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The Bounded Confidence Model Of Opinion Dynamics

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The bounded confidence model of opinion dynamics, introduced by Deffuant *et al*, is a stochastic model for the evolution of continuous-valued opinions within a finite group of peers. We prove that, as time goes to infinity, the opinions evolve globally into a random set of clusters too far apart to interact, and thereafter all opinions in every cluster converge to their barycenter. We then prove a mean-field limit result, propagation of chaos: as the number of peers goes to infinity in adequately started systems and time is rescaled accordingly, the opinion processes converge to i.i.d. nonlinear Markov (or McKean-Vlasov) processes; the limit opinion processes evolves as if under the influence of opinions drawn from its own instantaneous law, which are the unique solution of a nonlinear integro-differential equation of Kac type. This implies that the (random) empirical distribution processes converges to this (deterministic) solution. We then prove that, as time goes to infinity, this solution converges to a law concentrated on isolated opinions too far apart to interact, and identify sufficient conditions for the limit not to depend on the initial condition, and to be concentrated at a single opinion. Finally, we prove that if the equation has an initial condition with a density, then its solution has a density at all times, develop a numerical scheme for the corresponding functional equation, and show numerically that bifurcations may occur.

Keywords: Social networks; reputation; opinion; mean-field limit; propagation of chaos; nonlinear integro-differential equation; kinetic equation; numerical experiments.

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1. Introduction

Some models about opinion dynamics (or belief or gossip propagation, etc.) are based on binary values,[17,3,35,27,42,37] and often lead to attractors that display uniformity of opinions. These models are not valid for scenarios such as the social network of truck drivers interested in the quality of food of a highway restaurant or

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the critics' ratings about the new opening movies, for which it is required to have a continuous spectrum of opinions, as is also the case in politics when people are positioned on a scale going from extreme left-wing to right-wing opinions.[15]

The bounded confidence model introduced by Deffuant *et al.*[12] is a popular model for such scenarios. Peers have $[0,1]$-valued opinions; repeatedly in discrete steps, two peers are sampled, and if their opinions differ by at most a *deviation threshold* then both move closer, in barycentric fashion governed by a *confidence factor*. These parameters are the same for all peers, and the system is in binary mean-field interaction. The model has been studied and generalized, notably to other interaction graphs than the fully-connected one,[44,41,43,14,24] to vector-valued opinions,[34,44] and to peer-dependant deviation thresholds[43].

Reputation systems have lately emerged due to the necessity to measure trust about users while doing transactions over the internet; popular examples can be found in e-Bay[36] or Bizrate.[40] Some models for trust evolution and the potential effects of groups of liars attacking the system can be seen as a generalization of the bounded confidence model, in particular when there are no liars nor direct observations and the system evolves only by interaction between the peers.[28,9] The "Rendez-vous" model used by Blondel et al [8] has qualitative resemblance to the model used in this paper; like ours, it converges to a finite number of clusters in finite time for the finite N case. However, the interaction model is different, and our techniques (based on convexity and conservation of mean, see Proposition 3.2) do not seem to apply to this model.

The mean-field approximation method for large interacting systems has originated in statistical mechanics, notably after the seminal work of Ludwig Boltzmann.[38,22,31,13,6] It has been used, heuristically and rigorously, in many other fields, notably communication networks,[45,25,20,21,18,19,10] TCP connections,[5,39,4,23] robot swarms,[30] transportation systems,[2] and online reputation systems[28,33,32] in which is particularly appealing since the number of users may be very large (over 400 million for Facebook[1]).

This paper provides some rigorous proofs of old and new results on the Deffuant *et al.*[12] model, which has been studied intensively, but essentially by heuristic arguments and simulations. Notably, justifying the validity of the mean-field approach is not a simple matter, and classical methods do not apply, as seen below.

We prove that as time increases to infinity, opinions eventually group after some random finite time into a constant number of clusters, which are separated by more than the deviation threshold, and cannot influence one another. Thereafter, all opinions within every such cluster converge to their barycenter. The limit distribution of opinions is thus of a degenerate form, in which there are only a small number of fixed opinions which differ too much to influence each other, called a "partial consensus"; when it is constituted of one single opinion, it is called a "total consensus". Note that the limit distribution is itself random, i.e. different sample runs of the same model with same initial conditions always converge, but perhaps to different limiting distributions of opinions.

We then prove a mean-field limit result, called "propagation of chaos" in statistical mechanics: if the number of peers goes to infinity, the systems are adequately started, and time is rescaled accordingly, then the processes of the opinions converge in law to i.i.d. processes. Each of these is a so-called nonlinear Markov (or McKean-Vlasov) process, corresponding to an opinion evolving under the influence of opinions drawn independently from the marginal law of the opinion process itself, at a rate which is the limit of that at which a given peer in the finite system encounters its peers. Moreover, these marginals are the unique solution of an adequately started nonlinear integro-differential equation.

This implies a law of large numbers: the empirical measures of the interacting processes converge to the law of the nonlinear Markov process. Such process level results imply results for the marginal laws, but they are much stronger: limits are derived for functionals of the sample paths, such as hitting times or extrema. In particular, a functional law of large numbers holds for the marginal processes of these empirical measures, with limit the solution of the integro-differential equation.

The probabilistic structure of this limit equation is similar to that of kinetic equations such as the cutoff spatially-homogeneous Boltzmann or Kac equations, classically used in statistical mechanics to describe the limit of certain particle systems with binary interaction. Under quite general assumptions, satisfied here, it has long been known that it is well-posed in the space of probability laws, and that if the initial law has a density, then the solution has a density at all times satisfying a functional formulation of this equation.

Remark 1.1. There are two main difficulties in the propagation of chaos proof:

(1) the interaction is *binary* mean-field, since two opinions change simultaneously,
(2) the indicator functions related to the deviation threshold are discontinuous.

A system in which only one opinion would change at a time would be in *simple* mean-field interaction, and one could write equations for the opinions in almost closed form, which could be passed to the limit in various classical ways. This cannot be done for binary interaction, in which there is much more feedback between peers; moreover, this would require continuous coefficients. See Section 4.2 for details.

Such difficulties have been solved before[38]. In order to adapt results obtained for a class of interacting systems inspired by communication network models[20,21] using stochastic coupling techniques, which can be applied to various Boltzmann and Kac models[22,13], we introduce an intermediate *auxiliary* system, a continuous-time variant of the discrete-time model of Deffuant *et al.*[12] interacting at Poisson instants, which itself constitutes a relevant opinion model.

For this auxiliary system, we prove propagation of chaos, in total variation norm with estimates on any finite time interval. We then control the distance between this auxiliary system and the Deffuant *et al.* model, and prove propagation of chaos for a weaker topology, but still at the process level and allowing discontinuous measurable dependence on the $[0,1]$-values taken by the opinions.

The method can be easily generalized, for instance to vector-valued opinions, or to randomized interactions with a joint law governing whether one or both peers change opinion and by how much; for instance, choosing uniformly at random one peer to change opinion and leaving the other fixed would lead to a simple mean-field interacting model, and the limit model would be slowed down by a factor two.

To the best of our knowledge, this is the first rigorous mean-field limit result for this model. Similar integro-differential equations were used without formal justification before,[12,7,29] and appear to be incorrect by a factor 2 (perhaps by disregarding that two peers change opinion at once), which illustrates the interest of deriving the macroscopic equation from a microscopic description, as we do here.

We thank a referee to have brought to our attention the preprint Como-Fagnani.[11] It contains results for the marginal laws of a continuous-time variant of the model with two major simplifications: the interaction is simple mean-field (only one opinion changes at a time), and the indicator functions are replaced by Lipschitz-continuous functions; this removes difficulties (1) and (2) in Remark 1.1, to which its techniques do not apply. We have overcome these difficulties in the precise model of Deffuant *et al.*,[12] and have given much stronger results, for process laws in total variation norm and not for marginal laws in weak topologies.

One expects that the long-time behavior for the mean-field limit should be highly related to the behavior for an large number of peers of the long-time limit of the finite model. This heuristic inversion of long-time and large-number limits can be sometimes rigorously justified, for instance by a compactness-uniqueness method,[45,25,18,19] but here the limit nonlinear integro-differential equation may have multiple equilibria, and formal proof would constitute a formidable task.

We prove that the long-time behavior of the solution of the limit integro-differential equation is similar to that of the model with finitely many peers: it converges to a partial consensus constituted of a small number of fixed opinions which differ too much to influence each other.

We then develop a numerical method for the limit equation, and use it to explore the properties of the model. We observe phase transitions with respect to the number of limit opinions, while varying the deviation threshold for some fixed initial condition. We model the scenario of a company fusion, dividing the workers into an "undecided" group and two "extremist" factions, and obtain that having 20% of the workers "undecided" is enough to achieve consensus between all.

Last, we establish a bound on the deviation threshold, allowing to determine if there is total consensus or not, under the assumption of symmetric initial conditions.

In the sequel, Section 2 describes the finite model, and Section 3 studies some of its long time properties. Section 4 rigorously derives the mean-field limit, Section 5 studies some of its long time properties, and Section 6 is devoted to numerical results. The appendix contains some probabilistic complements in Section A, the details of the algorithm in Section B and all proofs in Section C.

2. Interacting system model, and reduced descriptions

The model for $N \geq 2$ interacting peers introduced by Deffuant *et al.*[12] is as follows. The random variable (r.v.) $X_i^N(k)$ with values in $[0,1]$ denotes the reputation record kept at peer $i \in \{1,\dots,N\}$ at time $k \in \mathbb{N} = \{0,1,\dots\}$, representing its opinion (or belief, etc.) about a given subject, the same for all peers. The discrete-time process of the states taken by the system of peers is

$$X^N = (X^N(k), k \in \mathbb{N}) , \qquad X^N(k) = (X_i^N(k))_{1 \leq i \leq N} ,$$

and evolves in function of the *deviation threshold* $\Delta \in (0,1]$ and the *confidence factor* $w \in (0,1)$. At each instant k, two peers i and j are selected uniformly at random without replacement, and:

- if $|X_i^N(k) - X_j^N(k)| > \Delta$ then $X^N(k+1) = X^N(k)$, the two peers' opinions being too different for mutual influence,
- if $|X_i^N(k) - X_j^N(k)| \leq \Delta$ then the values of peers i and j are updated to

$$\begin{cases} X_i^N(k+1) = wX_i^N(k) + (1-w)X_j^N(k) , \\ X_j^N(k+1) = wX_j^N(k) + (1-w)X_i^N(k) , \end{cases}$$

  and the values of the other peers do not change at time $k+1$, the two peers having sufficiently close opinions to influence each other.

Small values of Δ and large values of w mean that the peers trust very much their own opinions in comparison to the new information given by the other interacting peer. The extreme excluded values $\Delta = 0$ or $w = 1$ correspond to peers never changing opinion, and $w = 0$ to peers switching opinions if close enough. For $w = 1/2$, two close-enough peers would both end up with the average of their opinions.

A reduced, or macroscopic, description of the system is given by the *empirical measure* Λ^N, and by its marginal process $M^N = (M^N(k), k \in \mathbb{N})$ also called the *occupancy process*,

$$\Lambda^N = \frac{1}{N} \sum_{n=1}^N \delta_{X_i^N} = \frac{1}{N} \sum_{n=1}^N \delta_{(X_i^N(k), k\in\mathbb{N})} , \qquad M^N(k) = \frac{1}{N} \sum_{n=1}^N \delta_{X_i^N(k)} .$$

The random measure Λ^N has samples in $\mathcal{P}([0,1]^{\mathbb{N}})$, the space of probability measures on $[0,1]^{\mathbb{N}}$; its projection $M^N = (M^N(k), k \in \mathbb{N})$, which carries much less information, has sample paths in $\mathcal{P}([0,1])^{\mathbb{N}}$, the space of sequences of probability measures on $[0,1]$. For measurable $g : [0,1]^{\mathbb{N}} \to \mathbb{R}$ and $h : [0,1] \to \mathbb{R}$,

$$\langle g, \Lambda^N \rangle = \frac{1}{N} \sum_{n=1}^N g(X_i^N) , \qquad \langle h, M^N(k) \rangle = \frac{1}{N} \sum_{n=1}^N h(X_i^N(k)) .$$

We will also re-scale time as $t = \frac{k}{N}$, and consider in particular the rescaled occupancy process $\tilde{M}^N = (\tilde{M}^N(t), t \in \mathbb{R}_+)$ given by $\tilde{M}^N(t) = M^N(\lfloor Nt \rfloor)$, which in Section 4 will be shown to converge to a deterministic process $(m(t), t \in \mathbb{R}_+)$.

3. Long-time behavior of the finite N model

We consider a fixed finite number of peers and let time k go to infinity. We prove that the distribution of peer opinions $M^N(k)$ converges almost surely (a.s.) to a random distribution $M^N(\infty)$. Note that the limiting distribution $M^N(\infty)$ depends on chance as well as on the initial condition. We prove that $M^N(\infty)$ is a combination of at most $\lceil \frac{1}{\Delta} \rceil$ Dirac measures at points separated by at least Δ. A key observation here is that if h is any convex function then $\langle h, M^N(k) \rangle$ is non-increasing in k. Dittmer and Krause [14,26] obtained similar results, but for a deterministic model.

Definition 3.1. We say that $\nu \in \mathcal{P}[0,1]$ is a *partial consensus with c components* if $\nu = \sum_{m=1}^{c} \alpha_m \delta_{x_m}$ with $x_m \in [0,1]$, $|x_m - x_{m'}| > \Delta$ for $m \neq m'$, and $\alpha_m > 0$. Necessarily $c \leq \lceil \frac{1}{\Delta} \rceil$ and $\sum_{m=1}^{c} \alpha_m = 1$. If $c = 1$, *i.e.*, if ν is a Dirac measure, we say that ν is a *total consensus*.

If $M^N(k)$ is a partial consensus, then peers are grouped in a number of components too far apart to interact, and within one component all peers have the same value. Thus, a partial consensus is an absorbing state for M^N, and Theorem 3.9 below will show that $M^N(k)$ converges a.s., as $k \to \infty$, to one such state.

3.1. *Convexity and Moments*

We start with results about convexity and moments, which are needed to establish the convergence result and are also of independent interest.

Proposition 3.2. *For any convex function $h : [0,1] \to \mathbb{R}$, any x, y and w in $[0,1]$,*

$$h\left(wx + (1-w)y\right) + h\left(wy + (1-w)x\right) - h(x) - h(y) \leq 0$$

with equality when h is strictly convex possible only if $x = y$ or $w = 0$ or $w = 1$.

The following corollary is immediate from the interaction structure of the model.

Corollary 3.3. *If $h : [0,1] \to \mathbb{R}$ is a convex function, then $\langle h, M^N(k) \rangle$ is a non-increasing function of k along any sample path.*

Applying this to $h(x) = x$, $h(x) = -x$ and $h(x) = x^n$ yields that in any sample path, the first moment is constant and other moments are non-increasing with time.

Corollary 3.4. *For $n = 1, 2, \ldots$ and $k \in \mathbb{N}$, let $\mu_n^N(k) = \frac{1}{N} \sum_{i=1}^{N} X_i^N(k)^n$ denote the n-th moment of $M^N(k)$, and let $\sigma^N(k)$ be the standard deviation given by $\sigma^N(k)^2 = \mu_2^N(k) - \mu_1^N(k)^2$. Then:*

(1) The mean $\mu_1^N(k)$ is stationary in k, i.e., *$\mu_1^N(k) = \mu_1^N(0)$ for all k.*
(2) The moments and the standard deviation are non-increasing in k: if $k \leq k'$ then $\mu_n^N(k) \geq \mu_n^N(k')$ and $\sigma^N(k) \geq \sigma^N(k')$.

