

Numerical Integration of SDEs: A Short Tutorial

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1 Introduction

1.1 Itô and Stratonovich SDEs

1-dimensional stochastic differentiable equation (SDE) is given by [6, 7]

$$\frac{dX_t}{dt} = f(X_t, t)dt + g(X_t, t)dW_t \quad (1)$$

where $X_t = X(t)$ is the realization of a stochastic process or *random variable*. $f(X_t, t)$ is called the *drift* coefficient, that is the deterministic part of the SDE characterizing the local trend. $g(X_t, t)$ denotes the *diffusion* coefficient, that is the stochastic part which influences the average size of the fluctuations of X . The fluctuations themselves originate from the stochastic process W_t called *Wiener process* and introduced in Section 1.2. Interpreted as an integral, one gets

$$X_t = X_{t_0} + \int_{t_0}^t f(X_s, s)ds + \int_{t_0}^t g(X_s, s)dW_s \quad (2)$$

where the first integral is an ordinary Riemann integral. As the sample paths of a Wiener process are not differentiable, the Japanese mathematician K. Itô defined in 1940s a new type of integral called *Itô stochastic integral*. In 1960s, the Russian physicist R. L. Stratonovich proposed an other kind of stochastic integral called *Stratonovich stochastic integral* and used the symbol “ \circ ” to distinct it from the former Itô integral. (3) and (4) are the Stratonovich equivalents of (1) and (2) [1, 6].

$$\frac{dX_t}{dt} = f(X_t, t)dt + g(X_t, t) \circ dW_t \quad (3)$$

$$X_t = X_{t_0} + \int_{t_0}^t f(X_s, s)ds + \int_{t_0}^t g(X_s, s) \circ dW_s \quad (4)$$

The second integral in (2) and (4) can be written in a general form as [8]

$$\int_{t_0}^t g(X_s, s)dW_s = \lim_{h \rightarrow 0} \sum_{k=0}^{m-1} g(X_{\tau_k}, \tau_k)(W(t_{k+1}) - W(t_k)) \quad (5)$$

where $h = (t_{k+1} - t_k)$ with intermediary points $\tau_k = (1 - \lambda)t_k - \lambda t_{k+1}$, $\forall k \in \{0, 1, \dots, m - 1\}$, $\lambda \in [0, 1]$. In the stochastic integral of the Itô SDE given in (2), $\lambda = 0$ leads to $\tau_k = t_k$ and hence to evaluate the stochastic integral at the left-point of the intervals. In the definition of the Stratonovich integral, $\lambda = 1/2$ and so $\tau_k = (t_{k+1} - t_k)/2$, which fixes the evaluations of the second integral in (4) at the mid-point of each intervals [8].

To illustrate the difference between the Itô and Stratonovich calculi, lets have a closer look at the stochastic integral

$$\int_{t_0}^T W(s)dW_s = \lim_{m \rightarrow \infty} \sum_{k=0}^{m-1} W(\tau_k)(W(t_{k+1}) - W(t_k)) \quad (6)$$

$$= \frac{W(t)}{2} + (\lambda - \frac{1}{2})T \quad (7)$$

By combining the result of (7) with the respective values of λ discussed above for both interpretations, we obtain [8]

$$\int_{t_0}^T W(s)dW_s = \frac{1}{2}W(t) - \frac{1}{2}T \quad (8)$$

$$\int_{t_0}^T W(s) \circ dW_s = \frac{1}{2}W(t) \quad (9)$$

If we solve (2) and (4) of which the stochastic integrals (8) and (9) are respectively part of, one see that the Itô and Stratonovich representations *do not converge towards the same solution*. Conversions from Itô to Stratonovich calculus and inversely are possible in order to switch between the two different calculi. This is achieved by adding a correction term to the drift coefficients [1].

$$dX_t = f(X_t)dt + g(X_t)dW_t \quad (10)$$

$$dX_t = \underline{f}(X_t)dt + g(X_t) \circ dW_t \quad (11)$$

$$\underline{f} = f - \frac{1}{2}g'g \quad (12)$$

where $g' = \frac{dg(X_t)}{dX_t}$ is the first derivative of g . If the relation (12) is used (called the *Itô-Stratonovich drift correction formula*), the integration of the Stratonovich SDE (11) leads now to the same result as the integration of the Itô SDE (10) [1].

Both integrals have their advantages and disadvantages and which one should be used is more a modelling than mathematical issue. In financial mathematics, the Itô interpretation is usually used since Itô calculus only takes into account information about the past. The Stratonovich interpretation is the most frequently used within the physical sciences [6]. An excellent discussion of this subject can be found in [10], in particular see Chapter IX, Section 5: *The Itô-Stratonovich dilemma*.

