Single photons from coupled quantum modes: supplementary information

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APPENDIX A: SOLVING THE MASTER EQUATION IN THE PARTICLE NUMBER BASIS

In this section we give more detail on how we solved Eqs. 1 and 2. Although, we focus on the two mode case, the way to further extend to the three mode case (Eq. 6) should be clear.

The density matrix for two bosonic quantum modes can be expanded over a particle number (Fock) basis:

$$\boldsymbol{\rho} = |n_1, n_2\rangle \langle n_1', n_2'| \rho_{n_1,n_2,n_1',n_2'} \tag{A.1}$$

where the state $|n_1, n_2\rangle$ represents the state with n_1 particles in the first mode and n_2 particles in the second. The creation and annihilation operators fulfil the following relations:

$$\hat{a}_1 |n_1, n_2\rangle = \sqrt{n_1}\, |n_1 - 1, n_2\rangle \tag{A.2}$$

$$\hat{a}_1^\dagger |n_1, n_2\rangle = \sqrt{n_1 + 1}\, |n_1 + 1, n_2\rangle \tag{A.3}$$

$$\hat{a}_2 |n_1, n_2\rangle = \sqrt{n_2}\, |n_1, n_2 - 1\rangle \tag{A.4}$$

$$\hat{a}_2^\dagger |n_1, n_2\rangle = \sqrt{n_1 + 2}\, |n_1, n_2 + 1\rangle \tag{A.5}$$

With these relations, substitution of Eq. A.1 into the master equation (Eq. 2) with Hamiltonian (Eq. 1) yields:

$$\begin{aligned}
i\hbar\frac{d\rho_{n_1,n_2,n_1',n_2'}}{dt} &= \Big\{E_1\left(n_1 - n_1'\right) + E_2\left(n_2 - n_2'\right) \\
&+ \alpha\left(n_1(n_1-1) - n_1'(n_1'-1) + n_2(n_2-1)\right. \\
&\left.-n_2'(n_2'-1)\right) - \frac{i\Gamma}{2}\left(n_1 + n_1' + n_2 + n_2'\right)\Big\}\rho_{n_1,n_2,n_1',n_2'} \\
&- J\sqrt{n_1(n_2+1)}\rho_{n_1-1,n_2+1,n_1',n_2'} \\
&+ J\sqrt{(n_1'+1)n_2'}\rho_{n_1,n_2,n_1'+1,n_2'-1} \\
&- J\sqrt{n_2(n_1+1)}\rho_{n_1+1,n_2-1,n_1',n_2'} \\
&+ J\sqrt{(n_2'+1)n_1'}\rho_{n_1,n_2,n_1'-1,n_2'+1} \\
&+ F\sqrt{n_1}\rho_{n_1-1,n_2,n_1',n_2'} + F^*\sqrt{n_1+1}\rho_{n_1+1,n_2,n_1',n_2'} \\
&- F\sqrt{n_1'+1}\rho_{n_1,n_2,n_1'+1,n_2'} - F^*\sqrt{n_1'}\rho_{n_1,n_2,n_1'-1,n_2'} \\
&+ i\Gamma\sqrt{(n_1+1)(n_1'+1)}\rho_{n_1+1,n_2,n_1'+1,n_2'} \\
&+ i\Gamma\sqrt{(n_1+1)(n_1'+1)}\rho_{n_1+1,n_2,n_1'+1,n_2'}
\end{aligned} \tag{A.6}$$

Assuming that the population remains small, one can truncate at given values of n_1 and n_2 to yield a finite set of differential equations. For an initial condition, we take the population to be zero, that is, $\rho_{1,1,1,1} = 1$ with all other elements of the density matrix set to zero. Eq. A.6 can then be solved numerically (we used the Adams-Moulton-Bashforth procedure) until a steady state is reached. Note that one should also check at each time that there is negligible population in the truncated states.