Moreover, stationarity of moments is equivalent to reaching partial consensus:

Proposition 3.5. *If $M^N(k)$ is a partial consensus, then $\mu_n^N(k') = \mu_n^N(k)$ for all $n \geq 1$ and $k' \geq k$. Conversely, if for some $n \geq 2$ there exists a (random) instant k such that $\mu_n^N(k') = \mu_n^N(k)$ for all $k' \geq k$, then $M^N(k') = M^N(k)$ for all $k' \geq k$ and $M^N(k)$ is a partial consensus, almost surely.*

3.2. *Almost Sure Convergence to Partial Consensus*

Definition 3.6. We say that two peers i and j are *connected* at time k if their values x and y satisfy $|y - x| \leq \Delta$. We say that $F \subset \{1, 2, \dots, N\}$ is a *cluster* at time k if it is a maximal connected component.

In other words, a cluster is a maximal set of peers such that every peer can pass the deviation test with one neighbour in the cluster. The set of clusters at time k is a random partition of the set of peers. The following proposition states that a cluster can either split or stay constant, but cannot grow.

Proposition 3.7. *Let $\mathcal{C}^N(k) = \{C_1, \dots, C_\ell\}$ be the set of clusters at time k. Then either $\mathcal{C}^N(k+1) = \mathcal{C}^N(k)$ or $\mathcal{C}^N(k+1) = \big(\mathcal{C}^N(k) \setminus C_{\ell_1}\big) \cup \mathcal{C}'$ where $\mathcal{C}'$ is a partition of C_{ℓ_1}, for some $\ell_1 \in \{1, \dots, \ell\}$.*

The number of clusters is thus non decreasing, and since it is bounded by $\lceil \frac{1}{\Delta} \rceil$ it must be constant after some time, yielding the following:

Corollary 3.8. *There exists a random time K^N, a.s. finite, such that*

$$\mathcal{C}^N(k) = \mathcal{C}^N(K^N) \text{ for } k \geq K^N .$$

Finally, we prove that the occupancy measure converges to a partial consensus (see 1 for the usual weak topology on $\mathcal{P}[0, 1]$):

Theorem 3.9. *As k goes to infinity, $M^N(k)$ converges almost surely, for the weak topology on $\mathcal{P}[0, 1]$, to a random probability $M^N(\infty)$, which is a partial consensus with L^N components, where $L^N := \mathrm{Card}(\mathcal{C}^N(K^N))$ is the* final number of clusters.

Theorem 3.9 notably implies that there is convergence to total consensus if and only if $L^N = 1$. The probability $p^* := \mathbb{P}(L^N = 1)$ of convergence to total consensus is not necessarily 0 or 1, but:

(1) If the diameter of $M^N(0)$ is less than Δ (*i.e.*, $\max_{i,j} \left|X_i^N(0) - X_j^N(0)\right| < \Delta$) then $p^* = 1$ (obvious);
(2) If there is more than 1 cluster in $M^N(0)$ then $p^* = 0$ (see Proposition 3.7).

4. Mean-field limit results when N goes to infinity

This section is devoted to a rigorous statement for the following heuristic statistical mechanics limit picture: all peers act independently, as if each were influenced by an infinite supply of independent statistically similar peers, of which the instantaneous laws solve a nonlinear equation obtained by consistency from this feedback.

In statistical mechanics and probability theory, such convergence to an i.i.d. system is called *chaoticity*, and the fact that chaoticity at time 0 implies chaoticity at further times is called *propagation of chaos*.

Some other probabilistic definitions and facts are recalled in 1.

We introduce an intermediate auxiliary system, in which the peer meet at the instants of a Poisson process, which itself constitutes a relevant opinion model. We adapt results in Graham-Méléard[21,22] to prove that the sample paths of the auxiliary system are well approximated by the limit system. We eventually control the distance between the auxiliary system and the model of Deffuant *et al.*[12]

4.1. *Mean-field regime, rescaled and auxiliary systems*

The number N of peers is typically large, and we let it go to infinity. At each time-step two peers are possibly updated, and the empirical measures have jumps of order $\frac{1}{N}$, hence time must be rescaled by a factor N. This is a mean-field limit, in which time is usually rescaled by physical considerations (such as "the peers meet in proportion to their numbers"). It is also related to fluid limits.

For a Polish space $\mathcal{S}$, let $\mathcal{P}(\mathcal{S})$ denote the space of probability measures on $\mathcal{S}$, with the Borel σ-field, and $D(\mathbb{R}_+, \mathcal{S})$ denote the Skorohod space of paths from $\mathbb{R}_+$ to $\mathcal{S}$ which are right-continuous with left-hand limits, with the product σ-field.

A non-trivial continuous-time limit process is expected for the *rescaled* system

$$\widetilde{X}^N = (\widetilde{X}_i^N)_{1\le i\le N}\,, \qquad \widetilde{X}^N = (\widetilde{X}^N(t), t\in\mathbb{R}_+) = (X^N(\lfloor Nt\rfloor), t\in\mathbb{R}_+)\,, \tag{4.1}$$

with sample paths in $D(\mathbb{R}_+, [0,1]^N)$. The corresponding empirical measure $\widetilde{\Lambda}^N$ and the process $\widetilde{M}^N = (\widetilde{M}^N(t), t\in\mathbb{R}_+)$ constituted of its marginal laws are given by

$$\widetilde{\Lambda}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\widetilde{X}_i^N} = \frac{1}{N}\sum_{i=1}^N \delta_{(\widetilde{X}_i^N(t), t\in\mathbb{R}_+)}\,, \qquad \widetilde{M}^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{\widetilde{X}_i^N(t)}\,, \tag{4.2}$$

respectively with samples in $\mathcal{P}(D(\mathbb{R}_+, [0,1]))$ and sample paths in $D(\mathbb{R}_+, \mathcal{P}[0,1])$.

An *auxiliary* (rescaled) system is obtained by randomizing the jump instants of the original model by waiting i.i.d. exponential r.v. of mean $\frac{1}{N}$ between selections, instead of deterministic $\frac{1}{N}$ durations. A convenient construction using a Poisson process $(A(t), t\in\mathbb{R}_+)$ of intensity 1 is that, with sample spaces as above,

$$\begin{aligned}
&\widehat{X}^N = (\widehat{X}_i^N)_{1\le i\le N}\,, \quad \widehat{X}^N = (\widehat{X}^N(t), t\in\mathbb{R}_+) = (X^N(A(Nt)), t\in\mathbb{R}_+)\,, \\
&\widehat{\Lambda}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}_i^N}\,, \quad \widehat{M}^N = (\widehat{M}^N(t), t\in\mathbb{R}_+)\,, \ \ \widehat{M}^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}_i^N(t)}\,.
\end{aligned} \tag{4.3}$$

If T_k for $k\ge 0$ are given by $T_0 = 0$ and the jump instants of $(A(t), t\in\mathbb{R}_+)$, then

$$\widetilde{X}^N(t) = \widehat{X}^N(t') = X^N(k)\,, \qquad \frac{k}{N}\le t < \frac{k+1}{N}\,, \quad \frac{T_k}{N}\le t' < \frac{T_{k+1}}{N}\,. \tag{4.4}$$

Note that $\widetilde{M}^N(t) = M^N(\lfloor Nt \rfloor)$ and $\widehat{M}^N(t) = M^N(A(Nt))$, but that the relationship between $\widetilde{\Lambda}^N$ and $\widehat{\Lambda}^N$ and Λ^N is more involved.

The process $\widehat{X}^N$ is a pure-jump Markov process with rate bounded by N, at which two peers are chosen uniformly at random without replacement, say i and j at time t, and:

- if $|\widehat{X}_i^N(t-) - \widehat{X}_j^N(t-)| > \Delta$ then $\widehat{X}^N(t) = \widehat{X}^N(t-)$,
- if $|\widehat{X}_i^N(t-) - \widehat{X}_j^N(t-)| \leq \Delta$ then only the values of peers i and j change to

$$\begin{cases} \widehat{X}_i^N(t) = w\widehat{X}_i^N(t-) + (1-w)\widehat{X}_j^N(t-)\,, \\ \widehat{X}_j^N(t) = wX_j^N(t-) + (1-w)\widehat{X}_i^N(t-)\,. \end{cases}$$

Remark 4.1. Each of the $\frac{N(N-1)}{2}$ unordered pairs of peers is thus chosen at rate $\frac{2}{N-1} = N/\frac{N(N-1)}{2}$, and then both peers undergo a *simultaneous* jump in their values if these are close enough. Each peer is thus affected at rate $2 = (N-1)\frac{2}{N-1}$.

The generator $\mathcal{A}^N$ of $\widehat{X}^N = (\widehat{X}_n^N)_{1\leq n\leq N}$ acts on $f \in L^\infty([0,1]^N)$ (the Banach space of essentially bounded measurable functions on $[0,1]^N$) as

$$\mathcal{A}^N f((x_n)_{1\leq n\leq N}) = \frac{2}{N-1} \sum_{1\leq i<j\leq N} [f((x_n)_{1\leq n\leq N}^{i,j}) - f((x_n)_{1\leq n\leq N})] 1_{\{|x_i - x_j|\leq \Delta\}} \tag{4.5}$$

where $(x_n)_{1\leq n\leq N}^{i,j}$ is obtained from $(x_n)_{1\leq n\leq N}$ by replacing x_i and x_j with $wx_i + (1-w)x_j$ and $wx_j + (1-w)x_i$ and leaving the other coordinates fixed. Its operator norm is bounded by $2N$, and the law of the corresponding Markov process $\widehat{X}^N$ is well-defined in terms of the law of $\widehat{X}^N(0) = X^N(0)$.

For $1 \leq k \leq N$, this generator acts on $h_k \in L^\infty([0,1]^N)$ which depend only on the k-th coordinate, of the form $h_k((x_n)_{1\leq n\leq N}) = h(x_k)$ for some $h \in L^\infty[0,1]$, as

$$\frac{2}{N-1} \sum_{1\leq j\leq N: j\neq k} [h(wx_k + (1-w)x_j) - h(x_k)] 1_{\{|x_k - x_j|\leq \Delta\}}$$
$$:= \mathcal{A}\left(\frac{1}{N-1} \sum_{1\leq j\leq N: j\neq k} \delta_{x_j}(dy) \right) h(x_k) \tag{4.6}$$

where the generators $\mathcal{A}(\mu)$ act on $h \in L^\infty[0,1]$ as

$$\mathcal{A}(\mu)h(x) = 2\langle [h(wx + (1-w)y) - h(x)] 1_{\{|x-y|\leq \Delta\}}\,, \mu(dy)\rangle\,, \quad \mu \in \mathcal{P}[0,1]\,. \tag{4.7}$$

Heuristically, if the $\widehat{X}_i^N(0)$ converge in law to i.i.d. r.v. of law m_0, then the $\widehat{X}_i^N$ are expected to converge in law to i.i.d. processes of law Q, the law of a time-inhomogeneous Markov process $\widehat{X}$ with initial law m_0 and generator $\mathcal{A}(m(t))$ at time $t \in \mathbb{R}_+$, where $m(t) = \mathcal{L}(\widehat{X}_t) = Q_t$ is the instantaneous law of this same process, the marginal of Q. Such a process is called a nonlinear Markov process, or a McKean-Vlasov process. Considering the forward Kolmogorov equation for this

Markov process, $(m(t), t \in \mathbb{R}_+)$ should satisfy the following weak (or distributional-sense) formulation of a nonlinear integro-differential equation.

Definition 4.1 (Problem 1). We say that $m = (m(t), t \in \mathbb{R}_+)$ with $m(t) \in \mathcal{P}[0,1]$ is solution to Problem 1 with initial value $m_0 \in \mathcal{P}[0,1]$ if $m(0) = m_0$ and

$$\begin{aligned}\langle h, m(t)\rangle - \langle h, m(0)\rangle &= \int_0^t \langle \mathcal{A}(m(s))h, m(s)\rangle\, ds \\ &:= \int_0^t 2\Big\langle [h(wx+(1-w)y) - h(x)]1_{\{|x-y|\le\Delta\}}\,, m(s)(dy)m(s)(dx)\Big\rangle\, ds \quad (4.8)\end{aligned}$$

for all test functions $h \in L^\infty[0,1]$; this can be written more symmetrically as

$$\begin{aligned}\langle h, m(t)\rangle - \langle h, m(0)\rangle = \int_0^t &\Big\langle [h(wx+(1-w)y) + h(wy+(1-w)x) \\ &- h(x) - h(y)]1_{\{|x-y|\le\Delta\}}\,, m(s)(dy)m(s)(dx)\Big\rangle\, ds\,. \quad (4.9)\end{aligned}$$

The distance in total variation norm of μ and μ' in $\mathcal{P}(\mathcal{S})$ is given by

$$|\mu - \mu'| = \sup_{\|\phi\|_\infty \le 1} \langle \phi, \mu - \mu'\rangle = 2\sup\{\mu(A) - \mu'(A) : \text{measurable } A \subset \mathcal{S}\}\,. \quad (4.10)$$

Theorem 4.2. *Consider the generators $\mathcal{A}(\mu)$ given by* (4.7)*, and m_0 in $\mathcal{P}[0,1]$.*

(1) There is a unique solution $m = (m(t), t \in \mathbb{R}_+)$ to Problem 1 starting at m_0. For the total variation norm on $\mathcal{P}[0,1]$, $t \mapsto m(t)$ is continuous, and $m_0 \mapsto (m(t), t \in \mathbb{R}_+)$ is continuous for uniform convergence on bounded time sets.

(2) There is a unique law $Q = \mathcal{L}(\widehat{X})$ on $D(\mathbb{R}_+, [0,1])$ for an inhomogeneous Markov process $\widehat{X} = (\widehat{X}(t), t \in \mathbb{R}_+)$ with generator $\mathcal{A}(m(t))$ at time t and initial law $\mathcal{L}(\widehat{X}(0)) = m_0$. Its marginal $Q_t = \mathcal{L}(\widehat{X}_t)$ is given by $m(t)$.

Remark 4.2. Such nonlinear Markov processes and equations are well-known to probabilists. The equations 1 have same probabilistic structure as the weak forms (2.1), (2.2), (2.4) (with $\mathcal{L} = 0$) of the spatially homogeneous version (without x-dependence) of the Boltzmann equation (1.1) in Graham-Méléard[22], the weak form (1.7) of the (cutoff) Kac equation (1.1)-(1.2) in Desvillettes *et al.*,[13] the nonlinear Kolmogorov equation (2.7) in Graham,[18] and the kinetic equation (9.4.4) in Graham.[19] The weak formulation involves explicitly the generator of the underlying Markovian dynamics and allows to understand it more directly. The functional formulation (for probability density functions) of this integro-differential equation involves an adjoint expression of this generator, and will be seen in Section 6.

4.2. *Difficulties for classical mean-field limit proofs*

The system $\widehat{X}^N$ exhibits simultaneous jumps in two coordinates, and is in *binary mean-field interaction* in statistical mechanics terminology.

A system in which only one opinion would change at a time would be in *simple* mean-field interaction; the generator $\mathcal{A}^N$ in (4.5) would be replaced by a simpler expression, which could be written as a sum over i of terms acting only on the i-th coordinate in terms of the value x_i and of $\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}$. Consequently, the empirical measures would satisfy an equation in almost closed form, which could be exploited in various ways to prove convergence to a limit satisfying the closed nonlinear equation in which the empirical distribution is replaced by the law itself.

