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SINGULARLY PERTURBED PIECEWISE DETERMINISTIC GAMES*

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Abstract. In this paper we consider a class of hybrid stochastic games with the piecewise open-loop information structure. These games are indexed over a parameter ε which represents the time scale ratio between the stochastic (jump process) and the deterministic (differential state equation) parts of the dynamical system. We study the limit behavior of Nash equilibrium solutions to the hybrid stochastic games when the time scale ratio tends to 0. We also establish that an approximate equilibrium can be obtained for the hybrid stochastic games using a Nash equilibrium solution of a reduced order sequential discrete state stochastic game and a family of local deterministic infinite horizon open-loop differential games defined in the stretched out time scale. A numerical illustration of this approximation scheme is also developed.

Key words. n-person games, noncooperative games, stochastic games, differential games, time scale analysis, singular perturbations

AMS subject classifications. 91A06, 91A10, 91A15, 91A23, 93C70

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1. Introduction.

This paper deals with a class of piecewise deterministic stochastic games where the stochastic jump process has a slower time scale than the deterministic continuous time control systems that are defined between successive random jumps. These types of games may occur, for example, in imperfect competition models where the deterministic subsystem describes the productive capital accumulation of the firms competing on a market and where the market conditions are subject to infrequent random switches that are influenced by the actions of the economic agents. Situations where oligopolistic markets can be subject to abrupt modal changes are observed, for example, in the energy sector or in the new technology or telecommunication domains. Another interesting domain where this type of paradigm could be used is the modeling of economic dimensions of climate change. The fast modes would correspond to the competitive economic growth processes of different world economies, whereas the slow modes would be associated with different climate conditions. Indeed the transition from a climate mode to a different one would be influenced by the global emissions of greenhouse gases from all nations.

The information structure that we consider for these games is called *piecewise open-loop*; it has been introduced in [7] and consists in playing open-loop controls between successive jump times; the open-loop controls are adapted to the history of jump times and system states observed at jump times. It has been recognized that these piecewise deterministic games, when played under the piecewise open-loop information structure, are akin to the general class of stochastic sequential games

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with general (Borel) action and state spaces, considered in particular in [17] and [15], where existence and approximation of Nash equilibria have been studied. We also suppose that the players have separated deterministic dynamics and are linked only through the payoff rewards at that level. This means that between two successive modal changes each player $j = 1, \dots, m$ selects a deterministic trajectory $x_j(\cdot)$ that has a given initial state determined by the state at the time of the first jump and which will be in force until the second jump occurs. We assume that the m state variables x_j, $j = 1, \dots, m$, together influence the jump rates of the discrete modal stochastic process.

The aim of this paper is to develop a theory of approximation for a Nash equilibrium solution to this class of differential games when the time scale ratio between the (continuous) deterministic and (jump) stochastic dynamics tends to 0. This is the realm of *singular perturbation* theory in control. In [6] one can find a rather complete theory of singularly perturbed piecewise deterministic control systems and an illustration of the role played by "turnpikes" (i.e., global attractors for optimal trajectories in infinite horizon control problems) in the definition of the "limit control problem." The initial objective of this paper was to explore the possibility of extending the results obtained in [6] to the case of piecewise open-loop Nash equilibria. As is often the case when one passes from a single optimizer control formalism to the context of dynamic games with a Nash equilibrium solution, these results cannot be readily generalized, and we obtain substantially different and weaker limit theorems.

In the present paper we consider only the case of infinite horizon hybrid games with discounted payoffs. In brief the contribution of this paper can be stated as follows: (i) We define, in the slow time scale, a limit game problem in the form of a controlled Markov chain. A Nash equilibrium for this limit game, if it exists, will serve to build an approximate equilibrium solution of the original game. (ii) We prove that given a Nash equilibrium for the limit game, defined in terms of attractors for the players $x_j(\cdot)$-trajectories, one can construct a ς-equilibrium for any G^ε game, where ε is small enough by using strategies characterized by a uniform attractor property. (iii) Having solved the limit Nash game, we can use the associated potential function to characterize a set of local infinite horizon open-loop games, whose Nash equilibria satisfy the uniform attractor property. (iv) We thus derive a decomposition principle for this class of games and illustrate it on a numerical example.

The paper is organized as follows: in section 2 we recall the definition of a piecewise deterministic game played with piecewise open-loop strategies; in section 3 we study the limit game when the time scale ratio ε between fast and slow modes tends to 0; in section 4 we propose a method to construct a ς-equilibrium solutions for the hybrid game, using uniform attractor policies; in section 5 we study a class of local open-loop games for which the Nash equilibrium strategies satisfy the uniform attractor property; in section 6 we derive from these results a decomposition principle; and in section 7 we provide a numerical illustration of these limit properties and sketch an economic model of climate change policies having this two–time scale structure; in section 8 we summarize what has been achieved.

2. A class of piecewise deterministic games. In this section we define the class of dynamic games that are considered in this work. They are particular instances of piecewise deterministic games, as introduced in [7]. We use both a formalism of control systems and a formalism of calculus of variations, very much in the same way as in [4], where the so-called turnpike property for open-loop differential games with decoupled dynamics was established.

2.1. The dynamics. Consider m players, denoted $j \in M = \{1, \ldots, m\}$, controlling a system that has p discrete modes denoted $i \in I = \{0, \ldots, p-1\}$. Each player $j \in M$ also controls her own dynamical subsystem with mode-dependent dynamics,

$$\varepsilon \dot{x}_j(t) = f_j^i(x_j(t), u_j(t)), \tag{2.1}$$

$$u_j(t) \in U_j^i \subset \Re^{\nu_j}, \tag{2.2}$$

$$x_j(t) \in X_j \subset \Re^{n_j}, \tag{2.3}$$

where the control sets U_j^i are compacts and the state sets X_j are bounded. The functions $f_j^i(x_j, u_j)$ are supposed to satisfy the usual regularity assumptions made in control theory. Here ε is a parameter that will eventually be very small. We denote $\underline{x} = (x_j : j \in M) \in \underline{X}$ and $\underline{u} = (u_j : j \in M) \in \underline{U}$ to be the state and control vectors, respectively.

The "mode" dynamics is represented by a continuous time jump process $\xi(\cdot)$, with state set I and transition rates

$$q_{k\ell}(\underline{x}(t)) = \lim_{dt \to 0} \frac{1}{dt} \mathrm{P}[\xi(t+dt) = \ell \,|\, \xi(t) = k, \underline{x}(t)], \quad k, \ell \in I, \underline{x}(t) \in \underline{X},$$

that depend on the trajectory choice made by all players. Indeed we assume $q_{k\ell}(\underline{x}) \geq 0$ if $k \neq \ell$ and $q_{kk}(\underline{x}) = -\sum_{\ell \neq k} q_{k\ell}(\underline{x})$. As usual we introduce the notation

$$q^k(\underline{x}) \doteq \sum_{\ell \neq k} q_{k\ell}(\underline{x}).$$

Remark 1. The parameter ε represents the time scale ratio. Its inverse $1/\varepsilon$ is therefore a speed of adjustment factor for the deterministic part of the hybrid system; when $\varepsilon \to 0$ the deterministic part of the system is allowed to adjust much faster than the stochastic jumps occurrences.

It will be convenient to use a calculus of variations formalism, obtained in the following way: We assume that at time t the reward rate to Player j, when the mode is i, is given by a function $L_j^i(\underline{x}(t), u_j(t))$ which is C^1 in $\underline{x}$ and continuous in u_j. Let $F_j^i(z_j, x_j) = \{u_j \in U_j^i : z_j = f_j^i(x_j, u_j)\}$ be the set of controls for Player j that yield a velocity z_j at state x_j. We introduce a function $\mathcal{L}_j^i(\underline{x}, z_j)$ that associates a value in $\Re \cup \{-\infty\}$ with every $\underline{x} \in \Pi_{l=1}^m X_l \subset \Re^n$, $n = \sum_{j \in M} n_j$, and $z_j \in \Re^{n_j}$ as follows:

$$\mathcal{L}_j^i(\underline{x}, z_j) = \begin{cases} -\infty & \text{if } x_j \notin X_j \text{ or } F_j^i(z_j, x_j) = \emptyset; \\ \sup\{L_j^i(\underline{x}, u_j) : u_j \in F_j^i(z_j, x_j)\} & \text{otherwise.} \end{cases} \tag{2.4}$$

We can now consider a dynamic game where each player $j \in M = \{1, \ldots, m\}$ controls an absolutely continuous trajectory $x_j(\cdot)$ with state $x_j(t) \in X_j$ at time $t \in [0, \infty)$, where X_j is a compact subset in $\Re^{n_j}$.