If the pure dephasing term (Eq. 7) is included an additional term should be added to the right-hand side of Eq. A.6:

$$\left. i\hbar\,\frac{d\rho_{n_1,n_2,n_1',n_2'}}{dt}\right|_{\text{deph}} = -\frac{i\Gamma_P}{2}\left\{\left(n_1 - n_1'\right)^2 + \left(n_2 - n_2'\right)^2\right\}\rho_{n_1,n_2,n_1',n_2'} \tag{A.7}$$

Given the steady state density matrix, ρ_{n_1,n_2,n_1,n_2}, the mode populations are given by:

$$\langle N_1\rangle = \left\langle \hat{a}_1^\dagger \hat{a}_1 \right\rangle = \sum_{n_1,n_2} n_1 \rho_{n_1,n_2,n_1,n_2} \tag{A.8}$$

$$\langle N_2\rangle = \left\langle \hat{a}_2^\dagger \hat{a}_2 \right\rangle = \sum_{n_1,n_2} n_2 \rho_{n_1,n_2,n_1,n_2} \tag{A.9}$$

Equal time correlation functions are given by:

$$\left\langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \right\rangle = \sum_{n_1,n_2} n_1(n_1-1) \rho_{n_1,n_2,n_1,n_2} \tag{A.10}$$

$$\left\langle \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 \right\rangle = \sum_{n_1,n_2} n_2(n_2-1) \rho_{n_1,n_2,n_1,n_2} \tag{A.11}$$

$$\left\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 \right\rangle = \sum_{n_1,n_2} n_1 n_2 \rho_{n_1,n_2,n_1,n_2} \tag{A.12}$$

Unequal time correlation functions can be obtained from [19,32]:

$$\left\langle \hat{a}_1^\dagger(t') \hat{a}_1^\dagger(t) \hat{a}_1(t) \hat{a}_1(t') \right\rangle = \sum_{n_1,n_2} n_1 \theta^{\{11\}}_{n_1,n_2,n_1,n_2}(t,t') \tag{A.13}$$

$$\left\langle \hat{a}_2^\dagger(t') \hat{a}_2^\dagger(t) \hat{a}_2(t) \hat{a}_2(t') \right\rangle = \sum_{n_1,n_2} n_2 \theta^{\{22\}}_{n_1,n_2,n_1,n_2}(t,t') \tag{A.14}$$

$$\left\langle \hat{a}_1^\dagger(t') \hat{a}_2^\dagger(t) \hat{a}_2(t) \hat{a}_1(t') \right\rangle = \sum_{n_1,n_2} n_2 \theta^{\{11\}}_{n_1,n_2,n_1,n_2}(t,t') \tag{A.15}$$

where:

$$\theta^{\{11\}}(t,t') = \mathcal{U}_{t,t'}\left(\sum_{n_1,n_1'}\sqrt{n_1 n_1'} \sum_{n_2,n_2'}\rho_{n_1,n_2,n_1',n_2'}(t')\left|n_1-1,n_2\right\rangle\left\langle n_1'-1,n_2'\right|\right) \tag{A.16}$$

$$\theta^{\{22\}}(t,t') = \mathcal{U}_{t,t'}\left(\sum_{n_2,n_2'}\sqrt{n_2 n_2'} \sum_{n_1,n_1'}\rho_{n_1,n_2,n_1',n_2'}(t')\left|n_1,n_2-1\right\rangle\left\langle n_1',n_2'-1\right|\right) \tag{A.17}$$

$\mathcal{U}_{t,t'}$ is the evolution superoperator associated with the master equation (Eq. 2).

APPENDIX B: LINEARIZED FLUCTUATION THEORY

In this section we show how to derive analytic expressions for the second order correlators using linearized fluctuation theory. Our method is a straightforward extension of the work of Refs. [29-31] in which a single bosonic mode was considered. Using the positive-P representation the Fokker-Planck equation can be derived and transformed into a Langevin type equation. By expanding to first order in fluctuations about the mean-field one can then obtain the correlation matrix, yielding all second order correlators.