A binary mean-field interacting system is much more complex, since there is much more feedback between peers. It is impossible to relate it in a simple way to an independent system, in which the coordinates cannot jump simultaneously. Because of that, the coupling methods introduced by Sznitman,[38] see also Méléard[31] and Graham-Robert,[23] cannot be adapted here. Moreover, these use contraction techniques, and the metric used is too weak for the indicator functions.

Elaborate compactness-uniqueness methods are also used for proofs, see Sznitman,[38] and also Méléard,[31] Graham-Méléard[22] Section 4, and Graham,[18,19] but require weak topologies for compactness criteria, and continuity properties in order to pass to the limit; hence, the indicator functions prevent using them here.

Remark 4.3. The indicator functions require quite strong topologies. For instance, if $0 < a < b = a+\Delta < 1$ and $m_0 = \frac{1}{2}(\delta_a+\delta_b)$, then there exists $M_+^N(0)$ with support not intersecting $[a,b]$ and converging weakly to $m(0)$, and starting there $M_+^N(k)$ and $m_+^N(t)$ have at least two clusters and support outside $[a,b]$. There exists also $M_-^N(0)$ with support inside (a,b) and converging weakly to $m(0)$, and $M_-^N(k)$ and $m_+^N(t)$ have one cluster and support inside (a,b) for any $k\in\mathbb{N}$, and will be a total consensus after some random time.

4.3. *Rigorous mean-field limit results for the auxiliary system*

Systems of this type were studied in Graham-Méléard,[21,22] see also Ref. 13. The first paper studied a class of not necessarily Markovian multitype interacting systems, as a model for communication networks. The second studied Monte-Carlo methods for a class of Boltzmann models, and in particular expressed some notions and results of the first in this framework. Their results yield the following.

For $k\ge 1$ and $T\ge 0$ and laws P and P' on $D(\mathbb{R}_+,[0,1]^k)$, let $|P-P'|_T$ denote the distance in variation norm (4.10) of the restrictions of P and Q on $D([0,T],[0,1]^k)$. When clear, the processes will be restricted to $[0,T]$ without further mention.

Theorem 4.3. *Consider the auxiliary system* (4.3) *for* $N\ge 2$. *If the* $\widehat{X}_i^N(0) := X_i^N(0)$ *are i.i.d. of law* m_0, *then there is propagation of chaos. More precisely, let* $m=(m(t),t\in\mathbb{R}_+)$ *and* Q *be as in Theorem 4.2 for* $m(0)=m_0$, *and* $T>0$.

(1) For $1\le k\le N$,

$$|\mathcal{L}(\widehat{X}_1^N,\dots,\widehat{X}_k^N)-\mathcal{L}(\widehat{X}_1^N)\otimes\cdots\otimes\mathcal{L}(\widehat{X}_k^N)|_T \le 2k(k-1)\frac{2T+4T^2}{N-1}\,,$$

and

$$\left|\frac{1}{N}\sum_{i=1}^{N}\mathcal{L}(\widehat{X}_i^N)-Q\right|_T \le |\mathcal{L}(\widehat{X}_i^N)-Q|_T \le 6\frac{\exp(2T)-1}{N+1}\,.$$

(2) *For any* $\phi : D([0,T],[0,1]) \to \mathbb{R}$ *such that* $\|\phi\|_\infty \le 1$*,*

$$\mathbb{E}\left[\left\langle \phi, \widehat{\Lambda}^N - \frac{1}{N}\sum_{i=1}^{N}\mathcal{L}(\widehat{X}_i^N)\right\rangle^2\right] \le \frac{4+8T+16T^2}{N}\,.$$

Moreover

$$\widehat{\Lambda}^N \xrightarrow[N\to\infty]{\text{in probab.}} Q\,, \qquad \widehat{M}^N \xrightarrow[N\to\infty]{\text{in probab.}} m\,,$$

respectively for the weak topology on $\mathcal{P}(D(\mathbb{R}_+,[0,1]))$ *with the Skorohod topology on* $D(\mathbb{R}_+,[0,1])$*, and for the topology of uniform convergence on bounded time intervals on* $D(\mathbb{R}_+,\mathcal{P}[0,1])$ *with the weak topology on* $\mathcal{P}[0,1]$*.*

The assumption that the initial conditions are i.i.d. can be appropriately relaxed, as in Theorem 1.4 in Graham-Méléard[20].

These very strong results are obtained for a relevant opinion model given by the auxiliary (rescaled) system, and are of independent interest. In the next section we will derive from them some weaker results for the original discrete-time model.

Remark 4.4. The convergence result for $\widehat{\Lambda}^N$ is equivalent to convergence in law to Q (Ethier-Kurtz,[16] Corollary 3.3.3). The convergence result for $\widehat{M}^N$ implies convergence in law to m for test functions which are continuous, bounded, and measurable for the product σ-field (Ref. 16, Theorem 3.10.2). Separability issues restrict these results, see 1; in fact, convergence of $\widehat{\Lambda}^N$ holds for any convergence induced by a denumerable set of bounded measurable functions.

4.4. *From the auxiliary to the rescaled system*

For $k \ge 1$, let a_k denote the Skorohod metric on $D(\mathbb{R}_+,[0,1]^k)$ given by (3.5.21) in Ethier-Kurtz[16] for the atomic metric $(x,y) \mapsto 1_{\{x\neq y\}}$ on $[0,1]^k$ (which induces the topology of all subsets of $[0,1]^k$, for which any function is continuous). Note that a_k is measurable with respect to the usual Borel σ-field on $[0,1]^k \times [0,1]^k$.

A *time-change* is an increasing homeomorphism of $\mathbb{R}_+$, *i.e.*, a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ which is null at the origin and strictly increasing to infinity. Two paths are close for a_k if there is a time-change close to the identity such that the time-change of one path is equal to the other path.

Eq. (4.4) is the key to obtain the following quite general result showing that the rescaled system $\widetilde{X}^N$ is very close to the the auxiliary system $\widehat{X}^N$, up to a well-controlled (random) time-change.

Theorem 4.4. *Consider the rescaled system* (4.1) *and the auxiliary system* (4.3) *for* $N \ge 2$*. Then* $\lim_{N\to\infty} a_N(\widetilde{X}^N, \widehat{X}^N) = 0$ *in probability.*

This result and Theorem 4.3 now yield the main mean-field convergence result.

Theorem 4.5. *Consider the rescaled system (4.1) for $N \geq 2$. If the $\widetilde{X}_i^N(0) := X_i^N(0)$ are i.i.d. of law m_0, then there is propagation of chaos. More precisely, let $m = (m(t), t \in \mathbb{R}_+)$ and Q be as in Theorem 4.2 for $m(0) = m_0$.*

(1) For $1 \leq k \leq N$,

$$\lim_{N\to\infty} \mathcal{L}(\widetilde{X}_1^N, \dots, \widetilde{X}_k^N) = Q^{\otimes k}\,,$$

for the weak topology on $\mathcal{P}(D(\mathbb{R}_+, [0,1]^k))$ induced by test functions which are either uniformly continuous for the Skorohod metric a_k, bounded, and measurable for the usual product σ-field (for the usual Borel σ-field on $[0,1]^k$), or continuous for the usual Skorohod topology (for the usual metric on $[0,1]^k$) and bounded.

(2) For the usual topology of $[0,1]$,

$$\widetilde{\Lambda}^N \xrightarrow[N\to\infty]{\text{in probab.}} Q\,, \qquad \widetilde{M}^N \xrightarrow[N\to\infty]{\text{in probab.}} m\,,$$

respectively for the weak topology on $\mathcal{P}(D(\mathbb{R}_+, [0,1]))$ with the Skorohod topology on $D(\mathbb{R}_+, [0,1])$, and for the topology of uniform convergence on bounded time intervals on $D(\mathbb{R}_+, \mathcal{P}[0,1])$ with the weak topology on $\mathcal{P}[0,1]$.

The assumption that the initial conditions are i.i.d. may again be relaxed. For the second result, see again Remark 4.4.

5. Infinite N Model

We now study the mean-field limit $m = (m(t), t \in \mathbb{R}_+)$ obtained in Section 5 when N goes to infinity. As for the finite N model, we find that there is convergence to a partial consensus as t goes to infinity. The limit may depend on the initial conditions, as might the random limit when N is finite. We are able to say more, and notably find tractable sufficient conditions for the limit to be a total consensus.

5.1. *Convexity and Moments*

Applying Proposition 3.2 to the equivalent definition of Problem 1 given by (4.9) yields the following:

Corollary 5.1. *Let $m = (m(t), t \in \mathbb{R}_+)$ be a solution of Problem 1. If $h : [0,1] \to \mathbb{R}$ is convex, then $\langle h, m(t)\rangle$ is a non-increasing function of t. Moreover, for $n = 1, 2, \dots$ and $t \in \mathbb{R}_+$, let $\mu_n(t) = \int_0^1 x^n\, m(t)(dx)$ denote the n-th moment of $m(t)$, and $\sigma(t)$ its standard deviation (*i.e., *$\sigma(t)^2 = \mu_2(t) - \mu_1(t)^2$). Then:*

(1) The mean $\mu_1(t)$ is stationary: $\mu_1(t) = \mu_1(0)$ for all t.
(2) The moments $\mu_n(t)$ are non-increasing in t: if $t_1 \leq t_2$ then $\mu_n(t_1) \geq \mu_n(t_2)$.
(3) The standard deviation $\sigma(t)$ is also a non-increasing function of t.

Furthermore, we have some bounds.

Proposition 5.2. *For all $t \geq 0$, we have $\sigma(0) \geq \sigma(t) \geq \sigma(0)e^{-4w(1-w)t}$.*

Note that Corollary 5.1 and Proposition 5.2 generalize results of Ref 7, which established similar results for the case $w = 1/2$. However, the bound in Proposition 5.2 is different, as the equation considered in Ref. 7 misses a factor 2.

5.2. *Convergence to Partial Consensus*

It is immediate that a partial consensus is a stationary point for Problem 1, *i.e.*, if $(m(t), t \in \mathbb{R}_+)$ is solution of Problem 1 with initial value a partial consensus m_0, then $m(t) = m_0$ for all t. Conversely, we show, in Theorem 5.5 below, that any trajectory $(m(t), t \in \mathbb{R}_+)$ converges to a partial consensus.

It is useful to consider the essential sup and inf of $m(t)$, defined as follows.

Definition 5.3. For $\nu \in \mathcal{P}[0,1]$, let $\operatorname{ess\,sup}(\nu) = \inf\{b \in [0,1], \nu((b,1]) = 0\}$ and $\operatorname{ess\,inf}(\nu) = \sup\{a \in [0,1], \nu([0,a)) = 0\}$.

Note that if $\operatorname{ess\,inf}(\nu) = a$ and $\operatorname{ess\,sup}(\nu) = b$, then the support of ν is included in $[a,b]$, *i.e.*, for any measurable $B \subset [0,1]$, $\nu(B) = \nu(B \cap [a,b])$.

Proposition 5.4. *Let $(m(t), t \in \mathbb{R}_+)$ be solution of Problem 1. Then $\operatorname{ess\,sup}(m(t))$ [resp. $\operatorname{ess\,inf}(m(t))$] is a non-increasing [resp. non-decreasing] function of t.*

See Definition 3.1 for partial and total consensus.

Theorem 5.5. *Let $(m(t), t \in \mathbb{R}_+)$ be a solution of Problem 1. As t goes to infinity, $m(t)$ converges, for the weak topology on $\mathcal{P}[0,1]$, to some $m(\infty)$ which is a partial consensus for every $\Delta' < \Delta$,* i.e.*, of the form $m(\infty) = \sum_{m=1}^{c} \alpha_m \delta_{x_m}$ with $x_m \in [0,1]$, $|x_m - x_{m'}| \geq \Delta$ for $m \neq m'$, and $\alpha_m > 0$.*

Note that the limit $m(\infty)$ may depend on the initial condition m_0, and may or may not be a total consensus (as shown in the next section). We are in particular interested in finding initial conditions that guarantee that $m(\infty)$ is a total consensus. The following is an immediate consequence of Proposition 5.4.

Corollary 5.6. *If the diameter of m_0 is less than Δ,* i.e.*, if $\operatorname{ess\,sup}(m_0) - \operatorname{ess\,inf}(m_0) < \Delta$, then $m(\infty)$ is a total consensus.*

Note that the converse is not true: if the diameter of m_0 is larger or equal than Δ, there may be convergence to total consensus (see next section for an example).

5.3. *Convergence to Total Consensus*

We find sufficient criteria for guaranteeing some upper bounds on the number of components of $m(\infty)$, in particular, we find some sufficient conditions for convergence to total consensus. Although the bounds are suboptimal, to the best of our knowledge, they are the first of their kind. The bounds are based on Corollary 5.1.

First define, for $n \in \{1,2,3,\dots\}$ and $\mu_0 \in [0,1]$, the set $P_n(\mu_0)$ of partial consensus with n components and mean μ_0, *i.e.*, $\nu \in P_n(\mu)$ iff there is some sequence $0 \le x_1 < \cdots < x_n \le 1$ with $x_i + \Delta < x_{i+1}$, some sequence α_i for $i = 1, \dots, n$ with $0 < \alpha_i < 1$ and $\sum_{i=1}^n \alpha_i = 1$ such that $\nu = \frac{1}{n} \sum_{i=1}^n \alpha_i \delta_{x_i}$ and $\frac{1}{n} \sum_{i=1}^n \alpha_i x_i = \mu_0$.

Second, for any convex, continuous $h : [0,1] \to \mathbb{R}_+$, let $Q_n(\mu_0, h)$ be the set of strict lower bounds of the image by the mapping $\nu \mapsto \langle h, \nu \rangle$ of $P_n(\mu_0)$, *i.e.*, $q \in Q_n(\mu_0, h)$ iff for any consensus ν with n components and mean μ_0, it holds that $\langle h, \nu \rangle > q$. If $P_n(\mu_0)$ is empty, let $Q_n(\mu_0, h) = \mathbb{R}_+$.

Note that $Q_n(\mu_0, h)$ is necessarily an interval with lower bound 0. The following proposition states that Q_n is non decreasing with n.

Proposition 5.7. *For any $n \in \{1,2,3,\dots\}$ and $\mu_0 \in [0,1]$ and convex continuous $h : [0,1] \to \mathbb{R}_+$, it holds that $Q_n(\mu_0, h) \subset Q_{n+1}(\mu_0, h)$.*

Combining Proposition 5.7 with Corollary 5.1, we obtain:

Theorem 5.8. *Let $(m(t), t \in \mathbb{R}_+)$ be the solution of Problem 1 with initial condition m_0, and c be the number of components of the limiting partial consensus $m(\infty)$. Assume that, for some $n \in \{2,3,\dots\}$, some convex continuous $h : [0,1] \to \mathbb{R}_+$, and some $q \ge 0$, we have $q \in Q_n(\mu_0, h)$, where μ_0 is the mean of m_0.*

Under these assumptions, if $\langle h, m_0 \rangle \le q$ then $c \le n-1$.

Here is an example of use of the theorem, for $n = 2$ and $h(x) = |x - \mu_0|$.

Corollary 5.9. *Let $(m(t), t \in \mathbb{R}_+)$ be the solution of Problem 1 starting at m_0. Assume that $\Delta \ge \frac{1}{2}$ and $1 - \Delta \le \mu_0 \le \Delta$, where μ_0 is the mean of m_0. If*

$$\int_0^1 |x - \mu_0| \, m_0(dx) < \frac{2}{\Delta} \min \{\mu_0(\Delta - \mu_0),\ (1-\mu_0)(\Delta - 1 + \mu_0)\}$$

then $m(t)$ converges to total consensus.