The game is played as follows: at jump times $\tau^0 = 0, \tau^1, \ldots, \tau^\nu, \ldots$ of the $\xi(\cdot)$ process the players observe the state of the system, i.e., the pair $s^\nu = (\xi^\nu, \underline{x}^\nu)$, where $\xi^\nu = \xi(\tau^\nu)$ and $\underline{x}^\nu = \underline{x}(\tau^\nu)$. Then Player j selects an absolutely continuous function $y_j : [0, \infty) \to X_j$, with initial condition $y_j(0) = x_j^\nu$. The trajectory for Player j will thus be defined, between jump times τ^ν and $\tau^{\nu+1}$, by $x_j(t) = y_j(t - \tau^\nu)$. We denote by $\mathcal{X}_j$ the class of admissible functions $y_j : [0, \infty) \to X_j$ that serve to define the action set of Player j. A piecewise open-loop strategy for Player j is then defined as a mapping $\gamma_j : (\tau^\nu, s^\nu) \mapsto \mathcal{X}_j$.

At time t the reward rate to Player j is given by $\mathcal{L}_j^{\xi(t)}(\underline{x}(t), \varepsilon \dot{x}_j(t))$ and depends on the current mode $\xi(t)$, the state vector $\underline{x}(t) = (x_j(t))_{j \in M}$, and the time derivative $\dot{x}_j(\cdot)$ of Player j's own trajectory multiplied by the time scale ratio ε.

2.2. The hybrid game G^ε. We call *hybrid game* G^ε the game in normal form where the players select piecewise open-loop strategies as defined above and obtain payoffs defined as follows:

Let ρ_j be the discount rate of Player j. Associated with a strategy m-tuple $\underline{\gamma} = \{\gamma_j : j \in M\}$ the payoffs to the players are given by

$$V_j^\varepsilon(\underline{\gamma}; i, \underline{x}^o) = \mathrm{E}_{\underline{\gamma}}\left[\int_0^\infty e^{-\rho_j t}\mathcal{L}_j^{\xi(t)}(\underline{x}(t), \varepsilon\dot{x}_j(t))\, dt \mid (\xi(t^0) = i, \underline{x}(t^0) = \underline{x}^o\right]$$

$$j \in M, \quad (i, \underline{x}^o) \in I \times \underline{X}, \tag{2.5}$$

where $\mathrm{E}_{\underline{\gamma}}$ is the expectation given the probability measure induced by the strategy vector $\underline{\gamma}$.

Definition 2.1. (i) *A strategy m-tuple $\underline{\gamma}^*$ is a ς-equilibrium, with $\varsigma \geq 0$ given, if*

$$V_j^{\varepsilon *}(i, \underline{x}^o) = V_j^\varepsilon(\underline{\gamma}^*; i, \underline{x}^o) \geq V_j^\varepsilon([\underline{\gamma}^*_{M-j}, \gamma_j]; i, \underline{x}^o) - \varsigma \quad \forall \gamma_j \in \Gamma_j$$

$$j \in M, \quad (i, \underline{x}^o) \in I \times \underline{X}, \tag{2.6}$$

where $[\underline{\gamma}^*_{M-j}, \gamma_j]$ denotes the strategy vector obtained from $\underline{\gamma}^*$ when only Player j unilaterally changes her strategy to γ_j.

(ii) *A 0-equilibrium is also called a Nash equilibrium.*

The reader will note that we distinguish between $\varepsilon > 0$, which is the time scale ratio, and $\varsigma > 0$, which is the approximation used in the equilibrium conditions.

Indeed, when ε becomes very small this game will become ill-conditioned. In the rest of the paper we propose an approach for defining a limit game which is easier to solve and which can be used to construct approximate equilibria of the original game when ε is small.

3. The limit game G^0. In this section we introduce the so-called *limit game* G^0, which is defined as a multiagent controlled Markov chain with states in I and controls in X_j^i, $i \in I$, $j \in M$.

3.1. A discrete-state Markov game. The limit game G^0 is defined as a controlled Markov chain on the discrete set I where Player j's strategy is defined by a vector $\tilde{x}_j = (x_j^i : i \in I)$ with $x_j^i \in X_j$, $j \in M$. The controlled transition rates of the Markov chain are given by $q_{k,\ell}(\underline{x}^k)$, where we use the notation $\underline{x}^k = (x_j^k : j \in J)$. The payoff for Player j, when the game starts in state i and when the players use the strategy m-tuple $\underline{\tilde{x}} = (\tilde{x}_j : j \in M)$, is defined as follows:

$$V_j(\underline{\tilde{x}}; i) = \mathrm{E}_{\underline{\tilde{x}}}\left[\int_0^\infty e^{-\rho_j t}\mathcal{L}_j^{\xi(t)}(\underline{x}^{\xi(t)}, 0)\, dt \mid \xi(0) = i\right]. \tag{3.1}$$

3.2. Nash equilibrium in the limit game. We assume the following.

Assumption 1. *There exists an equilibrium $\underline{\tilde{x}}^*$ for the limit game. The equilibrium value function for Player j is given by*

$$V_j^*(i) = \max_{\tilde{x}_j} \mathrm{E}_{\underline{\tilde{x}}}\left[\int_0^\infty e^{-\rho_j t}\mathcal{L}_j^{*\xi(t)}([\underline{x}^{*\xi(t)}_{M-j}, \underline{x}_j^{*\xi(t)}], 0)\, dt \mid (\xi(0) = i\right],$$

$$i \in I,\ j \in M, \tag{3.2}$$

for each player j.

The Hamilton–Jacobi–Bellman (HJB) system of equations associated with this equilibrium is

$$\rho_j V_j^*(i) = \max_{x_j^i \in X_j} \mathcal{L}_j^i([\underline{x}_{M-j}^{*i}, x_j^i], 0) + \sum_{k\in I} q_{ik}([\underline{x}_{M-j}^{*i}, x_j^i]) V_j^*(k), \quad i \in I, j \in M. \tag{3.3}$$

Remark 2. The existence of an equilibrium for a sequential game has been proved in particular in [15] and [17]. A more general theory that covers the class of stochastic games considered here has been proposed in [2]. These theories could be applied to prove that a Nash equilibrium exists for the limit game. However, the assumption is more restrictive since it assumes that the equilibrium can be obtained in pure strategies.

3.3. Occupation measures. It will be convenient to use occupation measures to prove the main convergence results in the paper. Introducing the indicator function

$$\delta(i,k) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

we define, for any strategy m-tuple $\underline{\tilde{x}}$ and state k, given the initial state i, with $k, i \in I$, the occupation measures

$$\Pi_j^k(\underline{\tilde{x}}, i) = \mathrm{E}_{\underline{\tilde{x}}} \left[\int_0^\infty e^{-\rho_j t} \delta(k, \xi(t))\, dt \mid \xi(0) = i \right]. \tag{3.5}$$

For each $j \in M$ these occupation measures satisfy the coupled equations

$$\rho_j \Pi_j^\ell(\underline{\tilde{x}}; i) - \sum_{k\in I} q_{ki}(\underline{x}^k) \Pi_j^k(\underline{\tilde{x}}; i) = \delta(\ell, i), \quad i \in I. \tag{3.6}$$

We can also rewrite the payoff, given in (3.1), as

$$V_j(\underline{\tilde{x}}; i) = \sum_k \Pi_j^k(\underline{\tilde{x}}; i) \mathcal{L}_j^k(\underline{x}^k, 0). \tag{3.7}$$

4. Uniform attractor policies and ς-equilibria. One can make a change of time scale in the dynamical system (2.1)–(2.3) by introducing a *stretched out* time $\tau = \frac{t}{\varepsilon}$. We shall assume a uniform reachability condition in this extended time scale.

ASSUMPTION 2. *In the stretched out time scale (or when $\varepsilon = 1$), for any $\eta > 0$ and mode $i \in I$, any target state $\underline{x}^f$ in $\underline{X}$ can be reached within η by the dynamical system* (2.1)–(2.3) *from any initial state $\underline{x}^o \in \underline{X}$ in a uniformly bounded time. We call $\theta(\eta)$ the uniform bound on the η-reachability time. In summary we assume the following:*

> *$\forall \eta > 0, \exists \theta(\eta) > 0$ s.t. $\forall i \in I$, $\forall \underline{x}^o$ and $\underline{x}^f \in \underline{X}$, there exists a trajectory $\underline{x}(\cdot)$ s.t. $\underline{x}(0) = \underline{x}^o$, $\mathcal{L}_j^i(\underline{x}(t), \dot{x}_j(t)) < \infty$, $j \in M$ a.e. and $\forall t > \theta(\eta)$ $\|\underline{x}(t) - \underline{x}^f\| < \eta$ holds.*

We shall further assume that this reachability is achieved through the use of "admissible decentralized system behavior."

