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by

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ON THE BENEFITS OF A MONETARY UNION: DOES IT PAY TO BE BIGGER?*

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Abstract

A two area dynamic stochastic general equilibrium model is employed to investigate the welfare implications of losing monetary independence. Two policy regimes are compared: (i) in one area there is a common currency, while in the other area countries still retain their autonomous monetary policy; (ii) there are two monetary unions. When chosen by national authorities, monetary policy can stabilize optimally the effects of country-specific shocks. However, in that case, policy decisions internalize neither the spillover effects on consumers living in the same area nor their impact on the world economy. Thus the adoption of a common currency implies a trade-off between the cost of not tailoring monetary policy to single country economic conditions and the gains entailed by the improvement upon the conduct of national monetary policies. Our results show that under markup shocks and plausible calibrations, there may be welfare gains from adopting a common currency.

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1 Introduction

What are the costs and the benefits of a monetary union? Should independent countries abandon their own currency to delegate monetary policy to a common central bank? These questions are far from new[1] but have been revitalized by the debate on the creation of the European Monetary Union (EMU). On theoretical grounds the costs of losing monetary autonomy are well known: in presence of nominal rigidities countries that share the same currency cannot properly stabilize asymmetric shocks. By contrast, the sources of welfare benefits that can rationalize the existence of a currency area have not been identified[2], at least if we restrict ourselves to the new open economy macroeconomic literature in which the objectives of the policy makers are fully microfounded[3].

However there is a key aspect seemingly overlooked so far which can explain why a monetary union can be beneficial for its members. Especially, if we refer to the EMU experience, it is clear that the European Central Bank (ECB) sets the nominal interest rate for an economic area which is much *bigger* than each national economy. The difference in size may induce an improvement upon the conduct of the single country monetary policy given that, as stressed also by the recent literature[4], open economy policy makers seek to affect their terms of trade to other countries consumers' expense. Indeed on the one hand, by being concerned about the welfare of all consumers living in the area, the central bank of the monetary union internalizes the spillover effects that single country's policy makers would produce *inside* the area if there were monetary autonomy. On the other hand, by setting the monetary policy for the union as whole, the common authority better realizes the impact of its policy decisions on the *outside* world and the feedback effects on welfare in its own economy.

The contribution of this paper is to verify whether, once these channels are taken into account, the adoption of a common currency generates gains in terms of welfare that outweigh the costs of renouncing monetary policy independence. To this end, I develop a dynamic stochastic general equilibrium open economy model in which the world is constituted by a continuum of small open regions as in Galí and Monacelli (2005). Each region produces a bundle of differentiated goods. Preferences exhibit home bias and the elasticity of substitution between home and foreign bundles is different from one. Prices are staggered implying a cost for the adoption of a common currency due to the impossibility to properly stabilize asymmetric shocks.

The regions are split in two areas, H and F. In area F all regions belong to a single country (as in the U.S.). Conversely area H is formed by a collection of sovereign small open economies (as in Europe). In this setup two different policy regimes (called A and B) are considered. Under regime A, in area H there are flexible exchange rates

[1] See Mundell (1961).

[2] As emphasized by the so called Delors report (1989), there are microeconomic benefits from adopting a common currency like, for instance, saving in transaction costs. However it would be difficult to incorporate this kind of costs in a macroeconomics model.

[3] ...namely derived directly from the welfare of the representative household. See Rotemberg and Woodford (1997) and Benigno and Woodford (2005). There is a recent contribution of Corsetti (2007) on this issue who, in a model with heterogeneous countries, identifies the conditions under which monetary policy in a currency union is as efficient as under monetary autonomy.

[4] See for instance Corsetti and Pesenti (2001) and Epifani and Gancia (2005).

and each small open economy has its own autonomous central bank; under regime B in area H there is a single currency and monetary policy is under the control of a common central bank (ECB). Instead in area F, independently of the policy regimes, all regions share a common currency and monetary policy is delegated to a single authority (FED). Moreover, in both regimes A and B monetary policies are chosen under commitment and are optimal from the *timeless* perspective[5].

In this kind of setting optimal policy decisions of open economy authorities are biased by a free riding problem. Under the assumption of complete financial markets, consumption is highly correlated both across area and across regions. Because of this consumption sharing, single countries' policy makers have an incentive to seek to improve their terms of trade in their favor[6]. Through a terms of trade improvement, they try to squeeze the domestic/foreign output ratio to outsource labour effort. However their optimal monetary policies are affected in different ways by this incentive depending on the dimension of their own economy.

Under the baseline calibration small countries' policy makers perceive per-period domestic output as inefficiently high. They would rather prefer to lower home production and substitute consumption with leisure. Indeed since the economy is small, one unit decrease in domestic output - which does not affect world output - brings about a marginal decrease in labour that more than outweighs the consumption drop. This has a clear consequence for optimal monetary policy. In fact, by increasing the covariance between output and mark up shocks (which is typically negative), the authorities of a small open economy can induce domestic workers to enjoy more leisure contracting, by so doing, the per-period domestic production. In other words given that they regard home output as too high, these policy makers have a motive to focus more on output than inflation stabilization in response to a global mark up shock.

By contrast, under the baseline calibration the central bank of the monetary union considers per-period domestic production as too low. This is because its incentive to manipulate the terms of trade is much weaker than that of policy makers of the small open economy. Indeed, the authority of the currency union internalizes the feedback effects of its policy decisions stemming from the other area. Then it realizes that a terms of trade improvement is dampening the domestic/ foreign output ratio not only by squeezing the demand for domestic goods (reducing domestic production) but even by boosting that for foreign ones (thus increasing foreign output). As a result, this policy maker is willing to adopt a policy that weights more inflation than output stabilization allowing for a rise in the per-period domestic production.

These differences in incentives explain the differences in outcomes across policy regimes. In regime B policy makers are exactly symmetric; thus under global mark up shocks, they choose the same optimal monetary policy, thereby ensuring the same economic performance in the two areas. Conversely, in regime A, the monetary authorities of the two areas have opposite goals. By seeking to reduce per-period domestic

[5]See Benigno and Woodford (2005) and Woodford (2003).

[6]The effects of this externality are amplified by the hypothesis of home bias for both the bundles produced within the region and within the area as well as by the assumption that the elasticity of substitution between home and foreign bundles is different from one. For a discussion see Obstfeld and Rogoff (2000) , Benigno and Benigno (2003) and Pappa (2004). Notice that other policy instruments to affect terms of trade, such as tariffs, cannot be used in the WTO.

output, the central banks of the small open countries weighs more output than inflation stabilization. On the other hand given that from their perspective domestic output is on average too low, the policy maker of the monetary union focuses more on inflation than on output stabilization. Then in response, for instance, to a negative symmetric mark up shock, small open countries' authorities adopt a more restrictive policy than the policy maker of the monetary union. Therefore there is more deflation in area H and output in area F expands more than under regime B.

These differences across regimes explain why, despite the presence of idiosyncratic shocks, households of area H can be better off by sharing a common currency. Indeed, I show that, in the presence of mark up shocks, adopting the same currency may generate welfare benefits under reasonable calibrations. This finding is quite robust: even for relatively low level of the intertemporal elasticity of substitution between home and foreign bundles and high levels of the variance of the idiosyncratic shocks, welfare gains may be significant.

This paper is organized as follows. Section 2 describes the basic setup, section 3 determines the equilibrium conditions, section 4 formulates the optimal policy problems, section 5 describes the dynamic simulation and section 6 reports the results about the welfare evaluation.

2 The basic framework

The world consists of a continuum of small open regions indexed by $i \in [0,1]$[7]. The regions are subdivided in two areas, H and F. In area H, there is a continuum of regions indexed by $i \in [0, \frac{1}{2})$, which are independent countries. Area F consists of regions indexed by $i \in [\frac{1}{2}, 1]$, which belong to a single country. Each region produces a continuum of imperfect substitutable goods. Labour is immobile across both regions and areas.

2.1 Preferences

Agents are infinitely lived and maximize the expected value of the discounted sum of the period utility. Preferences of a generic region i representative household are defined over a private consumption bundle, C_t^i and labor $N_t^i(s)$:

$$\sum_{t=0}^{\infty} \beta^t E_0 \left[\frac{{C_t^i}^{1-\sigma}}{1-\sigma} - \frac{{N_t^i(s)}^{\varphi+1}}{\varphi+1} \right] \quad 0 < \beta < 1 \tag{1}$$

where β stands for the intertemporal preferences discount factor. Agents consume all the goods produced in the world economy. However, preferences exhibit home bias.

[7]This model is a general version of the basic framework layout by Galí and Monacelli (2005) and Galí and Monacelli (2007).