If we apply this to m_0 equal to the uniform distribution, we find the sufficient condition $\Delta > \frac{2}{3}$ for convergence to total consensus. In Corollary 5.15 we find a better result, obtained by exploiting symmetry properties.

Definition 5.10. We say that $\nu \in \mathcal{P}[0,1]$ is symmetric if the image measure of ν by $x \mapsto 1 - x$ is ν itself.

Note that if ν has a density f, this simply means that $f(x) = f(1-x)$. Necessarily, if ν is symmetric, the mean of ν is $\frac{1}{2}$. If a partial consensus $\nu = \frac{1}{n} \sum_{i=1}^n \alpha_i \delta_{x_i}$ (with $x_i < x_{i+1}$) is symmetric, then $x_{n+1-i} = 1 - x_i$ and $\alpha_{n+1-i} = \alpha_i$; in particular, if n is odd, $x_{\frac{n+1}{2}} = \frac{1}{2}$.

Proposition 5.11. *Let $(m(t), t \in \mathbb{R}_+)$ be a solution of Problem 1 with initial value m_0. If m_0 is symmetric, then $m(t)$ is symmetric for all $t \ge 0$.*

We can extend the previous method to the symmetric case as follows. Define SP_n as the set of *symmetric* partial consensus with n components, and let $q \in SQ_n(h)$

if and only if every symmetric consensus ν with n components satisfies $\langle h, \nu \rangle > q$. If SP_n is empty, then $SQ_n(h) = \mathbb{R}_+$. We have similarly:

Proposition 5.12. *For any $n \in \{1, 2, 3, \dots\}$ and convex continuous $h : [0, 1] \to \mathbb{R}_+$, it holds that $SQ_n(h) \subset SQ_{n+1}(h)$.*

Theorem 5.13. *Let $(m(t), t \in \mathbb{R}_+)$ be the solution of Problem 1 for a* symmetric *initial condition m_0, and c be the number of components of the limiting partial consensus $m(\infty)$. Assume that, for some $n \in \{2, 3, \dots\}$, some convex continuous $h : [0, 1] \to \mathbb{R}_+$, and some $q \geq 0$, we have $q \in SQ_n(h)$.*

Under these assumptions, if $\langle h, m_0 \rangle \leq q$ then $c \leq n - 1$.

We apply Theorem 5.13 with $h(x) = \left|x - \frac{1}{2}\right|$. It is easy to see that for $\nu \in SP_2$ we have $\langle h, \nu \rangle \geq \frac{\Delta}{2}$, which shows the following:

Corollary 5.14. *Let $(m(t), t \in \mathbb{R}_+)$ be the solution of Problem 1 for a* symmetric *initial condition m_0. If $\Delta > 2 \int_0^1 \left|x - \frac{1}{2}\right| \, m_0(dx)$ then $m(t)$ converges either to a total consensus or to a partial consensus with 3 or more components.*

In particular, if m_0 is the uniform distribution on $[0, 1]$, then $\int_0^1 \left|x - \frac{1}{2}\right| m_0(dx) = \frac{1}{4}$ and the condition in the previous corollary is $\Delta > \frac{1}{2}$, thus we have shown:

Corollary 5.15. *Let $(m(t), t \in \mathbb{R}_+)$ be the solution of Problem 1 with initial condition the uniform distribution on $[0, 1]$. If $\Delta > \frac{1}{2}$ then $m(t)$ converges to a total consensus.*

Corollary 5.16. *Let $(m(t), t \in \mathbb{R}_+)$ be a solution of Problem 1 with initial condition $m_0 = \left(\frac{1-\alpha}{2}\right)\delta_0 + \alpha\delta_{\frac{1}{2}} + \left(\frac{1-\alpha}{2}\right)\delta_1$. There is convergence to total consensus*

$$\text{for } \Delta > 1 - \alpha \text{ if } \alpha \leq \tfrac{1}{2}\,, \text{ or } \Delta > \tfrac{1}{2} \text{ if } \alpha \geq \tfrac{1}{2}\,.$$

6. Numerical Approach

In the mean-field limit, the dynamical behavior of the system of peers can be described by the integro-differential equation given in weak form in Definition 4.1 (Problem 1). This equation has no closed-form solution to our knowledge, and we have developed a numerical method for it.

We describe the algorithm, and analyze its precision and complexity. An important fact is that this algorithm requires considerably less running time than the probabilistic methods used in Neau[34] when N is large (which is not surprising in dimension 1). The program consists in 600 lines of C++ code, and the parsing and plotting of the results was done using Matlab.

6.1. *Functional formulation of Problem 1*

The numerical method is based on the functional formulation for probability density functions (PDFs) obtained by duality from the weak formulation in Definition 4.1

of Problem 1. The following result is fundamental in this aspect.

Theorem 6.1. *Let $(m(t), t \geq 0)$ be a solution of Problem 1. If the initial condition m_0 is absolutely continuous with respect to Lebesgue measure, then so is $m(t)$ for every $t \geq 0$, and moreover the densities $f(\cdot, t)$ of $m(t)$ satisfy the integro-differential equation*

$$\frac{\partial f(x,t)}{\partial t} = \frac{2}{w} \int_{x-\Delta w}^{x+\Delta w} f\left(\frac{x-(1-w)y}{w}, t\right) f(y,t)\, dy - 2f(x,t) \int_{x-\Delta}^{x+\Delta} f(y,t)\, dy\,. \tag{6.1}$$

Conversely, if $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is a solution of Eq.(6.1) such that $f(\cdot, t)$ is a PDF with support $[0, 1]$ for every $t \geq 0$, then the probability measures $m(t)(dx) = f(x, t)\, dx$ solve Problem 1.

This result and Theorem 4.2 yield an existence and uniqueness result for Eq.(6.1). This equation can be derived in statistical mechanics fashion by balance considerations. For the gain term, a particle in state $\frac{x-(1-w)y}{w}$ interacts at rate 2 (see Remark 4.1) with a particle in state y to end up in state x, and the joint density for this pre-interaction configuration at time t is $\frac{1}{w} f\big(\frac{x-(1-w)y}{w}, t\big) f(y, t)$ (particles are "independent before interacting"). The loss term is derived similarly.

Remark 6.1. As noted in Remark 4.2, this is a Boltzmann-like equation. This is made more obvious for $w \neq 1/2$ by the change of variables leading to post-interaction states x and y, which yields the equivalent formulation

$$\begin{aligned} \frac{\partial f(x,t)}{\partial t} = \frac{2}{2w-1} \int_{x-\Delta(2w-1)}^{x+\Delta(2w-1)} f\left(\frac{wx-(1-w)y}{2w-1}\right) f\left(\frac{wy-(1-w)x}{2w-1}\right) dy \\ - 2f(x,t) \int_{x-\Delta}^{x+\Delta} f(y,t)\, dy \end{aligned} \tag{6.2}$$

more reminiscent of Boltzmann or Kac equations such as (1.1) in Graham-Méléard[22] or (1.1)-(1.2) in Desvillettes *et al.*[13] In these, the fact that the gain term involves pre-collisional velocities is obscured by the physical symmetries between pre-collisional and post-collisional velocites, which are absent here.

In the rest of this section we assume that the hypothesis of the above theorem holds. We show next that if the PDF $f(\cdot, 0)$ is bounded then so is $f(\cdot, t)$ and we can control its growth over time. Let

$$M(t) \stackrel{\text{def}}{=} |f(\cdot, t)|_\infty \stackrel{\text{def}}{=} \sup_{x \in [0,1]} |f(x,t)|\,, \qquad t \geq 0\,.$$

Proposition 6.2. *Then $M(t) \leq e^{\left(\frac{2}{w} + \frac{2}{1-w}\right)t}(M(0) + 4) - 4$ for all $t \geq 0$.*

It follows that $f(\cdot, t)$ is bounded for all t, and iteratively, using (6.1), f is C^∞ on its second variable. Having controlled the growth of $f(x, t)$, it is easy to control the growth of its derivatives.

Proposition 6.3. *Then* $\left|\frac{\partial}{\partial t} f(\cdot,t)\right|_\infty \leq \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4)$ *for all* $t \geq 0$.

Iteratively, we obtain the following corollary.

Corollary 6.4. *If* $|f(\cdot,0)|_\infty < \infty$ *then* $\left|\frac{\partial^n}{\partial t^n} f(\cdot,t)\right|_\infty < \infty$ *for all* $n \in \mathbb{N}$ *and* $t \geq 0$.

6.2. *Numerical Solution of Eq.(6.1)*

Though Equation (6.1) for the PDF does not appear to have any tractable closed form solution, however, it lends itself well to numerical solution. We developed an algorithm that gives an approximate solution of Equation (6.1) over some finite time horizon T, given some initial condition $f(x,0)$, assumed to be piecewise constant. The algorithm is described in Appendix B. In the rest of this section, we present numerical results obtained with the algorithm.

We study different scenarios for the initial distribution: uniform, extremist versus undecided and beta. We find bifurcations as a function of Δ. Moreover, we compare the experimental results with the bounds obtained in section 5 and the probabilistic Monte Carlo simulations presented in Ref 12. The main results are summarized in the following table:

Scenario	*Parameters*	*Consensus*
Uniform	$\Delta \leq 0.27$, $w \in (0,1)$	Partial
Uniform	$\Delta > 0.27$, $w \in (0,1)$	Total
Extremists/Und.	(Δ,α) below black curve (fig. 4)	Partial
Extremists/Und.	(Δ,α) above black curve (fig. 4)	Total
Extremists/Und.	(Δ,α) above red curve (fig. 4)	Partial (theor. bound)
Beta	$\Delta > 0.25$, $w \in \{0.5, 0.75\}$	Total
Beta	$\Delta \leq 0.25$, $w \in \{0.5, 0.75\}$	Partial
Beta	$\Delta > 0.2$, $w = 0.9$	Total
Beta	$\Delta \leq 0.2$, $w = 0.9$	Partial

Table 1. Summary of the numerical experiments

6.2.1. *General Evolution of the System*

In order to illustrate the behavior of the system as time passes, we show how the system evolves from a uniform distribution to one (or more) components, depending on the deviation threshold Δ. We run those sets of experiments for 3 different values of w, specifically $0.5, 0.75$, and 0.9 and plot the probability function at times $t = 0$, $t = 20$ and $t = 100$. The simulations have been done with the parameters $I = 200, \Delta t = 0.1, T = 100$. Although the set of parameters might theoretically yield a big error, in practice this error is much smaller.

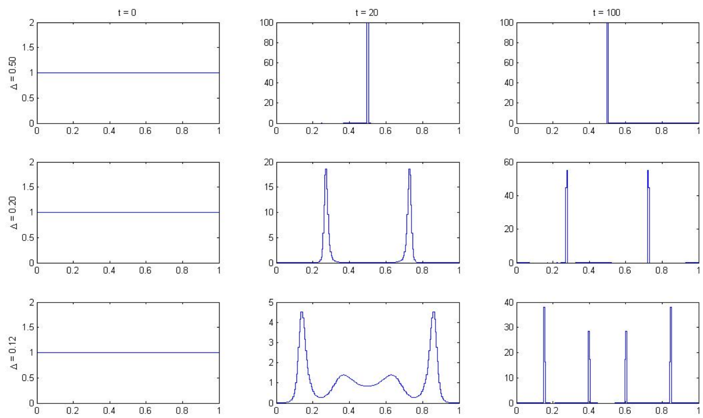


Fig. 1. $w = 0.5$. Evolution of $m(t)$ at times $t = 0, 20, 100$.

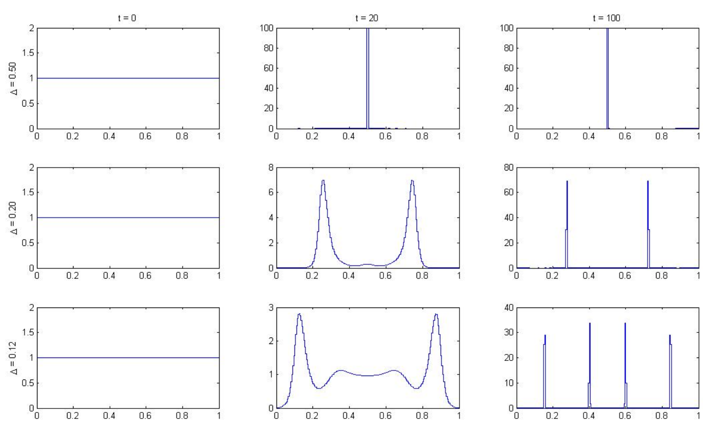


Fig. 2. $w = 0.75$. Evolution of $m(t)$ at times $t = 0, 20, 100$.

From the images, we see that w does not seem to impact the number of components of $m(\infty)$, but the weights do depend on w.

6.2.2. *Extremists versus Undecided*

We now present some common scenarios: imagine a company fusion and the opinion of the employees about the new company, or a rough categorization of voters in an election. We can characterize these opinions as extremists (either 0 or 1) or undecided (0.5). The proportion of opinions is α for the undecided and $\frac{1-\alpha}{2}$ for

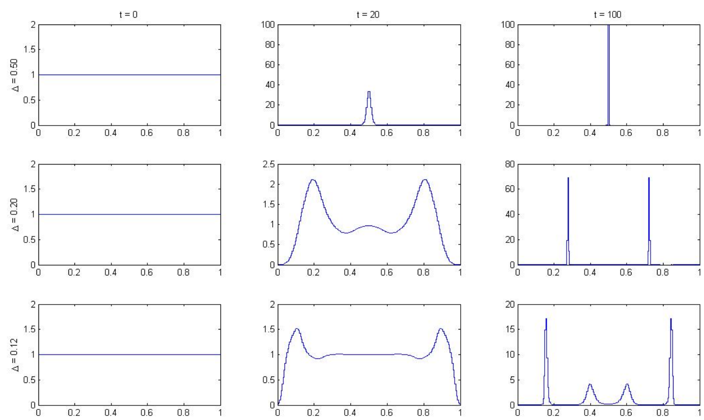


Fig. 3. $w = 0.9$. Evolution of $m(t)$ at times $t = 0, 20, 100$.

each of the extremist classes. To simulate this, we have approximated the initial conditions (Diracs) to constant splines of value $I\alpha$ and $I\frac{1-\alpha}{2}$ respectively, centered at their corresponding points, such that the initial condition has mass 1. We plot the result (1 component, i.e. total consensus, or 2 components) for each pair (α, Δ) in $[0,1] \times \left[\frac{1}{2}, 1\right]$ in Figure 4. We know from Corollary 5.16 that total consensus must occur for $\Delta \geq \alpha$ and we see that the region of convergence to total consensus is a bit larger, and slightly depends on w.

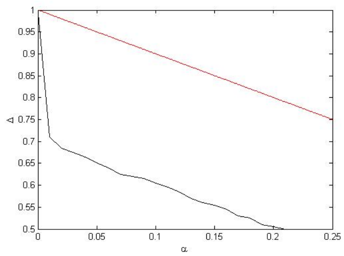
Note that values of Δ smaller than $\frac{1}{2}$ would result in no motion at all. We do this for the previous set of values for w and find that in every case, the fraction of undecided people necessary to achieve consensus is much smaller than what one would expect.

We also plot the center of masses of the first half of the distribution to show that it is not a smooth function of α and that close to the critical value $\Delta_c(\alpha)$ there is a jump. We did this for the previous 3 values of w but show only one result for brevity.