DEFINITION 4.1. *An admissible decentralized system behavior is a family of mappings $s_j^i(x_j)$ taking values in $\Re^{n_j}$, $i \in I$, $j \in J$, and such that the differential equations $\dot{x}_j(t) = s_j^i(x_j(t))$ admit a uniquely defined solution in X_j for $t \in [0,\infty)$, given $x(0) = x^o \in X_j$ and such that $\mathcal{L}_j^i(\underline{x}(t), s_j^i(x_j(t))) > -\infty$ a.e. on $[0,\infty)$.*

ASSUMPTION 3. *The uniform reachability Assumption* 2 *is achieved by a set of admissible decentralized system behavior* $s^i_j(x_j; \underline{x}^o, \underline{x}^f) = \dot{x}_j$ *such that*

$$\mathcal{L}^i_j(\underline{x}, s^i_j(x_j; \underline{x}^o, \underline{x}^f)) < \infty \quad \text{and} \quad s^i_j(x^f_j; \underline{x}^o, \underline{x}^f) = 0.$$

Remark 3. The notion of admissible decentralized system behavior is closely related to the concept of decentralized feedback control laws in the state equation formulation of the dynamical system.

We also make the following continuity assumption, which should be not too restrictive.

ASSUMPTION 4. *The control laws of admissible decentralized system behavior have the following continuity property:*

$$\lim_{\underline{x} \to \underline{x}^f} s^i_j(x_j; \underline{x}^o, \underline{x}) = s^i_j(x^f_j; \underline{x}^o, \underline{x}^f) = 0. \tag{4.1}$$

4.1. A correspondence mapping. Given a strategy $\underline{\tilde{x}}$ for the limit game G^0, we associate a strategy $\underline{\tilde{\gamma}}^\varepsilon = \sigma^\varepsilon(\underline{\tilde{x}})$ for the G^ε game defined as follows: For any discrete state $i \in I$ and any initial state $\underline{x}^o$ select the trajectory $\underline{x}^i(\cdot) : [0, \infty) \to \underline{X}$, where each component $x_j(t)$ is a solution of $s^i_j(x_j(t); \underline{x}^o, \underline{\tilde{x}}) = \varepsilon \dot{x}_j(t)$ with $x^i_j(0) = x^o_j$. This is always possible by Assumption 3, and the following holds:

$$\forall \eta > 0, \exists \theta(\eta) \text{ s.t. } \|x^i_j(t) - x^i_j\| < \eta \quad \forall j \in M \quad \forall t \geq \theta(\eta)\varepsilon. \tag{4.2}$$

Define the occupation measures associated with $\underline{\tilde{\gamma}}^\varepsilon$ in the G^ε game

$$\Pi^{k\varepsilon}_j(\underline{\tilde{\gamma}}^\varepsilon; i, x^o) = \mathrm{E}_{\underline{\tilde{\gamma}}^\varepsilon}\left[\int_0^\infty e^{-\rho_j t}\delta(k, \xi(t))\, dt \mid (\xi(0) = i, x(0)) = x^o\right]. \tag{4.3}$$

For any function $g(\underline{x})$ which is continuous the asymptotic reachability condition (4.2) implies that there exits $\theta'(\eta)$ such that

$$|g(\underline{x}^i(t)) - g(\underline{x}^i)| < \eta \quad \forall t \geq \theta'(\eta)\varepsilon. \tag{4.4}$$

For the sake of simplifying the notation we shall use simply θ instead of $\theta'(\eta)$ when there is no possibility of confusion. We can now prove the following.

PROPOSITION 4.2. *For any* $i \in I$ *and any* $\underline{x}^o \in \underline{X}$ *the following convergence holds for the occupation measures:*

$$\lim_{\varepsilon \to 0} \left|\Pi^k_j(\underline{\tilde{x}}; i) - \Pi^{k\varepsilon}_j(\sigma^\varepsilon(\underline{\tilde{x}}); i, x^o)\right| = 0. \tag{4.5}$$

Proof. The detailed proof, which is straightforward but lengthy, is given in Appendix A. We summarize here its general development. As $\delta(i,k)$ is an indicator function it is uniformly bounded and one has for any strategy γ for a game G^ε

$$\mathrm{E}_\gamma\left[\int_0^\infty e^{-\rho t}\delta(i, \xi(t))\, dt\right] = \lim_{T \to \infty} \mathrm{E}_\gamma\left[\int_0^T e^{-\rho t}\delta(i, \xi(t))\, dt\right], \tag{4.6}$$

and this convergence is uniform for all γ and ε. For any realization, the integral $\int_0^T e^{-\rho t}\delta(i, \xi(t, \omega))\, dt$ can be shown to be a continuous function of the sample path $\xi(t, \omega)$ in an appropriate norm $d(\cdot)$ (see Appendix A). Then to establish (4.5) it suffices to show that the weak convergence limit

$$P_\varepsilon[\sigma^\varepsilon(\underline{\tilde{x}})] \Rightarrow P[\underline{\tilde{x}}] \tag{4.7}$$

holds, where $P_\varepsilon[\sigma^\varepsilon(\underline{\tilde{x}})]$ and $P[\underline{\tilde{x}}]$ are the probability measures induced on the restriction of the sample space to functions defined over the interval $[0,T]$ by $\sigma^\varepsilon(\underline{\tilde{x}})$ and $\underline{\tilde{x}}$ for the games G^ε and G^0, respectively.

Applying[1] Theorem 15.4 from [1] we can say that the weak convergence property (4.7) holds if, for any finite set of sample times $t_1,\dots,t_p$, the probability measures induced on the p random variables $\xi(t_1),\dots,\xi(t_p)$ by the strategy $\sigma^\varepsilon(\underline{\tilde{x}})$ converge weakly to the probability measure induced by the limit game strategy $\underline{\tilde{x}}$. We summarize this by the expression

$$P_\varepsilon \pi^{-1}_{t_1,\dots,t_p} \Rightarrow P\pi^{-1}_{t_1,\dots,t_p}, \tag{4.8}$$

where $\pi^{-1}_{t_1,\dots,t_p}$ is the inverse image of the projection of the $\xi(\cdot)$ process on the p sample times. This property is shown to hold in the rest of the proof presented in the appendix. □

The next result will establish convergence for the payoff functionals.

PROPOSITION 4.3. *For any strategy $\underline{\tilde{x}}$ of the limit game G^0 the following holds true:*

$$\lim_{\varepsilon\to 0} \left| V_j^\varepsilon(\sigma^\varepsilon(\underline{\tilde{x}}); i, x^o) - V_j(\underline{\tilde{x}}; i)\right| = 0 \quad \forall i \in I. \tag{4.9}$$

Proof. Consider the sample paths of the process $\{(\xi(\cdot,\omega),\underline{x}(\cdot,\omega)) : [t,\infty) \to I \times \Re^n : \omega \in \Omega\}$ generated by a strategy $\underline{\tilde{\gamma}}^\varepsilon = \sigma^\varepsilon(\underline{\tilde{x}})$. Almost surely any sample path has a countable number of jump times denoted $t_l(\omega)$, $l = 0,\dots,\infty$. The following holds true:

$$\mathrm{E}_{\underline{\tilde{\gamma}}^\varepsilon}\left[\sum_{l=0}^{\infty} e^{-\rho_j t_l}\right] \le M. \tag{4.10}$$

Therefore we can write

$$\begin{aligned}
V_j^\varepsilon(\underline{\tilde{\gamma}}^\varepsilon; x^o, i) &= \mathrm{E}_{\underline{\tilde{\gamma}}^\varepsilon}\left[\int_0^\infty e^{-\rho_j t}\mathcal{L}_j^{\xi(t)}(\underline{x}(t),\dot{x}_j(t))\,dt \mid \underline{x}(0) = x^o; \xi(0) = i\right] \\
&= \mathrm{E}_{\underline{\tilde{\gamma}}^\varepsilon}\left[\sum_{l=0}^{\infty}\int_{t_l}^{t_{l+1}} e^{-\rho_j t}\mathcal{L}_j^{\xi_l}(\underline{x}(t),\dot{x}_j(t))\,dt \mid \underline{x}(0) = x^o; \xi(0) = i\right] \\
&= \mathrm{E}_{\underline{\tilde{\gamma}}^\varepsilon}\Bigg[\sum_{l=0}^{\infty}\int_{t_l}^{t_{l+1}} e^{-\rho_j t}\mathcal{L}_j^{\xi_l}(\underline{x}^{\xi_l},0)\,dt \\
&\qquad + \sum_{l=0}^{\infty}\int_{t_l}^{\min\{t_l+\varepsilon\theta,t_{l+1}\}} e^{-\rho_j t}\left(\mathcal{L}_j^{\xi_l}(\underline{x}(t),\dot{x}_j(t)) - \mathcal{L}_j^{\xi_l}(\underline{x}^{\xi_l},0)\right)dt \\
&\qquad + \sum_{l=0}^{\infty}\int_{\min\{t_l+\varepsilon\theta,t_{l+1}\}}^{t_{l+1}} e^{-\rho_j t}\left(\mathcal{L}_j^{\xi_l}(\underline{x}(t),\dot{x}_j(t)) - \mathcal{L}_j^{\xi_l}(\underline{x}^{\xi_l},0)\right)dt \\
&\qquad \mid \underline{x}(0) = x^o; \xi(0) = i\Bigg].
\end{aligned} \tag{4.11}$$

[1] For continuous time Markov chains the condition (15.11) in [1] holds as the probability of having more than one jump in an interval $[t, t+\delta]$ tends to zero as δ tends to zero. Furthermore the set of trajectories that have a jump at $t = T$ has a zero measure. Therefore we can apply Theorem 15.4 from [1].