The private consumption index is a CES aggregation of the following type:

$$C_t^i \equiv \left[\alpha_s^{\frac{1}{\eta}} C_{i,t}^i{}^{\frac{\eta-1}{\eta}} + (\alpha_b - \alpha_s)^{\frac{1}{\eta}} C_{H,t}^i{}^{\frac{\eta-1}{\eta}} + (1-\alpha_b)^{\frac{1}{\eta}} C_{F,t}^i{}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \qquad i \in \left[0, \frac{1}{2}\right) \quad (2)$$

$$C_t^i \equiv \left[\alpha_s^{\frac{1}{\eta}} C_{i,t}^i{}^{\frac{\eta-1}{\eta}} + (\alpha_b - \alpha_s)^{\frac{1}{\eta}} C_{F,t}^i{}^{\frac{\eta-1}{\eta}} + (1-\alpha_b)^{\frac{1}{\eta}} C_{H,t}^i{}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \qquad i \in \left[\frac{1}{2}, 1\right] \quad (3)$$

$\eta > 0$, $0 < \alpha_s < \alpha_b$ and $\frac{1}{2} < \alpha_b < 1$. α_s and α_b are the degrees of home bias for the goods produced within region i and the area to which region i belongs. Moreover, η denotes the elasticity of substitution among $C_{H,t}^i$ $C_{F,t}^i$, $C_{i,t}^i$ which are defined as:

$$C_{H,t}^i \equiv \left[2^{\frac{1}{\eta}} \int_0^{\frac{1}{2}} C_{j,t}^i{}^{\frac{\eta-1}{\eta}} dj\right]^{\frac{\eta}{\eta-1}} \quad C_{F,t}^i \equiv \left[2^{\frac{1}{\eta}} \int_{\frac{1}{2}}^1 C_{j,t}^i{}^{\frac{\eta-1}{\eta}} dj\right]^{\frac{\eta}{\eta-1}} \qquad (4)$$

$$C_{j,t}^i \equiv \left(\int_0^1 c_t^i(h^j)^{\frac{\varepsilon-1}{\varepsilon}} dh^j\right)^{\frac{\varepsilon}{\varepsilon-1}} \quad j \in \left[0, \frac{1}{2}\right) \quad C_{j,t}^i \equiv \left(\int_0^1 c_t^i(f^j)^{\frac{\varepsilon-1}{\varepsilon}} df^j\right)^{\frac{\varepsilon}{\varepsilon-1}} \quad i \in \left[\frac{1}{2}, 1\right] \quad (5)$$

where ε is the elasticity of substitution among goods produced in the same region. The definitions of the private consumption indexes (2), (4) and (5) enable us to determine consistent definitions of price indexes. In particular, $P_{C^i,t}$, the consumers' price index of region i, is:

$$P_{C^i,t} \equiv \left[\alpha_s P_{i,t}{}^{1-\eta} + (\alpha_b - \alpha_s) P_{H,t}{}^{1-\eta} + (1-\alpha_b) P_{F,t}{}^{1-\eta}\right]^{\frac{1}{1-\eta}} \qquad i \in \left[0, \frac{1}{2}\right) \quad (6)$$

$$P_{C^i,t} \equiv \left[\alpha_s P_{i,t}{}^{1-\eta} + (\alpha_b - \alpha_s) P_{F,t}{}^{1-\eta} + (1-\alpha_b) P_{H,t}{}^{1-\eta}\right]^{\frac{1}{1-\eta}} \qquad i \in \left[\frac{1}{2}, 1\right] \quad (7)$$

$$P_{H,t} \equiv \left[2 \int_0^{\frac{1}{2}} P_{j,t}{}^{1-\eta} dj\right]^{\frac{1}{1-\eta}} \qquad P_{F,t} \equiv \left[2 \int_{\frac{1}{2}}^1 P_{j,t}{}^{\frac{1}{1-\eta}} dj\right]^{\frac{1}{1-\eta}}$$

$$P_{j,t} \equiv \left(\int_0^1 p_t(h^j)^{1-\varepsilon} dh^j\right)^{\frac{1}{1-\varepsilon}} \quad j \in \left[0, \frac{1}{2}\right) \quad P_{j,t} \equiv \left(\int_0^1 p_t(f^j)^{1-\varepsilon} df^j\right)^{\frac{1}{1-\varepsilon}} \quad j \in \left[\frac{1}{2}, 1\right]$$

where all prices are denominated in the currency of the home country. Thus $P_{i,t}$, $P_{H,t}$ and $P_{F,t}$ are producers' price indexes. The law of one price is assumed to hold in all single good markets. However, given the home biased preferences, in general the purchasing power parity does not hold for indexes $P_{C^i,t}$.

2.2 Consumption demand, portfolio choices and labor supply

The consumption and price index definitions allow to solve the consumer problem in two stages. In a first stage, agents decide how much real net income to allocate to buy goods produced at home and abroad. According to the set of optimality conditions, it is possible to determine agents' demands as:

$$C_{i,t}^i = \alpha_s \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta} C_t^i \quad C_{H,t}^i = (\alpha_b - \alpha_s) \left(\frac{P_{H,t}}{P_{C^i,t}}\right)^{-\eta} C_t^i \quad C_{F,t}^i = (1-\alpha_b) \left(\frac{P_{F,t}}{P_{C^i,t}}\right)^{-\eta} C_t^i \quad i \in \left[0, \frac{1}{2}\right) \quad (8)$$

$$C^i_{i,t} = \alpha_s \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta} C^i_t \quad C^i_{F,t} = (\alpha_b - \alpha_s) \left(\frac{P_{F,t}}{P_{C^i,t}}\right)^{-\eta} C^i_t \quad C^i_{H,t} = (1-\alpha_b) \left(\frac{P_{H,t}}{P_{C^i,t}}\right)^{-\eta} C^i_t \quad i \in \left(\frac{1}{2}, 1\right] \tag{9}$$

and for $i \in \left[0, \frac{1}{2}\right)$:

$$C^i_{j,t} = 2\left(\frac{P_{j,t}}{P_{H,t}}\right)^{-\eta} C^i_{H,t} \;\; j \in \left[0, \frac{1}{2}\right) \;\; C^i_{j,t} = 2\left(\frac{P_{j,t}}{P_{F,t}}\right)^{-\eta} C^i_{F,t} \;\; j \in \left(\frac{1}{2}, 1\right] \tag{10}$$

$$c^i_t(h^j) = \left(\frac{p_t(h^j)}{P_{j,t}}\right)^{-\varepsilon} C^i_{j,t} \;\; j \in \left[0, \frac{1}{2}\right) \;\; c^i_t(f^j) = \left(\frac{p_t(f^j)}{P_{j,t}}\right)^{-\varepsilon} C^i_{j,t} \;\; j \in \left(\frac{1}{2}, 1\right] \tag{11}$$

while for $i \in \left(\frac{1}{2}, 1\right]$:

$$C^i_{j,t} = 2\left(\frac{P_{j,t}}{P_{F,t}}\right)^{-\eta} C^i_{F,t} \;\; j \in \left[0, \frac{1}{2}\right) \;\; C^i_{j,t} = 2\left(\frac{P_{j,t}}{P_{H,t}}\right)^{-\eta} C^i_{H,t} \;\; j \in \left(\frac{1}{2}, 1\right] \tag{12}$$

$$c^i_t(f^j) = \left(\frac{p_t(f^j)}{P_{j,t}}\right)^{-\varepsilon} C^i_{j,t} \;\; j \in \left[0, \frac{1}{2}\right) \;\; c^i_t(h^j) = \left(\frac{p_t(h^j)}{P_{j,t}}\right)^{-\varepsilon} C^i_{j,t} \;\; j \in \left(\frac{1}{2}, 1\right] \tag{13}$$

The second stage coincides with the standard consumer problem. Agents maximize (1) with respect to C^i_t, D^i_{t+1} and $N^i_t(s)$ subject to the following sequence of budget constraints:

$$E_t\{Q^i_{t,t+1} D^i_{t+1}\} = D^i_t + W_{i,t}(s) N^i_t(s) - P_{C^i,t} C^i_t + T^i_t \tag{14}$$

$$N^i_t(s) = \left(\frac{W_{i,t}(s)}{W_{i,t}}\right)^{-\upsilon^i_t} N^i_t \tag{15}$$

where:

$$W_{i,t} \equiv \left[\int_0^1 W_{i,t}(s)^{1-\upsilon^i_t} ds\right]^{\frac{1}{1-\upsilon^i_t}} \tag{16}$$

Condition (14) is the budget constraint which states that nominal saving, net of lump sum transfers, has to equalize the nominal value of a state contingent portfolio. In fact, $W_{i,t}(s)$ stands for the per hour nominal wage, $Q^i_{t,t+1}$ denotes what is usually called the stochastic discount factor and D^i_{t+1} is the payoff of one period maturity portfolio of firm shares.

Constraint (15) is a consequence of a CES aggregation of labor inputs which will be specified below and states that the labour market is monopolistically competitive. Indeed each agent offers a different kind of labour service. Thus υ^i_t stands for the elasticity of demand of labor which is time-varying and region-specific as in Clarida, Galí and Gertler (2002). Finally, (16) is simply the aggregate wage index. Domestic and international markets are assumed to be complete.

By the optimality conditions of the household problem:

$$(1 + \mu^i_t) N^i_t(s)^{\varphi} C^{i\sigma}_t = \frac{W_{i,t}}{P_{C^i,t}} \tag{17}$$

$$\beta \left(\frac{C^i_{t+1}}{C^i_t}\right)^{-\sigma} \left(\frac{P_{C^i,t}}{P_{C^i,t+1}}\right) = Q^i_{t,t+1} \tag{18}$$

which hold in all states of nature and at all periods and where $\mu_t^i \equiv \frac{1}{v_t^i - 1}$. According to (17), workers set the real wage as mark up over the marginal rate of substitution between consumption and leisure, while the value of the intertemporal marginal rate of substitution of consumption should equalize the stochastic discount factor expressed in terms of the currency of region i. Notice that since wages are perfectly flexible, $N_t^i(s) = N_t^i$ and $W_{i,t}(s) = W_{i,t}$ for all s and t.