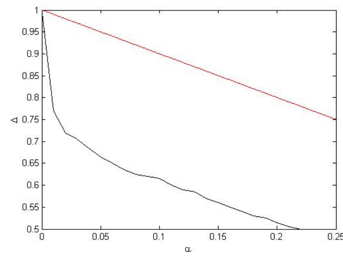
6.2.3. *Initial Uniform Conditions, Impact of* Δ

We present here the evolution of the number of components with respect to Δ, using as initial condition a uniform distribution. Note that we have capped the situations with more than 7 components into the category "7 or more", which are represented by 7 in the graph. For a component to be considered as such, we require that it has at least 1% of the total mass. Otherwise we consider it as a zero. Again, the results are plotted for the 3 different values of w.

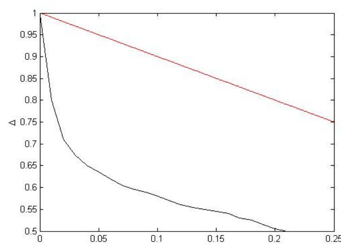
We observe that the results are almost independent of w, as there is almost no difference between the 3 curves (see Figure 6 for the combined plot of all 3



(a) $w = 0.5$



(b) $w = 0.75$



(c) $w = 0.9$

Fig. 4. Bifurcation diagram for extremists and undecided. The curly line separates the region of convergence to total consensus (above) from convergence to a partial consensus with two components. The straight line is the sufficient condition in Corollary 5.16.

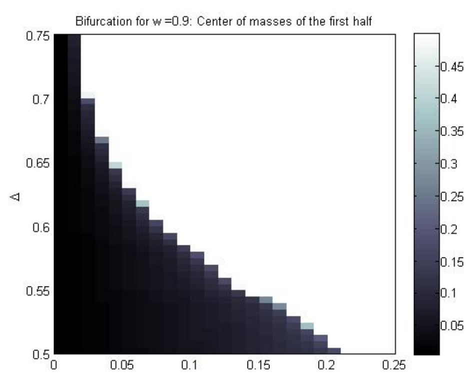


Fig. 5. $w = 0.9$. Center of masses of the first half, showing that the transition is abrupt.

functions). Another interesting thing to remark is that if we compare our results for $w = 0.5$ with the deterministic model with the ones in Ref 12 with the probabilistic model, the intervals of Δ in which they have a high probability of convergence to

n components correspond to the same intervals in which we have convergence to n components. This suggests that the approximation for $N = \infty$ is good enough to preserve properties such as the final state.

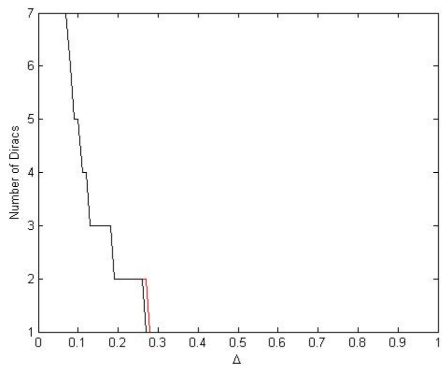


Fig. 6. Δ vs Number of components of $m(\infty)$. Uniform initial conditions. Blue - $w = 0.5$ (below black), Red - $w = 0.75$, Black - $w = 0.9$

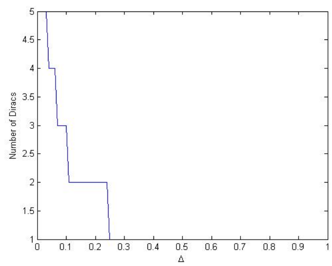
6.2.4. *Beta Distribution as Initial Condition*

Here we study the evolution of the number of components with respect to Δ, using as initial condition a Beta(1,6) distribution. The functions that have 5 or more components have been put into the category represented with a 5. Again, we consider a component if it has 1% of the total mass or more. We present the results for the 3 different values of w.

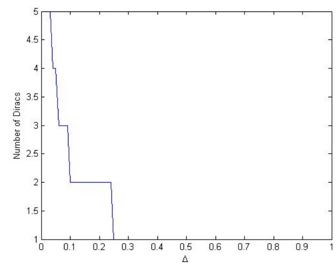
We can observe again the same phenomenon as in the uniform case, namely that the influence of w is negligible. If we compare the results from the ones in Subsection 7.3, we can conclude that the final result depends on the initial condition, even for the same parameters w and Δ. Moreover, we can see that for a fixed (w, Δ), if we start with a Beta distribution, the number of components will be smaller or equal than if we start with a uniform one. This is explained by the fact that with the Beta distribution the mass is more concentrated than with the Uniform distribution (in our case: to the left) and therefore it should be harder (i.e, Δ should be smaller) to split in the same number of components.

Appendix A. Probabilistic, Topological and Measurability issues

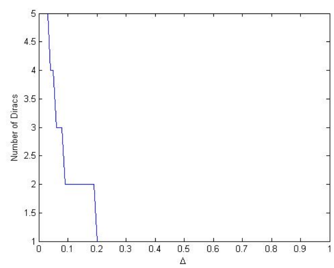
In the particular case of probability measures on $[0, 1]$, a sequence ν_n converges weakly to ν if and only if $\langle f, \nu_n \rangle$ converges to $\langle f, \nu \rangle$ for any continuous (and hence bounded) $f : [0, 1] \to \mathbb{R}$. Equivalently, the cumulative distribution function (CDF) of ν_n converges to the CDF of ν at all continuity points of the limit.



(a) $w = 0.5$



(b) $w = 0.75$



(c) $w = 0.9$

Fig. 7. Δ vs Number of components of $m(\infty)$. Initial condition Beta(1,6).

More generally, Ethier-Kurtz[16] will be the main reference.

Let $\mathcal{S}$ be a metric space with a σ-field (not necessarily the Borel σ-field), $\mathcal{P}(\mathcal{S})$ the space of probability measures on $\mathcal{S}$ (for this σ-field), and $D(\mathbb{R}_+, \mathcal{S})$ the Skorohod space of right-continuous paths with left-hand limits (for this metric).

When $\mathcal{S}$ is given the Borel σ-field, the weak topology of $\mathcal{P}(\mathcal{S})$ corresponds to the convergences

$$P_n \xrightarrow[n\to\infty]{\text{weak}} P \Leftrightarrow \langle f, P_n \rangle \xrightarrow[n\to\infty]{} \langle f, P \rangle \,, \; \forall f \in C_b(\mathcal{S}, \mathbb{R})$$

where $C_b(\mathcal{S}, \mathbb{R})$ denotes the space of continuous bounded functions. Convergence in law of random elements, defined possibly on distinct probability spaces but having common sample space $\mathcal{S}$, is defined as weak convergence of their laws:

$$Y_n \xrightarrow[n\to\infty]{\text{law}} Y \Leftrightarrow \mathcal{L}(Y_n) \xrightarrow[n\to\infty]{\text{weak}} \mathcal{L}(Y) \Leftrightarrow \mathbb{E}(f(Y_n)) \xrightarrow[n\to\infty]{} \mathbb{E}(f(Y)) \,, \; \forall f \in C_b(\mathcal{S}, \mathbb{R}) \,.$$

If $\mathcal{S}$ is separable and is given the Borel σ-field, then the weak topology is metrizable and $\mathcal{P}(\mathcal{S})$ is separable (Ref. 16, Theorems 3.3.1 and 3.1.7).

If $\mathcal{S}$ is not separable, then the Borel σ-field is usually too strong to sustain reasonable probability measures, and $\mathcal{S}$ must be given a weaker, separable, σ-field. This causes problems between topological and measure-theoretic issues, and classic results such as the Portmanteau theorem (Ref. 16, Theorem 3.3.1) may fail to hold.

The natural σ-field on $D(\mathbb{R}_+, \mathcal{S})$ is the product (or projection) σ-field of the σ-field on $\mathcal{S}$, and will always be used in the sequel. The classical topology given $D(\mathbb{R}_+, \mathcal{S})$ is the Skorohod topology, which can be metrized by (3.5.2) or (3.5.21) in Ref. 16. If $\mathcal{S}$ is separable then $D(\mathbb{R}_+, \mathcal{S})$ is separable (Ref. 16, Theorem 3.5.6) and then, if $\mathcal{S}$ is given the Borel σ-field, the Borel σ-field of the Skorohod topology and the product σ-field coincide. For weak convergence with a continuous limit process, uniform convergence on bounded time intervals may be used with adequate measurability assumptions on the test functions (Ref. 16, Theorem 3.10.2).

Appendix B. Algorithm

In this section we present an algorithm for the numerical solution of Equation (6.1). The algorithm takes as input the initial condition $f^r(x,0)$, assumed to be a piecewise constant function and the time horizon T up to which which we want to calculate an approximate solution. It outputs an approximation of the solution $f^r(x,T)$. It works as follows.

First, in steps of Δt we approximate $f^r(x, t+\Delta t)$ by using a forward Euler method:

$$f^e(x, t+\Delta t) \stackrel{\text{def}}{=} f^r(x,t) + \Delta t \partial_t f^r(x,t)$$

Here we exploit the fact that $f^r(x,t)$ is a piecewise constant function, so that we can calculate analytically the derivative which is a piecewise linear function. The expression for the derivative is explained later. Hence, $f^e(x, t+\Delta t)$ is also piecewise linear (in x), as it is the sum of a piecewise linear and a piecewise constant function. Then, we approximate $f^e(x, t+\Delta t)$ with another piecewise constant function (which we will call $f^r(x, t+\Delta t)$ for simplicity) of I intervals, so that we can reuse the same scheme and we can compute explicitly the expression for the derivative. The constants are chosen so that the integral of $f^r(., t)$ is equal to 1 (i.e. it is probability density). We perform this loop until we calculate $f^r(x,T)$ in steps of Δt. The algorithm is given next.

Algorithm 1 Numerical Solution of Equation (6.1)

Input $f^r(x,0), T, \Delta t, I$
Output $f^r(x,T)$
for $t \leftarrow 0$ **to** T **step** Δt **do**
  $f^e(x, t+\Delta t) \leftarrow f^r(x,t) + \Delta t \partial_t f^r(x,t)$
  $f^r(x, t+\Delta t) \leftarrow$ `PiecewiseConstantApproximation`$(f^e(x, t+\Delta t), I)$
end for

The method `PiecewiseConstantApproximation` returns a piecewise constant approximation such that the total integral equals 1.

B.1. *Piecewise constant approximation*

We choose as piecewise constant approximation for $f^e(x,t)$ on any interval $X = [x_s, x_e]$, $M = \frac{\int_X f^e(x,t)dx}{x_e - x_s}$, i.e. the center of mass.

Proposition B.1. *For any $t \geq 0$ it holds that $\int_0^1 f^r(x,t)\,dx = 1$.*

B.2. *Analytical expression of $\partial_t f^r(x,t)$*

Now we will give an exact expression for the derivative, using the fact that $f^r(x,t)$ is piecewise constant. This helps to understand how the calculation of the derivative is implemented and its asymptotic cost. We can write, for any t, that

$$f^r(x,t) = \sum_{i=1}^{I} a_i[H(x - x_{i+1}) - H(x - x_i)]$$

where $H(x)$ is the Heaviside step function. Let us set, for any x_i and x_j, that

$$\begin{aligned} I_1^{i,j}(x) &\stackrel{\text{def}}{=} \int_{x-\Delta}^{x+\Delta} H(x - x_i)H(z - x_j)dz = \int_{-\Delta}^{\Delta} H(x - x_i)H(x + u - x_j)du\,, \\ I_2^{i,j}(x) &\stackrel{\text{def}}{=} \frac{1}{w}\int_{x-w\Delta}^{x+w\Delta} H(z - x_i)H\left(\frac{x - (1-w)z - wx_j}{w}\right)dz \\ &= \int_{-\Delta}^{\Delta} H(x + wu - x_i)H(x - (1-w)u - x_j)du\,. \end{aligned}$$

The expression of $I_1^{i,j}(x)$ and $I_2^{i,j}(x)$ depends on the relative order between x_i and x_j and $m = \max\{(1-w)x_i + wx_j, x_i - w\Delta\}$ and is summarized in Tables 2 and 3. Finally, we can calculate $\partial_t f^r(x,t)$ as:

$$\begin{aligned} \partial_t f^r(x,t) = -2\sum_{i,j} a_i a_j (I_1^{i,j}(x) + I_1^{i+1,j+1}(x) - I_1^{i,j+1}(x) - I_1^{i+1,j}(x)) \\ + 2\sum_{i,j} a_i a_j (I_2^{i,j}(x) + I_2^{i+1,j+1}(x) - I_2^{i,j+1}(x) - I_2^{i+1,j}(x))\,. \end{aligned}$$

Case	$I_1^{i,j}(x)$
$x_i \leq x_j - \Delta \leq x_j + \Delta$	$\begin{cases} 0 & \text{if } x \leq x_j - \Delta \\ x - (x_j - \Delta) & \text{if } x_j - \Delta \leq x \leq x_j + \Delta \\ 2\Delta & \text{if } x_j + \Delta \leq x \end{cases}$
$x_j - \Delta \leq x_i \leq x_j + \Delta$	$\begin{cases} 0 & \text{if } x \leq x_i \\ x - (x_j - \Delta) & \text{if } x_i \leq x \leq x_j + \Delta \\ 2\Delta & \text{if } x_j + \Delta \leq x \end{cases}$
$x_j - \Delta \leq x_j + \Delta \leq x_i$	$\begin{cases} 0 & \text{if } x \leq x_i \\ 2\Delta & \text{if } x_i \leq x \end{cases}$

Table 2. Expression for $I_1^{i,j}(x)$

Case	$I_2^{i,j}(x)$
$m \leq x_i + w\Delta \leq x_j - (1-w)\Delta \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_j - (1-w)\Delta \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$x_i + w\Delta \leq m \leq x_j - (1-w)\Delta \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_j - (1-w)\Delta \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$m \leq x_j - (1-w)\Delta \leq x_i + w\Delta \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} - \frac{x_i-x}{w} & \text{if } x_j - (1-w)\Delta \leq x \leq x_i + w\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_i + w\Delta \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$x_i + w\Delta \leq x_j - (1-w)\Delta \leq m \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq m \\ \frac{x-x_j}{1-w} + \Delta & \text{if } m \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$m \leq x_j - (1-w)\Delta \leq x_j + (1-w)\Delta \leq x_i + w\Delta$	$\begin{cases} 0 & \text{if } x \leq x_j - (1-w)\Delta \\ \frac{x-x_j}{1-w} - \frac{x_i-x}{w} & \text{if } x_j - (1-w)\Delta \leq x \leq x_j + (1-w)\Delta \\ \Delta - \frac{x_i-x}{w} & \text{if } x_j + (1-w)\Delta \leq x \leq x_i + w\Delta \\ 2\Delta & \text{if } x_i + w\Delta \leq x \end{cases}$
$x_i + w\Delta \leq x_j - (1-w)\Delta \leq x_j + (1-w)\Delta \leq m$	$\begin{cases} 0 & \text{if } x \leq m \\ 2\Delta & \text{if } m \leq x \end{cases}$
$x_j - (1-w)\Delta \leq m \leq x_i + w\Delta \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq m \\ \frac{x-x_j}{1-w} - \frac{x_i-x}{w} & \text{if } m \leq x \leq x_i + w\Delta \\ \frac{x-x_j}{1-w} + \Delta & \text{if } x_i + w\Delta \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$x_j - (1-w)\Delta \leq x_i + w\Delta \leq m \leq x_j + (1-w)\Delta$	$\begin{cases} 0 & \text{if } x \leq m \\ \frac{x-x_j}{1-w} + \Delta & \text{if } m \leq x \leq x_j + (1-w)\Delta \\ 2\Delta & \text{if } x_j + (1-w)\Delta \leq x \end{cases}$
$x_j - (1-w)\Delta \leq m \leq x_j + (1-w)\Delta \leq x_i + w\Delta$	$\begin{cases} 0 & \text{if } x \leq m \\ \frac{x-x_j}{1-w} + \frac{x_i-x}{w} & \text{if } m \leq x \leq x_j + (1-w)\Delta \\ \Delta - \frac{x_i-x}{w} & \text{if } x_j + (1-w)\Delta \leq x \leq x_i + w\Delta \\ 2\Delta & \text{if } x_i + w\Delta \leq x \end{cases}$
$x_j - (1-w)\Delta \leq x_i + w\Delta \leq x_j + (1-w)\Delta \leq m$	$\begin{cases} 0 & \text{if } x \leq m \\ 2\Delta & \text{if } m \leq x \end{cases}$
$x_j - (1-w)\Delta \leq x_j + (1-w)\Delta \leq m \leq x_i + w\Delta$	$\begin{cases} 0 & \text{if } x \leq m \\ \Delta - \frac{x_i-x}{w} & \text{if } m \leq x \leq x_i + w\Delta \\ 2\Delta & \text{if } x_i + w\Delta \leq x \end{cases}$
$x_j - (1-w)\Delta \leq x_j + (1-w)\Delta \leq x_i + w\Delta \leq m$	$\begin{cases} 0 & \text{if } x \leq m \\ 2\Delta & \text{if } m \leq x \end{cases}$