As $\mathcal{L}_j^i$ is bounded in $X \times U_j$, the second term in (4.11) tends to zero as ε tends to zero. As $\mathcal{L}_j^i$ is continuous, the last term in (4.11) can be made as small as desired as ε tends to zero. It remains to show that the first term in (4.11) tends to $V_j(\underline{\tilde{x}}; i)$ when ε tends to zero, to conclude that (4.9) holds. Since

$$\begin{aligned} & \mathrm{E}_{\underline{\tilde{\gamma}}^\varepsilon}\left[\sum_{l=0}^{\infty}\int_{t_l}^{t_{l+1}} e^{-\rho_j t}\mathcal{L}_j^{\xi_l}(\underline{x}^l,0)\,dt \mid \underline{x}(0)=x^o;\xi(0)=i\right] \\ (4.12)\qquad & = \sum_{k\in I}\Pi^k(\underline{\tilde{\gamma}}^\varepsilon; i, x^o)\mathcal{L}_j^k(\underline{x}^k,0), \end{aligned}$$

the result holds true by Proposition 4.2. ◻

4.2. The auxiliary ε-control problems. Given the strategy $\sigma^\varepsilon(\underline{\tilde{x}})$ in the game G^ε and an initial state x^o, we define for each player $j \in M$ an auxiliary ε-control problem as follows:

$$(4.13)\quad W_j^\varepsilon(\gamma_j^\varepsilon; i, \underline{x}^o|\sigma^\varepsilon(\underline{\tilde{x}})) = \mathrm{E}_{\gamma_j}\left[\int_0^\infty e^{-\rho_j t}L_j^{\xi(t)}(\underline{x}(t), u_j(t))\,dt \mid \xi(0)=i, \underline{x}(0)=x^o\right],$$

where the Markov jump process $\xi(\cdot)$ is still characterized by jump rates $q_{k\ell}(\underline{x}(t))$ and the state equations in mode $k \in I$ are

$$(4.14)\qquad \varepsilon\dot{x}_j(t) = f_j^k(x_j(t), u_j(t)),$$

$$(4.15)\qquad \varepsilon\dot{x}_\ell(t) = s_\ell^k(x_\ell(t); \underline{x}^k, \underline{\tilde{x}}) \quad \text{if } \ell \neq j.$$

In the equations above $(k, \underline{x}^k) = (\xi(t^{k+}), \underline{x}(t^{k+}))$ is the state of the system right after the last jump time and $s_\ell^k(x_\ell(t); \underline{x}^k, \underline{\tilde{x}})$ is the admissible decentralized system behavior associated with strategy $\sigma^\varepsilon(\underline{\tilde{x}})$ for Player ℓ. This control problem for Player j is thus obtained by fixing the dynamics of the other players to their admissible decentralized system behavior, and therefore the following holds:

$$(4.16)\qquad W_j^\varepsilon(\gamma_j^\varepsilon; i, \underline{x}^o \mid \sigma^\varepsilon(\underline{\tilde{x}})) = V_j^\varepsilon([\sigma_{M-j}^\varepsilon(\underline{\tilde{x}}), \gamma_j^\varepsilon]; i, \underline{x}^o).$$

Using these auxiliary control problems we will be able to exploit existing results from the theory of singularly perturbed systems, in particular those established in [6].

4.3. The auxiliary limit-control problem. For a given $\underline{\tilde{x}}$ and for a given set of potential vectors $\tilde{w}_j = (w_j^i)_{i\in I}$ for each player j, define the Hamiltonians

$$H_j^i(\tilde{w}_j; \underline{\tilde{x}}) = \max_{x_j, u_j}\left\{L_j^i([\tilde{x}_{-j}^i, x_j], u_j) + \sum_{k\in I} q_{ik}(\tilde{x}_{-j}^i, x_j)w_j^i \mid \text{s.t. } f_j^i(x_j, u_j) = 0\right\}.$$

For each player j we consider then the solution of the algebraic equations

$$(4.17)\qquad \rho_j w_j^{i*} = H_j^i(\tilde{w}_j^*; \underline{\tilde{x}}), \quad i \in I.$$

Note that these problems, for all $j \in M$, correspond to the solution of the limit game introduced in section 3, with the HJB equations given in (3.3).

4.4. Convergence of the auxiliary ε-control problem. These control problems have been studied in [6], where the following convergence result is established.

THEOREM 4.4. *Let the assumptions of Theorems* 3 *and* 4 *in* [6] *be satisfied (see Appendix B for a reminder of these assumptions). Let $\gamma_j^{\varepsilon *}$ be the optimal strategy in the ε-control problem and let v_j^* be the optimal control in the limit-control problem; then there exists a constant C such that*

$$\left|W_j^\varepsilon(\gamma_j^{\varepsilon *}; i, \underline{x}^o \mid \sigma^\varepsilon(\underline{\tilde{x}}^*)) - w_j^{i*}\right| \leq C\varepsilon^{\frac{\alpha}{1+\alpha}}. \tag{4.18}$$

Remark 4. Notice that we have $w_j^{i*} = V_j(\underline{\tilde{x}}^*; i) = V_j^*(i)$.

4.5. Convergence of the ε-game. We can now establish the following result.

THEOREM 4.5. *Let $\underline{\tilde{x}}^*$ be an equilibrium in the limit game G^0. Then for all positive ς there exists ε_0 such that for all $0 < \varepsilon \leq \varepsilon_0$, the strategy m-tuple $\sigma^\varepsilon(\underline{\tilde{x}}^*)$ defines a ς-Nash equilibrium for the game G^ε.*

Proof. Let $\underline{\tilde{x}}^*$ be an equilibrium in the limit game. Let γ_j^ε be a strategy for Player j in the game G^ε. Given $\sigma_{M-j}^\varepsilon(\underline{\tilde{x}}^*)$, let $\gamma_j^{\varepsilon *}$ be the optimal strategy in the ε-control problem.

Given ς, there exist ε_0 such that for $\varepsilon < \varepsilon_0$ we have

$$V_j^\varepsilon([\sigma_{M-j}^\varepsilon(\underline{\tilde{x}}^*), \gamma_j^\varepsilon]; i, \underline{x}^o) = W_j^\varepsilon(\gamma_j^\varepsilon; i, \underline{x}^o \mid \sigma^\varepsilon(\underline{\tilde{x}}^*)) \tag{4.19}$$

$$\leq W_j^\varepsilon(\gamma_j^{\varepsilon *}; i, \underline{x}^o \mid \sigma^\varepsilon(\underline{\tilde{x}}^*)) \tag{4.20}$$

$$\leq V_j(\underline{\tilde{x}}^*; i) + \varsigma/2 \tag{4.21}$$

$$\leq V_j^\varepsilon(\sigma^\varepsilon(\underline{\tilde{x}}^*); i, x^o) + \varsigma. \tag{4.22}$$

The first equality is valid by definition. The first inequality comes from the fact that $\gamma_j^{\varepsilon *}$ is the optimal strategy in the ε-control problem. The second inequality comes from Theorem 4.4 and Remark 4. The last inequality comes from Proposition 4.3. ∎

We have thus proved that the correspondence mapping introduced in section 4.1 tends to define an approximate Nash equilibrium when the time scale ratio tends to 0.

5. The local infinite horizon open-loop games. We have established that a correspondence mapping based on a property of uniform reachability of steady states defines an approximate equilibrium for the G^ε game when ε is small. We can now go one step further and show that such a correspondence mapping can be obtained from the equilibrium solutions of a class of local infinite horizon open-loop differential games (IHOLDGs), with the overtaking optimality criterion.

