7^5 M_0^2}{m_W^3}\frac{1}{2\beta\gamma}\left(\frac{1}{c} + \mathcal{O}(c)\right)\,. \tag{65}$$

Numerical solutions for $c \to 0$ and $\alpha = \frac{1}{2}$ converge to the following approximate parameter values

$$\begin{aligned} \beta &= 3.266281\ , \\ \gamma &= 0.0468665\ , \\ \beta^2\gamma &= 0.4999995\ , \\ M_0 &= 0.959721\ . \end{aligned} \tag{66}$$

The above values were obtained by extrapolation of solutions to $c = 0$, see Figs. 13 to 16. Therefore the relative errors are of order 10^{-6} which is considerably higher than average, relative errors from the integration which were at least 10^{-8}.

Note that $\beta^2\gamma$ tends to $\frac{1}{2}$ in the fine-monopole limit (see Fig. 15). Neglecting all orders different from $\frac{1}{c}$ in the above expression for M_P^2 and using (61) one has

$$m_W = 1.1 \cdot\, 10^{-5}\, \frac{\kappa^{5/2}}{\xi^{1/2}}\mathrm{GeV}\,, \tag{67}$$

where r_0 has been parametrized by $r_0 = \frac{0.2\,\mathrm{mm}}{\xi} \approx 10^{12}\,\mathrm{GeV}^{-1} \cdot \frac{1}{\xi}$.

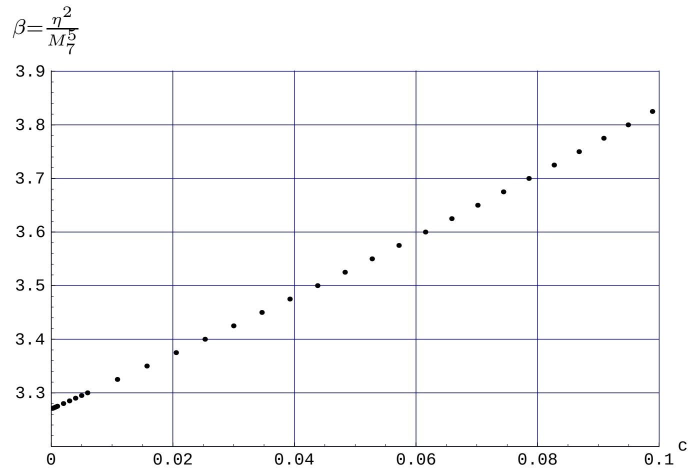

Figure 13: *Behavior of β in the fine-monopole limit $c \to 0$ for $\alpha = \frac{1}{2}$.*

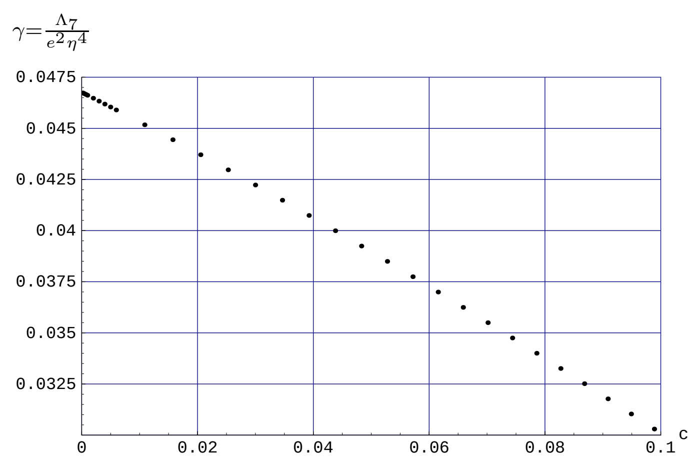

Figure 14: *Behavior of γ in the fine-monopole limit $c \to 0$ for $\alpha = \frac{1}{2}$.*