2.3 Firms, technology and price setting

In each region i there is a continuum of firms. Each of them produces a single differentiated good with a constant return to scale technology of the type:

$$y_t(h^i) = A_t^i N_t(h^i) \tag{19}$$

with $N_t(h^i) = \left[\int_0^1 N_t^i(s)^{\frac{v_t^i - 1}{v_t^i}} ds\right]^{\frac{v_t^i}{v_t^i - 1}}$ being the labor input and A_t^i the region-specific technology shock. Given (19) and the fact that $N_t^i = N_t^i(s)$ for all h^i, the aggregate relationship between output and labor can be read as:

$$N_t^i = \frac{Y_t^i}{A_t^i} Z_t^i \tag{20}$$

where $Y_t^i \equiv \left[\int_0^1 y_t^i(h)^{\frac{\varepsilon-1}{\varepsilon}} dh\right]^{\frac{\varepsilon}{\varepsilon-1}}$ and $Z_t^i \equiv \int_0^1 \frac{y_t(h^i)}{Y_t^i} dh^i$, and $N_t^i \equiv \int_0^1 N_t(h^i) dh^i$. Using (10) and (11) I will show below that $Z_t \equiv \int_0^1 \left(\frac{p_t(h^i)}{P_{i,t}}\right)^{-\varepsilon} dh^i$; thus Z_t^i can be interpreted as an index of the relative price dispersion across firms. We assume that good prices adjust according to a staggered mechanism *à la* Calvo. Therefore, in each period a given firm can reoptimize its price only with probability $1-\theta$. As a result, the fraction of firms that set a new price is fixed and the aggregate producer price index of the intermediate goods evolves accordingly to:

$$P_{i,t}^{(1-\varepsilon)} = \theta P_{i,t-1}^{(1-\varepsilon)} + (1-\theta)\tilde{p}_{i,t}(h^i)^{(1-\varepsilon)} \tag{21}$$

with $\tilde{p}_t(h^i)$ being the optimal price. Firms maximize the discounted expected sum of the future profits that would be collected if the optimal price could not be changed.

$$\sum_{s=0}^{\infty} (\theta)^s E_t \left\{ Q_{t,t+s}^i y_{t+s}(h^i) \left[\tilde{p}_t(h^i) - MC_{i,t+s}^n\right] \right\} \tag{22}$$

where $y_t(h^i) = (\frac{p_t(h^i)}{P_{i,t}})^{-\varepsilon} Y_t^i$ and $MC_{i,t}^n = \frac{(1-\tau^i)W_{i,t}}{A_t^i}$ is the nominal marginal cost with τ^i denoting a constant labor subsidy. Taking into account (18) and that $MC_{i,t} \equiv \frac{MC_{i,t}^n}{P_{i,,t}}$, the optimality condition of the firm problem can be written as:

$$\sum_{s=0}^{\infty} (\beta\theta)^s E_t \left\{ {C_{t+s}^i}^{-\sigma} \left(\frac{\tilde{p}_t(h^i)}{P_{i,t+s}}\right)^{-\varepsilon} Y_{t+s}^i \frac{P_{i,t}}{P_{C^i,t+s}} \left[\frac{\tilde{p}_t(h^i)}{P_{i,t}} - \frac{\varepsilon}{\varepsilon-1}\frac{P_{i,t+s}}{P_{i,t}} MC_{i,t+s}\right] \right\} = 0 \tag{23}$$

Condition (23) states implicitly that firms reset their prices as a mark up over a weighted average of the current and expected marginal costs, where the weight of the expected marginal cost at some date $t+s$ depends on the probability that the price is still effective at that date.

3 Equilibrium

International risk sharing

The assumption of complete markets implies:

$$\frac{C_t^{i\,-\sigma}}{P_{C^i,t}} = \frac{C_t^{j\,-\sigma}}{\mathcal{E}_{ij,t} P_{C^j,t}} \tag{24}$$

for all t, $i \in \left[0, \frac{1}{2}\right)$ and $j \in \left(\frac{1}{2}, 1\right]$. According to (24), the value of marginal utility of consumption is equalized across regions. However, given the home bias in consumption, even if the law of one price holds, the purchasing power parity does not. As a consequence, consumption can be different across both regions and areas.

By properly integrating this equation we obtain:

$$\frac{C_t^{i\,-\sigma}}{P_{C^i,t}} = \frac{C_{H,t}^{*\ -\sigma}}{\mathcal{E}_{iH,t} P_{H,t}^*} \quad i \in \left[0, \frac{1}{2}\right) \quad \frac{C_t^{i\,-\sigma}}{P_{C^i,t}} = \frac{C_{F,t}^{*\ -\sigma}}{\mathcal{E}_{iF,t} P_{F,t}^*} \quad i \in \left(\frac{1}{2}, 1\right] \quad \frac{C_{H,t}^{*\ -\sigma}}{P_{H,t}^*} = \frac{C_{F,t}^{*\ -\sigma}}{\mathcal{E}_{HF,t} P_{F,t}^*} \tag{25}$$

for all i, where $\mathcal{E}_{ij,t}$ stands for the nominal exchange rate of region j currency to region i currency[8]. Here $C_{H,t}^* \equiv \left[2 \int_0^{\frac{1}{2}} C_t^{i\,-\sigma(1-\eta)} di\right]^{\frac{-1}{\sigma(1-\eta)}}$, $C_{F,t}^* \equiv \left[2 \int_{\frac{1}{2}}^1 C_t^{j\,-\sigma(1-\eta)} dj\right]^{\frac{-1}{\sigma(1-\eta)}}$, $P_{H,t}^* \equiv \left[2 \int_0^{\frac{1}{2}} \left(\mathcal{E}_{Hj,t} P_{C^j,t}\right)^{(1-\eta)} dj\right]^{\frac{1}{(1-\eta)}}$ and $P_{F,t}^* \equiv \left[2 \int_{\frac{1}{2}}^1 P_{C^j,t}{}^{(1-\eta)} dj\right]^{\frac{1}{(1-\eta)}}$.

Regarding conditions (25), notice the following. Within area F, there is always a common currency, independently of the policy regime. Thus, $\mathcal{E}_{Fi,t} = 1$ for all $i \in [\frac{1}{2}, 1]$. Conversely within area H, $\mathcal{E}_{Hi,t} = 1$ for all $i \in [0, \frac{1}{2})$ only under regime B when there is a common currency and the exchange rates are fixed. Finally, in general, $\mathcal{E}_{HF,t}$ is floating under both regimes A and B. As shown in the appendix, it follows from to (25) and (24) that:

$$\frac{P_{i,t}}{P_{C^i,t}} = \left[\gamma_s + (\gamma_b - \gamma_s)\left(\frac{C_{H,t}^*}{C_t^i}\right)^{-\sigma(1-\eta)} + (1-\gamma_b)\left(\frac{C_{F,t}^*}{C_t^i}\right)^{-\sigma(1-\eta)}\right]^{\frac{1}{1-\eta}} \tag{26}$$

for $i \in \left[0, \frac{1}{2}\right)$ and where $\gamma_s \equiv \frac{1}{\alpha_s}$ and $\gamma_b \equiv \frac{-\alpha_b}{1-2\alpha_b}$. A corresponding condition can be retrieved for area F:

$$\frac{P_{i,t}}{P_{C^i,t}} = \left[\gamma_s + (\gamma_b - \gamma_s)\left(\frac{C_{F,t}^*}{C_t^i}\right)^{-\sigma(1-\eta)} + (1-\gamma_b)\left(\frac{C_{H,t}^*}{C_t^i}\right)^{-\sigma(1-\eta)}\right]^{\frac{1}{1-\eta}} \tag{27}$$

[8]... and $\mathcal{E}_{Hj,t}$ stands for the nominal exchange rate of region j currency to a common unit of account of area H.

for all $i \in \left(\frac{1}{2}, 1\right]$.

At the same time, (6) and (7) can be log-linearized as:

$$\hat{p}_{i,t} - \hat{p}^i_{c,t} = -\left(\alpha_b - \alpha_s\right)\hat{s}_{iH,t} - \left(1 - \alpha_b\right)\hat{s}_{iF,t} \qquad i \in \left[0, \frac{1}{2}\right) \tag{28}$$

$$\hat{p}_{i,t} - \hat{p}^i_{c,t} = -\left(\alpha_b - \alpha_s\right)\hat{s}_{iF,t} - \left(1 - \alpha_b\right)\hat{s}_{iH,t} \qquad i \in \left[\frac{1}{2}, 1\right] \tag{29}$$

where $\hat{s}_{iH,t} \equiv e_{iH,t} + \hat{p}_{H,t} - \hat{p}_{i,t}$ and $\hat{s}_{iF,t} \equiv e_{iF,t} + \hat{p}_{F,t} - \hat{p}_{i,t}$ denote the terms of trade of a small open economy i and areas H and F respectively[9] and where $\hat{c}_{H,t} \equiv 2\int_0^{\frac{1}{2}} \hat{c}^j_t dj$ and $\hat{c}_{F,t} \equiv 2\int_{\frac{1}{2}}^1 \hat{c}^j_t dj$[10]. By combining (6) and (7) with (28) and (29) and using (25) :

$$\hat{s}_{iH,t} = -\frac{\sigma}{\alpha_s}(\hat{c}_{H,t} - \hat{c}^i_t) \quad i \in \left[0, \frac{1}{2}\right) \qquad \hat{s}_{iF,t} = -\frac{\sigma}{\alpha_s}(\hat{c}_{F,t} - \hat{c}^i_t) \quad i \in \left[\frac{1}{2}, 1\right] \tag{30}$$