Let $V_j^*(i) : i \in I$ be the potential function associated with Player j in the equilibrium solution for the limit game. For any discrete state i and initial continuous state $\underline{x}(0) = \underline{x}^o$ define in the stretched out time scale the open-loop differential game with rewards over the time interval $[0, \theta)$ given by

$$J_j^\Theta[\underline{x}^o; \underline{x}(\cdot)] = \int_0^\Theta \left\{ \mathcal{L}_j^i(\underline{x}(\tau), \dot{x}_j(\tau)) + \sum_{\ell \in I} q_{i\ell}(\underline{x}(\tau)) V_j^*(\ell) \right\} d\tau, \quad j \in M, \tag{5.1}$$

where each player j selects an absolutely continuous trajectory $\{x_j(\tau) : \tau \geq 0\}$ with $x_j(0) = x_j^o$.

DEFINITION 5.1. *An overtaking equilibrium for the open-loop game defined by the payoff functionals* (5.1) *is an M-trajectory $(\underline{x}^*(\tau), \tau \geq 0)$ such that, for each player $j \in M$ and trajectory $(x_j(\tau), \tau \geq 0)$ the following holds:*

$$\liminf_{\Theta \to \infty} \left(J_j^\Theta[\underline{x}^o; \underline{x}^*(\cdot)] - J_j^\Theta[\underline{x}^o; [\underline{x}^{*-j}(\cdot), x_j(\cdot)]] \right) \geq 0. \tag{5.2}$$

Assumption 5. *The jump rates* $q_{i\ell}(\underline{x})$ *are affine in* $\underline{x}$. *The reward rates* $\mathcal{L}^i_j(\underline{x}, z_j)$ *are concave in* x_j *and* u_j *for each* $j \in M$ *and satisfy globally the following condition, also called "strict diagonal concavity" in* [16] *and* [4]:

$$\begin{aligned} &\forall \underline{x}^a,\ \underline{x}^b \in \underline{X},\ \underline{x}^a \neq \underline{x}^b \\ &\forall \zeta^a_j \in \partial_{x_j}\mathcal{L}^i_j(\underline{x}^a, z_j),\ \zeta^b_j \in \partial_{x_j}\mathcal{L}^i_j(\underline{x}^a, z_j), \\ &\sum_{j\in M}(\zeta^a_j - \zeta^b_j)(x^a_j - x^b_j) > 0. \end{aligned} \tag{5.3}$$

Under this assumption we can establish the following.

- An overtaking equilibrium for this game exists and is unique. It is characterized by a "turnpike," which is an attractor for all the equilibrium trajectories emanating from different initial states $\underline{x}^o$.
- This attractor corresponds to the equilibrium control associated with $i \in I$ in the limit game.
- From the overtaking equilibrium solutions to these open-loop games defined for all $i \in I$ and all initial state $\underline{x}^o$ we can construct an approximate (ς) equilibrium for the hybrid game problem.

The existence and uniqueness results with the turnpike property have been proved in [4]. The correspondence between the turnpikes and the equilibrium solutions in the limit game is easily obtained in the following lemma.

Lemma 5.2. *The turnpike attractor for the IHOLDG defined in* (5.1) *coincides with the equilibrium solution to the limit game.*

The existence and uniqueness results with the turnpike property have been proved in Carlson and Haurie [4]. The correspondence between the turnpikes and the equilibrium solution in the limit game is easily obtained in the following lemma.

Lemma 5.3. *The turnpike attractor for the open-loop game defined in* (5.1) *coincides with the equilibrium solution to the limit game.*

Proof. Introduce for each player $j \in M$ the Hamiltonians $\mathcal{H}^i : \Re^n \times \Re^{n_j} \to \Re \cup \{-\infty\}$ defined as

$$\mathcal{H}^i(\underline{x}, p_j) = \sup_{z_j}\left\{\mathcal{L}^i_j(\underline{x}, z_j) + \sum_{\ell\in I} q_{i\ell}(\underline{x})V^*_j(\ell) + p_j z_j\right\}. \tag{5.4}$$

If $\underline{x}^*(\cdot)$ is an overtaking equilibrium at $\underline{x}^o$, then there exists, for each player $j \in M$, an absolutely continuous function $p^*_j(\cdot)$ such that

$$\dot{x}_j(t) \in \partial_{p_j}\mathcal{H}^i(\underline{x}^*(t), p^*_j(t)), \tag{5.5}$$

$$\dot{p}_j(t) \in -\partial_{x_j}\mathcal{H}^i(\underline{x}^*(t), p^*_j(t)). \tag{5.6}$$

The turnpike is a solution of

$$0 \in \partial_{p_j}\mathcal{H}^i(\bar{\underline{x}}^i, \bar{p}^i_j), \tag{5.7}$$

$$0 \in -\partial_{x_j}\mathcal{H}^i(\bar{\underline{x}}^i, \bar{p}^i_j), \tag{5.8}$$

$$j \in M. \tag{5.9}$$

Under the strict diagonal concavity assumption there exists a unique solution to (5.7)–(5.8), and any solution to (5.5)–(5.6) which remains bounded is such that

$$\lim_{t\to\infty}\underline{x}^*(t) = \bar{\underline{x}}^i, \quad \lim_{t\to\infty} p^*_j(t) = \bar{p}^i_j, \quad j \in M.$$

Now it is an easy matter to check that the conditions (5.7)–(5.8) correspond to the sufficient optimality conditions for the problem

$$\max_{x_j^i}\left[\mathcal{L}_j^i([\underline{x}_{M-j}^{*i}, x_j^i], 0) + \sum_{\ell\in I} q_{i\ell}([\underline{x}_{M-j}^{*i}, x_j^i])V_j^*(\ell)\right], \quad j\in M, \tag{5.10}$$

which is also the right-hand side for mode i of the HJB equations in the limit game. Therefore the unique turnpike defines also an equilibrium in the limit game G^0. □

It is established, in the theory of turnpikes for infinite horizon control or open-loop equilibrium problems under the overtaking optimality criterion that the uniform ς-reachability condition is satisfied by the trajectories converging toward their respective attractors; see the book [5] for a complete discussion of these topics. In order to link these trajectories with a ς-equilibrium of the G^ε game we need this last assumption.

ASSUMPTION 6. *In mode i, the overtaking trajectories of Player j, emanating from different initial states x_j^o, can be synthesized, i.e., they are obtained as solutions to a system of state equations*

$$\dot{x}_j(t) = f_j^i(x_j(t), \mu_j^{*i}(x_j(t))); \quad x_j(0) = x^o, \tag{5.11}$$

where $\mu_j^{*i}(\cdot)$ is an admissible and continuous decentralized feedback law.

We can summarize the developments in the following.

PROPOSITION 5.4. *Given the potential functions associated with the Nash equilibrium of the limit game G^0, one can construct a family of IHOLDGs, with payoffs defined in* (5.1). *If Assumptions* 5 and 6 *are satisfied, the Nash equilibria of these IHOLDGs, under the overtaking optimality criterion, define a piecewise open-loop strategy for the G^ε game which is a ς-equilibrium if ε is small enough.*

Proof. It suffices to apply Theorem 4.5 with a strategy m-tuple $\sigma^\varepsilon(\underline{\tilde{x}}^*)$ obtained from the admissible decentralized system behavior (5.11). □

6. A decomposition principle. The result obtained can be interpreted as a decomposition principle for this two–time scale game. At a higher level one solves the limit stochastic game G^0 and one obtains for each player an equilibrium steady state $\tilde{x}_j$ and an equilibrium potential function $V_j^*(k)) : k \in I$. These potential functions are transmitted to all players. The G^ε game is then played as follows:

> *At a jump time t^o of the process $\xi(\cdot)$, the players observe the state $(\xi(t^{o+}), \underline{x}(t^{o+})) = (i, \underline{x}^i)$. Making a time translation to get $t^o = 0$, the players solve an IHOLDG where for each player $j \in M$ the payoff is defined for any $\Theta > 0$ by*
>
> $$J_j^\Theta[\underline{x}^o; \underline{x}(\cdot)] = \int_0^\Theta \left\{\mathcal{L}_j^i(\underline{x}(\tau), \dot{x}_j(\tau)) + \sum_{\ell\in I} q_{i\ell}(\underline{x}(\tau))V_j^*(\ell)\right\} d\tau.$$
>
> *The players find the unique Nash overtaking equilibrium for this open-loop game and they follow this trajectory, as long as the jump process remains in state i.*

This way of playing the game is close to being a piecewise open-loop Nash equilibrium when the time scale ratio ε is small.