1. Positive-P representation & the Fokker-Planck equation

For the two mode problem, the density matrix can be expanded on a basis of coherent state projection operators [30]:

$$\boldsymbol{\rho} = \int \mathcal{P}(\alpha_1,\beta_1,\alpha_2,\beta_2)\boldsymbol{\Lambda}(\alpha_1,\beta_1,\alpha_2,\beta_2)d\mu \tag{B.1}$$

$$\boldsymbol{\Lambda}(\alpha_1,\beta_1,\alpha_2,\beta_2) = \frac{\left|\alpha_1,\alpha_2\right\rangle\left\langle\beta_1^*,\beta_2^*\right|}{\left\langle\beta_1^*,\beta_2^*\right|\alpha_1,\alpha_2\rangle} \tag{B.2}$$

This is known as the positive-P representation, which differs from the Glauber-Sudarshan representation as it uses non-diagonal coherent state projectors. α_1, α_2, β_1, and β_2 are independent variables covering the whole complex plane. The integration measure, $d\mu = d^2\alpha_1 d^2\alpha_2 d^2\beta_1 d^2\beta_2$. For ease of notation, let $\vec{\alpha} = \alpha_1,\beta_1,\alpha_2,\beta_2$.

Annihilation and creation operators fulfil the following identities when acting on $\boldsymbol{\Lambda}(\vec{\alpha})$:

$$\hat{a}_n\boldsymbol{\Lambda}(\vec{\alpha}) = \alpha_n\boldsymbol{\Lambda}(\vec{\alpha}) \tag{B.3}$$

$$\hat{a}_n^\dagger\boldsymbol{\Lambda}(\vec{\alpha}) = \left(\beta_n + \frac{\partial}{\partial\alpha_n}\right)\boldsymbol{\Lambda}(\vec{\alpha}) \tag{B.4}$$

$$\boldsymbol{\Lambda}(\vec{\alpha})\hat{a}_n^\dagger = \beta_n\boldsymbol{\Lambda}(\vec{\alpha}) \tag{B.5}$$

$$\boldsymbol{\Lambda}(\vec{\alpha})\hat{a}_n = \left(\alpha_n + \frac{\partial}{\partial\beta_n}\right)\boldsymbol{\Lambda}(\vec{\alpha}) \tag{B.6}$$

Using these relations one obtains the Fokker-Planck equation from the master equation (Eq. 2):

$$\begin{aligned}\frac{\partial\mathcal{P}(\vec{\alpha})}{\partial t} =& \\ &\frac{i}{\hbar}\sum_{n=1,2}\left\{\frac{\partial}{\partial\alpha_n}\left[\left(E_n + 2\alpha\alpha_n\beta_n - \frac{i\Gamma}{2}\right)\alpha_n\mathcal{P}(\vec{\alpha})\right]\right. \\ &\quad - \frac{\partial}{\partial\beta_n}\left[\left(E_n + 2\alpha\alpha_n\beta_n + \frac{i\Gamma}{2}\right)\beta_n\mathcal{P}(\vec{\alpha})\right] \\ &\quad \left. -\alpha\left[\frac{\partial^2}{\partial\alpha_n^2}\left(\alpha_n^2\mathcal{P}(\vec{\alpha})\right) - \frac{\partial^2}{\partial\beta_n^2}\left(\beta_n^2\mathcal{P}(\vec{\alpha})\right)\right]\right\} \\ &- \frac{iJ}{\hbar}\left(\alpha_1\frac{\partial\mathcal{P}(\vec{\alpha})}{\partial\alpha_2} + \alpha_2\frac{\partial\mathcal{P}(\vec{\alpha})}{\partial\alpha_1}\right. \\ &\quad \left. -\beta_1\frac{\partial\mathcal{P}(\vec{\alpha})}{\partial\beta_2} - \beta_2\frac{\partial\mathcal{P}(\vec{\alpha})}{\partial\beta_1}\right)\end{aligned} \tag{B.7}$$