Moreover, by properly integrating the log-linear approximation of (26) and (28), it is easy to show that:

$$\hat{s}_{HF,t} = -\sigma\left(\frac{1}{2\alpha_b - 1}\right)(\hat{c}_{F,t} - \hat{c}_{H,t}) \tag{31}$$

where $\hat{s}_{HF,t} \equiv \hat{e}_{HF,t} + \hat{p}_{F,t} - \hat{p}_{H,t}$ stands for the terms of trade between area F and area H. According to (31), in equilibrium a rise in the terms trade of the two areas reduces their relative consumption ratio as long as $\alpha_b > 1 - \alpha_b$[11]. A terms of trade worsening[12] makes home consumers substitute the goods produced in area F with the goods produced in area H and increase their overall consumption because they relatively prefer the bundle produced in their own area. Notice that the impact of an improvement on the terms of trade on consumption differentials depends critically on the household relative risk adversion (or the inverse of the intertemporal elasticity of substitution of consumption) σ. The higher is σ, the lower is the difference in average consumption across areas associated with a movement in the terms of trade. More risk adverse households are more willing to share risk across different states of the world (or less willing to shift consumption across periods). Finally, by taking (30) in differences, it follows:

$$\Delta e_{iH,t} + \pi_{H,t} - \pi_{i,t} = -\sigma\gamma_s(\Delta\hat{c}_{H,t} - \Delta\hat{c}^i_t) \qquad i \in \left[0, \frac{1}{2}\right) \tag{32}$$

$$\pi_{F,t} - \pi_{i,t} = -\sigma\gamma_s(\Delta\hat{c}_{F,t} - \Delta\hat{c}^i_t) \qquad i \in \left[\frac{1}{2}, 1\right] \tag{33}$$

Equation (33), and in regime B also equation (32), can be interpreted as a constraint imposed by the adoption of a common currency according to which, in response to

[9]...namely the average price of the goods produced in the small open economy i relative to the average price of the goods produced in areas H and F. With a notational abuse $\hat{p}_{F,t}$ indicates the log-deviation of the average price in area F expressed in terms of the common currency of that area. Similar interpretation applies to $\hat{p}_{H,t}$.

[10]We will use this as a general notation. For a given variable $\hat{x}_t$, $\hat{x}_{H,t} \equiv 2\int_0^{\frac{1}{2}} \hat{x}^j_t dj$ and $\hat{x}_{F,t} \equiv 2\int_{\frac{1}{2}}^1 \hat{x}^j_t dj$.

[11]That is $\alpha_b > \frac{1}{2}$ as previously assumed.

[12]...namely an increase of $\hat{s}_{HF,t}$.

asymmetric shocks, the terms of trade cannot adjust instantaneously because of the sluggish price adjustment and the fixed exchange rates. Conversely under regime A in area H, when there is monetary autonomy, the fluctuations of the nominal exchange rates assure that condition (32) is always satisfied.

3.1 IS curve

Given (18) and (25), we can recover the following condition for area F:

$$\frac{1}{1+r_{F,t}} = \beta E_t \left\{ \left(\frac{C^*_{F,t+1}}{C^*_{F,t}} \right)^{-\sigma} \Pi^{*\;-1}_{F,t+1} \right\} \tag{34}$$

where $\frac{1}{1+r_{F,t}} = E_t\{Q^i_{t,t+1}\}$. When markets are complete, the expected value of the intertemporal marginal rate of substitution of private consumption, namely the price of a riskless portfolio, equalizes the price of the riskless bond, being $r_{F,t}$ the nominal interest rate. The analogue of (34) for area H is

$$\frac{1}{1+r^i_t} = \beta E_t \left\{ \left(\frac{C^i_{t+1}}{C^i_t} \right)^{-\sigma} \Pi^{-1}_{C^i,t+1} \right\} \tag{35}$$

under regime A and:

$$\frac{1}{1+r_{H,t}} = \beta E_t \left\{ \left(\frac{C^*_{H,t+1}}{C^*_{H,t}} \right)^{-\sigma} \Pi^{*\;-1}_{H,t+1} \right\} \tag{36}$$

otherwise. The log-linear approximation of conditions (34), (35) and (36) leads to:

$$r_{F,t} - \rho \;= E_t\{\pi_{F,t+1}\} - \sigma E_t\{\Delta\hat{c}_{F,t+1} + (1-\gamma_b)(\Delta\hat{c}_{H,t+1} - \Delta\hat{c}_{F,t+1})\} \tag{37}$$

$$\begin{aligned} r^i_t - \rho \;&= E_t\{\pi_{i,t+1}\} - \sigma E_t\{\Delta\hat{c}^i_{t+1} + (\gamma_b - \gamma_s)(\Delta\hat{c}_{H,t+1} - \Delta\hat{c}^i_{t+1}) \\ &\quad +(1-\gamma_b)(\Delta\hat{c}_{F,t+1} - \Delta\hat{c}^i_{t+1})\} \end{aligned} \tag{38}$$

$$r_{H,t} - \rho \;= E_t\{\pi_{H,t+1}\} - \sigma E_t\{\Delta\hat{c}_{H,t+1} + (1-\gamma_b)(\Delta\hat{c}_{F,t+1} - \Delta\hat{c}_{H,t+1})\} \tag{39}$$

where $\rho \equiv -log(\beta)$. Conditions (37), (38) and (39) are the so called IS curves. Notice that under regime A, r^i_t can be different across the regions in area H being national central banks independent in their policy decisions. Conversely under regime B, $r^i_t = r_{H,t}$ for all i, being the nominal interest of area H set by the common central bank of the monetary union.

3.2 Aggregate demand

In each region i of area H the demand for a specific good, $y_t(h^i)$, is determined by the demand of home and foreign consumers namely:

$$y^i_t(h) = c^i_{i,t}(h) + \int_0^{\frac{1}{2}} c^j_{i,t}(h)dj + \int_{\frac{1}{2}}^1 c^j_{i,t}(h)dj \tag{40}$$

for all $i \in \left[0, \frac{1}{2}\right)$. Given (8), condition (40) can be read as:

$$Y_t^i = \alpha_s \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta} C_t + 2(\alpha_b - \alpha_s) \int_0^{\frac{1}{2}} \left(\frac{P_{i,t}}{P_{C^j,t}}\right)^{-\eta} C_t^j dj + 2(1 - \alpha_b) \int_{\frac{1}{2}}^1 \left(\frac{P_{i,t}}{P_{C^j,t}}\right)^{-\eta} C_t^j dj \tag{41}$$

with $Y_t^i \equiv \left[\int_0^1 y_t^i(h)^{\frac{\varepsilon-1}{\varepsilon}} dh\right]^{\frac{\varepsilon}{\varepsilon-1}}$ Because of (24), the aggregate demand for region i can be written as:

$$Y_t^i = \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta} \left[\alpha_s C_t^i + (\alpha_b - \alpha_s) C_t^{i\sigma\eta} \mathcal{C}_{H,t} + (1 - \alpha_b) C_t^{i\sigma\eta} \mathcal{C}_{F,t}\right] \tag{42}$$

with:

$$\mathcal{C}_{H,t} \equiv 2 \int_0^{\frac{1}{2}} C_t^{j\,1-\sigma\eta} dj \qquad \mathcal{C}_{F,t} \equiv 2 \int_{\frac{1}{2}}^1 C_t^{j\,1-\sigma\eta} dj \tag{43}$$

for all $i \in \left[0, \frac{1}{2}\right)$. A symmetric condition can be stated for all $i \in \left(\frac{1}{2}, 1\right]$, namely:

$$Y_t^i = \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta} \left[\alpha_s C_t^i + (\alpha_b - \alpha_s) C_t^{i\sigma\eta} \mathcal{C}_{F,t} + (1 - \alpha_b) C_t^{i\sigma\eta} \mathcal{C}_{H,t}\right] \tag{44}$$

It is easy to show that the first order approximation of (42) and (44) corresponds to:

$$\hat{y}_t^i = \hat{c}_t^i + (\delta_b - \delta_s)(\hat{c}_{H,t} - \hat{c}_t^i) + (1 - \delta_b)(\hat{c}_{F,t} - \hat{c}_t^i) \quad i \in \left[0, \frac{1}{2}\right) \tag{45}$$

$$\hat{y}_t^i = \hat{c}_t^i + (\delta_b - \delta_s)(\hat{c}_{F,t} - \hat{c}_t^i) + (1 - \delta_b)(\hat{c}_{H,t} - \hat{c}_t^i) \quad i \in \left[\frac{1}{2}, 1\right] \tag{46}$$

where $\delta_s \equiv \gamma_s \eta\sigma + \alpha_s(1 - \eta\sigma)$ and $\delta_b \equiv \gamma_b \eta\sigma + \alpha_b(1 - \eta\sigma)$. According to (45), the aggregate demand of goods produced in region i depends directly on the terms of trade (through (30)). Any terms of trade improvement[13] between region i and areas H or F switches the expenditure of both home and foreign households toward foreign goods. Aggregating (45) and (46), we obtain:

$$\hat{y}_{H,t} = \hat{c}_{H,t} + (1 - \delta_b)(\hat{c}_{F,t} - \hat{c}_{H,t}) \quad i \in \left[0, \frac{1}{2}\right) \tag{47}$$

$$\hat{y}_{F,t} = \hat{c}_{F,t} + (1 - \delta_b)(\hat{c}_{H,t} - \hat{c}_{F,t}) \quad i \in \left[\frac{1}{2}, 1\right] \tag{48}$$