1.2 Standard Wiener process

A scalar *standard Brownian motion*, or *standard Wiener process*, over $[t_0, T]$ is a random variable $W(t)$ that depends continuously on $t \in [t_0, T]$. For $t_0 \leq s < t \leq T$, the random variable given by the increment $W(t) - W(s)$ is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = t - s$. Equivalently, $W(t) - W(s) \sim \sqrt{t - s}\mathcal{N}(0, 1)$ with $W(t_0 = 0) = 0$ [4]. The conditions for the stochastic process $W(t)$ to be a Wiener process are [6]

1. $[W(t), t \geq 0]$ has stationary independent increments dW
2. $W(t)$ is normally distributed for $t \geq 0$
3. $\langle W(t) \rangle = t$ for $t \geq 0$
4. $W(0) = 0$

1.3 Discretized Brownian motion

Lets take $t_0 = 0$ and divide the interval $[0, T]$ into N steps such as: $h = T/N$. Lets also denote $W_j = W(t_j)$ with $t_j = jh$ [4].

$$W_j = W_{j-1} + dW_j \quad W_0 = 0 \quad j = 1, 2, \dots, N \quad (13)$$

where each dW_j is an independent random variable of the form $\sqrt{h}\mathcal{N}(0, 1)$ ¹. The figure below displays the realizations of three independent Wiener processes.

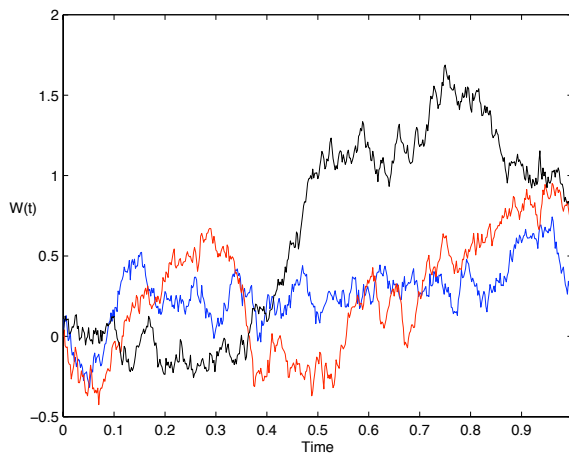


Figure 1: Three discretized, 1-dimensional Brownian paths with $T = 1$ and $N = 500$. It is worth noting that when $t \rightarrow \infty$, the process has an infinite variance but still an expectation equal to zero. The trajectories have been simulated with the Java tool *libSDE* (see Section 4).

¹ $\mathcal{N}(0, h) = \sqrt{h}\mathcal{N}(0, 1)$

2 Numerical integration

2.1 Iterative methods

It is difficult to deal with the SDEs analytically because of the highly non-differentiable character of the realization of the Wiener process. There are different, iterative methods that can be used to integrate SDE systems. The most widely-used ones are introduced in the following sections.

- **Explicit order 0.5 strong Taylor scheme**
Euler-Maruyama (EM) and Euler-Heun (EH)
- **Explicit order 1.0 strong Taylor scheme**
Milstein and derivative-free Milstein (Runge-Kutta approach)
- **Explicit order 1.5 strong Taylor scheme**
Stochastic Runge-Kutta (SRK)

2.2 Explicit order 0.5 strong Taylor scheme

2.2.1 Euler-Maruyama method

The simplest stochastic numerical approximation is the Euler-Maruyama method that requires the problem to be described using the Itô scheme. For Stratonovich interpretation, one can use the Euler-Heun method, see Section 2.2.2.

This approximation is a continuous time stochastic process that satisfy the iterative scheme [9]

$$Y_{n+1} = Y_n + f(Y_n)h_n + g(Y_n)\Delta W_n \quad Y_0 = x_0 \quad n = 0, 1, \dots, N-1 \quad (14)$$

$$\Delta W_n = [W_{t+h} - W_t] \sim \sqrt{h}\mathcal{N}(0, 1) \quad (15)$$

where $Y_n = Y(t_n)$, $h_n = t_{n+1} - t_n$ is the step size, $\Delta W_n = W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, h_n)$ with $W(t_0) = 0$. From now on, the following notation is used: $h = h_n$ (fixed step size), $f_n = f(Y_n)$ and $g_n = g(Y_n)$. (14) becomes

$$Y_{n+1} = Y_n + f_n h + g_n \Delta W_n \quad (16)$$

As the order of convergence for the Euler-Maruyama method is low (strong order of convergence 0.5, weak order of convergence 1), the numerical results are inaccurate unless a small step size is used. In fact, Euler-Maruyama represents the *order 0.5 strong Taylor scheme*. By adding one more term from the stochastic Taylor expansion, one obtains a 1.0 strong order of convergence scheme known as *Milstein scheme* [9].