2. Linearized Langevin equation

According to the Ito calculus, a Fokker-Planck equation in the form:

$$\frac{\partial\mathcal{P}(\vec{\alpha})}{\partial t} = -\frac{\partial\left(f_n(\vec{\alpha})\mathcal{P}(\vec{\alpha})\right)}{\partial\alpha_n} + \frac{\partial^2\left(M_{nm}(\vec{\alpha})\mathcal{P}(\vec{\alpha})\right)}{\partial\alpha_n\partial\alpha_m} \tag{B.8}$$

is equivalent to the Langevin type equation:

$$\frac{\partial\alpha_n}{\partial t} = f_n(\vec{\alpha}) + \sqrt{2M_{nm}(\vec{\alpha})}\eta_m \tag{B.9}$$

where η_m are stochastic Gaussian noise terms. In our case:

$$f_1 = \frac{i}{\hbar}\left[\left(-E_1 - 2\alpha\alpha_1\beta_1 + \frac{i\Gamma}{2}\right)\alpha_1 + J\alpha_2\right] \tag{B.10}$$

$$f_2 = \frac{i}{\hbar}\left[\left(E_1 + 2\alpha\alpha_1\beta_1 + \frac{i\Gamma}{2}\right)\beta_1 - J\beta_2\right] \tag{B.11}$$

$$f_3 = \frac{i}{\hbar}\left[\left(-E_2 - 2\alpha\alpha_2\beta_2 + \frac{i\Gamma}{2}\right)\alpha_2 + J\alpha_1\right] \tag{B.12}$$

$$f_4 = \frac{i}{\hbar}\left[\left(E_2 + 2\alpha\alpha_2\beta_2 + \frac{i\Gamma}{2}\right)\beta_2 - J\beta_1\right] \tag{B.13}$$

and

$$\boldsymbol{M} = \frac{i\alpha}{\hbar}\begin{pmatrix} -\alpha_1^2 & 0 & 0 & 0 \\ 0 & \beta_1^2 & 0 & 0 \\ 0 & 0 & -\alpha_2^2 & 0 \\ 0 & 0 & 0 & \beta_2^2 \end{pmatrix} \tag{B.14}$$

By linearizing the equations to first order in fluctuations, ($\alpha_1 = \bar{a}_1 + \delta\alpha_1$, $\beta_1 = \bar{a}_1^* + \delta\beta_1$,$\alpha_2 = \bar{a}_2 + \delta\alpha_2$, $\beta_2 = \bar{a}_2^* + \delta\beta_2$), where $\bar{a}_i$ is the mean-field solution, we obtain:

$$\frac{\partial}{\partial t}\begin{bmatrix}\delta\alpha_1\\ \delta\beta_1\\ \delta\alpha_2\\ \delta\beta_2\end{bmatrix} = -\boldsymbol{A}\begin{bmatrix}\delta\alpha_1\\ \delta\beta_1\\ \delta\alpha_2\\ \delta\beta_2\end{bmatrix} + \boldsymbol{D}\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{bmatrix} \tag{B.15}$$

where the linearized drift matrix $\boldsymbol{A}$ is:

$$\boldsymbol{A} = -\frac{i}{\hbar}\begin{bmatrix} -E_1'^* & -2\alpha\bar{a}_1^2 & J & 0\\ 2\alpha\bar{a}_1^{*2} & E_1' & 0 & -J\\ J & 0 & -E_2'^* & -2\alpha\bar{a}_2^{*2}\\ 0 & -J & 2\alpha\bar{a}_2^2 & E_2'\end{bmatrix} \tag{B.16}$$

$\boldsymbol{D}$ is the square root of the diffusion array evaluated with the mean-field:

$$\boldsymbol{D} = \sqrt{\frac{2i\alpha}{\hbar}\begin{pmatrix} -\bar{a}_1^2 & 0 & 0 & 0\\ 0 & \bar{a}_1^{*2} & 0 & 0\\ 0 & 0 & -\bar{a}_2^2 & 0\\ 0 & 0 & 0 & \bar{a}_2^{*2}\end{pmatrix}} \tag{B.17}$$

and we have defined $E_n' = E_n + 4\alpha|\bar{a}_n|^2 + \frac{i\Gamma}{2}$. The calculation of the mean-field solution becomes a separate problem and is dealt with in Ref. [27].