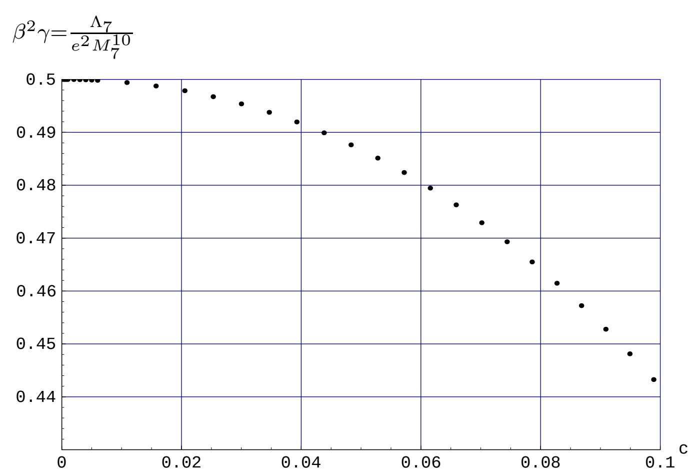

Figure 15: *Behavior of $\beta^2\gamma = \frac{\Lambda_7}{e^2\, M_7^{10}}$ in the fine-monopole limit $c \to 0$ for $\alpha = \frac{1}{2}$.*

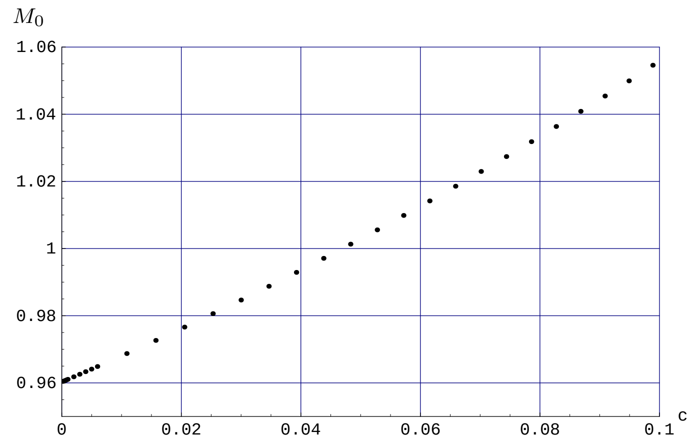

Figure 16: *Behavior of M_0 in the fine-monopole limit $c \to 0$ for $\alpha = \frac{1}{2}$.*

2. Since Newtons law is established down to 0.2 mm [40] we simply need to have $\xi > 1$.

3. In order to have classical gravity applicable in the bulk we require that curvature at infinity is negligible with respect to the corresponding power of the fundamental scale. By looking at eqs. (49)-(52) we see that we have to impose

$$c^2\, m_W^2 \ll M_7^2 \quad \text{and} \quad \frac{m_W^2}{\mathcal{L}_0^2} \ll M_7^2\,. \tag{68}$$

   Using (64) and the first of the above relations, the second one can immediately be transformed into

$$\beta\,\gamma\, m_W^2 \ll M_7^2\,. \tag{69}$$

   One then finds

$$2 \cdot 10^{-15} \frac{\xi}{\kappa} \ll 1 \quad \text{and} \quad 4 \cdot 10^{-3} \left(\frac{\kappa^3}{\xi}\right)^{1/2} \ll 1\,, \tag{70}$$

   where β and γ have again been replaced by their limiting values (66) for $c \to 0$.

4. Classical gravity is applicable in the monopole core whenever the curvature invariants R^2, $R_{AB}R^{AB}$, $R_{ABCD}R^{ABCD}$ and $C_{ABCD}C_{ABCD}$ are small compared to the forth power of the fundamental gravity scale. Since these quantities are of the order of the mass m_W^4 (see (42)) we have

$$m_W^4 \ll M_7^4\,. \tag{71}$$

   Using (62) and (67) one finds

$$1.1 \cdot 10^{-8} \left(\frac{\kappa^3}{\xi}\right)^{1/2} \ll 1\,. \tag{72}$$

Finally, the fine-monopole limit requires

$$c = \frac{2}{r_0\, m_W} \ll 1\,. \tag{73}$$

Again (67) implies

$$1.8 \cdot 10^{-7} \left(\frac{\xi^3}{\kappa^5}\right)^{1/2} \ll 1\,. \tag{74}$$

It is now easy to see that for a wide range of parameter combinations all these requirements on ξ and κ can simultaneously be satisfied. One possible choice is $\xi = 100$ and $\kappa = 1$. This shows that already in the case $\alpha = \frac{1}{2}$ there are physical solutions corresponding to a fine-monopole in the sense of eq. (73) respecting all of the above requirements (1)-(4). Fig. 17 shows the fine-monopole solution corresponding to the lowest c-value in the sequence shown in Figs. 13 to 16.

