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A Note on Natural Risk Statistics

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Abstract

Recently Heyde, Kou and Peng (2007) proposed the notion of a natural risk statistic associated with a finite sample that relaxes the subadditivity assumption in the classical coherent risk statistics. In this note we use convex analysis to provide alternate proofs of the representation results regarding natural risk statistics.

Key words: natural and coherent risk statistics, convex analysis.

1 Introduction

Over the past years the field of risk measurement has become of great importance to financial industry. In their seminal paper on risk measures, Artzner et al. [1] introduced the notion of a *coherent risk statistic*, defined as a function $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfying the following axioms:

C1 Translation-invariance: $\rho(X + a\mathbf{1}) = \rho(X) + a$ for all $a \in \mathbb{R}$, where $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^n$.

C2 Positive homogeneity: $\rho(tX) = t\rho(X)$ for all $t \geq 0$.

C3 Monotonicity: $\rho(X) \leq \rho(Y)$ whenever $X \leq Y$ which means that $x_i \leq y_i$ for all $i = 1, \dots, n$ where x_i denotes the i-th coordinate of X and y_i the i-th coordinate of Y.

C4 Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathbb{R}^n$.

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The above axioms imply the well-known representation theorem [1] that a risk statistic ρ is coherent if and only if

$$\rho(X) = \sup_{W \in \mathcal{W}} \langle W, X \rangle, \quad \forall X \in \mathbb{R}^n \tag{1.1}$$

where $\mathcal{W} \subseteq \mathcal{P} := \{W \in \mathbb{R}^n \mid \sum_{i=1}^n w_i = 1\,,\ w_i \geq 0, i = 1, \ldots, n\}$ is a closed, convex set.

In a recent fundamental work, Heyde, Kou and Peng [2] extended the notion of coherence, and introduced a new data-based class of risk measures called *natural risk statistics* which prove to be robust and thus particularly suitable for external risk measurement. A natural risk statistic is a function $\hat{\rho} : \mathbb{R}^n \to \mathbb{R}$ that satisfies axioms C1–C3 along with:

C4' Comonotonic subadditivity: $\hat{\rho}(X+Y) \leq \hat{\rho}(X) + \hat{\rho}(Y)$ whenever X and Y are comonotonic (X and Y are comonotonic if $(x_i - x_j)(y_i - y_j) \geq 0$ for any $i, j \in \{1, \ldots, n\}$).

C5 Permutation invariance: $\hat{\rho}(X) = \hat{\rho}(X_\pi)$ for every permutation $\pi \in S_n$, where S_n is the set of all permutations of $\{1, \ldots, n\}$ and X_π denotes the permuted vector, i.e. $X_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$.

A natural risk statistic serves as a risk measure of the observed data $X = (x_1, \ldots, x_n)$. For a justification of the concept and a thorough comprehensive study of natural risk statistics as well as a detailed comparison to other classes of risk measures we refer to [2]. Among the main results of [2] we have the following two representation theorems.

Theorem 1. *Let $D := \{X \in \mathbb{R}^n \mid x_1 \leq x_2 \leq \ldots \leq x_n\}$ and denote by X_{os} the order statistics of X, i.e. $X_{os} := (x_{(1)}, \ldots, x_{(n)}) := X_\pi$ for some $\pi \in S_n$ such that $X_\pi \in D$.*

(i) For an arbitrarily given set of weights $\mathcal{W} \subset \mathcal{P}$, the function

$$\hat{\rho}(X) := \sup_{W \in \mathcal{W}} \langle W, X_{os} \rangle, \quad \forall X \in \mathbb{R}^n \tag{1.2}$$

is a natural risk statistic.

(ii) Conversely, if $\hat{\rho}$ is a natural risk statistic, then there exists a closed convex set of weights $\mathcal{W} \subset \mathcal{P}$ such that

$$\hat{\rho}(X) := \sup_{W \in \mathcal{W}} \langle W, X_{os} \rangle, \quad \forall X \in \mathbb{R}^n. \tag{1.3}$$

Theorem 2. *Let D be as in Theorem 1.*

(i) For an arbitrarily given set of weights $\mathcal{W} \subset \mathcal{P} \cap D$, the function

$$\hat{\rho}(X) := \sup_{W \in \mathcal{W}} \langle W, X_{os} \rangle, \quad \forall X \in \mathbb{R}^n \tag{1.4}$$

is a subadditive natural risk statistic, i.e. satisfies C4.