3. Correlation Matrix

Following the method of Ref.[31] we now define the correlation matrix:

$$\boldsymbol{G} = \begin{bmatrix} \langle\delta\alpha_1\delta\alpha_1\rangle & \langle\delta\alpha_1^*\delta\alpha_1\rangle & \langle\delta\alpha_2\delta\alpha_1\rangle & \langle\delta\alpha_2^*\delta\alpha_1\rangle\\ \langle\delta\alpha_1\delta\alpha_1^*\rangle & \langle\delta\alpha_1^*\delta\alpha_1^*\rangle & \langle\delta\alpha_2\delta\alpha_1^*\rangle & \langle\delta\alpha_2^*\delta\alpha_1^*\rangle\\ \langle\delta\alpha_1\delta\alpha_2\rangle & \langle\delta\alpha_1^*\delta\alpha_2\rangle & \langle\delta\alpha_2\delta\alpha_2\rangle & \langle\delta\alpha_2^*\delta\alpha_2\rangle\\ \langle\delta\alpha_1\delta\alpha_2^*\rangle & \langle\delta\alpha_1^*\delta\alpha_2^*\rangle & \langle\delta\alpha_2\delta\alpha_2^*\rangle & \langle\delta\alpha_2^*\delta\alpha_2^*\rangle \end{bmatrix} \tag{B.18}$$

The equal time second order correlation function for mode 1 is then given by Eq. 4. Upon solving formally the linearized Langevin equation, one obtains in the steady state [31]:

$$\boldsymbol{G} = \int_0^\infty e^{-\boldsymbol{A}t}\boldsymbol{D}^2 e^{-\boldsymbol{A}^T t}dt \tag{B.19}$$

We now note that the characteristic equation for a 4×4 matrix, $\boldsymbol{A}$, is:

$$\begin{aligned}&\lambda^4 - \lambda^3\mathrm{tr}\{\boldsymbol{A}\} - \frac{\lambda^2}{2}\left(\mathrm{tr}\{\boldsymbol{A}^2\} - \mathrm{tr}\{\boldsymbol{A}\}^2\right)\\ &- \frac{\lambda}{6}\left(2\mathrm{tr}\{\boldsymbol{A}^3\} - 3\mathrm{tr}\{\boldsymbol{A}^2\}\mathrm{tr}\{\boldsymbol{A}\} + \mathrm{tr}\{\boldsymbol{A}\}^3\right)\\ &+ \mathrm{Det}\{\boldsymbol{A}\} = 0\end{aligned} \tag{B.20}$$

where Det denotes the determinant. Since this equation contains powers up to order 4 and every matrix obeys its own characteristic equation, $e^{-\boldsymbol{A}t}$ must be a polynomial in $\boldsymbol{A}$ of degree 3; higher order terms can always be reduced using the characteristic equation:

$$\boldsymbol{A}^4 = \boldsymbol{A}^3\sigma_0 + \frac{\boldsymbol{A}^2}{2}\sigma_1 + \frac{\boldsymbol{A}}{6}\sigma_2 - \sigma_3 \tag{B.21}$$

where

$$\sigma_0 = \mathrm{tr}\{\boldsymbol{A}\} \tag{B.22}$$

$$\sigma_1 = \left(\mathrm{tr}\{\boldsymbol{A}^2\} - \mathrm{tr}\{\boldsymbol{A}\}^2\right) \tag{B.23}$$

$$\sigma_2 = \left(2\mathrm{tr}\{\boldsymbol{A}^3\} - 3\mathrm{tr}\{\boldsymbol{A}^2\}\mathrm{tr}\{\boldsymbol{A}\} + \mathrm{tr}\{\boldsymbol{A}\}^3\right) \tag{B.24}$$

$$\sigma_3 = \mathrm{Det}\{\boldsymbol{A}\} \tag{B.25}$$

The correlation matrix can therefore be expanded in the form:

$$\begin{aligned}\boldsymbol{G} &= c_1\boldsymbol{D}^2 + c_2\left(\boldsymbol{A}\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{A}^T\right) + c_3\left(\boldsymbol{A}^2\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{A}^{T2}\right)\\ &+ c_4\left(\boldsymbol{A}^3\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{A}^{T3}\right) + c_5\left(\boldsymbol{A}\boldsymbol{D}^2\boldsymbol{A}^T\right)\\ &+ c_6\left(\boldsymbol{A}^2\boldsymbol{D}^2\boldsymbol{A}^T + \boldsymbol{A}\boldsymbol{D}^2\boldsymbol{A}^{T2}\right)\\ &+ c_7\left(\boldsymbol{A}^3\boldsymbol{D}^2\boldsymbol{A}^T + \boldsymbol{A}\boldsymbol{D}^2\boldsymbol{A}^{T3}\right)\\ &+ c_8\left(\boldsymbol{A}^2\boldsymbol{D}^2\boldsymbol{A}^{T2}\right) + c_9\left(\boldsymbol{A}^3\boldsymbol{D}^2\boldsymbol{A}^{T2} + \boldsymbol{A}^2\boldsymbol{D}^2\boldsymbol{A}^{T3}\right)\\ &+ c_{10}\left(\boldsymbol{A}^3\boldsymbol{D}^2\boldsymbol{A}^{T3}\right)\end{aligned} \tag{B.26}$$

Note that in the limit $\alpha \mapsto 0$ the matrix $\boldsymbol{D}$ vanishes and therefore the correlation matrix $\boldsymbol{G}$ also vanishes. In such a case $g_{2,11}$ must become equal to 1, so one sees that nonlinearity is essential to observe $g_{2,11} \neq 1$. From Ref.[31] (Eq. 5) we can re-write $\boldsymbol{D}$ in terms of $\boldsymbol{G}$ and $\boldsymbol{A}$:

$$\boldsymbol{A}\boldsymbol{G} + \boldsymbol{G}\boldsymbol{A}^T = \boldsymbol{D}^2 \tag{B.27}$$

This introduces terms of order 4 in $\boldsymbol{A}$ into Eq. B.26, which can be reduced using Eq.B.21. Eq. B.26 then gives a complete set of equations for the coefficients c_i:

$$\begin{aligned}c_1 &= c_7\sigma_3 - c_4\frac{\sigma_2}{6}\\ c_2 &= c_9\sigma_3 - c_4\frac{\sigma_1}{2}\\ c_3 &= c_{10}\sigma_3 - c_4\sigma_0\\ c_4 &= -\frac{1}{2\sigma_3}\\ c_5 &= -c_3 - c_7\frac{\sigma_1}{2} - c_9\frac{\sigma_2}{6}\\ c_6 &= -c_9\frac{\sigma_1}{2}\\ c_7 &= -\frac{6c_2}{\sigma_2}\\ c_8 &= -c_7 - c_9\sigma_0 - \frac{c_{10}\sigma_1}{2}\\ c_9 &= -c_{10}\sigma_0\\ c_{10} &= -6\frac{c_4 + c_6 + c_7\sigma_0}{\sigma_2}\end{aligned} \tag{B.28}$$

Solving this system of equations yields:

$$
\begin{aligned}
c_1 &= \frac{1}{12}\left(\frac{\sigma_2}{\sigma_3} - \frac{18\left(\sigma_1\sigma_2 + 3\sigma_0\left(\sigma_1^2 - 4\sigma_3\right)\right)}{\sigma_4}\right)\\
c_2 &= \frac{\sigma_1}{4\sigma_3} - \frac{3\sigma_0\left(3\sigma_0\sigma_1 + \sigma_2\right)}{\sigma_4}\\
c_3 &= \frac{\sigma_0}{2\sigma_3} + \frac{3\left(3\sigma_0\sigma_1 + \sigma_2\right)}{\sigma_4}\\
c_4 &= -\frac{1}{2\sigma_3}\\
c_5 &= \frac{3\sigma_1^2\left(3\sigma_0\sigma_1 + \sigma_2\right) - 12\left(6\sigma_0\left(\sigma_0^2 + \sigma_1\right)\right)\sigma_3}{4\sigma_3\sigma_4}\\
c_6 &= \frac{3\sigma_0\sigma_1\left(3\sigma_0\sigma_1 + \sigma_2\right)}{2\sigma_3\sigma_4}\\
c_7 &= -\frac{3\sigma_1\sigma_2 + 9\sigma_0\left(\sigma_1^2 - 4\sigma_3\right)}{2\sigma_3\sigma_4}\\
c_8 &= \frac{3\sigma_0\left(\sigma_0\left(3\sigma_0\sigma_1 + \sigma_2\right) - 6\sigma_3\right)}{\sigma_3\sigma_4}\\
c_9 &= -\frac{3\sigma_0\left(3\sigma_0\sigma_1 + \sigma_2\right)}{\sigma_3\sigma_4}\\
c_{10} &= \frac{3\left(3\sigma_0\sigma_1 + \sigma_2\right)}{\sigma_3\sigma_4}
\end{aligned}
\tag{B.29}
$$

where

$$\sigma_4 = 3\sigma_0\sigma_1\sigma_2 + \sigma_2^2 + 36\sigma_0^2\sigma_3 \tag{B.30}$$