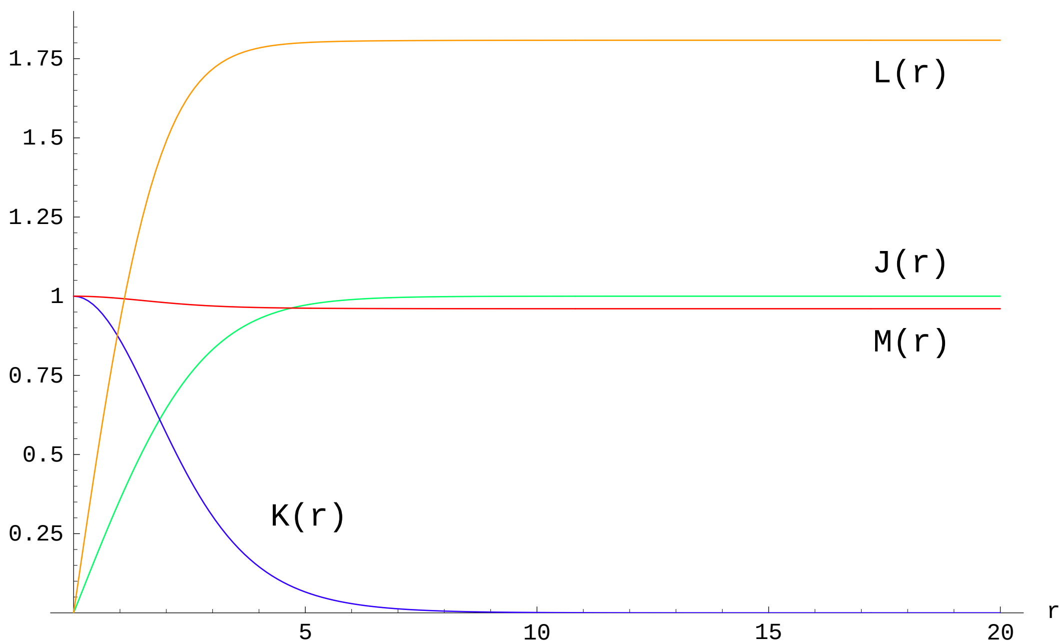

Figure 17: *Gravity-localizing solution in the fine-monopole limit for* $c = 1.728 \cdot 10^{-5}$ *and corresponding parameter values of* ($\alpha = 0.50000000$, $\beta = 3.27000000$, $\gamma = 0.04676000$)

7 Conclusion

We have demonstrated in this paper that it is possible to generalize the idea of warped compactification on a topological defect in a higher dimensional spacetime to $n = 3$ transverse dimensions by considering a specific field theoretical model. Numerical solutions were found for the case of a monopole realized as a 't Hooft-Polyakov monopole. This generalization turned out to be non-trivial since (at least) in the rotational invariant transverse space setup considered, Einstein's equations don't seem to admit strictly local defect solutions. Even though the transverse space still approaches a constant curvature space at spatial infinity, it is not necessarily an anti-de-Sitter space. Both signs of the bulk cosmological constant are possible in order to localize gravity. We considered a fine monopole limit in the case $\alpha = 1/2\,(m_W = m_H)$ and verified that the model proposed is not in conflict with Newtons law, that it leads to a possible solution of the hierarchy problem and that classical gravity is applicable in the bulk and in the core of the defect. Even though stability should be guaranteed by topology, small perturbations around the monopole background should be considered as well as quadratic corrections to the Einstein-Hilbert action.

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A Derivation of Fine-tuning relations

By taking linear combinations of Einstein equations (15) to (17) one can easily derive the following relations:

$$\frac{[M(r)^3 M'(r)\mathcal{L}(r)^2]'}{M(r)^4\mathcal{L}(r)^2} = -\frac{\beta}{5}\left(2\gamma + \epsilon_0 - \epsilon_\rho - 2\epsilon_\theta\right)\,, \tag{75}$$

$$\frac{[M(r)^4\mathcal{L}(r)\mathcal{L}'(r)]'}{M(r)^4\mathcal{L}^2(r)} - \frac{1}{\mathcal{L}(r)^2} = -\frac{\beta}{5}\left(2\gamma - 4\epsilon_0 - \epsilon_\rho + 3\epsilon_\theta\right)\,. \tag{76}$$