2.2.2 Euler-Heun method

If a problem is described using the Stratonovich scheme, then the Euler-Heun method has to be used instead of the Euler-Maruyama method that is only valid for Itô SDEs [3, 6].

$$Y_{n+1} = Y_n + f_n h + \frac{1}{2} [g_n + g(\bar{Y}_n)] \Delta W_n \quad (17)$$

$$\bar{Y}_n = Y_n + g_n \Delta W_n \quad (18)$$

$$\Delta W_n = [W_{t+h} - W_t] \sim \sqrt{h} \mathcal{N}(0, 1) \quad (19)$$

2.3 Explicit order 1.0 strong Taylor scheme

2.3.1 Milstein method

The Milstein scheme is slightly different whether it is the Itô or Stratonovich representation that is used [3, 6, 7]. It can be proved that Milstein scheme converges strongly with order 1 (and weakly with order 1) to the solution of the SDE. The Milstein scheme represents the *order 1.0 strong Taylor scheme*.

$$Y_{n+1} = Y_n + f_n h + g_n \Delta W_n + \frac{1}{2} g_n g'_n [(\Delta W_n)^2 - h] \quad (20)$$

$$Y_{n+1} = Y_n + f_n h + g_n \Delta W_n + \frac{1}{2} g_n g'_n (\Delta W_n)^2 \quad (21)$$

$$\Delta W_n = [W_{t+h} - W_t] \sim \sqrt{h} \mathcal{N}(0, 1) \quad (22)$$

where $g'_n = \frac{dg(Y_n)}{dY_n}$ is the first derivative of g_n . The iterative method defined by (20) must be used with Itô SDEs whether (21) has to be applied to Stratonovich SDEs. Note that when *additive noise* is used, i.e. when g_n is constant and not anymore a function of Y_n , then both Itô and Stratonovich interpretations are equivalent ($g'_n = 0$).

2.3.2 Derivative-free Milstein method

The drawback of the previous method is that it requires the analytic specification of the first derivative of $g(Y_n)$, analytic expression that can become quickly highly complex. The following implementation approximates this derivative thanks to a Runge-Kutta approach [6].

$$Y_{n+1} = Y_n + f_n h + g_n \Delta W_n + \frac{1}{2\sqrt{h}} [g(\bar{Y}_n) - g_n] [(\Delta W_n)^2 - h] \quad (23)$$

$$Y_{n+1} = Y_n + f_n h + g_n \Delta W_n + \frac{1}{2\sqrt{h}} [g(\bar{Y}_n) - g_n] (\Delta W_n)^2 \quad (24)$$

$$\bar{Y}_n = Y_n + f_n h + g_n \sqrt{h} \quad (25)$$

$$\Delta W_n = [W_{t+h} - W_t] \sim \sqrt{h} \mathcal{N}(0, 1) \quad (26)$$

where (23) and (24) must be applied respectively to Itô and Stratonovich SDEs.

2.4 Explicit order 1.5 strong Taylor scheme

2.4.1 Definition

By adding more terms from a stochastic Taylor expansion than in Milstein scheme, higher strong orders can be obtained. A method to generate a strong

order 1.5 method is introduced by Burrage & Platen [2, 5]. For the need of this method, a random variable ΔZ_n is introduced.

$$\Delta Z_n = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{\tau_{s_2}} dW_{s_1} ds_2 \quad (27)$$

which is a Gaussian distributed with mean zero, variance $\frac{1}{3}h^3$ and correlation $E(\Delta W_n \Delta Z_n) = \frac{1}{2}h^2$ [2, 5].

2.4.2 Stochastic Runge-Kutta

This implementation allows to achieve a 1.5 strong order of converge. This is the highest strong order obtained with a Runge-Kutta approach that keeps a “simple” structure. This implementation makes use of the ΔZ_n introduced in (27) [2, 5]. Note that this method has been designed for Itô SDEs.

$$\Delta Y_{n+1} = Y_n + f_n h + g_n \Delta W_n + \frac{1}{2} g_n g'_n [(\Delta W_n)^2 - h] \quad (28)$$

$$+ f'_n g_n \Delta Z_n + \frac{1}{2} \left[f_n f'_n + \frac{1}{2} g_n^2 f''_n \right] h^2 \quad (29)$$

$$+ \left[f_n g'_n + \frac{1}{2} g_n^2 g''_n \right] [\Delta W_n h - \Delta Z_n] \quad (30)$$

$$+ \frac{1}{2} g_n [g_n g''_n + (g'_n)^2] \left[\frac{1}{3} (\Delta W_n)^2 - h \right] \Delta W_n \quad (31)$$