The linearized drift (Eq. B.16) and squared diffusion array (Eq. B.17) are fully defined once the mean-field solution is given. The coefficients in Eq.B.29 then give the correlation matrix of the system using Eq. B.26.

APPENDIX C: DERIVATION OF EQ. 5

We have derived Eq. 5 to demonstrate that the coupling of a well to a second well can have an effect on the second order correlation function, $g_{2,11}$. To derive Eq. 5, we first note that the Heisenberg equation for an operator $\hat{O}$, including the Lindblad dissipation is:

$$i\hbar\frac{d\hat{O}}{dt} = \left[\hat{O}, \mathcal{H}\right] + i\frac{\Gamma}{2}\sum_{n=1}^{2}\left(2\hat{a}^\dagger\hat{O}\hat{a} - \hat{a}^\dagger\hat{a}\hat{O} - \hat{O}\hat{a}^\dagger\hat{a}\right) \tag{C.1}$$

Using the Hamiltonian in Eq. 1, we obtain the following equations for the field operators $\hat{a}_1$ and $\hat{a}_1^\dagger$:

$$i\hbar\frac{d\hat{a}_1}{dt} = \left(E_1 - \frac{i\Gamma}{2} + 2\alpha\hat{a}_1^\dagger\hat{a}_1\right)\hat{a}_1 - J\hat{a}_2 + F \tag{C.2}$$

$$i\hbar\frac{d\hat{a}_1^\dagger}{dt} = \left(E_1 + \frac{i\Gamma}{2} + 2\alpha\hat{a}_1^\dagger\hat{a}_1\right)\hat{a}_1^\dagger - J\hat{a}_2^\dagger + F^* \tag{C.3}$$

Linearizing these equations ($\hat{a}_1 = \bar{a}_1 + \delta\hat{\alpha}_1$, $\hat{a}_1^\dagger = \bar{a}_1^* + \delta\hat{\alpha}_1^\dagger$), we obtain:

$$i\hbar\frac{d\delta\hat{a}_1}{dt} = \left(E_1 - \frac{i\Gamma}{2} + 4\alpha|\bar{a}_1|^2\right)\delta\hat{a}_1 + 2\alpha\bar{a}_1^2\delta\hat{a}_1^\dagger - J\delta\hat{a}_2 \tag{C.4}$$

$$i\hbar\frac{d\delta\hat{a}_1^\dagger}{dt} = \left(E_1 + \frac{i\Gamma}{2} + 4\alpha|\bar{a}_1|^2\right)\delta\hat{a}_1^\dagger + 2\alpha\bar{a}_1^{*2}\delta\hat{a}_1 - J\delta\hat{a}_2^\dagger \tag{C.5}$$

In the steady state, these equations yield a relationship between different second order correlators:

$$\left(E_1 - \frac{i\Gamma}{2} + 4\alpha|\bar{a}_1|^2\right)\bar{a}_1^{*2}\left\langle\delta\hat{a}_1^2\right\rangle + \alpha|\bar{a}_1|^4\left(2\left\langle\delta\hat{a}_1^\dagger\delta\hat{a}_1\right\rangle + 1\right) = J\bar{a}_1^{*2}\left\langle\delta\hat{a}_1\delta\hat{a}_2\right\rangle \tag{C.6}$$

$$\left(E_1 + \frac{i\Gamma}{2} + 4\alpha|\bar{a}_1|^2\right)\bar{a}_1^{2}\left\langle\delta\hat{a}_1^{\dagger 2}\right\rangle + \alpha|\bar{a}_1|^4\left(2\left\langle\delta\hat{a}_1^\dagger\delta\hat{a}_1\right\rangle + 1\right) = J\bar{a}_1^{2}\left\langle\delta\hat{a}_1^\dagger\delta\hat{a}_2^\dagger\right\rangle \tag{C.7}$$