By multiplying with $M(r)^4\mathcal{L}(r)^2$, integrating from 0 to ∞ and using the definition of the brane tensions (26) we obtain

$$M(r)^3 M'(r)\mathcal{L}(r)^2|_0^\infty = -\frac{2\beta\gamma}{5}\int_0^\infty M(r)^4\mathcal{L}(r)^2 dr + \frac{\beta}{5}\left(\mu_0 - \mu_\rho - 2\mu_\theta\right)\,, \tag{77}$$

$$M(r)^4\mathcal{L}(r)\mathcal{L}'(r)|_0^\infty = \int_0^\infty M(r)^4 dr - \frac{2\beta\gamma}{5}\int_0^\infty M(r)^4\mathcal{L}(r)^2 dr - \frac{\beta}{5}\left(4\mu_0 + \mu_\rho - 3\mu_\theta\right)\,. \tag{78}$$

Using the boundary conditions for the metric functions (31) and (46), and taking the difference of the eqs. (77) and (78) then establishes the first part of eq. (58). To prove the second part of (58) one starts right from the general expressions of the stress-energy components ϵ_i, relations (23)-(25):

$$\mu_0 - \mu_\theta = \int_0^\infty dr M(r)^4\mathcal{L}(r)^2\left(\epsilon_\theta - \epsilon_0\right) = \int_0^\infty dr M(r)^4\mathcal{L}(r)^2\left\{\frac{K'(r)^2}{\mathcal{L}(r)^2} + \frac{\left[1 - K(r)^2\right]^2}{\mathcal{L}(r)^4} + \frac{J(r)^2K(r)^2}{\mathcal{L}(r)^2}\right\}\,. \tag{79}$$

Multiplying the equation of motion for the gauge field (19) by $K(r)$ and substituting the $J(r)^2K(r)^2$ term gives

$$\mu_0 - \mu_\theta = \int_0^\infty dr M(r)^4 \left\{ K'(r)^2 + \frac{1-K(r)^2}{\mathcal{L}(r)^2} + \frac{\left[M(r)^4 K'(r)\right]' K(r)}{M(r)^4} \right\} . \tag{80}$$

Integration by parts in the last term of the above equation leads to

$$\mu_0 - \mu_\theta = \int_0^\infty dr M(r)^4 \left[\frac{1-K(r)^2}{\mathcal{L}(r)^2} \right] + K(r) M(r)^4 K'(r)|_0^\infty . \tag{81}$$

which together with the behavior of the gauge field at the origin $K'(0) = 0$ (see (39)) finishes the proof of relation (58).

The proof of relation (59) is simply obtained by rewriting (77) with vanishing left hand side.

To establish (60) we start directly from the definitions of the stress-energy tensor components, relations (23)-(25):

$$\mu_0 + \mu_\rho + 2\mu_\theta = \int_0^\infty dr M(r)^4 \mathcal{L}(r)^2 \left[J'(r)^2 + \frac{2\, J(r)^2 K(r)^2}{\mathcal{L}(r)^2} + \alpha \left(J(r)^2 - 1 \right)^2 \right] . \tag{82}$$

Collecting derivatives in the equation of motion for the scalar field (18) and multiplying by $J(r)$ gives

$$\frac{2\, J(r)^2 K(r)^2}{\mathcal{L}(r)^2} = \frac{\left[M(r)^4 \mathcal{L}(r)^2 J'(r)\right]'}{M(r)^4 \mathcal{L}(r)^2} J(r) - \alpha J(r)^2 \left(J(r)^2 - 1 \right) . \tag{83}$$

Eliminating now the second term in the equation (82) leads to

$$\mu_0 + \mu_\rho + 2\mu_\theta = \int_0^\infty dr M(r)^4 \mathcal{L}(r)^2 \left[J'(r)^2 + \frac{\left[M(r)^4 \mathcal{L}(r)^2 J'(r)\right]'}{M(r)^4 \mathcal{L}(r)^2} J(r) + \alpha \left(1 - J(r)^2 \right) \right] . \tag{84}$$

If we now expand and integrate the second term by parts we are left with

$$\mu_0 + \mu_\rho + 2\mu_\theta = \alpha \int_0^\infty dr \left(1 - J(r)^2 \right) M(r)^4 \mathcal{L}(r)^2 + M(r)^4 \mathcal{L}(r)^2 J(r) J'(r)|_0^\infty , \tag{85}$$

which reduces to (60) when the boundary conditions for the metric (31) and (46) are used.