7. Examples. In this section we provide two illustrations of the application of the theory developed above. The first one is a complete numerical computation realized on a dynamic duopoly model. In this example one shows how one can easily solve the different dynamic games used in this approximation theory. The second illustration is the reformulation as an hybrid stochastic system of a well-known integrated assessment model of climate change with noncooperative behavior of the economic agents (groups of nations). The stochastic jump process represents modifications in climate modes. The complete development and exploitation of this model would take too much space and will be the subject of another article. We nevertheless indicate how the theory of approximation developed herein would permit an important simplification in the type of game to solve.

7.1. A simple numerical example. As a complete illustrative example, let us first consider a simple but nontrivial model and compute the equilibrium turnpike values of the limit game and the equilibrium trajectories of the local IHOLDG that exhibit the turnpike property. We consider a duopoly ($M = \{1, 2\}$) with two slow market modes ($I = \{0, 1\}$).

The fast economic dynamics is defined by the state equations that describe the accumulation of production capacities (x_j) through investment (u_j) by the two firms

$$f^i_j(x_j, u_j) = u^i_j - x^i_j, \quad i \in I, j \in M. \tag{7.1}$$

The slow dynamics is described by the two transition rates between market modes

$$q_{01}(x) = x^0_1 + x^0_2, \tag{7.2}$$

$$q_{10}(x) = 1, \tag{7.3}$$

and $\xi(0) = 0$. Typically mode $\xi = 0$ would represent a "strong" market and mode $\xi = 1$ would represent a "depressed" market. The transition from strength to depression is influenced by the total supply on the market. The return from depression to strength is random and not controlled.

The reward functions are the firms' profits expressed as $L^i_j(x, u) = a^i_j x^i_j - (u^i_j)^2$. A common discount rate is fixed at $\rho_j = \rho = 0.05$. Payoffs are total expected discounted rewards over an infinite time horizon.

This dynamic duopoly model is similar (it has normalized parameter values) to the model proposed in [9], where a theory of stochastic duopoly with modal jumps is developed and a numerical solution method is proposed. We shall solve now the limit game problem and the local IHOLDGs for this singularly perturbed dynamic game.

7.1.1. Solving the limit game. In that particular case, it is possible to find an explicit solution to (3.6). Using Maple we obtain the following expressions for the occupation measures:

$$\Pi^0_j(0; \underline{\tilde{x}}) = \frac{(1 + \rho)}{\rho(1 + \rho + x^0_1 + x^0_2)}, \tag{7.4}$$

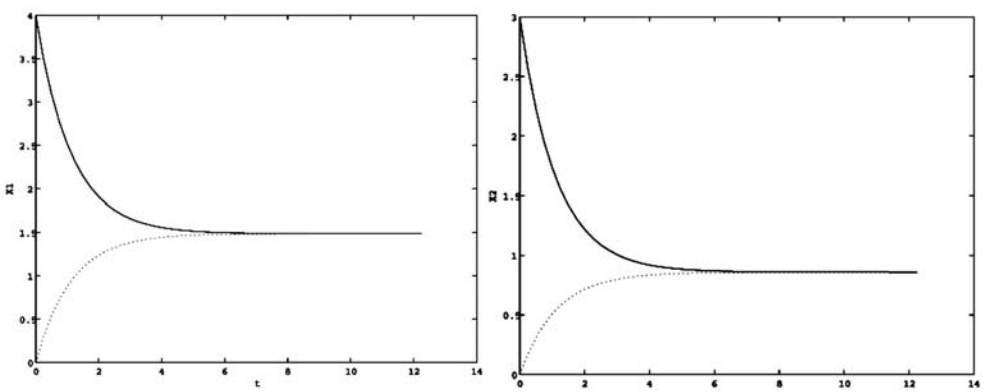
$$\Pi^1_j(0; \underline{\tilde{x}}) = \frac{x^0_1 + u^0_2}{\rho(1 + \rho + x^0_1 + x^0_2)}. \tag{7.5}$$

Now using the expression (3.7) for the payoffs associated with the strategy $\underline{x}$ in the limit game G^0, we reduce the search for a Nash equilibrium to the solution of a variational inequality which can be solved with an algorithm given by Konnov [11]. The results are displayed in Table 7.1, which provides the equilibrium steady state values for different sets of parameters a^i_j used in the reward function. These steady state values, provided by the equilibrium solutions of the limit game, indicate the target production capacity to which the duopolists should aim depending on the prevailing market mode.

Table 7.1
Equilibrium policy for different values of a_j^i.

a_1^0	a_2^0	a_1^1	a_2^1	x_1^0	x_2^0	x_1^1	x_2^1
2.00	2.00	2.00	2.00	1.0000	1.0000	1.0000	1.0000
2.00	2.00	0.50	0.50	0.8325	0.8325	0.2500	0.2500
2.00	2.00	1.00	0.25	0.8665	0.8261	0.5000	0.1250
4.00	2.00	1.00	0.25	1.4865	0.8580	0.5000	0.1250
4.00	2.00	0.25	1.00	1.4572	0.8914	0.1250	0.5000

7.1.2. Solving the local IHOLDG. We also computed an approximation of the equilibrium trajectories of the local IHOLDG. This is done by discretizing the time scale and taking a finite, but large, time horizon. Doing so, we reduce the problem to a variational inequality that can be solved with the same algorithm [11]. Figure 7.1 displays two trajectories with different initial states x, for the state $i = 0$ and for the case where $a_1^0 = 4.00$, $a_2^0 = 2.00$, $a_1^1 = 1.00$, and $a_2^1 = 0.25$. Note that in this case, the potential values (equilibrium payoffs in the limit game) are $V_1(0) = 26.5682$, $V_2(0) = 6.2776$, $V_1(1) = 25.5411$, $V_2(1) = 5.9936$, respectively.



(a) $x_1(t)$ (b) $x_2(t)$

Fig. 7.1. *Optimal trajectories for $i = 0$ when $a_1^0 = 4$, $a_2^0 = 2$, $a_1^1 = 1$, $a_2^1 = 0.25$, and $\rho = 5\%$. Solid line: $x_1(0) = 4$ and $x_2(0) = 3$. Dotted line: $x_1(0) = 0$ and $x_2(0) = 0$.*

As expected, we distinctly see that the trajectories are attracted by the turnpike values given in Table 7.1.

This provides a complete illustration of the results obtained in this paper. The limit game equilibrium tells the player what they should do in the “slow” time scale; the local IHOLDG tells them what they could do in the “fast” time scale in order to be consistent with the long-term equilibrium solution.

7.2. A model of competitive economic growth with climate change thresholds. A stochastic control approach to climate change modeling has been advocated in [8] and applied in [10]. These models extended the formalism proposed by Nordhaus [12] by introducing a stochastic jump process representing sudden switches in climate or global environment conditions. Nordhaus and Yang proposed in [14] a deterministic differential game model which represents the noncooperative behavior of different groups of nations involved in the climate change process. We propose here to extend the model by introducing a description of climate dynamics in the form of discrete modal changes. More precisely let us assume that there exist different climate modes, denoted ξ, which correspond to different patterns of the general circulation which determines climate dynamics. For example, we distinguish the current pattern ($\xi = 0$) from a second pattern ($\xi = 1$) where the thermo-haline circulation has been stopped and a third pattern ($\xi = 2$) where, in addition, the West-Antarctic ice sheet has collapsed (this type of threshold event was considered in the stochastic control model proposed in [10]).

7.2.1. The model equations. Let us denote the following.

Parameters.

$j = 1, \dots, m$	the m groups of nations, also called players
μ_j	capital depreciation rate in country j (typically 10% per year)
ν	greenhouse gas (GHG) natural elimination rate
ρ_j	long-term discount rate for country j (typically 0.08% per decade)

State variables.

K_j	productive capital stock of country j
M	GHG concentration
ξ	climate mode (discrete value $\xi \in \{0, 1, 2\}$)

Control variables.

I_j	investment in productive capital stock of country j
C_j	consumption in country j
u_j	abatement effort in country j
E_j	GHG emission in country j

Production and emission functions.

$F_j^{\xi(t)}(K_j, u_j)$	economic output of country; this function is increasing and concave in K_j, decreasing and concave in u_j; its shape depends on ξ
$G_j^{\xi(t)}(K_j, u_j)$	GHG emissions of country; increasing and convex in K_j, decreasing and convex in u_j; its shape depends on ξ

State equations.

$\dot{K}_j(t) = I_j(t) - \mu_j K_j(t)$	capital accumulation process
$\dot{M}(t) = \sum_{j=1,\dots,m} E_j(t) - \nu M(t)$	GHG concentration process

Modal jump rates.

$q_{k\ell}(M) = \lim_{dt \to 0} \frac{\mathrm{P}[\xi(t+dt)=\ell \mid \xi(t)=k]}{dt}$	transition rate from mode k to mode ℓ

Constraints.

