Medial Axis Approximation of Planar Shapes from Union of Balls: A Simpler and more Robust Algorithm

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Abstract

Given a dense sampling S of the smooth boundary of a planar shape O. We show that the medial axis of the union of Voronoi balls centered at Voronoi vertices inside O has a particularly simple structure and can be computed more efficiently and robustly than for a general union of balls.

1 Introduction

The medial axis is a powerful shape descriptor: Lieutier [6] has shown that the medial axis of a shape (bounded open subset of Euclidean space) is always homotopy equivalent to the shape itself. Unfortunately, efficient algorithms for the exact computation of the medial axis are only known for a small number of shapes —among them unions of open balls. On the other hand, balls are powerful modeling primitives since any shape can be well approximated by a union of balls. Often this approximation can be obtained algorithmically: for example, given a shape O in $\mathbb{R}^3$ with smooth boundary ∂O and a sufficiently dense sample S of ∂O, then the union of polar balls (certain Voronoi balls) whose centers are contained in O—inner polar balls—provide a homotopic approximation of O that is also geometrically close to O [1]. In the plane the situation is even simpler [4]: Given a shape $O \subset \mathbb{R}^2$ with smooth boundary ∂O, the Voronoi vertices of a dense sample S of ∂O are either close to the medial axis of O or close to the medial axis of $\mathbb{R}^2 \setminus O$. The two types of Voronoi vertices (inner Voronoi vertices if they reside inside O and outer Voronoi vertices otherwise) can be separated easily. If the sampling S is dense enough, then the union U of Voronoi balls centered at the inner Voronoi vertices (inner Voronoi balls) provide a good approximation of O and the medial axis of the union of U can be used to approximate the medial axis of O. In this paper we show that (provided the sampling S of ∂O is dense enough) the medial axis of the inner Voronoi balls has a particularly simple structure that can be computed efficiently and robustly provided. The key observation to prove the correctness of the new algorithm is that in our scenario the vertices in the boundary of U—the boundary of U is made up from circular arcs that meet in vertices—all need to be sample points.

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2 Preliminaries

Shape. A shape is a bounded open subset of $\mathbb{R}^d$. We call a shape O *smooth*, if its boundary ∂O is a smooth submanifold of $\mathbb{R}^d$.

Medial Axis of a Shape. The medial axis $M(O)$ of a shape O is the closure of the set of points in O that have at least two closest points in the boundary ∂O.

Sampling. A sampling of (the boundary of) a smooth shape is a subset $S \subset \partial O$. We call S an ε-sampling if every point $x \in \partial O$ has a point in S at distance at most $\varepsilon f(x)$, where the *feature size* function

$$f : \partial O \to (0, \infty),\ x \mapsto \inf_{y \in M(O)} \|x - y\|$$

measures the distance from ∂O to the medial axis of O.

Voronoi diagram. Let $S \subset \mathbb{R}^d$ be a finite set of (sample) points. The *Voronoi cell* of $p \in S$ is given as the convex polyhedron

$$V_p = \{x \in \mathbb{R}^d : \forall q \in S, \|x - p\| \leq \|x - q\|\}.$$

The faces of Voronoi cells are called *Voronoi faces*. A *Voronoi vertex* is a Voronoi 0-face. If S is a sample of a shape O, we distinguish two types of Voronoi vertices: *inner vertices* that reside inside O and *outer vertices* that reside in $\mathbb{R}^d \setminus O$.

Delaunay triangulation. Dual to the Voronoi diagram of S is the Delaunay triangulation of S. the faces of the Delaunay triangulation are given by the convex hulls of all subsets $T \subseteq S$ whose dual Voronoi cells have a non-empty common intersection, i.e., $\bigcap_{p \in T} V_p \neq \emptyset$. The non-empty intersection is the dual Voronoi face of the Delaunay face conv(T).

Ball. A ball $B_{c,r}$ with center $c \in \mathbb{R}^d$ and radius $r \in (0, \infty)$ is given as the set

$$B_{c,r} = \{x \in \mathbb{R}^d : \|x - c\| < r\}.$$

A *Voronoi ball* is centered at a Voronoi vertex and its radius is given by the distance to the closest sample point. If S is sample of a shape O, we also distinguish two types of Voronoi balls: *inner balls* centered at inner vertices and *outer balls* centered at outer vertices.