B Numerics

As already pointed out, the numerical problem encountered is to find those solutions to the system of differential equations (15)-(19) and boundary conditions for which the integral defining the 4-dimensional Planck-scale (34) is finite. This is a two point boundary value problem on the interval $r = [0, \infty)$ depending on three independent parameters (α, β, γ). Independently of the numerical method employed, the system of equations (15)-(19) was rewritten in a different way in order for the integration to be as stable as possible. By introducing the derivatives of the unknown functions $J(r)$, $K(r)$, $M(r)$ and $\mathcal{L}(r)$ as new dependent variables one obtains a system of ordinary first order equations. In the case of $M(r)$ it has proven to be convenient to define $y_7 = M'(r)/M(r)$ as a new unknown function (rather than $M'(r)$) since the boundary condition for y_7 at infinity then simply reads

$$\lim_{r\to\infty} y_7 = -\frac{c}{2}\,. \tag{86}$$

With the following definitions we give the form of the equations which is at the base of several numerical methods employed:

$$\begin{aligned}
y_1(r) &= J(r)\,, & y_1' &= y_5\,,\\
y_2(r) &= K(r)\,, & y_2' &= y_6\,,\\
y_3(r) &= M(r)\,, & y_3' &= y_3\,y_7\,,\\
y_4(r) &= \mathcal{L}(r)\,, & y_4' &= y_8\,,\\
y_5(r) &= J'(r)\,, & y_5' &= -2\left(2\,y_7+\tfrac{y_8}{y_4}\right)y_5+2\,\tfrac{y_1y_2^2}{y_4^2}+\alpha\,y_1\left(y_1^2-1\right)\,,\\
y_6(r) &= K'(r)\,, & y_6' &= -\tfrac{y_2\left(1-y_2^2\right)}{y_4^2}-4y_6y_7+y_1^2y_2\,,\\
y_7(r) &= M'(r)/M(r)\,, & y_7' &= -4y_7^2-2y_7\tfrac{y_8}{y_4}-\tfrac{\beta}{5}\left(2\gamma+\epsilon_0-\epsilon_\rho-2\epsilon_\theta\right)\,,\\
y_8(r) &= \mathcal{L}'(r)\,, & y_8' &= -\tfrac{y_8^2}{y_4}-4y_7y_8+\tfrac{1}{y_4}-\tfrac{\beta y_4}{5}\left(2\gamma-4\epsilon_0-\epsilon_\rho+3\epsilon_\theta\right)\,,
\end{aligned} \tag{87}$$

with

$$\epsilon_0-\epsilon_\rho-2\epsilon_\theta = \frac{\alpha}{2}\left(y_1^2-1\right)^2-\frac{\left(1-y_2^2\right)^2}{y_4^4}-2\,\frac{y_6^2}{y_4^2}\,, \tag{88}$$

$$-4\epsilon_0-\epsilon_\rho+3\epsilon_\theta = \frac{\alpha}{2}\left(y_1^2-1\right)^2+4\,\frac{\left(1-y_2^2\right)^2}{y_4^4}+5\,\frac{y_1^2\,y_2^2}{y_4^2}+3\,\frac{y_6^2}{y_4^2}\,. \tag{89}$$

This is an autonomous ordinary system of coupled differential equations depending on the parameters $(\alpha,\,\beta,\,\gamma)$. The boundary conditions are

$$\begin{aligned}
y_1(0) &= 0\,, & y_2(0) &= 1\,, & y_3(0) &= 1\,, & y_4(0) &= 0\,,\\
\lim_{r\to\infty} y_1(r) &= 1\,, & \lim_{r\to\infty} y_2(r) &= 0\,, & y_7(0) &= 0\,, & y_8(0) &= 1\,.
\end{aligned} \tag{90}$$

In order to find solutions with the desired metric asymptotics at infinity it is useful to define either one (or more) of the parameters $(\alpha,\,\beta,\,\gamma)$ or the constants $J'(0)$ and $K''(0)$ as additional dependent variables, e.g. $y_9(r) = \alpha$ with $y_9'(r) = 0$, see [41]. Before discussing the different methods that were used we give some common numerical problems encountered.

- Technically it is impossible to integrate to infinity. The possibility of compactifying the independent variable r was not believed to simplify the numerics. Therefore the integration has to be stopped at some upper value of $r = r_{\max}$. For most of the solutions this was about $20 \sim 30$.