$F_j^{\xi(t)}(K_j, u_j) \geq I_j(t) + C_j(t)$	the economic output of country j can be consumed or saved as an investment
$G_j^{\xi(t)}(K_j, u_j) \leq E_j(t)$	GHG emissions of country j are bounded below by the emission function

Payoffs.

$J_j = \int_0^\infty e^{-\rho_j t} U_j(C_j(t))\, dt$	the long-term welfare of country j; $U_j(\cdot)$ is a utility function

Initial conditions.

$K_j(0) = K_j^o$	initial physical capital stock of each nation
$M(0) = M^o$	initial GHG concentration
$\xi(0) = 0$	initial climate mode

7.2.2. The dynamic game. This model summarizes the situation of economies where the production activity generates GHG emissions which accumulate and may trigger a climate modal change. An abatement effort can be made by each country at a cost represented by a loss of output in the production function. The climate mode has also a direct influence, also expressed in terms of loss of output, on the economic production function. In this model the m groups of nations play a noncooperative dynamic game where the objective of the player is to reach an equilibrium for the long-term expected welfare,

$$V_j(\gamma; 0, K^o, M^o) = \mathrm{E}_\gamma \left[\int_0^\infty e^{-\rho_j t} U_j(C_j(t))\, dt \right], \tag{7.6}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ is the vector of piecewise open-loop strategies, and the expectation is taken with respect to the measure induced by γ and the economic and climate dynamics described above. This model retains the cost-benefit analysis framework proposed in [13] and [14] and represents climate change as a stochastic modal switch process.

7.2.3. The limit game. Notice that this model combines decoupled dynamics (for the K_j capital stocks) and a fully coupled state equation (for the GHG concentration M). However, M is a "passive" state variable which influences only the jump rates. It could be shown (see the paper [3] for a discussion of differential games with active and passive variables) that the turnpike property will also hold for differential games having this structure.

The model proposed here introduces the cost of climate change as a modification of the economic production function triggered by a random change of climate mode. The limit game will be a Markov game played over the discrete climate modes and where the controls are the long-term steady state values of the economies. The local IHOLDGs are infinite horizon differential games representing optimal economic growth with an adapted reward and zero discount rate.

8. Conclusion. We have considered here a class of hybrid games, under the piecewise open-loop information structure. We have given conditions under which, when the time scale ratio between the stochastic jump process and the deterministic part tend to 0, the Nash equilibrium solutions can be approximated by playing a family of auxiliary infinite horizon open-loop differential games (IHOLDGs). These local games are constructed using the potential functions obtained from the Markov Nash equilibrium of a simplified sequential game. This theory uses the asymptotic stability properties established for IHOLDGs under the strict diagonal concavity assumption. These conditions are the usual ones when one deals with Cournot solutions in dynamic imperfect competition models [16]. The results established in this paper are therefore useful for studying dynamic economic competition, when the market conditions change randomly but relatively seldom, compared with the adjustment speed of the economic decision variables. As indicated in the introduction we can envision such a situation in the framework of competitive economic growth with global environmental impact, like climate change triggered by greenhouse gas emissions, for example.

The evolution of the environmental state, and therefore the evaluation of the environmental damage, is random but evolving slowly. The economies of the world have the possibility to adjust at a much faster speed than the global environment modifications (see [8] for a discussion of control models with different time scales in the climate change modeling domain).

Appendix A. Continuity and convergence of the occupation measure. Let D be the space of functions on $[0,1]$ that are right-continuous and have left-hand limits. Adapting results from [1] we prove that the integral $\int_0^1 e^{-\rho t}\delta(i,\xi(t,\omega))\,dt$ is a continuous function of $\xi(t,\omega)$ in D. Let Λ denote the class of strictly increasing, continuous mappings from $\lambda(\cdot):[0,1]\to[0,1]$. For $\zeta(\cdot)$ and $\xi(\cdot)$ in D, define $d(\zeta(\cdot),\zeta(\cdot))$ to be the infimum of the $\varepsilon>0$ for which there exists in Λ a $\lambda(\cdot)$ such that

$$\sup_t |\lambda(t)-t|\le\varepsilon \tag{A.1}$$

and

$$\sup_t |\zeta(t)-\xi(\lambda(t))|\le\varepsilon. \tag{A.2}$$

We are now ready to prove that under the norm d, the function

$$\int_0^1 e^{-\rho t}\delta(i,\xi(t,\omega))\,dt$$

is continuous. We know that $\xi(\cdot,\omega)$ has at most countably many discontinuities. Say that $\xi(\cdot,\omega)$ has N discontinuities. Recall that $\xi(\cdot,\omega)$ takes value in $I=\{1,2,\dots,p\}$. Therefore, given $\xi(\cdot,\omega)\in D$ for any $\xi(\cdot,\tilde\omega)\in D$ such that $d(\xi(\cdot,\omega),\xi(\cdot,\tilde\omega))<\varepsilon<1$, both $\xi(\cdot,\omega)$ and $\xi(\cdot,\tilde\omega)$ must have the same sequence of jumps since otherwise $d(\xi(\cdot,\omega),\xi(\cdot,\tilde\omega))\ge 1$. Let $\{t_1,\dots,t_N\}$ (respectively, $\{\tilde t_1,\dots,\tilde t_N\}$) be the jump times of $\xi(\cdot,\omega)$ (respectively, $\xi(\cdot,\tilde\omega)$). By definition of the norm $|t_n-\tilde t_n|<\varepsilon$ for all $n\in\{1,\dots,N\}$. We have therefore

$$\begin{aligned}\left|\int_0^1 e^{-\rho t}\left(\delta(i,\xi(t,\omega))-\delta(i,\xi(t,\tilde\omega))\right)dt\right| &\le \int_0^1 |\delta(i,\xi(t,\omega))-\delta(i,\xi(t,\tilde\omega))|\,dt\\ &\le N(p-1)\varepsilon.\end{aligned} \tag{A.3}$$

Since p and N are finite and ε can be taken as small as desired, the continuity is proved.

Proof of Proposition 4.2. For $\xi(0)=i$, let us consider in the limit game G^0 and for a game G^ε, the probability to have no jump in the interval $[0,t]$ induced by $P[\underline{\tilde x}]$ and $P_\varepsilon[\sigma^\varepsilon(\underline{\tilde x})]$, respectively. For the limit game G^0 this probability is given by

$$\mathcal{P}^0[t,0,i,i]=e^{-\int_0^t q_{ii}(\underline{x}^i)\,ds}. \tag{A.4}$$

For the game G^ε, under the strategy $\sigma^\varepsilon(\underline{\tilde x})]$ this probability is given by

$$\mathcal{P}^\varepsilon[t,0,i,i]=e^{-\int_0^t q_{ii}(\underline{x}(s))\,ds}, \tag{A.5}$$

which can be rewritten as

$$\mathcal{P}^\varepsilon[t,0,i,i]=e^{-\int_0^t q_{ii}(\underline{x}^i)\,ds-\int_0^{\min(t,\varepsilon\theta)}(q_{ii}(\underline{x}(s))-q_{ii}(\underline{x}^i))\,ds-\int_{\min(t,\varepsilon\theta)}^t(q_{ii}(\underline{x}(s))-q_{ii}(\underline{x}^i))\,ds}.$$

As the jump rates are bounded over X, the second integral in the expression above converges to 0 when $\varepsilon \to 0$. When $\varepsilon \to 0$, the absolute value of the integral in the third term is bounded by ηt, which can be made as small as desired by choosing θ sufficiently large. Only the first term remains in the exponent, and this corresponds to (A.5). Therefore this establishes the convergence of $\mathcal{P}^\varepsilon[t,0,i,i]$ to $\mathcal{P}^0[t,0,i,i]$ when $\varepsilon \to 0$.