3 Convergence

An approximation Y converges with strong order $\gamma > 0$ if there exists a constant K such that [2]

$$E(|X_T - Y_N|) \leq K \cdot h^\gamma \quad (32)$$

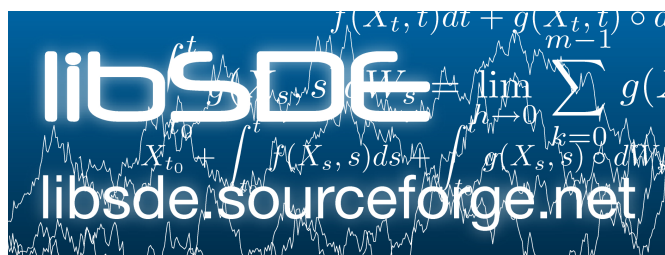
for step sizes $h \in (0, 1)$, with X_T being the true solution at time T and Y_N the approximation. The symbol E stands for *expectation*. It appears that Euler-Maruyama scheme converges only with strong order $\gamma = 0.5$. Strong approximation is tightly linked to the use of the original increments of the Wiener process [2]. However in several applications, it is not needed to simulate a pathwise approximation of a Wiener process. For instance, one could be only interested in the moments of the solution of a SDE. A discrete time approximation Y converges with weak order $\beta > 0$ if for any polynomial $g(\cdot)$ there exists a constant K_g such that

$$|E(g(X_T)) - E(g(Y_N))| \leq K_g \cdot h^\beta \quad (33)$$

for step sizes $h \in (0, 1)$. It turns out that Euler-Maruyama scheme converges with weak order $\beta = 1.0$ [2].

If a numerical method is convergent with order γ and the step size is made k times smaller, then the approximation error decreases by a factor k^γ . For instance, if the order equals 1 and we want to decrease the error 100 times, we have to make the step size 100 times smaller. If the order equals 0.5 and we still want to decrease the error 100 times, we have to make the step size $100^2 = 10000$ times smaller.

4 libSDE



libSDE is an open-source Java library to simulate Stochastic Differential Equations (SDEs). More in detail, the user can:

- simulate *Itô* and *Stratonovich SDEs*
- specify the integration method to be used among *Euler-Maruyama*, *Euler-Heun*, *derivative-free Milstein*, and *Stochastic Runge-Kutta methods*.
- set the integration step-size
- set the number of trajectories (time series) to be simulated
- set the time interval to be considered $[t_0, T]$
- set the number of time points per time serie
- use the libSDE framework to implement other integration methods and functions to be simulated

libSDE has been implemented to simulate intrinsic noise affecting gene expression in cells (both transcription and translation were modeled) and is now part of GeneNetWeaver (GNW), a tool for the automatic generation of *in silico* gene networks and reverse engineering benchmarks (<http://gnw.sf.net>).

5 Credits

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The Java library libSDE has been released open source under an MIT license. If this software or the present tutorial was useful for your scientific work, please cite:

T. Schaffter, *Numerical Integration of SDEs: A Short Tutorial*, Swiss Federal Institute of Technology in Lausanne (EPFL), Switzerland, 2010, Unpublished manuscript.

References

- [1] K.E.S. Abe, W. Shaw, and M. Giles, *Pricing Exotic Options using Local, Implied and Stochastic Volatility obtained from Market Data*, (2004).
- [2] P.M. Burrage, *Runge-Kutta methods for stochastic differential equations*, (1999).
- [3] H. Gilsing and T. Shardlow, *SDELab: A package for solving stochastic differential equations in MATLAB*, *Journal of Computational and Applied Mathematics* **205** (2007), no. 2, 1002–1018.
- [4] D.J. Higham, *An algorithmic introduction to numerical simulation of stochastic differential equations*, *SIAM review* (2001), 525–546.
- [5] R.L. Honeycutt, *Stochastic runge-kutta algorithms. I. White noise*, *Physical Review A* **45** (1992), no. 2, 600–603.
- [6] P.E. Kloeden, E. Platen, and H. Schurz, *Numerical solution of SDE through computer experiments*, Springer, 1994.
- [7] H. Lamba, *Stepsize control for the Milstein scheme using first-exit-times*.
- [8] L. Panzar and E.C. Cipu, *Using of stochastic Ito and Stratonovich integrals derived security pricing*, (2005).
- [9] U. Picchini, *Sde toolbox: Simulation and estimation of stochastic differential equations with matlab*.
- [10] N. G. Van Kampen, *Stochastic processes in physics and chemistry*, North-Holland, 2007.