- Forward integration with arbitrary but fixed values of $(\alpha,\, \beta,\, \gamma,\, J'(0)\,, K''(0))$ turned out to be very unstable. This means that even before some integration routine (e.g. the Runge-Kutta method [41]) reached $r_{\max}$, the values of some y_i went out of range which was due to the presence of terms $\frac{1}{(\mathcal{L})^n}$ or quadratic and cubic (positive coefficient) terms in y_i.

- Some right hand sides of (87) contain terms singular at the origin such that their sums remain regular. Starting the integration at $r = 0$ is therefore impossible. To overcome this problem the solution in terms of the power series (36)-(39) was used within the interval $[0,\, \epsilon]$ (for $\epsilon = 0.01$).

- The solutions are extremely sensitive to initial conditions which made it unavoidable to pass from single precision to double precision (from about 7 to about 15 significant digits). However this didn't completely solve the problem. Even giving initial conditions (corresponding to a gravity localizing solution) at the origin to machine precision is in general not sufficient to obtain satisfactory precision at $r_{\max}$ in a single Runge-Kutta forward integration step from ϵ to $r_{\max}$.

The method used for exploiting the parameter space $(\alpha,\, \beta,\, \gamma)$ for gravity localizing solutions was a generalized version of the shooting method called the multiple shooting method [41], [42], [43], [44]. In the shooting method a boundary value problem is solved by combining a root-finding method (e.g. Newton's method [41]) with forward integration. In order to start the integration at one boundary, the root finding routine specifies particular values for the so-called shooting parameters and compares the results of the integration with the boundary conditions at the other boundary. This method obviously fails whenever the "initial guess" for the shooting parameters is too far from a solution such that the forward integration does not reach the second boundary. For this reason the multiple shooting method was used in which the interval $[\epsilon\,, r_{\max}]$ was divided into an variable number of sub-intervals in each of which the shooting method was applied. This of course drastically increased the number of shooting parameters and as a result the amount of computing time. However, it resolved two problems:

- Since the shooting parameters were specified in all sub-intervals the precision of the parameter values at the origin was no longer crucial for obtaining high precision solutions.

- Out of range errors can be avoided by augmenting the number of shooting intervals (at the cost of increasing computing time).

Despite all these advantages of multiple shooting, a first combination of parameter values $(\alpha,\, \beta,\, \gamma,\, J'(0)\,, K''(0))$ leading to gravity localization could not be found by this method, since convergence depends strongly on how close initial shooting parameters are to a real solution. This first solution was found by backward integration combined with the simplex method for finding zeros of one real function of several real variables. For a discussion of the simplex method see e.g. [41]. This solution, corresponds to the parameter values ($\alpha = 1.429965428$, $\beta = 3.276535576$, $\gamma = 0.025269415$).

Once this solution was known, it was straightforward to investigate with the multiple shooting method which subset of the (α, β, γ)-space leads to the desired metric asymptotics. We used a known solution (α_1, β_1, γ_1) to obtain starting values for the shooting parameters of a closeby other solution ($\alpha_1 + \delta\alpha$, $\beta_1 + \delta\beta$, $\gamma_1 + \delta\gamma$). This lead in general to rapid convergence of the Newton-method. Nevertheless, $\delta\alpha$, $\delta\beta$, $\delta\gamma$ still had to be small. By this simple but time-consuming operation the fine-tuning surface in parameter-space, presented in section 4.2 and shown in Fig. 9 was found.

References

[1] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **B125** (1983) 136.

[2] K. Akama, in *Proceedings of the Symposium on Gauge Theory and Gravitation*, Nara, Japan, eds. K. Kikkawa, N. Nakanishi and H. Nariai (Springer-Verlag, 1983), [hep-th/0001113].

[3] *Modern Kaluza-Klein Theories*, eds.T. Appelquist, A. Chodos and P. G. Freund, (Addison-Wesley, 1987).

[4] G. Dvali, M. Shifman, Phys. Lett. **B396** (1997) 64-69; [Erratum-ibid. **B407** (1997) 452]

[5] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **429** (1998) 263 [arXiv:hep-ph/9803315].

[6] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 4690[hep-th/9906064].

[7] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999)3370 [hep-ph/9905221].

[8] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **125** (1983) 139.