In the following we deal with sampled smooth shapes in $\mathbb{R}^2$ that we want to approximate by the set of inner balls. We are interested in the medial axis of such a shape and its approximation by the medial axis of the union of inner balls. It is well known how to compute the medial axis of a union of balls. Here we show that in our setting this medial axis has a particularly simple form and can be computed by a simpler and numerically (more) robust algorithm than in the general case, provided the shape is sampled sufficiently dense.

Union of Balls. Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a set of balls in $\mathbb{R}^2$. The union of these balls

$$U = \bigcup_{i=1}^{n} B_i$$

is a shape whose boundary ∂U is made up from circular arcs. Circular arcs join at vertices. We denote the set of these vertices by $V(U)$ and call it the *vertex set* of the union of balls.

Medial Axis of Union of Balls. Attali and Montanvert [3] and Amenta and Kolluri [2] have characterized the medial axis of a union of balls. Their characterization makes use of the dual of the union of balls, which is the regular triangulation of the balls restricted to the union itself. The regular triangulation is dual to the *power diagram* of the balls. The power diagram generalizes the notion of Voronoi diagram to sets of weighted points. A ball $B_{c,r}$ can be considered as a special case of a weighted point. The weight in this case is r^2. The distance of a point $x \in \mathbb{R}^2$ to a weighted point $p = (y, w)$, where $y \in \mathbb{R}^2$ is the point and $w \in \mathbb{R}$ its weight, is called *power distance* and given as $\pi_p(x) = \|x - y\|^2 - w$. The power diagram is now defined just as the Voronoi diagram only with the Euclidean distance replaced by the power distance. The dual complex of the power diagram is called *regular triangulation* though it does not necessarily need to be a triangulation in degenerate cases. Here we restrict the regular triangulation to a union $U = \bigcup_{B \in \mathcal{B}} B$ of balls that induces the power diagram and call it $D(\mathcal{B})$. Restricting the diagram means we only include faces whose duals in the power diagram have a non-empty intersection with U. A face in $D(\mathcal{B})$ is called singular, if it belongs to the boundary of $D(\mathcal{B})$ and is not a face of a 2-face of $D(\mathcal{B})$.

Theorem 1 (Amenta and Kolluri [2]) *Let $\mathcal{B}$ be a set of balls in $\mathbb{R}^2$ and $D(\mathcal{B})$ be the induced restricted regular triangulation. The medial axis of $U = \bigcup_{B \in \mathcal{B}} B$ consists of*

(1) the singular faces of $D(\mathcal{B})$, and
(2) the intersection of the 1-skeleton of the Voronoi diagram of $V(U)$ with the non-singular faces of $D(\mathcal{B})$.

This characterization of the medial axis of a union of balls immediately gives rise to an algorithm to compute the medial axis. This algorithm involves the computation of the regular triangulation induced by $\mathcal{B}$ and the computation of the Voronoi diagram of $V(U)$.

Here we will show that the medial axis of the union of inner balls of a densely sampled smooth shape has a much simpler structure that yields a simpler and more robust algorithm that essentially only needs to compute one Voronoi diagram.

3 Structural Results

In the following we assume that $O \subset \mathbb{R}^2$ is a connected shape with smooth boundary ∂O and $S \subset \partial O$ is a finite sampling. Let $\mathcal{B}$ be the set of inner Voronoi balls, $U = \bigcup_{B \in \mathcal{B}} B$ its underlying shape and $V(U)$ be the vertex set of U. Our main result is the following theorem.

Theorem 2 *If S is an ε-sampling of ∂O with $\varepsilon < 1/(2 + 2\sqrt{2}) \simeq 0.207$, then for every $p \in S$ exactly one of the following three statements is true.*

(1) $p \in V(U)$.
(2) p corresponds to an endpoint of $M(U)$.
(3) p can be removed from S without changing $\mathcal{B}$.

The key to our proof of Theorem 2 is the following lemma that relates $V(U)$ and S.

Lemma 3 *If S is an ε-sampling of ∂O with $\varepsilon < 1/(2 + 2\sqrt{2})$, then $V(U) \subseteq S$.*

3.1 Medial Axis Structure

Using Lemma 3 we get a characterization of the medial axis of U in form of three lemmas. That is, in the statement of these lemmas we assume that Lemma 3 holds, i.e., $V(U) \subseteq S$. To formulate the lemmas we need one more notation: let $E(S)$ be the 1-skeleton of the Voronoi diagram of the sample point set S (plus a point ∞ at infinity). We classify the edges in $E(S)$ into three classes: the endpoints of *inner edges* are both in U, the endpoints of *outer edges* are both in $(\mathbb{R}^2 \cup \{\infty\}) \setminus U$, and *mixed edges* have one endpoint in U and one endpoint in $(\mathbb{R}^2 \cup \{\infty\}) \setminus U$. Note that the definition of inner and outer here is with respect to U and not to O as it is for the Voronoi vertices/balls.