(ii) Conversely, suppose the natural risk statistic $\hat{\rho}$ is subadditive. Then there exists a closed convex set of weights $\mathcal{W} \subset \mathcal{P} \cap D$ such that

$$\hat{\rho}(X) := \sup_{W \in \mathcal{W}} \langle W, X_{os} \rangle, \quad \forall X \in \mathbb{R}^n. \tag{1.5}$$

The above theorems correspond to Theorems 1 and 4 of [2] respectively. However note that the original statements in [2] did not specify the conditions of closedness and convexity on the set $\mathcal{W}$. Using the representation (1.1) for a coherent risk statistic via supremum over a closed convex set of weights and from Theorem 1 we observe the following important connection between natural and coherent risk statistics:

> A function $\hat{\rho}$ is a natural risk statistic if and only if there exists a coherent risk statistic ρ such that $\hat{\rho}(X) = \rho(X_{os})$ for all $X \in \mathbb{R}^n$.

The assertions of Theorems 1 (i) and 2 (i) are easily verified. The non-trivial parts are Theorem 1 (ii) and Theorem 2 (ii). In this note we give alternate proofs of Theorems 1 (ii) and 2 (ii) using convex duality theory. This illustrates the strength of convex duality theory when dealing with risk measures having some kind of convexity property, in our case this is axiom C4'. The reader might find our proofs a lot shorter than the original ones presented in [2], but this fact most certainly does not disqualify the original approaches. To the contrary, we like to point out that in our approach we draw heavily on fundamental results from convex analysis, which themselves rely on comprehensive proofs, whereas the authors of [2] prove things almost from scratch.

The remainder of this note is organized as follows. In Section 2 we collect some fundamental results from convex analysis which form the basis of our proofs of Theorems 1 (ii) and 2 (ii). These proofs are then presented in Sections 3 and 4 respectively.

2 Some Facts from Convex Analysis

An introduction to convex analysis can be found in Rockafellar's book [3]. All results presented in this section are stated therein.

Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a convex function. f is said to be *proper* if its *domain* is non-empty, i.e. $\operatorname{dom} f := \{X \in \mathbb{R}^n \mid f(X) < \infty\} \neq \emptyset$. Any proper convex function is continuous over the interior of its domain. We call f *lower semi-continuous* (l.s.c.) if $f(X) \leq \liminf_{n\to\infty} f(X_n)$ whenever $(X_n)_{n\in\mathbb{N}} \subset \mathbb{R}^n$ is a sequence converging to $X \in \mathbb{R}^n$. Fenchel's Theorem states that f is proper, convex, and l.s.c. if and only if

$$f(X) = \sup_{Z \in \mathbb{R}^n} \langle Z, X \rangle - f^*(Z), \quad X \in \mathbb{R}^n, \tag{2.6}$$

where

$$f^* : \mathbb{R}^n \to (-\infty, \infty], \; f^*(Z) = \sup_{X \in \mathbb{R}^n} \langle Z, X \rangle - f(X), \quad Z \in \mathbb{R}^n, \tag{2.7}$$

is the so called *dual* (or conjugate) function of f and $\langle\cdot,\cdot\rangle$ denotes the Euclidian scalar product on $\mathbb{R}^n$. Note that f^* is itself a proper convex l.s.c. function and that (2.6) is equivalent to $f = f^{**}$. For a proof of (2.6) we refer to [3] Theorem 12.2.

The set of maximizers of (2.6) is called the *subgradient* at X. It is denoted by

$$\partial f(X) := \{Z \in \mathbb{R}^n \mid f(X) + f^*(Z) = \langle Z, X\rangle\} \tag{2.8}$$

and may be empty, e.g. if $X \notin \operatorname{dom} f$. However, it can be shown that

$$\partial f(X) \neq \emptyset \text{ for all } X \in \operatorname{ri}(\operatorname{dom} f) \tag{2.9}$$

where $\operatorname{ri}(A)$ denotes the relative interior of a set $A \subset \mathbb{R}^n$. For a comprehensive discussion of subdifferentiability and a proof of (2.9) we again refer to [3], in particular Theorem 23.4.