3.3 Aggregate supply

Given condition (23), the optimal price is determined as:

$$\frac{\tilde{p}_t(h^i)}{P_{i,t}} = \frac{K_t^i}{F_t^i} \tag{49}$$

[13] namely a decrease of $\hat{s}_{iH,t}$ or $\hat{s}_{iF,t}$.

with:

$$K_t^i \equiv \sum_{s=0}^{\infty} (\beta\theta)^s E_t \left[{C_{t+s}^i}^{-\sigma} Y_{t+s}^i \left(\frac{P_{i,t+s}}{P_{i,t}} \right)^{\varepsilon} \frac{P_{i,t+s}}{P_{C^i,t+s}} \frac{\varepsilon}{\varepsilon - 1} MC_{i,t+s} \right] \tag{50}$$

$$F_t^i \equiv \sum_{s=0}^{\infty} (\beta\theta)^s E_t \left[{C_{t+s}^i}^{-\sigma} Y_{t+s}^i \left(\frac{P_{i,t+s}^i}{P_t^i} \right)^{\varepsilon-1} \frac{P_{i,t+s}}{P_{C^i,t+s}} \right] \tag{51}$$

which can be read as:

$$K_t^i = C_t^{i-\sigma} Y_{i,t}^i \frac{P_{i,t}}{P_{C^i,t}} \frac{\varepsilon}{\varepsilon - 1} MC_{i,t} + \beta\theta E_t \left\{ \Pi_{i,t+s}^{\varepsilon} K_{t+1}^i \right\} \tag{52}$$

$$F_t^i = C_t^{i-\sigma} Y_t^i \frac{P_{i,t}}{P_{C^i,t}} + \beta\theta E_t \left\{ \Pi_{i,t+1}^{\varepsilon-1} F_{t+1}^i \right\} \tag{53}$$

where $\Pi_{i,t} \equiv \frac{P_{i,t}}{P_{i,t-1}}$. Following Benigno and Woodford (2005), from (49) and (21) we can retrieve the next conditions:

$$\frac{1 - \theta \Pi_{i,t}^{\varepsilon-1}}{1 - \theta} = \left(\frac{F_t^i}{K_t^i} \right)^{\varepsilon - 1} \tag{54}$$

$$Z_t^i = \theta Z_{t-1}^i \Pi_{i,t}^{\varepsilon} + (1 - \theta) \left(\frac{1 - \theta \Pi_{i,t}^{\varepsilon-1}}{1 - \theta} \right)^{\frac{\varepsilon}{\varepsilon - 1}} \tag{55}$$

By the log-linear approximation of (17) (50) (51) and (55):

$$\pi_{i,t} = \lambda \widehat{mc}_t^i + \beta E_t \{ \pi_{i,t+1} \} \tag{56}$$

with $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$ and where:

$$\widehat{mc}_t^i = \left(\hat{w}_t^i - \hat{p}_{c,t}^i \right) - \left(\hat{p}_{i,t} - \hat{p}_{c,t}^i \right) - \hat{a}_t^i \tag{57}$$

for all t and i. Condition (56) is the New Keynesian Phillips Curve which results from the Calvo mechanism. As usual, current *domestic* inflation depends on the expectation on future *domestic* inflation and the current real marginal cost of producing goods. Being the economy open in equilibrium this cost is determined by the real wage, which is equal to the marginal rate of substitution between consumption and leisure, the labour productivity and the product price index relative to the consumption price index (26) and (27). By substituting(28) and log-linear approximation of (26) we obtain:

$$\begin{aligned} \widehat{mc}_t^i &= \varphi \hat{y}_t^i + \sigma \hat{c}_t^i + (\alpha_b - \alpha_s) \hat{s}_{iH,t} + (1 - \alpha_b) \hat{s}_{iF,t} - (1 + \varphi) \hat{a}_t^i + \hat{\mu}_t^i \\ &= \varphi \hat{y}_t^i + \sigma \hat{c}_t^i + \sigma \left[(\gamma_b - \gamma_s) (\hat{c}_{H,t} - \hat{c}_t^i) + (1 - \gamma_b) (\hat{c}_{F,t} - \hat{c}_t^i) \right] - (1 + \varphi) \hat{a}_t^i + \hat{\mu}_t^i \end{aligned} \tag{58}$$

for all $i \in \left[0, \frac{1}{2}\right)$. According to (58), an improvement of the terms of trade of region i[14] lowers firms' real marginal costs. Given (58), we can rewrite condition (56) for

[14]namely a decrease of $\hat{s}_{iH,t}$ or $\hat{s}_{iF,t}$.

$i \in \left[0, \frac{1}{2}\right)$ and its symmetric condition for $i \in \left[\frac{1}{2}, 1\right]$ as:

$$\pi_{i,t} = \lambda \left[\varphi \hat{y}_t^i + \sigma \left(\gamma_s \hat{c}_t^i + (\gamma_b - \gamma_s)\, \hat{c}_{H,t} + (1 - \gamma_b)\, \hat{c}_{F,t}\right) - (1 + \varphi) \hat{a}_t^i + \hat{\mu}_t^i\right] + \beta E_t\{\pi_{i,t+1}\} \tag{59}$$

$$\pi_{i,t} = \lambda \left[\varphi \hat{y}_t^i + \sigma \left(\gamma_s \hat{c}_t^i + (\gamma_b - \gamma_s)\, \hat{c}_{F,t} + (1 - \gamma_b)\, \hat{c}_{H,t}\right) - (1 + \varphi) \hat{a}_t^i + \hat{\mu}_t^i\right] + \beta E_t\{\pi_{i,t+1}\} \tag{60}$$

Under regime A the rational expectation stochastic equilibrium is characterized by (38), (45) and (59) for all $i \in \left[0, \frac{1}{2}\right)$ and by (33), (37), (46) and (60) for all $i \in \left[\frac{1}{2}, 1\right]$, while under regime B by (32), (39), (45) and (59) for all $i \in \left[0, \frac{1}{2}\right)$ and by (33), (37), (46) and (60) for all $i \in \left[\frac{1}{2}, 1\right]$.

It remains to determine to determine the optimal monetary policy.

4 Optimal monetary policy problems

As anticipated in the introduction, the main objective of this paper is to compare in terms of welfare costs and benefits of a monetary union in a fully new-keynesian micro-founded model. For this purpose we consider two policy regimes. Under regime A, while there is a common currency in area F, countries in area H retain their own central banks; by contrast under regime B, there are two monetary unions, one in the area F and the other in the area H. Independently of the policy regimes we assume that all monetary authorities (the central banks of the monetary unions and those of the small open economies) are benevolent, take as given other policy makers' choices and can commit credibly to past and future promises[15]. These hypotheses allow to find the Nash equilibrium policies by using the linear quadratic approach pioneered by Benigno and Woodford (2005) and Benigno and Woodford (2006). Thanks to the optimal policies, it is possible to quantify the difference in welfare for the households of area H across the two policy regimes and to identify which regime is preferable depending on the parameters of the model.

The linear quadratic approach is implemented as follows[16]. First the non-linear optimal policy problems are specified. Second, the zero inflation deterministic steady state of these problems is determined. Then, a purely quadratic approximation to the objectives for both the small open economy and the monetary unions authorities employing the second order approximation of the structural equations are retrieved. Finally the optimal policies can be found by maximizing these quadratic approximations subject to the equilibrium conditions approximated to the first order.

4.1 The deterministic steady state

The steady state level of output is determined by a constant and generic labour subsidy τ. We assume τ equal across countries and across regimes. As shown in the appendix, under these assumptions, for any τ there exists a symmetric deterministic steady state

[15]In other words policies are supposed optimal from the timeless perspective.

[16]For more specific details on the non-linear optimal policy problem, on the zero inflation steady state and on the quadratic approximation, see the appendix.

at which zero inflation is a Nash equilibrium policy for all policy makers in areas H and F under both regimes A and B. Thus, the equilibrium equations and the objectives of the policy makers are approximated around the following steady state:

$$Y = (1-\tilde{\tau})^{-\frac{1}{\sigma+\varphi}} \tag{61}$$

$$C = Y \tag{62}$$

$$F = K = \frac{YC^{-\sigma}}{1-\theta} = \frac{Y^{\varphi+1}(1-\tilde{\tau})}{1-\theta} \tag{63}$$

$$\Pi_H = \Pi_F = 1 \qquad Z = 1 \tag{64}$$

where

$$\tilde{\tau} \equiv 1-(1-\tau)(1+\mu)\frac{\varepsilon}{\varepsilon-1}$$

Allowing for different level of labor subsidies enable us to emphasize two special cases of interest. In particular, as clarified below it is possible to identify the steady state levels of domestic output considered efficient from the viewpoint of both the small open economy authorities and the central banks of the monetary union. Given their different incentives, at the steady state these policy makers consider efficient two different levels of domestic output. As a consequence, they have different "perceptions" of the steady state distortion. As it will be made clear in the next sections, this same divergence is crucial in explaining the different monetary policies over the business cycle.