Table 3. Expression for $I_2^{i,j}(x)$

B.3. ***Error Bound***

To calculate the error made by our approximation, define

$$g^s(x,t) \stackrel{\text{def}}{=} f(x,t) \quad \text{if } t \geq s \geq 0, \quad g^s(x,t) \stackrel{\text{def}}{=} f^r(x,t) \text{ if } 0 \leq t < s,$$

and let $\nu_e^t(dx)$, $\nu_r^t(dx)$ and $\mu_s^t(dx)$ be the measures associated to $f^e(x,t)$, $f^r(x,t)$ and $g^s(x,t)$ respectively. Note that $\nu_r^t(dx) = \mu_t^t(dx)$. Thus, we want to bound

$$\varepsilon_{tot} = |\mu_0^T(dx) - \nu_r^T(dx)|_T = \left| \sum_{k=1}^{T/(\Delta t)} \mu_{(k-1)\Delta t}^T(dx) - \mu_{k\Delta t}^T(dx) \right|_T \leq \sum_{k=1}^{T/(\Delta t)} |\mu_{(k-1)\Delta t}^T(dx) - \mu_{k\Delta t}^T(dx)|_T.$$

We can bound the error done in each iteration of the loop by decomposing it as

$$|\mu_{k\Delta t}^{k\Delta t}(dx) - \mu_{(k-1)\Delta t}^{k\Delta t}(dx)|_T \leq |\nu_r^{k\Delta t}(dx) - \nu_e^{k\Delta t}(dx)|_T + |\nu_e^{k\Delta t}(dx) - \mu_{(k-1)\Delta t}^{k\Delta t}(dx)|_T \stackrel{\text{def}}{=} \varepsilon_{c.s} + \varepsilon_{eu}.$$

Proposition B.2. *Let $M(t) = |f^r(x,t)|_\infty$. If $|f^r(x,0)|_\infty = M(0) = M < \infty$ then the following uniform bound holds:*

$$|\partial_t f^r(x, k\Delta t)|_\infty \leq c(M,T)\,, \quad \forall\, 0 \leq k \leq \frac{T}{\Delta t} - 1\,.$$

This results in the following bound for $\varepsilon_{c.s}$:

$$\varepsilon_{c.s} \leq c(M,T)\Delta t. \tag{B.1}$$

The constant $c(M,T)$ needs to be evaluated empirically; in practice, numerical experiments have shown that it is of the order $10^{-6} - 10^{-7}$. Then, with the known fact that

Proposition B.3.

$$\varepsilon_{eu} = O((\Delta t)^2), \tag{B.2}$$

using Equations (B.1) and (B.2) yields

$$|\mu_{k\Delta t}^{k\Delta t}(dx) - \mu_{(k-1)\Delta t}^{k\Delta t}(dx)|_T \leq \varepsilon_{c.s} + \varepsilon_{eu} = c\Delta t + O\left((\Delta t)^2\right).$$

Finally, we bound $|\mu_{(k-1)\Delta t}^T(dx) - \mu_{k\Delta t}^T(dx)|_T$ in terms of $|\mu_{k\Delta t}^{k\Delta t}(dx) - \mu_{(k-1)\Delta t}^{k\Delta t}(dx)|_T$.

Proposition B.4. *For all $1 \leq k \leq \frac{T}{\Delta t}$ and for all $t \geq k\Delta t$ we have that*

$$\left|\mu_{k\Delta t}^t(dx) - \mu_{(k-1)\Delta t}^t(dx)\right|_T \leq e^{8(t-k\Delta t)} \left|\mu_{k\Delta t}^{k\Delta t}(dx) - \mu_{(k-1)\Delta t}^{k\Delta t}(dx)\right|_T.$$

Combining the previous propositions, we obtain:

Theorem B.5. *For any fixed T, the error of the method is $C + O(\Delta t)$, where C is a constant that depends on c and T.*

B.4. *Complexity*

We will now give the complexity analysis of the algorithm. The computation of the derivative takes $O(I^2)$, where I is the number of intervals, since there is a double sum over I intervals. Also, this produces $O(I^2)$ splines because every $I_k^{i,j}(x), k = 1, 2$ is composed of at most 4 splines. Since the splines are not produced in increasing order of x, we need to sort them, which takes $O(I^2 \log I)$ time. Finally, we only need one pass to make the piecewise constant spline approximation since now everything is sorted. This takes $O(I^2)$ time. Since all this loop is executed $\frac{T}{\Delta t}$ times, the running time has complexity $O\left(\frac{1}{\Delta t} I^2 \log I\right)$.

Appendix C. Proofs

C.1. *Proof of Proposition 3.2*

By definition, since h is convex,

$$
\begin{aligned}
h(wx + (1-w)y) &\leq wh(x) + (1-w)h(y)\,, \\
h(wy + (1-w)x) &\leq wh(y) + (1-w)h(x)\,,
\end{aligned}
$$

with strict inequalities if h is strictly convex except when $x = y$ or $w \in \{0, 1\}$, and summing these two inequalities yields the result.

C.2. *Proof of Proposition 3.5*

The first statement is obvious, since a partial consensus is an absorbing state.

We prove the second statement. It follows from the second statement in Corollary 3.4 that, if the two peers, say (i, j) chosen at any time slot k' are such that $\left|X_i^N(k') - X_j^N(k')\right| \leq \Delta$ and $X_i^N(k') \neq X_j^N(k')$, then $\mu_n^N(k'+1) < \mu_n^N(k')$. Assume now that the hypothesis of the second statement holds. It follows that all peers chosen for interaction at times $k' \geq k$ have reputation values that either differ by more than Δ, or are equal, thus, at any time slot $k' \geq k$, the interaction has no effect. It follows that $M^N(k) = M^N(k')$ for $k' \geq k$.

Further, assume that $M^N(k)$ is not a partial consensus. Thus, there exists a pair of peers (i, j) such that $\left|X_i^N(k) - X_j^N(k)\right| \leq \Delta$ and $X_i^N(k) \neq X_j^N(k)$. The pair (i, j) is never chosen in a interaction at times $k' \geq k$, for otherwise this would contradict the fact that $M^N(k')$ is stationary. But this occurs with probability 0.

C.3. *Proof of Proposition 3.7*

Let i and j be the peers selected for interaction at time $k+1$. If at time k they were in different clusters, then nothing happens and the proposition holds. Assume now that at time k they were in the same cluster, say C_ℓ. Let i' be a peer not in C_ℓ at time k; at time $k+1$ after interaction, the opinions of i and j have moved closer, hence farther from i' to which they are still not connected. Hence, the only difference between connections at time k and $k+1$ concern pairs of peers that that are both in the same cluster, and the result easily follows.

C.4. *Proof of Theorem 3.9*

Let $\sigma^2(k)$ be the variance of $M^N(k)$ (we drop superscript N in the notation local to this proof). By Corollary 3.4, $\sigma(k)$ is non-decreasing and non-negative, and thus converges to some $\sigma(\infty)$.

For $k \geq K^N$ the set of clusters remains the same, $\mathcal{C}^N(k) = \{C_1, .., C_\ell\}$, and we can thus define the diameter of cluster $\ell_1 \in \{1, \ldots, L^N\}$ by

$$\delta_{\ell_1}(k) = \max_{i,j \in C_{\ell_1}} \left| X_i^N(k) - X_j^N(k) \right| \tag{C.3}$$

and set

$$\delta_{\ell_1} = \limsup_{k \geq K^N} \delta_{\ell_1}(k)$$

Assume that $\delta_{\ell_1} > 0$ for some ℓ_1. Since the sequence $\sigma^2(k)$ converges, there exists some random time $K_1 \geq K^N$ such that for all $k, k' > K_1$ we have

$$\left| \sigma^2(k') - \sigma^2(k) \right| < \frac{2w(1-w)}{N} \left(\frac{\delta_{\ell_1}}{2} \right)^2 , \tag{C.4}$$

and there is an infinite subsequence of time slots $K_2(n) \geq K_1$ for $n \in \mathbb{N}$ such that

$$\delta_{\ell_1}(K_2(n)) > \frac{\delta_{\ell_1}}{2} > 0 \,.$$

For $k \geq K^N$, let $(I(k), J(k))$ be a pair of peers that achieves the maximum in (C.3) and let E_k be the event "the pair of peers selected for interaction at time k is $(I(k), J(k))$". The probability of E_k, conditional to all past up to time slot k, is $\frac{2}{N(N-1)}$, thus is constant and positive. Thus the probability that E_k occurs infinitely often is 1, *i.e.*, with probability 1 we can extract an infinite subsequence of time slots $K_3(n)$ of $K_2(n)$ such that $E_{K_3(n)}$ is true. The following lemma then implies that

$$\sigma^2\left(K_3(n)+1\right) - \sigma^2\left(K_3(n)\right) > \frac{2w(1-w)}{N} \left(\frac{\delta_{\ell_1}}{2} \right)^2$$

which contradicts (C.4), which proves by contradiction that $\delta_{\ell_1} = 0$.

Lemma C.1. *Let (i, j) be the pair of peers chosen for interaction at time slot k. Assume that $\left| X_i^N(k) - X_j^N(k) \right| \leq \Delta$. Then the reduction in variance is $\sigma^2(k+1) - \sigma^2(k) = \frac{2w(1-w)}{N} \left(X_i^N(k) - X_j^N(k) \right)^2$.*

Proof. By direct computation. □

Let $\mu_{\ell_1}(k)$ be the empirical mean of cluster ℓ_1 at time $k \geq K^N$. Since interactions that modify the state of the process at times $k \geq K^N$ are all intra-cluster, it follows that $\mu_{\ell_1}(k) = \mu_{\ell_1}(K^N) := \mu_{\ell_1}(\infty)$ for all $k \geq K^N$. For $i \in C_{\ell_1}$ it holds that

$\left|X_i^N(k) - \mu_{\ell_1}(k)\right| \le \delta_{\ell_1}(k) \to 0$, and hence $X_i^N(k) \to \mu_{\ell_1}(\infty)$ as $k \to \infty$. Thus, for any continuous $f : [0,1] \to \mathbb{R}$:

$$\lim_{k\to\infty} \langle f, M^N(k)\rangle = \frac{1}{N}\sum_{\ell_1=1}^{L^N} N_{\ell_1} f\left(\mu_{\ell_1}(\infty)\right)$$

where N_{ℓ_1} is the cardinality of C_{ℓ_1}. This shows that, with probability 1, $M^N(k)$ converges to $M^N(\infty) = \frac{1}{N}\sum_{\ell_1=1}^{L^N} N_{\ell_1}\delta_{\mu_{\ell_1}(\infty)}$.

It remains to show that $M^N(\infty)$ is a partial consensus. This follows from the fact that if i and j are not in the same cluster at time slot k, then $\left|X_i^N(k) - X_j^N(k)\right| > \Delta$, which implies that $|\mu_{\ell_1}(k) - \mu_{\ell_2}(k)| > \Delta$ if $\ell_1 \neq \ell_2$ and, since, $\mu_{\ell_1}(k)$ is stationary for k large enough, that $|\mu_{\ell_1}(\infty) - \mu_{\ell_2}(\infty)| > \Delta$.

C.5. *Proof of Theorem 4.2*

We write (4.7) and (4.8) in the notation of Section 2.2 in Graham[18], in which the corresponding equations are (2.5) and (2.7), and

$$\begin{aligned}\mathcal{A}(\mu)h(x) &= 2\left\langle [h(wx+(1-w)y) - h(x)]1_{\{|x-y|\le\Delta\}}\,, \mu(dy)\right\rangle \\ &= \int (h(z)-h(x))J(\mu,x,dz)\end{aligned}$$

for $J(\mu,x,dz)$ the image measure of $1_{\{|x-y|\le\Delta\}}2\mu(dy)$ by $y \mapsto wx+(1-w)y$. Since $|J(\mu,x,\cdot)| \le 2$ and $|J(\mu,x,\cdot) - J(\nu,x,\cdot)| \le 2|\mu-\nu|$, the assumptions of Proposition 2.3 in Ref. 18 are satisfied, yielding the results. The family (4.7) is uniformly bounded by 4 in operator norm, and thus there is a well-defined inhomogeneous Markov process with generator $\mathcal{A}(m(t))$ at time t and arbitrary initial law.

C.6. *Proof of Theorem 4.3*

First, the proof of *(1)*. The generator $\mathcal{A}^N$ corresponds to the "binary mean-field model" (2.6) in Graham-Méléard[22] with N instead of n and $\mathcal{L}_i = 0$, and (using $\sum_{1\le i\neq j\le N} = 2\sum_{1\le i<j\le N}$) "jump kernel"

$$\widehat{\mu}(x,y,dh,dk) = 1_{\{|x-y|\le\Delta\}}2\delta_{\{(w-1)x+(1-w)y,(w-1)y+(1-w)x\}}(dh,dk)$$

which is uniformly bounded in total mass by $\Lambda = 2$. We conclude with Theorem 3.1 in Ref. 22 and the triangular inequality $|\frac{1}{N}\sum_{i=1}^N \mathcal{L}(\widehat{X}_i^N) - Q|_T \le |\mathcal{L}(\widehat{X}_i^N) - Q|_T$ (the $\widehat{X}_i^N$ are exchangeable).

Now, the proof of *(2)*. As in the proof of Theorem 3.1 in Ref. 22,

$$\left\langle \phi, \widehat{\Lambda}^N - \frac{1}{N}\sum_{i=1}^N \mathcal{L}(\widehat{X}_i^N)\right\rangle^2 = \frac{1}{N^2}\left[\sum_{i=1}^N (\phi(\widehat{X}_i^N) - \mathbb{E}[\phi(\widehat{X}_i^N)])\right]^2$$

in which

$$\left[\sum_{i=1}^{N}(\phi(\widehat{X}_i^N) - \mathbb{E}[\phi(\widehat{X}_i^N)])\right]^2 = \sum_{i=1}^{N}(\phi(\widehat{X}_i^N) - \mathbb{E}[\phi(\widehat{X}_i^N)])^2 + \sum_{1\le i\ne j\le N}(\phi(\widehat{X}_i^N) - \mathbb{E}[\phi(\widehat{X}_i^N)])(\phi(\widehat{X}_j^N) - \mathbb{E}[\phi(\widehat{X}_j^N)])$$

where the first sum on the r.h.s. has N terms, the second $N(N-1)$, and

$$\mathbb{E}\big[(\phi(\widehat{X}_i^N) - \mathbb{E}[\phi(\widehat{X}_i^N)])(\phi(\widehat{X}_j^N) - \mathbb{E}[\phi(\widehat{X}_j^N)])\big] = \mathbb{E}[\phi(\widehat{X}_i^N)\phi(\widehat{X}_j^N)] - \mathbb{E}[\phi(\widehat{X}_i^N)]\mathbb{E}[\phi(\widehat{X}_j^N)]\,,$$

and we conclude to the first formula in *(2)* using *(1)* for $k = 2$.