3 Proof of Theorem 1 (ii)

From C1 and C3 it follows that $\hat\rho$ is (Lipschitz) continuous with respect to the maximum-norm $\|\cdot\|_\infty$, because for any $X, Y \in \mathbb{R}^n$ we have that

$$\hat\rho(X) = \hat\rho(X - Y + Y) \leq \hat\rho(\|X - Y\|_\infty \cdot \mathbf{1} + Y) = \|X - Y\|_\infty + \hat\rho(Y),$$

so $\hat\rho(X) - \hat\rho(Y) \leq \|X - Y\|_\infty$, and similarly we obtain $\hat\rho(Y) - \hat\rho(X) \leq \|X - Y\|_\infty$. Recall that $D = \{X \in \mathbb{R}^n \mid x_1 \leq x_2 \leq \ldots \leq x_n\}$. Clearly, D is a translation-invariant (i.e. $X \in D \Rightarrow X + a \cdot \mathbf{1} \in D$ for all $a \in \mathbb{R}$) closed convex cone. We introduce the following auxiliary function

$$\rho(X) := \hat\rho(X) + \delta(X \mid D) \in (-\infty, \infty], \quad X \in \mathbb{R}^n,$$

where $\delta(\cdot \mid D)$ denotes the indicator function, i.e. $\delta(X \mid D) = 0$ if $X \in D$ and $\delta(X \mid D) = \infty$ if $X \notin D$. So $\rho = \hat\rho$ on D. Moreover, ρ is a proper, strictly positive homogeneous, subadditive (thus convex), and translation-invariant function which is monotone on $\operatorname{dom}\rho = D$, and l.s.c. on $\mathbb{R}^n$. Note that the subadditivity follows from the fact that any $X, Y \in D$ are comonotonic and that $\rho = \infty$ outside D, whereas l.s.c. is due to continuity of $\hat\rho$ and the fact that D is closed convex. Now by (2.6)

$$\rho(X) = \sup_{Z \in \mathbb{R}^n} \langle Z, X\rangle - \rho^*(Z), \quad X \in \mathbb{R}^n.$$

In the following we show that $\operatorname{dom}\rho^* \subset \{Z \in \mathbb{R}^n \mid \sum_{i=1}^n z_i = 1\}$ and $\rho^* = \delta(\cdot \mid \operatorname{dom}\rho^*)$. First of all, since the constant vector $k\mathbf{1} \in D$ for any $k \in \mathbb{R}$ and by translation-invariance, we derive for any $Z \in \mathbb{R}^n$ that

$$\rho^*(Z) \quad = \quad \sup_{X \in \mathbb{R}^n} \langle Z, X\rangle - \rho(X) \quad \geq \quad \sup_{k \in \mathbb{R}} k(\langle Z, \mathbf{1}\rangle - 1).$$

So either $\langle Z, \mathbf{1}\rangle = 1$, i.e. $\sum_i z_i = 1$, or $\rho^*(Z) = \infty$. Secondly, positive homogeneity yields

$$\rho^*(Z) = \sup_{X \in \mathbb{R}^n} \langle Z, X\rangle - \rho(X) = \sup_{X \in \mathbb{R}^n} \langle Z, tX\rangle - t\rho(X) = t\rho^*(Z)$$

for all $t > 0$. Hence,

$$\rho^* = \delta(\cdot \mid \operatorname{dom} \rho^*) \quad \text{and} \quad \rho(X) = \sup_{Z \in \operatorname{dom} \rho^*} \langle Z, X\rangle. \tag{3.10}$$

According to (2.9) we have $\partial\rho(X) \neq \emptyset$ for all $X \in \operatorname{int} D$. Now fix any $X \in \operatorname{int} D$ and let $Z \in \partial\rho(X)$. Then it follows from (3.10) that $\rho(X) = \langle Z, X\rangle$. Denoting by $e_1, \ldots, e_n$ the canonical basis of $\mathbb{R}^n$, for each i there is an $\epsilon > 0$ small enough such that $X - \epsilon e_i \in D$. By monotonicity of ρ on D we obtain

$$\langle Z, X - \epsilon e_i\rangle \leq \rho(X - \epsilon e_i) \leq \rho(X) = \langle Z, X\rangle$$