In the case of the small open economy i, the efficient level of steady state output can be retrieved by maximizing:

$$\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{C_t^{i\,1-\sigma}}{1-\sigma}-\frac{1}{\varphi+1}\left(\frac{Y_t^i}{A_t^i}\right)^{\varphi+1}\right]$$

with respect to C_t^i and Y_t^i, subject to (42) where $P_{i,t}/P_{C^i,t}$ are determined consistently with (26), while $C^*_{H,t}$, $C^*_{F,t}$, $\mathcal{C}_{H,t}$ and $\mathcal{C}_{F,t}$ are taken as given. According to the first order conditions at the symmetric deterministic steady state:

$$Y_s = \delta_s^{\frac{-1}{\sigma+\varphi}} \tag{65}$$

where, as above, $\delta_s \equiv \gamma_s\eta\sigma + \alpha_s(1-\eta\sigma)$ which is always greater than 1 as long as $\sigma\eta > 1$. The optimal labour subsidy that allows to implement this allocation is given by:

$$\tilde{\tau}_s = 1-\delta_s \tag{66}$$

Thus, the small open policy makers would not like to reach the Pareto efficient steady state at which the monopolistic distortions are exactly eliminated[17]. They would rather prefer a lower level of steady state production. This is because financial markets are complete and consumption is highly correlated across regions. So domestic utility rises

[17] The Pareto efficient allocation corresponds to $Y = 1$ which can be achieved by setting $\tilde{\tau} = 0$.

if domestic production falls (relative to other countries' production) and the terms of trade improve[18]. Indeed even if a terms of trade improvement causes consumption to drop, its contraction is more than compensated in terms of welfare by the corresponding increase in leisure. In other words, by manipulating the terms of trade in their favor small open economy policy makers (as those of the monetary union) attempt to externalize labour effort to other countries' workers.

Notice that in the case of the small open economy the incentive to outsource production is stronger the higher is δ_s which depends positively on η, the elasticity of substitution between home and foreign goods, σ, the inverse of the intertemporal elasticity of substitution of consumption and $1-\alpha_s$ the degree of openness of the small country. Indeed the higher are η and σ, the more home households are inclined to substitute consumption of the domestic goods with that of foreign goods (i.e. the higher is the switching effect), and then the less the overall consumption falls because of the reduction in the domestic production.

In the case of the policy maker of the monetary union, the desired steady state output can be determined by maximizing:

$$\sum_{t=0}^{\infty}\beta^t E_0\left[\int_0^{\frac{1}{2}}\left(\frac{{C_t^i}^{1-\sigma}}{1-\sigma}-\frac{1}{\varphi+1}\left(\frac{Y_t^i}{A_t^i}\right)^{\varphi+1}\right)di\right] \tag{67}$$

with respect to C_t^i and Y_t^i for all $i \in [0,1]$ subject to:

$$\frac{P_{i,t}}{P_{C^i,t}}=\frac{(1-\tilde{\tau})}{A_t^{i\varphi+1}}\frac{Y_t^{i\varphi}}{C_t^{i-\sigma}} \tag{68}$$

for all $i \in [\frac{1}{2},1]$, (42) and (44) and where $P_{i,t}/P_{C^i,t}$, $\mathcal{C}_{H,t}$ and $\mathcal{C}_{F,t}$ are determined according to (26), (27) and (43)[19]. F rom the first order conditions of this problem it follows that at the symmetric steady state:

$$Y_b=\left[1-\frac{(1-\delta_b)(\sigma+\varphi)}{(\delta_b\varphi+\gamma_b\sigma)}\right]^{\frac{-1}{\sigma+\varphi}} \tag{69}$$

This allocation can be achieved by setting the labour subsidy:

$$\tilde{\tau}_b=\frac{(1-\delta_b)(\sigma+\varphi)}{(\delta_b\varphi+\gamma_b\sigma)} \tag{70}$$

where $\delta_b \equiv \gamma_b\eta\sigma+\alpha_b(1-\eta\sigma)$ which is always greater than 1 as long as $\sigma\eta>1$. According to these conditions, even in the case of the big economy, the policy makers seek to improve the terms of trade by reducing domestic production with respect to what would be Pareto efficient. However by the comparing (69) and (65), it can be shown that under reasonable calibrations:

$$1>Y_b>Y_s \tag{71}$$

[18]And in fact the optimal labour subsidy is set equal to $1-\delta_s$, a parameter related with the average elasticity of the domestic goods demand with respect to the terms of trade of the small open economy. As made clear by (45) and (59), two are the relevant terms of trade from the small open economy point of view: those of the small open economy and areas H and F.

[19]Implicitly (68) states that the policy maker of area H takes as given the strategy $\tilde{\tau}$ chosen by a symmetric policy maker in area F.

Thus, at the symmetric steady state, the policy maker of the monetary union would choose a level of domestic output higher than that considered efficient by the small open economy authorities. The reasons for this outcome are threefold. First of all, bigger countries are less open. Then the incentive of their policy makers to improve the terms of trade is weaker. Secondly, big economy authorities realize to hold monopoly power only on the terms of trade across areas[20] and they internalize the external effects produced within the monetary union. Finally they take into account the impact of their policies on the foreign economy. In particular they are aware that a terms of trade improvement causes an increase in foreign production thanks to the boost in foreign good demand[21]. So they recognize that a lower labor tax rate (lower than that set by the small open economy policy maker which take as given foreign output) allows to reach the same desired level of domestic/foreign output ratio. All these motives contribute to weaken the desire of influencing their terms of trade.

Summing up, the difference in size between small and big countries affects the incentives of their policy makers and thus the desired steady state level of domestic output. Specifically in the case of the monetary union this level is closer to Pareto efficiency than in the case of the small open economy. As explained in the next sections, these different "perceptions" in the steady state distortion are key even for optimal policy decisions over the business cycle.

4.2 The case of a closed economy

In this section we step behind doing a small digression to explain how and why the steady state distortion influences optimal monetary policy decisions in a closed economy[22]. To this end consider the approximation of objective of the small open economy policy maker[23] in the limiting case of $\alpha_s = \alpha_b = 1$:

$$-\frac{1}{2}Y^{\varphi+1}\sum_{t=0}^{\infty}\beta^t E_0\left[\varpi_{c,1}\pi_t^2 + \varpi_{c,2}(\tilde{y}_t^c)^2 - 2\varpi_{c,3}\tilde{c}_t^c\tilde{y}_t^c + \varpi_{c,4}(\tilde{c}_t^c)^2\right] + t.i.p. \tag{72}$$

with:

$$\begin{aligned}
\varpi_{c,1} &\equiv [1-\zeta_c(\varphi+1)]\,\frac{\varepsilon}{\lambda} \\
\varpi_{c,2} &\equiv [1-\zeta_c(\varphi+1)]\,\varphi \\
\varpi_{c,3} &\equiv \zeta_c\sigma \\
\varpi_{c,4} &\equiv \left[(1-\tilde{\tau})(\sigma-1)+\zeta_c\sigma^2+(1-\zeta_c\varphi)\right] \\
\zeta_c &\equiv \frac{\tilde{\tau}}{\varphi+\sigma}
\end{aligned}$$

[20] This explains the dependence of $\tilde{\tau}_b$ on $1-\delta_b$, a parameter that governs the elasticity of aggregate demand for the domestic goods to the terms of trade across areas.

[21] ...at least under flexible prices which is the relevant case for the steady state analysis.

[22] On this topic the seminal contributions have been those of Benigno and Woodford (2005) and Benigno and Woodford (2006).

[23] See the appendix.

and where t.i.p. stands for terms independent of policy. (72) expresses the utility losses (approximated up to second order) as function of inflation and the welfare-relevant consumption and output gap. In fact $\tilde{x}_t^c \equiv \hat{x}_t - \hat{x}_t^c$ denotes the deviations of x_t from the target of monetary authority when the economy is closed. This target can be retrieved by equations (123) -(123) under the assumption that $\alpha_s = \alpha_b = 1$ and satisfies the next conditions:

$$[1 - \zeta_c(\varphi + 1)] \left(\widehat{mrs}_t^c - \widehat{mrt}_t^c\right) = \zeta_c(\varphi + 1)\hat{\mu}_t \tag{73}$$

$$\hat{y}_t^c = \hat{c}_t^c \tag{74}$$

with

$$\widehat{mrs}_t^c - \widehat{mrt}_t^c = \varphi \hat{y}_t^c + \sigma \hat{c}_t^c - (\varphi + 1)\hat{a}_t$$

$\widehat{mrs}_t^c$ and $\widehat{mrt}_t^c$ represent the log-deviation of the marginal rate of substitution and of the marginal rate of transformation between consumption and output.

Thanks to conditions (73) and (74) we can achieve the following conclusions about the goals of the monetary authority:

1. When the steady state is efficient (i.e. $\tilde{\tau} = 0$ and $\zeta_c = 0$), we go back to the standard result of closed economy literature[24] for which the central bank would like to close the gap between $\widehat{mrs}_t^c$ and $\widehat{mrt}_t^c$ because in this way it reaches the first best allocation.

2. Under technological shocks, the monetary authority still wishes to close that gap even if the steady state is inefficient (i.e. $\zeta_c \neq 0$). Indeed in that case she cannot influence the distortions due to the steady state inefficiency. Therefore she seeks to replicate the fluctuations of the first best allocation. In other words, under technological shocks, the target of the monetary authority is not affected by the steady state inefficiency because the flexible price allocation is constrain efficient.