The first lemma states that if a point in the shape underlying the union of balls is not contained in the Voronoi 1-skeleton of the vertex set of the union of balls, then it is not on the medial axis of the shape.

Lemma 4 *If $x \in U$ but $x \notin E(S)$, then $x \notin M(U)$.*

Proof. Grow a ball B around x until it touches ∂U. We distinguish two cases: either B touches a ball from $\mathcal{B}$ tangentially in a single point, or not. If B touches a ball from $\mathcal{B}$ tangentially in a single point y, then y is the unique closest point to x on ∂U and thus x cannot be on the medial axis of U. If B does not touch a ball from $\mathcal{B}$ tangentially in a single point, then it touches ∂U in a vertex of the of the union of balls, i.e., in a point in $V(U) \subseteq S$ (the inclusion holds under the assumptions of Lemma 3). There can be only one such point since by assumption x cannot have two or more closest points in S. Hence also in this case x cannot be on the medial axis of U. □

The next lemma states that a point in U that is contained in an inner edge is on the medial axis of U.

Lemma 5 *If $x \in U$ is a point on an inner edge, then $x \in M(U)$.*

Proof. Let B_1 and B_2 be the inner Voronoi balls that correspond to the endpoints of the inner edge that contains x, and let p and q be the two intersection points of the boundary spheres of B_1 and B_2. By the definition of Voronoi edges: $p, q \in V(U) \subseteq S$ (the inclusion holds under the assumptions of Lemma 3), p and q are equidistant from x, and p and q cannot be covered by any other (inner) Voronoi ball, i.e., $p, q \in \partial U$. The ball centered at x touching p and q is contained in $B_1 \cup B_2$. Hence x has at least two closest points in ∂U, i.e., $x \in M(U)$. □

The last lemma states that a point in the shape underlying the union of balls that is contained in the relative interior of a mixed edge is not on the medial axis of U.

Lemma 6 *If $x \in U$ is a point in the interior of a mixed edge, then $x \notin M(U)$.*

Proof. Let $z \in U$ be the endpoint in U of the mixed edge that contains x, let B_1 be the inner Voronoi ball centered at z, and let B_2 be the outer Voronoi ball centered at the second endpoint of the mixed edge. The closest point to x in ∂B_1 is the intersection point of the mixed edge with ∂B_1. Let y be this intersection point. If $y \in \partial U$, then y is the unique closest point to x in ∂U and thus x cannot be in $M(U)$. It remains to consider the case $y \notin \partial U$. In this case there exists an inner Voronoi ball B' that covers y. Let p and q be the two intersection points of the boundary spheres of B_1 and B_2, respectively. We have $p, q \in V(U) \subseteq S$ (the inclusion holds under the assumptions of Lemma 3). Let γ be the arc between p and q on ∂B_1 that is covered by B_2. By construction $y \in \gamma$ and B' covers a sub-segment of γ that contains y. This sub-segment cannot contain p or q by the definition of Voronoi balls, but that means that B' is completely covered by $B_1 \cup B_2$ which is also not possible by Voronoi ball properties. Hence B' cannot exist and y actually is the the unique closest point to x in ∂U and thus x cannot be in $M(U)$. □

Obviously points on outer edges cannot be on the medial axis of U, since $M(U) \subset U$.

3.2 Proof of Key Lemma

To prove the key lemma we need to collect some well known facts from ε-sampling theory.

Lemma 7 *If S is an ε-sampling of ∂O with $\varepsilon < 1$, then*

(1) The radius of any Voronoi ball B is at least $f(p)/2$, where $p \in S \cap \partial B$.

(2) Voronoi edges in the Voronoi diagram of O either correspond to consecutive sample points on ∂O and intersect ∂O in a single point, or they correspond to non-consecutive sample points and do not intersect ∂O.

(3) For any two points $x, y \in \partial O$ such that there is an arc on ∂O that does not contain a point from S it holds that $\|x - y\| \leq \frac{2\varepsilon}{1-2\varepsilon} f(x)$.

(4) Outer Voronoi vertices cannot be covered by inner Voronoi balls, i.e., inner, outer and mixed edges have their respective property not only with respect to U but also with respect to O.