Classically, the weak topology in the Polish space $\mathcal{P}(D(\mathbb{R}_+, [0,1]))$ has a convergence-determining sequence $(g_m)_{m\ge1}$ of continuous functions bounded by 1 (such a sequence is constructed in the proof of Proposition 3.4.4 in Ethier-Kurtz[16]), and can thus be metrized by $d(P,Q) = \left(\sum_{i\ge1} 2^{-i}\langle g_m, P-Q\rangle^2\right)^{1/2}$. Moreover, the first formula in *(2)* and the second in *(1)* imply that $\mathbb{E}(d(\widehat{\Lambda}^N, Q)^2)$ goes to 0, which proves convergence in probability for $\widehat{\Lambda}^N$.

The result for $\widehat{\Lambda}^N$ implies the result for its marginal process $\widehat{M}^N$ as a quite general topological fact, since the limit marginal process m is continuous and the spaces are Polish (Theorem 4.6 in Graham-Méléard,[22] Section 4.3 in Méléard[31]); proofs first use the Skorohod topology, and then Theorem 3.10.2 in Ref. 16.

C.7. *Proof of Theorem 4.4*

Let $\lambda_N : \mathbb{R}_+ \to \mathbb{R}_+$ be the (random) time-change given by the linear interpolation of $\lambda_N(\frac{k}{N}) = \frac{T_k}{N}$, *i.e.*, by

$$t \in \left[\frac{k}{N}, \frac{k+1}{N}\right] \mapsto \lambda_N(t) = (k+1-tN)\frac{T_k}{N} + (tN-k)\frac{T_{k+1}}{N}\,, \qquad k\in\mathbb{N}\,.$$

Then (4.4) implies that

$$\widetilde{X}^N(t) = \widehat{X}^N(\lambda_N(t))\,, \qquad t\in\mathbb{R}_+\,,$$

so that their atomic distance is null. The triangular inequality yields, for $k\in\mathbb{N}$,

$$|\lambda_N(t) - t| \le \left|\frac{T_k}{N} - \frac{k}{N}\right| + \frac{1}{N}(T_{k+1} - T_k) + \frac{1}{N}\,, \qquad t \in \left[\frac{k}{N}, \frac{k+1}{N}\right],$$

and hence, for any $T > 0$,

$$\sup_{0\le t\le T}|\lambda_N(t) - t| \le \frac{1}{N}\sup_{0\le k\le\lfloor NT\rfloor}|T_k - k| + \frac{1}{N}\sup_{0\le k\le\lfloor NT\rfloor}(T_{k+1} - T_k) + \frac{1}{N}\,.$$

For $\varepsilon > 0$, Kolmogorov's maximal inequality implies that

$$\mathbb{P}\left(\frac{1}{N}\sup_{0\le k\le\lfloor NT\rfloor}|T_k - k| \ge \varepsilon\right) \le \frac{1}{\varepsilon^2N^2}\sum_{i=1}^{\lfloor NT\rfloor}\mathrm{var}(T_i - T_{i-1}) = \frac{\lfloor NT\rfloor}{\varepsilon^2N^2}\,,$$

and classically

$$\mathbb{P}\left(\frac{1}{N}\sup_{0\le k\le \lfloor NT\rfloor}(T_{k+1}-T_k)\ge\varepsilon\right)=1-(1-\mathrm{e}^{-N\varepsilon})^{\lfloor NT\rfloor+1}\le(\lfloor NT\rfloor+1)\mathrm{e}^{-N\varepsilon}\,.$$

Hence, for all $\delta>0$,

$$\lim_{N\to\infty}\mathbb{P}\left(\sup_{0\le t\le T}|\lambda_N(t)-t|\ge\delta\right)=0\,,$$

from which the result follows.

C.8. *Proof of Theorem 4.5*

Result *(1)* follows from the previous convergence in probability result and Theorem 4.3, using either the uniform continuity of the test functions (for the atomic metric) or Corollary 3.3.3 in Ethier-Kurtz[16] (for the usual metric). Result *(2)*, which involves Polish spaces, follows as for Theorem 4.3.

C.9. *Proof of Proposition 5.2*

For $0<b$ and $t\in[0,b]$ define $u(t):=\sigma^2(b-t)-\sigma^2(b)$. Note that $\mu_1(t)$ is a constant thus $u(t)=\mu_2(b-t)-\mu_2(b)$. By the alternative definition of Problem 1

$$\begin{aligned}u(t)=-\int_{b-t}^{b}\int_{[0,1]^2}&\left[(wx+(1-w)y)^2+(wy+(1-w)x)^2-x^2-y^2\right]\\&1_{\{|x-y|\le\Delta\}}m(s)(dx)m(s)(dy)ds\end{aligned}$$

By Proposition 3.2, the bracket is nonpositive, and the indicator function is upper bounded by 1 thus

$$\begin{aligned}u(t)\le-\int_{b-t}^{b}\int_{[0,1]^2}&\left[(wx+(1-w)y)^2+(wy+(1-w)x)^2-x^2-y^2\right]\\&m(s)(dx)m(s)(dy)ds\\=K\int_{b-t}^{t}&\sigma^2(s)ds=K\left(\sigma^2(b)+\int_0^t u(s)ds\right)\end{aligned}$$

with $K=4w(1-w)$. By Grönwall's lemma:

$$u(t)\le K\sigma^2(b)t+K^2\sigma^2(b)e^{Kt}\int_0^t se^{-Ks}ds=\sigma^2(b)\left(e^{Kt}-1\right)$$

Let $t=b$ and the proposition follows.

C.10. *Proof of Proposition 5.4*

Fix some $t_0\ge0$; we will show that $\operatorname{ess\,inf}(m(t))\ge\operatorname{ess\,inf}(m(t_0))$ for every $t\ge t_0$. Clearly, it is sufficient to consider the case $\operatorname{ess\,inf}(m(t_0))>0$. Take some arbitrary

$a < \operatorname{ess\,inf}(m(t_0))$. Let $h(x) = 1_{\{x \le a\}}$ and $\varphi(t) = \langle h, m(t)\rangle$. We have $\varphi(t_0) = 0$ and, by definition of Problem 1:

$$\varphi(t) \le 2\int_{t_0}^{t} \langle |h(wx + (1-w)y) - h(x)|\,, m(s)(dx)m(s)(dy)\rangle ds$$

Note that $|h(wx + (1-w)y) - h(x)| \le 1$ and that $h(wx + (1-w)y) - h(x) \neq 0$ requires either $x \le a, y > a$ or $x > a, y \le a$. Thus

$$\varphi(t) \le 2\int_{t_0}^{t} 2\varphi(s)(1-\varphi(s))ds \le 4\int_{t_0}^{t} \varphi(s)ds$$

By Grönwall's lemma, this shows that $\varphi(t) = 0$ for $t \ge t_0$. Thus $m(t)[0,a] = 0$ for all $t \ge t_0$ and this is true for any $a < \operatorname{ess\,inf}(m(t_0))$ thus $\operatorname{ess\,inf}(m(t)) \ge \operatorname{ess\,inf}(m(t_0))$. This shows $\operatorname{ess\,inf}(m(t))$ is non decreasing. The proof is similar by analogy for the $\operatorname{ess\,sup}$.

C.11. *Proof of Theorem 5.5*

1. We show that $m(t)$ converges to some probability $m(\infty)$. This follows from Proposition 3.2 applied for example to the family of functions $h_\omega : x \to e^{-\omega x}$ indexed by $\omega \in [0,\infty)$. For any fixed ω, $\langle h_\omega, m(t)\rangle$ is a nondecreasing function of t and is nonnegative, thus converges as $t \to \infty$. The limit is a probability (apply convergence to the constant equal to 1).
2. We would like to conclude that $m(\infty)$ is a stationary point, i.e. $\langle \mathcal{A}(m(\infty))h, m(\infty)\rangle = 0$ for any $h \in L^\infty[0,1]$, however there is a technical difficulty since the definition of $\mathcal{A}$ involves the discontinuous function $1_{\{|x-y|\le\Delta\}}$. We circumvent the difficulty as follows. For $\varepsilon > 0$ and smaller than Δ, let $\ell_\varepsilon(x)$ be the continuous function of $x \in \mathbb{R}^+$ equal to 1 for $x \le \Delta - \varepsilon$, 0 for $x \ge \Delta$, and the linear interpolation in-between. We have $1_{\{x\le\Delta-\varepsilon\}} \le \ell_\varepsilon(x) \le 1_{\{x\le\Delta\}}$ for all $x \ge 0$. Let $h(x) = x^2$. By the alternative definition of Problem 1, for t and $u \ge 0$:

$$\begin{aligned}&\langle h, m(t+u)\rangle - \langle h, m(t)\rangle\\ &\le -2w(1-w)\int_t^{t+u} \langle (x-y)^2\ell_\varepsilon(|x-y|), m(s)(dx)m(s)dy\rangle ds\end{aligned}$$

Fix $u \ge 0$ and let $t \to \infty$. By weak convergence of the product measure $m(t) \otimes m(t)$ it follows that

$$0 \le -2w(1-w)u\langle (x-y)^2\ell_\varepsilon(|x-y|), m(\infty)(dx)m(\infty)dy\rangle$$

and thus $\langle (x-y)^2\ell_\varepsilon(|x-y|), m(\infty)(dx)m(\infty)dy\rangle = 0$ from where we conclude that

$$\langle (x-y)^2 1_{\{|x-y|\le\Delta-\varepsilon\}}, m(\infty)(dx)m(\infty)dy\rangle = 0 \tag{C.5}$$

for all $\varepsilon \in (0,\Delta)$.
3. Fix some $\varepsilon > 0$ and integrate the previous equation with respect to y; it comes that $\langle r(x), m(\infty)(dx)\rangle = 0$ with $r(x) \stackrel{\text{def}}{=} \langle (y-x)^2 1_{\{|y-x|\le\Delta-\varepsilon\}}, m(\infty)(dy)\rangle$, thus there is a set $\Omega_1 \subset [0,1]$ with $m(\infty)(\Omega_1) = 1$ and $r(x) = 0$ for every $x \in \Omega_1$. Let x_1

be an element of Ω_1 (which is not empty since $m(\infty)(\Omega_1) = 1$). Then $r(x_1) = 0$ and thus $m(\infty)\left([(x_1 - \Delta + \varepsilon, x_1) \cup (x_1, x_1 + \Delta - \varepsilon)] \cap [0,1]\right) = 0$ and the restriction of $m(\infty)$ to $(x_1 - \Delta + \varepsilon, x_1 + \Delta - \varepsilon) \cap [0,1]$ is a dirac mass at x_1. Apply the same reasoning to the complement of $(x_1 - \Delta + \varepsilon, x_1 + \Delta - \varepsilon)$, this shows recursively that $m(\infty)$ is a finite sum of Dirac masses, i.e. $m(\infty) = \sum_{i=1}^{I} \alpha_i \delta_{x_i}$ for some $I \in \mathbb{N}$, $\alpha_i > 0$, $\sum_{i=1}^{I} \alpha_i = 1$ and $x_i \in [0,1]$.

Assume that $|x_i - x_j| < \Delta$ for some $i \neq j$. Apply Eq.(C.5) with $\varepsilon = \frac{\Delta - |x_i - x_j|}{2}$. The right-handside of Eq.(C.5) is lower bounded by $\alpha_i \alpha_j (x_i - x_j)^2 > 0$, which is a contradiction. Therefore $|x_i - x_j| \geq \Delta$ for all $i \neq j$.

C.12. *Proof of Proposition 5.7*

First we show that if $\nu \in P_{n+1}(\mu_0)$ then there exists some $\nu' \in P_n(\mu_0)$ with $\langle h, \nu' \rangle \leq \langle h, \nu \rangle$, which will clearly show the proposition.

We are given $\nu = \sum_{i=1}^{n+1} \alpha_i \delta_{x_i} \in P_{n+1}(\mu_0)$. Let $x'_n = \frac{\alpha_n x_n + \alpha_{n+1} x_{n+1}}{\alpha_n + \alpha_{n+1}}$ and

$$\nu' = \sum_{i=1}^{n-1} \alpha_i \delta_{x_i} + \left((\alpha_n + \alpha_{n+1})\, \delta_{x'_n} \right)$$

We have $\nu' \in P_n(\mu)$ and by convexity of h:

$$(\alpha_n + \alpha_{n+1})\, h(x'_n) \leq \alpha_n h(x_n) + \alpha_{n+1} h(x_{n+1})$$

thus $\langle h, \nu' \rangle \leq \langle h, \nu \rangle$ as required.

C.13. *Proof of Theorem 5.8*

By hypothesis $\langle h, m_0 \rangle \leq q$ and since h is continuous, by Theorem 5.5, $\langle h, m(\infty) \rangle \leq q$. Since the mean of $m(\infty)$ is also μ_0 (again by Theorem 5.5 applied to $h(x) = x$), it follows that q is not in $Q_d(h, \mu_0)$. Together with the hypothesis $q \in Q_n(h, \mu_0)$, Proposition 5.7 implies that $c < n$.

C.14. *Proof of Proposition 5.11*

Let $m'(t)$ be the image measure of $m(t)$ by $x \mapsto 1 - x$. By direct computation and the alternative form of Problem 1, it follows that $m'(t)$ is solution to Problem 1 with initial condition $m'(0) = m(0)$. By uniqueness, $m'(t) = m(t)$.

C.15. *Proof of Proposition 5.12*

Let ν be a symmetric partial consensus with $n + 1$ components. We do as in the proof of Proposition 5.7: If $n + 1$ is even, we replace the two middle components by their weighted averages. If $n + 1$ is odd, we replace the three middle components $x_{m-1}, x_m = 0.5, x_{m+1}$ (with $m = n/2 + 1$) by two components $(\alpha_{m-1} x_{m-1} + 0.5\alpha_m x_m)/(\alpha_{m-1} + 0.5\alpha_m)$ and $(0.5\alpha_m x_m + \alpha_{m+1} x_{m+1})/(0.5\alpha_m + \alpha_{m+1})$ with weights $\alpha_{m-1} + 0.5\alpha_m$ and $0.5\alpha_m + \alpha_{m+1}$. We obtain some $\nu' \in SP_n$

and $\langle h, \nu' \rangle \leq \langle h, \nu \rangle$ for any convex h, thus if $q \in SQ_n(h)$ we must also have $q \in SQ_{n+1}(h)$.

C.16. *Proof of Theorem 5.13*

The proof is similar to Theorem 5.8.

C.17. *Proof of Theorem 6.1*

Assuming that m_0 is absolutely continuous, the fact that $m(t)$ is absolutely continuous can be proved by probabilistic arguments which use representations by inhomogeneous Markov processes with uniformly bounded jump rates.

More precisely, the proof of Theorem 2.1 in Desvillettes *et al.*,[13] for a class of equations (the generalized cutoff Kac equation) with the same probabilistic structure as ours, extends immediately to the present situation. It is an extension of Theorem 4.2 proved using only its hypotheses.