or equivalently $-\epsilon z_i \leq 0$. The latter inequality implies that all coordinates of Z must be non-negative. Since ρ^* is l.s.c. the set $\mathcal{W} := \operatorname{dom} \rho^* \cap \mathbb{R}^n_+ \subset \mathcal{P}$ is closed and convex. We obtain $\rho(X) = \sup_{W \in \mathcal{W}} \langle W, X\rangle$ for all $X \in \operatorname{int} D$. For any boundary point X of D, we choose a sequence $(X_k)_{k \in \mathbb{N}} \subset \operatorname{int} D$ converging to X. Then, recalling that $\rho = \hat{\rho}$ on D and by continuity of $\hat{\rho}$, we have

$$\rho(X) = \lim_{k \to \infty} \rho(X_k) = \lim_{k \to \infty} \sup_{W \in \mathcal{W}} \langle W, X_k\rangle = \sup_{W \in \mathcal{W}} \lim_{k \to \infty} \langle W, X_k\rangle = \sup_{W \in \mathcal{W}} \langle W, X\rangle$$

in which the third equality follows from the Cauchy-Schwartz-Inequality, because for all $W \in \mathcal{W}$:

$$|\langle W, X_k\rangle - \langle W, X\rangle| \leq \|W\|_2 \|X_k - X\|_2 \leq \|X_k - X\|_2$$

where $\|\cdot\|_2$ denotes the Euclidian norm. So finally we arrive at

$$\rho(X) = \sup_{W \in \mathcal{W}} \langle W, X\rangle \quad \forall X \in D.$$

Consequently, since $\hat{\rho}$ is permutation invariant and $X_{os} \in D$ for every $X \in \mathbb{R}^n$, we obtain

$$\hat{\rho}(X) = \hat{\rho}(X_{os}) = \rho(X_{os}) = \sup_{W \in \mathcal{W}} \langle W, X_{os}\rangle \quad \text{for all } X \in \mathbb{R}^n.$$

□

4 Proof of Theorem 2 (ii)

Suppose the natural risk statistic $\hat{\rho}$ is subadditive, so in particular it is convex. Being a proper, continuous (see beginning of Section 3), translation-invariant, and convex function, we know from (2.6) and by arguments already presented in the proof of Theorem 1 that

$$\hat{\rho}(X) = \sup_{W \in \operatorname{dom} \hat{\rho}^*} \langle W, X \rangle \quad \forall X \in \mathbb{R}^n, \tag{4.11}$$

where $\operatorname{dom} \hat{\rho}^* \subset \{Z \in R^n \mid \sum_{i=1}^n z_i = 1\}$. Suppose $Z \in R^n$ is such that there is a $i \in \{1, \ldots, n\}$ with $z_i < 0$. Monotonicity yields

$$\hat{\rho}^*(Z) \geq \sup_{t>0} -t\langle Z, e_i \rangle - \hat{\rho}(-te_i) \geq \sup_{t>0} -tz_i = \infty$$

because $\hat{\rho}(-te_i) \leq 0$. Consequently, we have $\operatorname{dom} \hat{\rho}^* \subset \mathcal{P}$. Next we show that $\hat{\rho}^*$ is permutation invariant. Let $Z \in \mathbb{R}^n$. Then, for any $\pi \in S_n$ we obtain

$$\hat{\rho}^*(Z_\pi) = \sup_{X \in \mathbb{R}^n} \langle Z_\pi, X \rangle - \hat{\rho}(X) = \sup_{X \in \mathbb{R}^n} \langle Z, X_{\pi^{-1}} \rangle - \hat{\rho}(X_{\pi^{-1}}) = \hat{\rho}^*(Z)$$

because $\hat{\rho}$ is permutation invariant and thus $\hat{\rho}(X_{\pi^{-1}}) = \hat{\rho}(X)$. Hence, $\operatorname{dom} \hat{\rho}^*$ is a permutation invariant set. Therefore, as $\langle Z, X \rangle \leq \langle Z_{os}, X_{os} \rangle$ for all $X, Z \in \mathbb{R}^n$, it follows from (4.11) that

$$\hat{\rho}(X) = \sup_{W \in \mathcal{W}} \langle W, X_{os} \rangle \quad \text{for all } X \in \mathbb{R}^n,$$

where $\mathcal{W} := \operatorname{dom} \hat{\rho}^* \cap D$ is closed and convex. □

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Svindland and Filipović first submitted a proof of Theorems 1 and 4 in Heyde, Kou and Peng [2] to Operations Research Letters. Five months later, while their paper has been under review, Ahmed also submitted a similar proof independently. At the suggestion of the area editor we wrote this joint paper together.

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