Proof. [Lemma 3] We show for $\varepsilon < 1/(2 + 2\sqrt{2})$ all points in the vertex set $V(U)$ of the union of balls are sample points. The proof makes use of the following decomposition of the boundary circle of an inner Voronoi ball B: the edges incident to the Voronoi vertex corresponding to the ball can be classified as either inner or mixed. Each mixed edge corresponds to two consecutive sample points on ∂B. We label the arc on ∂B between two sample points exactly as the Voronoi edge, i.e., either as inner or mixed. By definition the relative interior of an inner arc is contained in U and thus cannot contain a point from $V(U)$. Hence the points in $V(U)$ must be on mixed arcs.

The balls centered at the endpoints of a Voronoi edge intersect in a *lens*. We call such a lens corresponding to a mixed edge a *mixed lens*. We can conclude that any point in $V(U)$ must be contained in the intersection of two mixed lenses. From our sampling condition constraints on the shape of a mixed lens it follows: Lemma 7 implies that the endpoints of a mixed lens need to be consecutive sample points on ∂O. It also implies an upper bound on the distance of these two sample points and a lower bound (dependent of ε) on the radius of the Voronoi balls corresponding to the endpoints of the mixed edge. The lenses get thinner when ε gets smaller. A point in $V(U)$ that is not a sample point must be an intersection point on the boundary of two mixed lenses that is not one of the endpoints (which are sample points). Note that the union of the two Voronoi balls that define a mixed edge (or lens) can by definition not contain any sample point in their interior. We call this union also the *forbidden region* of a mixed lens. The extrema case of two intersecting mixed lenses whose endpoints are not contained in the forbidden region of the other lens and that are as *fat* as the chosen ε allows, is given as follows: given two interior balls B_1 and B_2 that touch each other in a single point $p \in S$ and both have radius $f(p)/2$, and a third outer ball B_3 with radius $f(p)/2$ that also touches p and is tangential to the line through the center points of B_1 and B_2. The intersections of B_1 and B_3 and B_2 and B_3, respectively, define two mixed lenses. In the extreme case both lenses have length $2\varepsilon/(1-2\varepsilon)f(p)$ and can intersect only in p, which provides an upper bound on the allowable ε (for larger ε intersection of the lenses different from p are possible and thus points in $V(U)$ but not in S cannot be ruled out). We get for this upper bound on ε:

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{2\varepsilon}{1-2\varepsilon},$$

which gives $\varepsilon = 1/(2+2\sqrt{2})$. □

The proof of Theorem 2 follows almost immediately from Lemma 3.

4 Algorithm

Our structural results give rise to an algorithm to compute the medial axis of the union U of the inner Voronoi balls. The algorithm is correct provided the sampling condition in Lemma 3 is satisfied.

The main step in the algorithm is the classification of the vertices of the Voronoi diagram of the set of sample points S. For the classification we use a curve reconstruction algorithm that picks Delaunay edges from the Delaunay triangulation of S that connect the points in S in exactly the same order as they are connected along ∂O. A convenient choice of reconstruction algorithm is the one-step algorithm due to Gold and Snoeyink [5]—a variant of the Crust algorithm [1]. The curve reconstruction can then be used to classify the Voronoi vertices by traversing the 1-skeleton of the Voronoi diagram of S. Label the vertex at infinity as *outer* and proceed to incident, so far unlabeled Voronoi vertices and label them. Whenever the incidence is via a Voronoi edge that is dual to a Delaunay edge from the reconstruction change the label, i.e., from *outer* to *inner*, or vice versa. Once all the Voronoi vertices are labeled just output all Voronoi edges whose endpoints are both labeled *inner*. The union of all these edges is the medial axis of the union U of inner Voronoi balls (provided the sampling condition holds) and in general a reasonable approximation of the medial axis of the original shape O. For an implementation and illustrations of the structures mentioned in this paper see Mesecina [7].

5 Conclusions

We have presented structural proofs which lead to a simpler algorithm for computing the medial axis of the union of inner Voronoi balls of a planar shape. This algorithm is faster than the standard algorithm, since it saves the computation of the restricted regular triangulation of $\mathcal{B}$ and the computation of the Voronoi diagram of the vertex set $V(U)$. The set of inner Voronoi balls tends to be highly degenerate, i.e., there are often many spheres bounding the balls that pass through the same point. Therefore the standard algorithm is prone to numerical errors since it needs to compute $D(\mathcal{B})$. Our algorithm only needs to compute the Voronoi diagram of S and hence is not only faster, but also more robust.