Defining $\Delta = E_1 + \frac{i\Gamma}{2} + 4\alpha|\bar{a}_1|^2$ we can recast these equations as:

$$|\Delta|^2\bar{a}_1^{*2}\left\langle\delta\hat{a}_1^2\right\rangle + \Delta\alpha|\bar{a}_1|^4 = \Delta J\bar{a}_1^{*2}\left\langle\delta\hat{a}_1\delta\hat{a}_2\right\rangle \tag{C.8}$$

$$|\Delta|^2\bar{a}_1^{2}\left\langle\delta\hat{a}_1^{\dagger 2}\right\rangle + \Delta^*\alpha|\bar{a}_1|^4 = \Delta^* J\bar{a}_1^{2}\left\langle\delta\hat{a}_1^\dagger\delta\hat{a}_2^\dagger\right\rangle \tag{C.9}$$

Therefore:

$$2|\Delta|^2\Re e\left\{\bar{a}_1^{*2}\left\langle\delta\hat{a}_1^2\right\rangle\right\} + 2\alpha|\bar{a}_1|^4\Re e\left\{\Delta\right\} = 2J\Re e\left\{\Delta\bar{a}_1^{*2}\left\langle\delta\hat{a}_1\delta\hat{a}_2\right\rangle\right\} \tag{C.10}$$

Assuming that $\bar{a}_1$ is real, we obtain after some algebra:

$$\Re e\left\{\left\langle\delta\hat{a}_1^2\right\rangle\right\} = \frac{-\alpha n_1 + J\Re e\left\{\left\langle\delta\hat{a}_1\delta\hat{a}_2\right\rangle\right\} - \zeta J\Im m\left\{\left\langle\delta\hat{a}_1\delta\hat{a}_2\right\rangle\right\}}{\left(E_1 + 4\alpha n_1\right)\left(1 + \zeta^2\right)} \tag{C.11}$$

where $\zeta = \Gamma/\left(2(E_1 + 4\alpha n_1)\right)$. Using Eq. 4, we then arrive at Eq. 5.

Figure Ia shows the value of $g_{2,11}$ calculated within the linear fluctuation theory (solid curve) and full density matrix approach (dashed curve) for a range of values of E_2. Although the linear fluctuation theory is an approximation, it captures the main behaviour of the second order correlation function. Figure Ib shows a comparison of the terms in Eq. 5 that contribute to $g_{2,11}$. Clearly, for slightly positive values of E_2, the last term in Eq. 5 makes the dominant contribution and results in the low value of $g_{2,11}$ seen in Fig. Ia. This correlated fluctuation term is a result of the interplay between dissipation and tunnelling.

Finally, we note that the slight detuning between the pump energy and well energy levels is essential for our observed effect. In the case of zero detuning we obtain $g_{2,11}$ close to 1, in agreement with Ref. [25].

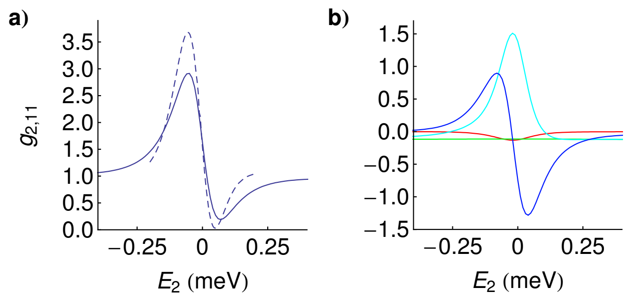

FIG. I: a) Dependence of $g_{2,11}$ on E_2 calculated with linearized fluctuation theory (solid) and full density matrix approach (dashed). b) Values of the different terms in Eq. 5: second term, $\frac{2}{n_1}\left[\langle\delta\alpha_1^*\delta\alpha_1\rangle\right]$ (red); third term (green); fourth term (cyan); last term (blue). $F = 0.1$meV, $E_1 = 0.07$meV.