3. Conversely under mark up shocks the monetary authority is willing to bear a difference between $\widehat{mrs}_t^c$ and $\widehat{mrt}_t^c$, where this difference depends on the mark up shocks themselves and on the size of the steady state distortion. In particular if the steady state output is inefficiently high (i.e. $\zeta_c > 0$), the monetary authority wants output to negatively comove with these shocks. As a result, the central bank would focus more on inflation than output stabilization (more than what would do if the steady state were efficient) given that a positive mark up shock tends to reduce output. Viceversa[25] an inefficient low level of steady state output (i.e. $\zeta_c < 0$) would imply a monetary policy that weighs more output than inflation stabilization.

At a first glance the third result is quite puzzling: mark up shocks generate inefficient fluctuations in consumption and output. Intuitively we could expect that then the central bank would like to completely stabilize output and consumption (as in fact it is willing to do when the steady state is efficient). Instead, it wants output and consumption to react to these shocks. Why? The underling reason can be understood

[24]See among others Galí (2008) and Woodford (2003).

[25]under the parametric restriction: $\tilde{\tau} < \frac{\sigma+\varphi}{\varphi+1}$.

by considering condition (17) (in its closed economy case counterpart), when prices are flexible and there are no shocks to technology:

$$E\left\{\frac{W_t}{P_t}\right\} = E\left\{Y_t^{\varphi+\sigma}\right\} E\left\{(1+\mu_t)\right\} + Cov\left\{Y_t^{\varphi+\sigma}(1+\mu_t)\right\} \tag{75}$$

According to (75), the lower is the covariance between mark up shocks and output, the higher is the average per-period output for a given level of per-period real wage. Indeed, if output fluctuates more in response to mark up shocks - which corresponds to a decrease in the covariance between output and the mark up shocks themselves - consumers have to rise on average their labour effort in order to get the same real wage. Then if output is on average inefficiently low because of the steady state distortion, allowing for negative comovements between output and mark up shocks have beneficial effects because it shifts downward the average supply curve engendering an efficient increase in the average level of per-period output.

4.3 The case of the small open economy

As shown in the appendix, the objective of the small open economy policy maker of country i in area H can be approximated up to the second order as:

$$-\frac{1}{2}Y^{\varphi+1}\sum_{t=0}^{\infty}\beta^t E_0\left[\varpi_{s,1}(\pi_{i,t})^2 + \varpi_{s,2}(\tilde{y}_t^{i,s})^2 - 2\varpi_{s,3}(\tilde{c}_t^{i,s}\tilde{y}_t^{i,s}) + \varpi_{s,4}(\tilde{c}_t^{i,s})^2\right] + t.i.p. \tag{76}$$

where

$$\begin{aligned}
\varpi_{s,1} &\equiv [1-\zeta_s(\varphi+1)]\frac{\varepsilon}{\lambda} \\
\varpi_{s,2} &\equiv [1-\zeta_s(\varphi+1)]\varphi \\
\varpi_{s,3} &\equiv \zeta_s\gamma_s\sigma \\
\varpi_{s,4} &\equiv (1-\tilde{\tau})(\sigma-1) + \zeta_s\gamma_s^2\sigma^2 + (1-\zeta_s\varphi)(\delta_s+\omega_1) \\
\zeta_s &\equiv \frac{\delta_s-(1-\tilde{\tau})}{\delta_s\varphi+\gamma_s\sigma}
\end{aligned}$$

ω_1 is properly defined in the appendix and $\tilde{x}_t^{i,s} \equiv \hat{x}_t^i - \hat{x}_t^{i,s}$. $\hat{x}_t^{i,s}$ indicates the target of the small open economy monetary authority which is determined by (123)[26]. Notice that in the case of the small open economy the $t.i.p.$, the terms independent of monetary policy, include the aggregate variables of both areas H and F.

The welfare approximation in (76) contains the output gap, the consumption gap and inflation like in a closed economy. What is crucially different are the weights attached to these variables and the target that the authority would like to implement. This divergence with respect to the closed economy case is rationalized again by the

[26]This target can be interpreted as the constrained efficient allocation from the small open economy viewpoint, namely the allocation that would be chosen by a small open economy policy maker that has as objective the maximization of (76) subject *exclusively* to constraint (45).

desire of open economy policy makers to manipulate the terms of trade in their favour. In fact, on the one hand, this incentive works even over the business cycle and gives reason, for instance, for the higher weight attributed to consumption gap volatility: policy makers realize that fluctuations in consumption are associated with fluctuations in the terms of trade. On the other hand, this same incentive explains why from the small open economy policy makers viewpoint, the steady state is efficient as long as $Y = Y_s$ - which implies $\tilde{\tau} = \tilde{\tau}_s$ and thus $\zeta_s = 0$ - and not when $Y = Y_c = 1$ as in a closed economy. This has clear consequences for the weights in (76) (given that ζ_s depends critically on the difference between $\tilde{\tau}$ and $\tilde{\tau}_s$) and for channels through which openness modifies the conduct of small open economy central banks.

To better investigate these channels consider the subsequent conditions:

$$[1-\zeta_s(\varphi+1)](\widehat{mrs}^s_{H,t} - \widehat{mrt}^s_{H,t}) = \zeta_s(\varphi+1)\hat{\mu}_{H,t} + \kappa_s \hat{s}^s_{HF,t} \tag{77}$$

$$\hat{y}^s_{H,t} = \hat{c}^s_{H,t} + \frac{(1-\delta_b)(2\alpha_b-1)}{\sigma}\hat{s}^s_{HF,t} \tag{78}$$

where

$$\widehat{mrs}^s_{H,t} - \widehat{mrt}^s_{H,t} = \varphi\hat{y}^s_{H,t} + \sigma\hat{c}^s_{H,t} - (\varphi+1)\hat{a}_{H,t} + \sigma(1-\gamma_b)(\hat{c}_{F,t} - \hat{c}^s_{H,t})$$

$$\hat{s}^s_{HF,t} = -\frac{\sigma}{2\alpha_b - 1}(\hat{c}_{F,t} - \hat{c}^s_{H,t})$$

$$\kappa_s \equiv \frac{(2\alpha_b-1)}{\sigma\delta_s}\left[(1-\zeta_s\varphi)(\sigma(1-\gamma_b)\delta_s - \omega_2) + \zeta_s\gamma_s((1-\delta_b) - \sigma\eta(1-\gamma_b))\right]$$

Conditions (77) and (78) are recovered by properly rearranging and integrating (122), the equations determining the target[27] of the small open economy authority. The comparison with their akin of the closed economy, namely (73) and (74), allows to stress the following findings:

1 As indicated by the terms $\kappa_s\hat{s}_{HF,t}$, even when the steady state is efficient from the small open economy viewpoint (i.e. $\zeta_s = 0$), in general, the target does not coincide with the flexible price allocation. This is because small country policy makers try to manipulate their terms of trade even over the business cycle[28].

2 The target reacts to domestic mark up shocks if and only if there is a steady state inefficiency from the small open economy perspective (i.e. $\zeta_s \neq 0$).

This second result confirms our intuition that from the small country viewpoint the welfare relevant distortion is determined by the difference between $\tilde{\tau}$, the actual steady state labour subsidy, and $\tilde{\tau}_s = 1-\delta_s$, its desired level (which in turn governs the value of ζ_s). As long as $\tilde{\tau} \neq \tilde{\tau}_s$, the small open policy maker considers inefficient the average per-period wedge between the marginal rate of substitution between consumption and labour and its marginal rate of transformation. In particular, under the baseline calibration the per-period output is regarded as inefficiently high (i.e $\tilde{\tau} > 1-\delta_s$ and $\zeta_s > 0$). Indeed, once the aggregate world variables are taken as given, at the margin

[27]under the assumption that the target is *implemented* which ensures that $\int_0^{\frac{1}{2}} \hat{x}^{i,s}_t di = \hat{x}_{H,t}$.

[28]Notice that not surprisingly if $\alpha_b = 1$ then $\kappa_b = 0$. In fact if the area is closed, then on average the effects due to the terms of trade externality disappear.

an increase in leisure rises utility by more than an increase in consumption. This generates a motive for the central banks of the small open economies to seek to squeeze the average per-period output and to modify their inflation output trade-off. In fact, by focusing more on output than on inflation stabilization[29] in response to mark up shocks, these authorities can induce domestic households to work more. In this way per-period domestic output, which is perceived as too high, can fall.