6 Acknowledgments

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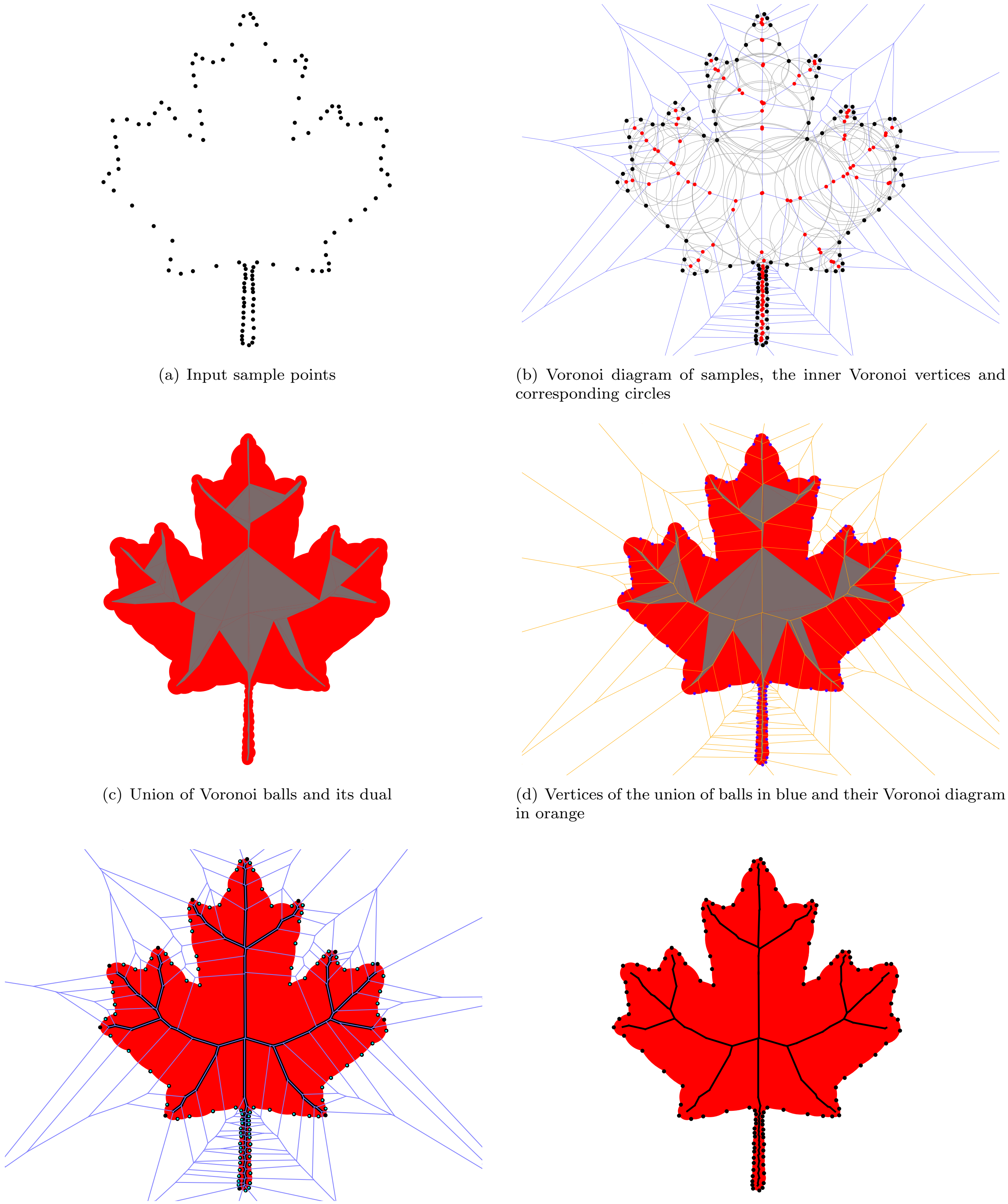

(a) Input sample points

(b) Voronoi diagram of samples, the inner Voronoi vertices and corresponding circles

(c) Union of Voronoi balls and its dual

(d) Vertices of the union of balls in blue and their Voronoi diagram in orange

(e) Sample points in black, vertices of union of balls in light blue. Notice how the vertices of the union are a subset of samples and the inner Voronoi diagram in blue overlaps the medial axis in black.

(f) Final result for both our and the standard algorithm: medial axis of the inner Voronoi balls. The standard algorithm needs the steps illustrated on Figure 1(c) and 1(d), our method computes it directly from Figure 1(b)

Figure 1: Illustrations of the steps of the standard and our algorithm to compute the medial axis of the union of inner Voronoi balls in 2D