We can prove similarly that the probability $\mathcal{P}^\varepsilon[t,1,i,k]$ of having $\xi(t)=k$ and having exactly one jump in the interval $[0,t]$, induced by $\sigma^\varepsilon(\tilde{\underline{x}})$ for the game G^ε, converges to the probability $\mathcal{P}^0[t,1,i,k]$ of having $\xi(t)=k$ with exactly one jump in the interval $[0,t]$, induced by $\tilde{\underline{x}}$ for the game G^0. $\mathcal{P}^0[t,1,i,k]$ is given by

$$\mathcal{P}^0[t,1,i,k] = \int_0^t q_{ik}(\underline{x}^i) e^{-\int_0^s q_{ii}(\underline{x}^i)\,dv} e^{-\int_s^t q_{kk}(\underline{x}^k)\,dv} ds, \tag{A.6}$$

whereas the probability $\mathcal{P}^\varepsilon[t,1,i,k]$ is given by

$$\mathcal{P}^\varepsilon[t,1,i,k] = \int_0^t q_{ik}(\underline{x}(s)) e^{-\int_0^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_s^t q_{kk}(\underline{x}(v))\,dv}\, ds. \tag{A.7}$$

For a given t, if ε is small enough we have $\varepsilon\theta < t/2$ and thus can write

$$\begin{aligned}
&\left|\mathcal{P}^\varepsilon[t,1,i,k] - \mathcal{P}^0[t,1,i,k]\right| \\
&\le \int_0^{\varepsilon\theta} \left| q_{ik}(\underline{x}(s)) e^{-\int_0^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_s^t q_{kk}(\underline{x}(v))\,dv} - q_{ik}(\underline{x}^i) e^{-\int_0^s q_{ii}(\underline{x}^i)\,dv} e^{-\int_s^t q_{kk}(\underline{x}^k)\,dv} \right| ds \\
&+ \int_{t-\varepsilon\theta}^{t} \left| q_{ik}(\underline{x}(s)) e^{-\int_0^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_s^t q_{kk}(\underline{x}(v))\,dv} - q_{ik}(\underline{x}^i) e^{-\int_0^s q_{ii}(\underline{x}^i)\,dv} e^{-\int_s^t q_{kk}(\underline{x}^k)\,dv} \right| ds \\
&+ \int_{\varepsilon\theta}^{t-\varepsilon\theta} \left| q_{ik}(\underline{x}(s)) e^{-\int_0^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_s^t q_{kk}(\underline{x}(v))\,dv} - q_{ik}(\underline{x}^i) e^{-\int_0^s q_{ii}(\underline{x}^i)\,dv} e^{-\int_s^t q_{kk}(\underline{x}^k)\,dv} \right| ds.
\end{aligned}$$

The first two terms tend to zero as ε tends to zero, whereas the last term can be rewritten as

$$\begin{aligned}
&\int_{\varepsilon\theta}^{t-\varepsilon\theta} \Bigg| \left(q_{ik}(\underline{x}(s)) - q_{ik}(\underline{x}^i)\right) e^{-\int_0^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_s^t q_{kk}(\underline{x}(v))\,dv} \\
&- q_{ik}(\underline{x}^i)\left(e^{-\int_0^s q_{ii}(\underline{x}^i)\,dv} e^{-\int_s^t q_{kk}(\underline{x}^k)\,dv} - e^{-\int_0^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_s^t q_{kk}(\underline{x}(v))\,dv} \right) \Bigg|\, ds \\
&\le \int_{\varepsilon\theta}^{t-\varepsilon\theta} \left| q_{ik}(\underline{x}(s)) - q_{ik}(\underline{x}^i) \right| \left| e^{-\int_0^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_s^t q_{kk}(\underline{x}(v))\,dv} \right| ds \\
&+ \int_{\varepsilon\theta}^{t-\varepsilon\theta} \left| q_{ik}(\underline{x}^i) \right| \left| e^{-\int_0^s q_{ii}(\underline{x}^i)\,dv} e^{-\int_s^t q_{kk}(\underline{x}^k)\,dv} - e^{-\int_0^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_s^t q_{kk}(\underline{x}(v))\,dv} \right| ds.
\end{aligned}$$

When ε tends to zero, the first term on the right-hand side of the inequality above can be made as small as desired, choosing θ sufficiently big. The second term can be rewritten as

$$\int_{\varepsilon\theta}^{t-\varepsilon\theta} \left| e^{-\int_0^s q_{ii}(\underline{x}^i)\,dv} e^{-\int_s^t q_{kk}(\underline{x}^k)\,dv} - e^{-\int_0^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_s^t q_{kk}(\underline{x}(v))\,dv} \right| ds$$

$$= \int_{\varepsilon\theta}^{t-\varepsilon\theta} \left| e^{-\int_0^{\varepsilon\theta} q_{ii}(\underline{x}^i)\,dv} e^{-\int_{\varepsilon\theta}^s q_{ii}(\underline{x}^i)\,dv} e^{-\int_s^{s+\varepsilon\theta} q_{kk}(\underline{x}^k)\,dv} e^{-\int_{s+\varepsilon\theta}^t q_{kk}(\underline{x}^k)\,dv} \right.$$

$$\left. - e^{-\int_0^{\varepsilon\theta} q_{ii}(\underline{x}(v))\,dv} e^{-\int_{\varepsilon\theta}^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_s^{s+\varepsilon\theta} q_{kk}(\underline{x}(v))\,dv} e^{-\int_{s+\varepsilon\theta}^t q_{kk}(\underline{x}(v))\,dv} \right| ds$$

$$\leq \int_{\varepsilon\theta}^{t-\varepsilon\theta} \left| e^{-\int_0^{\varepsilon\theta} q_{ii}(\underline{x}^i)\,dv} e^{-\int_s^{s+\varepsilon\theta} q_{kk}(\underline{x}^k)\,dv} \right|$$

$$\left| e^{-\int_{\varepsilon\theta}^s q_{ii}(\underline{x}^i)\,dv} e^{-\int_{s+\varepsilon\theta}^t q_{kk}(\underline{x}^k)\,dv} - e^{-\int_{\varepsilon\theta}^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_{s+\varepsilon\theta}^t q_{kk}(\underline{x}(v))\,dv} \right| ds$$

$$+ \int_{\varepsilon\theta}^{t-\varepsilon\theta} \left| e^{-\int_{\varepsilon\theta}^s q_{ii}(\underline{x}(v))\,dv} e^{-\int_{s+\varepsilon\theta}^t q_{kk}(\underline{x}(v))\,dv} \right|$$

$$\left| e^{-\int_0^{\varepsilon\theta} q_{ii}(\underline{x}^i)\,dv} e^{-\int_s^{s+\varepsilon\theta} q_{kk}(\underline{x}^k)\,dv} - e^{-\int_0^{\varepsilon\theta} q_{ii}(\underline{x}(v))\,dv} e^{-\int_s^{s+\varepsilon\theta} q_{kk}(\underline{x}(v))\,dv} \right| ds.$$

By continuity of the exponential and $q(\cdot)$ functions and (4.2), this expression can be made as small as desired, when ε tends to zero. We can now extend this approach via an induction argument. Suppose that for $n-1$, $\mathcal{P}^\varepsilon[t, n-1, i, k]$ tends to $\mathcal{P}^0[t, n-1, i, k]$ when ε tends to zero. We have

$$\mathcal{P}^\varepsilon[t, n, i, k] = \sum_{l \neq k} \int_0^t \mathcal{P}^\varepsilon[s, n-1, i, l] \mathcal{P}^\varepsilon[t-s, 1, l, k]\, ds. \tag{A.8}$$

Using the result proved for $n = 1$, we can easily conclude that the following also holds:

$$\lim_{\varepsilon \to 0} \mathcal{P}^\varepsilon[t, n, i, k] = \sum_{l \neq k} \int_0^t \mathcal{P}^0[s, n-1, i, l] \mathcal{P}^0[t-s, 1, l, k]\, ds = \mathcal{P}^0[t, n, i, k]. \tag{A.9}$$

We can thus prove, by induction, that for each finite n , $\mathcal{P}^\varepsilon[t, n, i, k]$ tends to $\mathcal{P}^0[t, n, i, k]$ when ε tends to zero. Knowing that the probability of having more than n jumps tends to zero as n tends to infinity, it follows that

$$P_\varepsilon \pi^{-1}_{t_1,\ldots,t_k} \Rightarrow P \pi^{-1}_{t_1,\ldots,t_k}, \tag{A.10}$$

and this establishes the convergence result (4.5), as was indicated at the beginning of the proof.

Appendix B. Results of [6]. In [6] the single player case is solved and the convergence result used in this paper as Theorem 4.4 is based on the following assumption.

ASSUMPTION 7. *For any vector* $\underline{v} = \{v(i)\}_{i \in I}$ *consider the family of deterministic optimal control problems*

$$\begin{aligned} H^i(\theta, x^o, \underline{v}) &= \inf \frac{1}{\theta} \int_0^\theta \left(L^i(x(t), u(t)) + \sum_{i \in I} q_{ik}(x(t)) v(k) \right), \\ \frac{dx(t)}{dt} &= f^i(x(t), u(t)), \\ u(t) &\in U^i, \\ x(0) &= x^o. \end{aligned}$$

One assumes that there exist two constants $A > 0$ and $\alpha \in (0, 1]$, and for each $i \in I$ a function $H^i(\underline{v})$, such that for each $i \in I$, $x^o \in X$, and $\underline{v}$ in a bounded set Ω

$$|H^i(\theta, x^o, \underline{v}) - H^i(\underline{v})| \leq \frac{1}{\theta^\alpha}. \tag{B.1}$$

Under this assumption Theorem 4.4 is established.

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