[9] S. Randjbar-Daemi and C. Wetterich, Phys. Lett. B **166** (1986) 65.

[10] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and R. Sundrum, Phys. Lett. B **480** (2000) 193 [arXiv:hep-th/0001197].

[11] G. Dvali, G. Gabadadze and M. Shifman, arXiv:hep-th/0202174.

[12] J. Polchinski, Phys. Rev. Lett. **75** (1995) 4724 [hep-th/9510017].

[13] O. DeWolfe, D. Z. Freedman, S. S. Gubser and A. Karch, Phys. Rev. D **62** (2000) 046008 [arXiv:hep-th/9909134].

[14] A. G. Cohen and D. B. Kaplan, Phys. Lett. **B470** (1999) 52 [hep-th/9910132].

[15] A. Chodos and E. Poppitz, Phys. Lett. **B471** (1999) 119 [hep-th/9909199].

[16] I. Olasagasti and A. Vilenkin, Phys. Rev. D **62** (2000) 044014 [arXiv:hep-th/0003300].

[17] R. Gregory, Phys. Rev. Lett. **84** (2000) 2564 [hep-th/9911015].

[18] T. Gherghetta and M. E. Shaposhnikov, Phys. Rev. Lett. **85** (2000) 240 [arXiv:hep-th/0004014].

[19] M. Giovannini, H. Meyer and M. E. Shaposhnikov, Nucl. Phys. B **619** (2001) 615 [arXiv:hep-th/0104118].

[20] P. Tinyakov, K. Zuleta, Phys. Rev. D **64** (2001) 025022.

[21] S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. B **214** (1983) 491.

[22] S. Randjbar-Daemi, A. Salam and J. Strathdee, Phys. Lett. B **132** (1983) 56.

[23] T. Gherghetta, E. Roessl, M. E. Shaposhnikov, Phys. Lett. **B491** (2000) 353.

[24] G. R. Dvali, arXiv:hep-th/0004057.

[25] K. Benson, I. Cho Phys. Rev. **D64** (2001) 065026

[26] S. Randjbar-Daemi and M. E. Shaposhnikov, Phys. Lett. B **491** (2000) 329 [arXiv:hep-th/0008087].

[27] S. Randjbar-Daemi and M. E. Shaposhnikov, Phys. Lett. B **492** (2000) 361 [arXiv:hep-th/0008079].

[28] G. 't Hooft, Nucl. Phys. **B79** (1974) 276.

[29] A.M. Polyakov, JETP Lett. **20**, (1974) 194

[30] P. van Nieuwenhuizen, D. Wilkinson and M. J. Perry, Phys. Rev. **D13** (1976) 778.

[31] F. A. Bais and R. J. Russell Phys. Rev. **D11** (1975) 2692.

[32] K. Lee, V.P. Nair and E.J. Weinberg, Phys. Rev. **D45** (1992) 2751.

[33] M. E. Ortiz, Phys. Rev. **D45** (1992) R2586.

[34] P. Breitenlohner, P. Forgcs and D. G. Maison, Nucl. Phys. **B383** (1992) 357.

[35] C. W. Misner, K. S. Thorne and J. A. Wheeler, "Gravitation" , Freeman and Company, San Francisco (1973)

[36] M. Georgi, S. L. Glashow, Phys. Rev. Lett. **28** (1972) 1494.

[37] A. Vilenkin, E. P. S. Shellard, "Cosmic Strings and Other Topological Defects", Cambridge Monographs on Mathematical Physics (1994)

[38] R. Rajaraman, "Solitons and Instantons, An Introduction to Solitons and Instantons in Quantum Field Theory", North Holland (1984)

[39] M.K. Prasad, C.H. Sommerfield, Phys. Rev. Lett. **35** (1975) 760

[40] C. D. Hoyle et al., Phys. Rev. Lett. **86**, 1418 (2001).

[41] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery, "Numerical Recipes in C, The Art of Scientific Computing", Cambridge University Press, Second Edition (1992)

[42] F. S. Acton, "Numerical Methods That Work", Washington: Mathematical Association of America, corrected edition 1990.

[43] H. B. Keller, "Numerical Methods For Two-Point Boundary-Value Problems", Waltham, MA: Blaisdell, 1968

[44] J. Stoer, R. Bulirsch, "Introduction to Numerical Analysis" New York: Springer-Verlag, 1980