If $m = (m(t), t \in \mathbb{R}_+)$ is a solution of Problem 1 and $m(t)(dx) = f(x,t)\,dx$ then, for any bounded h, an elementary change of variables yields

$$\begin{aligned}
&\int h(x) f(x,t)\,dx - \int h(x) f(x,0)\,dx \\
&\quad = 2\int_0^t \iint h(wx + (1-w)y) 1_{\{|x-y|\leq\Delta\}} f(x,s) f(y,s)\,dxdy\,ds \\
&\qquad - 2\int_0^t \iint h(x) 1_{\{|x-y|\leq\Delta\}} f(x,s) f(y,s)\,dxdy\,ds \\
&\quad = \frac{2}{w}\int_0^t \int h(x') \left[\int_{x'-\Delta w}^{x'+\Delta w} f\left(\frac{x'-(1-w)y}{w}, s\right) f(y,s)\,dy\right] dx'\,ds \\
&\qquad - 2\int_0^t \int h(x) f(x,s) \left[\int_{x-\Delta}^{x+\Delta} f(y,s)\,dy\right] dx\,ds
\end{aligned}$$

from which (6.1) readily follows.

The converse statement follows by integrating Eq.(6.1) by $h(x)\,dx$, which after the reverse change of variables yields Problem 1 as a weak formulation.

Eq.(6.2) is obtained similarly using the change of variables $x' = \frac{wx-(1-w)y}{2w-1}$ and $y' = \frac{wy-(1-w)x}{2w-1}$.

C.18. *Proof of Proposition 6.2*

Since $f(x,t)$ is non-negative, we have:

$$\frac{\partial f(x,t)}{\partial t} \leq \frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t) f\left(\frac{x-(1-w)y}{w}, t\right) dy.$$

For a fixed arbitrary t, let $A_i = \{x \in \operatorname{Supp}(f(x,t)) | i-1 < f(x,t) \leq i\},\ i > 0$ be the level sets. Note that $A_j = \emptyset$ for all $j > \lceil M(t) \rceil$ and that the A_i are disjoint.

For any x, we have that:

$$\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy$$

$$\leq \frac{2}{w}\sum_{i,j}\mu\left(\left\{y\,\middle|\,y\in A_i,\frac{x-(1-w)y}{w}\in A_j\right\}\right)\max\{i,j\}^2.$$

Using the fact that the A_i are disjoint we can get that:

$$\frac{2}{w}\sum_{i,j}\mu\left(\left\{y\,\middle|\,y\in A_i,\frac{x-(1-w)y}{w}\in A_j\right\}\right)\max\{i,j\}^2$$

$$= \frac{2}{w}\sum_{i}\mu\left(\left\{y\,\middle|\,y\in A_i,\frac{x-(1-w)y}{w}\in \bigcup_{k\leq i}A_k\right\}\right)i^2$$

$$+\frac{2}{w}\sum_{i}\mu\left(\left\{y\,\middle|\,y\in \bigcup_{k<i}A_k,\frac{x-(1-w)y}{w}\in A_i\right\}\right)i^2 = I_1+I_2.$$

We can bound I_1 and I_2 now as:

$$I_1 \leq \frac{2}{w}\sum_i \mu(A_i)i^2, \qquad I_2 \leq \frac{2}{1-w}\sum_i \mu(A_i)i^2,$$

subject to the following restrictions:

$$\sum_i \mu(A_i)\leq 1, \qquad \sum_i (i-1)\mu(A_i)\leq \int_0^1 f(x,t)dx = 1.$$

Plugging the second restriction into the bound of I_1 and I_2, we get that:

$$\sum_{i=1}^{\lceil M(t)\rceil}\mu(A_i)i^2 \leq \frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil -1} + \sum_{i=1}^{\lceil M(t)\rceil -1}\mu(A_i)\left(i^2 - \frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil -1}(i-1)\right)$$

$$= \frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil -1} + \frac{1}{\lceil M(t)\rceil -1}\sum_{i=1}^{\lceil M(t)\rceil -1}\mu(A_i)\left(\lceil M(t)\rceil i - \lceil M(t)\rceil - i)(i-\lceil M(t)\rceil\right).$$

The maximum of the RHS is attained when $\mu(A_i)=0\quad\forall\, i>1$ and $\mu(A_1)$ is as big as possible. By the first restriction, $\mu(A_1)=1$. In that case, we have that:

$$\sum_{i=1}^{\lceil M(t)\rceil}\mu(A_i)i^2 \leq \frac{\lceil M(t)\rceil^2}{\lceil M(t)\rceil -1}+1 \leq \lceil M(t)\rceil + 3 \leq M(t)+4.$$

Therefore:

$$\sup_{A_i}\left\{\sum_i \mu(A_i)i^2\right\} \leq M(t)+4.$$

Finally, for any x we have:

$$\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy \leq I_1 + I_2 \leq \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4),$$

which means that:

$$M'(t) \leq \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4).$$

Integrating, we get the result.

C.19. *Proof of Proposition 6.3*

Again, since $f(x,t)$ is non-negative, for all x,

$$\begin{aligned}&\left|\frac{\partial}{\partial t}f(x,t)\right| \\ &\leq \max\left\{\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy,\ 2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(y,t)dy\right)\right\}.\end{aligned}$$

On the one hand,

$$2f(x,t)\left(\int_{x-\Delta}^{x+\Delta} f(y,t)dy\right) \leq 2M(t)\int_0^1 f(y,t)dy \leq 2M(t),$$

on the other, using Proposition 6.2,

$$\frac{2}{w}\int_{x-w\Delta}^{x+w\Delta} f(y,t)f\left(\frac{x-(1-w)y}{w},t\right)dy \leq \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4),$$

therefore

$$\left|\frac{\partial}{\partial t}f(\cdot,t)\right|_\infty \leq \left(\frac{2}{w}+\frac{2}{1-w}\right)(M(t)+4).$$

C.20. *Proof of Proposition B.1*

Let $f^r(x,t)$ be defined piecewise in the intervals $X_i = [x_i, x_{i+1}]$ and let M_i be constant chosen for the piecewise constant approximation on the interval X_i. We have that, independently of t:

$$\int_0^1 f^r(x,t)dx = \int_0^1 \sum_{i=1}^{I} M_i 1_{X_i} dx = \sum_{i=1}^{I}\int_{X_i} \frac{\int_{x_i}^{x_{i+1}} f^e(y,t)dy}{x_{i+1}-x_i}dx = \int_0^1 f^e(y,t)dy.$$

C.21. *Proof of Proposition B.2*

Keeping in mind that for any interval, the slope of $f^e(x, k\Delta t)$ is bounded by $\frac{2\Delta t|\partial_t f^r(x,(k-1)\Delta t)|_\infty}{1/I}$, yielding:

$$\varepsilon_{c.s.}(I) \leq I \frac{\text{Max. Slope}}{2}\left(\frac{1}{I}\right)^2 = \Delta t|\partial_t f^r(x,(k-1)\Delta t)|_\infty. \qquad \text{(C.6)}$$

On the other hand:

$$M(\Delta t) = |f^r(x,\Delta t)|_\infty \leq |f^e(x,\Delta t)|_\infty \leq |f^r(x,0)|_\infty + \Delta t|\partial_t f^r(x,0)|_\infty$$

$$\leq M + \Delta t K_1 M + \Delta t K_2 = (1+\Delta t K_1)M + \Delta t K_2,$$

where $K_1 = \frac{2}{w} + \frac{2}{1-w}$, $K_2 = \frac{8}{w} + \frac{8}{1-w}$. The first inequality is true because when we approximate by piecewise constant splines, the maximum of the function decreases and the third is true by Proposition 6.2. Note that in order to be able to apply it we are implicitly using Proposition B.1 as the total mass is conserved. By induction:

$$M\left(\frac{T}{\Delta t}\Delta t\right) \leq (1+\Delta t K_1)^{\frac{T}{\Delta t}} M + \Delta t K_2 \sum_{i=0}^{T/\Delta t - 1}(1+\Delta t K_1)^i$$

$$= (1+\Delta t K_1)^{\frac{T}{\Delta t}} M + \frac{K_2}{K_1}((1+\Delta t K_1)^{\frac{T}{\Delta t}} - 1) \leq \frac{K_2}{K_1}(1+\Delta t K_1)^{\frac{T}{\Delta t}}\left(M + \frac{K_2}{K_1}\right).$$

We can now bound $M(k\Delta t)$ in the following way. As K_1 and K_2 are positive, taking into account that $(1+K_1\Delta t)^{\frac{T}{\Delta t}}$ is decreasing with Δt, we have for any k:

$$M(k\Delta t) \leq (1+\Delta t K_1)^{\frac{T}{\Delta t}} M + \frac{K_2}{K_1}(1+\Delta t K_1)^{\frac{T}{\Delta t}} \leq e^{K_1 T}\left(M + \frac{K_2}{K_1}\right).$$

Using Proposition 6.3:

$$|\partial_t f^r(x,(k-1)\Delta t)|_\infty \leq K_1 M((k-1)\Delta t) + K_2 \leq K_1 e^{K_1 T}\left(M + \frac{K_2}{K_1}\right) + K_2 \equiv c.$$

Combining this equation with equation (C.6) we get the desired result.

C.22. *Proof of Proposition B.3*

We have that:

$$\begin{aligned}
\varepsilon_{eu} &= |\nu_e^{k\Delta t}(dx) - \mu_{(k-1)\Delta t}^{k\Delta t}(dx)|_T \\
&= \int_0^1 |g^{(k-1)\Delta t}(x,k\Delta t) - g^{(k-1)\Delta t}(x,(k-1)\Delta t) - \Delta t\partial_t g^{(k-1)\Delta t}(x,(k-1)\Delta t)|dx \\
&\leq \frac{1}{2}(\Delta t)^2|\partial_{tt}^2 g^{(k-1)\Delta t}(x,(k-1)\Delta t)|_\infty + O\left((\Delta t)^3\right).
\end{aligned}$$

By Corollary 6.4, we can bound, for any k:

$$|\partial_{tt}^2 g^{(k-1)\Delta t}(x,(k-1)\Delta t)|_\infty \leq 16\Delta|\partial_t g^{(k-1)\Delta t}(x,(k-1)\Delta t)|_\infty |g^{(k-1)\Delta t}(x,(k-1)\Delta t)|_\infty$$

$$\leq 16\Delta \left(K_1 e^{K_1 T} \left(M + \frac{K_2}{K_1} \right) + K_2 \right) e^{K_1 T} \left(M + \frac{K_2}{K_1} \right) = C_2,$$

therefore:

$$\varepsilon_{eu} \leq \frac{C_2}{2} (\Delta t)^2 + O((\Delta t)^3) = O((\Delta t)^2). \tag{C.7}$$

C.23. ***Proof of Proposition B.4***

$$\begin{aligned}
&\frac{\partial}{\partial t} \int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)| dx \leq \int_0^1 |\partial_t g^{k\Delta t}(x,t) - \partial_t g^{(k-1)\Delta t}(x,t)| \\
&\leq \int_0^1 2 \left| -g^{k\Delta t}(x,t) \int_{x-\Delta}^{x+\Delta} g^{k\Delta t}(y,t) dy + g^{(k-1)\Delta t}(x,t) \int_{x-\Delta}^{x+\Delta} g^{(k-1)\Delta t}(y,t) dy \right| \\
&\quad + \int_0^1 \frac{2}{w} \left| \int_{x-w\Delta}^{x+w\Delta} g^{k\Delta t}(y,t) g^{k\Delta t} \left(\frac{x-(1-w)y}{w}, t \right) dy \right. \\
&\quad \left. - \int_{x-w\Delta}^{x+w\Delta} g^{(k-1)\Delta t}(y,t) g^{(k-1)\Delta t} \left(\frac{x-(1-w)y}{w}, t \right) dy \right| = I + J.
\end{aligned}$$

We will first bound I. We have that:

$$I \leq 2 \int_0^1 |g^{(k-1)\Delta t}(x,t) - g^{k\Delta t}(x,t)| \int_{x-\Delta}^{x+\Delta} g^{(k-1)\Delta t}(y,t) dz dx$$

$$+2 \int_0^1 g^{k\Delta t}(x,t) \int_{x-\Delta}^{x+\Delta} |g^{(k-1)\Delta t}(y,t) - g^{k\Delta t}(y,t)| dz dx = I_1 + I_2.$$

On the one hand:

$$I_1 \leq 2 \int_0^1 |g^{(k-1)\Delta t}(x,t) - g^{k\Delta t}(x,t)| dx,$$

on the other:

$$\begin{aligned}
I_2 &\leq 2 \int_0^1 g^{k\Delta t}(x,t) \int_0^1 |g^{(k-1)\Delta t}(y,t) - g^{k\Delta t}(y,t)| dz dx \\
&\leq 2 \int_0^1 |g^{(k-1)\Delta t}(x,t) - g^{k\Delta t}(x,t)| dx.
\end{aligned}$$

Now we will bound J:

$$\begin{aligned}
J \leq\ & \frac{2}{w} \int_0^1 \int_{x-w\Delta}^{x+w\Delta} g^{k\Delta t}(y,t) \\
& \times \left| g^{k\Delta t} \left(\frac{x-(1-w)y}{w}, t \right) - g^{(k-1)\Delta t} \left(\frac{x-(1-w)y}{w}, t \right) \right| dz dx \\
& + \frac{2}{w} \int_0^1 \int_{x-w\Delta}^{x+w\Delta} \left| g^{k\Delta t}(y,t) - g^{(k-1)\Delta t}(y,t) \right| \\
& \times g^{(k-1)\Delta t} \left(\frac{x-(1-w)y}{w}, t \right) dz dx = J_1 + J_2
\end{aligned}$$

$$J_1 = 2\int_0^1 \int_{x-\Delta}^{x+\Delta} g^{k\Delta t}(x,t) \left| g^{k\Delta t}(y,t) - g^{(k-1)\Delta t}(y,t) \right| dzdx$$

$$\leq 2\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)| dx$$

$$J_2 = 2\int_0^1 \int_{x-\Delta}^{x+\Delta} \left| g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t) \right| g^{(k-1)\Delta t}(y,t)\, dzdx$$

$$\leq 2\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)| dx.$$

Adding all the equations together we get that:

$$\frac{\partial}{\partial t}\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)| dx \leq I + J \leq I_1 + I_2 + J_1 + J_2$$

$$\leq 8\int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)| dx.$$

Integrating:

$$\left| \mu^t_{k\Delta t}(dx) - \mu^t_{(k-1)\Delta t}(dx) \right|_T = \int_0^1 |g^{k\Delta t}(x,t) - g^{(k-1)\Delta t}(x,t)| dx$$

$$\leq e^{8(t-k\Delta t)} \int_0^1 |g^{k\Delta t}(x,k\Delta t) - g^{(k-1)\Delta t}(x,k\Delta t)| dx$$

$$= e^{8(t-k\Delta t)} |\mu^{k\Delta t}_{k\Delta t}(dx) - \mu^{k\Delta t}_{(k-1)\Delta t}(dx)|_T,$$

as we wanted to prove.

C.24. *Proof of Theorem B.5*

$$\varepsilon_{tot} \leq \sum_{k=1}^{T/(\Delta t)} \left| \mu^T_{(k-1)\Delta t}(dx) - \mu^T_{k\Delta t}(dx) \right|_T \leq e^{8T} \sum_{k=1}^{T/(\Delta t)} \left| \mu^{k\Delta t}_{(k-1)\Delta t}(dx) - \mu^{k\Delta t}_{k\Delta t}(dx) \right|_T$$
$$= e^{8T} \frac{T}{\Delta t} \left(c\Delta t + O\left((\Delta t)^2 \right) \right) = C + O\left(\Delta t\right).$$

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