The *timelessly* optimal monetary policy can be retrieved by maximizing (76) with respect to $\tilde{y}_t^{i,s}$, $\tilde{c}_t^{i,s}$ and $\pi_{i,t}$ subject to the following sequence of constraints:

$$\tilde{y}_t^i = \delta_s \tilde{c}_t^i \tag{79}$$

$$\pi_{i,t} = \lambda \left[\varphi \tilde{y}_t^{i,s} + \sigma \gamma_s \tilde{c}_t^{i,s} \right] + \lambda \upsilon_t^{i,s} + \beta E_t \{ \pi_{i,t+1} \} \tag{80}$$

for all t where:

$$\upsilon_t^{i,s} = \varphi \hat{y}_t^{i,s} + \sigma \gamma_s \hat{c}_t^{i,s} + \sigma \left(\gamma_b - \gamma_s \right) \hat{c}_{H,t} + \sigma \left(1 - \gamma_b \right) \hat{c}_{F,t} - (1+\varphi) \hat{a}_t^i + \hat{\mu}_t^i.$$

4.4 The case of the monetary union

As shown in the appendix, if there is a monetary union in area H, the objective of the monetary policy maker can be approximated in a purely quadratic way as:

$$-\frac{1}{2} Y^{\varphi+1} \sum_{t=0}^{\infty} \beta^t E_0 \Big[\varpi_{b,1} (\pi_{H,t})^2 + \varpi_{b,2} (\tilde{y}_{H,t}^b)^2 + 2\varpi_{b,3} \tilde{c}_{H,t}^b \tilde{y}_{H,t}^b + 2\varpi_{b,4} \tilde{c}_{F,t}^b \tilde{y}_{H,t}^b + \varpi_{b,5} (\tilde{y}_{F,t}^b)^2$$
$$+ 2\varpi_{b,6} \tilde{c}_{F,t}^b \tilde{y}_{F,t}^b + 2\varpi_{b,7} \tilde{c}_{H,t}^b \tilde{y}_{F,t}^b + \varpi_{b,8} (\tilde{c}_{F,t}^b)^2 + \varpi_{b,9} (\tilde{c}_{H,t}^b)^2 + 2\varpi_{b,10} \tilde{c}_{H,t}^b \tilde{c}_{F,t}^b \Big] + t.i.p. \tag{81}$$

where

$$\begin{aligned}
\varpi_{b,1} &\equiv [1 - \zeta_b(\varphi+1)] \frac{\varepsilon}{\lambda} \\
\varpi_{b,2} &\equiv [1 - \zeta_b(\varphi+1)] \varphi \\
\varpi_{b,3} &\equiv -\zeta_b \sigma \gamma_b \\
\varpi_{b,4} &\equiv -\zeta_b \sigma \left(1 - \gamma_b\right) \\
\varpi_{b,5} &\equiv -(\xi - \zeta_b)(\varphi+1)\varphi \\
\varpi_{b,6} &\equiv -(\xi - \zeta_b) \sigma \gamma_b \\
\varpi_{b,7} &\equiv (\xi - \zeta_b) \sigma \left(1 - \gamma_b\right)
\end{aligned}$$

[29] as long as $\frac{\delta_s \varphi + \gamma_s \sigma}{\varphi+1} > \delta_s - (1 - \tilde{\tau})$.

$$
\begin{aligned}
&\varpi_{b,8} \equiv -(1-\zeta_b(\varphi+1))(1-\delta_b) - (\xi-\zeta_b)\varphi\delta_b + \zeta_b\sigma^2(1-\gamma_b)^2 + (1-\zeta_b\varphi)\eta\sigma^2\gamma_b(1-\gamma_b)\\
&\varpi_{b,9} \equiv (\sigma-1)(1-\tilde{\tau}) + (1-\zeta_b(\varphi+1))\delta_b - (\xi-\zeta_b)\varphi(1-\delta_b) - (1-\zeta_b\varphi)\eta\sigma^2\gamma_b(1-\gamma_b)\\
&+\zeta_b\sigma^2\gamma_b^2 + (\xi-\zeta_b)\sigma^2(1-\gamma_b)^2\\
&\varpi_{b,10} \equiv (1-\zeta_b\varphi)\eta\sigma^2\gamma_b(1-\gamma_b) + \zeta_b\sigma^2\gamma_b(1-\gamma_b) + (\xi-\zeta_b)\sigma^2(1-\gamma_b)\gamma_b\\
&\zeta_b \equiv \frac{1}{2}\frac{\tilde{\tau}}{\sigma+\varphi} - \frac{\delta_b - 1 + (1/2)\tilde{\tau}}{(1-2\gamma_b)\sigma + (1-2\delta_b)\varphi}\\
&\xi \equiv \frac{\tilde{\tau}}{\sigma+\varphi}
\end{aligned}
$$

and $\tilde{x}_t^b \equiv \hat{x}_t - \hat{x}_t^b$. $\hat{x}_t^b$ denotes the target of the monetary union central bank which can be determined from (47)-(48) and (139)-(143)[30]. In addition *t.i.p.*, the terms independent of policy, include the state contingent path of $\pi_{F,t}$ decided by the policy maker of the monetary union in area F and the differentials between country specific and average union variables[31].

To grasp some insights the incentives driving the optimal monetary policy of the monetary union consider the following "targeting" condition:

$$
\begin{aligned}
&[1-\zeta_b(\varphi+1)](\widehat{mrs}_{H,t}^b - \widehat{mrt}_{H,t}^b) - (\xi-\zeta_b)(\varphi+1)(\widehat{mrs}_{F,t}^b - \widehat{mrt}_{F,t}^b) =\\
&\zeta_b(\varphi+1)\hat{\mu}_{H,t} + (\xi-\zeta_b)(\varphi+1)\hat{\mu}_{F,t} + \kappa_b\hat{s}_{HF,t}^b
\end{aligned} \tag{82}
$$

where:

$$
\begin{aligned}
&\widehat{mrs}_{H,t}^b - \widehat{mrt}_{H,t}^b = \varphi\hat{y}_{H,t}^b + \sigma\hat{c}_{H,t}^b - (\varphi+1)\hat{a}_{H,t} + \sigma(1-\gamma_b)(\hat{c}_{F,t}^b - \hat{c}_{H,t}^b)\\
&\widehat{mrs}_{F,t}^b - \widehat{mrt}_{F,t}^b = \varphi\hat{y}_{F,t}^b + \sigma\hat{c}_{F,t}^b - (\varphi+1)\hat{a}_{F,t} + \sigma(1-\gamma_b)(\hat{c}_{F,t}^b - \hat{c}_{F,t}^b)\\
&\kappa_b \equiv (2\alpha_b - 1)\sigma^{-1}(1 - \zeta_b\left[(\varphi+\sigma))((1-\delta_b) - \sigma(1-\gamma_b)) - (\xi-\zeta_b)(\varphi+\sigma)(\delta_b - \sigma\gamma_b)\right]
\end{aligned}
$$

Thus $\widehat{mrs}_{H,t}^b$ and $\widehat{mrt}_{H,t}^b$ (as $\widehat{mrs}_{F,t}^b$ and $\widehat{mrt}_{F,t}^b$) stand for the average marginal rate of substitution and transformation between consumption and output in area H (in area F). Like its analogue (77), condition (82) is derived from the equations that determine the target of the monetary union policy makers, namely (47)-(48) and (139)-(143).

Condition (82) leads to the next conclusions:

1 Differently from the case of the small open economy, the common central bank in area H wants to stabilize a weighted average between the gap between $\widehat{mrs}_{H,t}^b$ and $\widehat{mrt}_{H,t}^b$ in area H and this same gap in area F. In fact, the monetary authority of the currency area takes into account how its decisions affect the demand and the supply of foreign goods and the related feedback effects on its own economy.

2 However the central bank of the monetary union attaches different weights to home and foreign variables.

[30]namely the constraint efficient allocation from the perspective of the policy maker of area H. This allocation corresponds to the allocation chosen by a policy maker that maximizes (81) subject *exclusively* to constraints (47) and (48). See the appendix.

[31]Indeed, by choosing the average union inflation, the common central bank can influence only the average union performance. However, these terms have to be taken into account for the welfare evaluation.

3 Moreover that authority balances the need to stabilize the gaps between the marginal rates of substitution and transformation with a twofold desire: on the one hand, as indicated by the term κ_b, it wants to manipulate the terms of trade in its favor over the business cycles; on the other hand, in the presence of domestic mark up shocks, it seeks to influence the average per-period levels of both domestic and foreign output.

These features are direct consequence of the desire to improve the terms of trade already emphasized above. This incentive stems from a free riding problem. Under the assumption of complete financial markets, consumption is highly correlated across countries. Given that labour effort lowers utility, this risk sharing in consumption generates a conflict on where to produce output. Indeed, the higher is the substitutability between home and foreign bundles, the more countries wish to outsource production and squeeze domestic output *relatively* to foreign output. In this manner they can reduce labour effort without decreasing too much consumption.

This mechanism gives reason of why the size of the economy shapes optimal monetary policy decisions. In the limiting case in which the economy is small, the only way monetary policy can lessen domestic/foreign relative output ratio is through a contraction of domestic production. In fact, the economic performance of a single small open economy is irrelevant for aggregate output behavior. Conversely when the economy is big, policy makers realize that their decisions can diminish the domestic/foreign per-period output ratio through either a reduction of the domestic output or an increase of the foreign one. As consequence, and as highlighted by condition (82), the target of the monetary union central bank depends on foreign mark up shocks and attaches asymmetric weights to domestic and foreign economic performances. In particular under the baseline calibration, per-period foreign production is considered inefficiently low even more than domestic one[32]. In other words, policy makers of the big economies want households abroad to work more than consumers at home.

The optimal monetary policy problem of the common central bank in area H can be formulated as maximizing (81) with respect to $\tilde{y}^b_{H,t}$, $\tilde{y}^b_{F,t}$, $\tilde{c}^b_{H,t}$, $\tilde{c}^b_{F,t}$ and $\pi_{H,t}$ subject to the following sequence of constraints:

$$\tilde{y}^b_{H,t} = \tilde{c}^b_{H,t} + (1-\delta_b)(\tilde{c}^b_{F,t} - \tilde{c}^b_{H,t})$$

$$\tilde{y}^b_{F,t} = \tilde{c}^b_{F,t} + (1-\delta_b)(\tilde{c}^b_{H,t} - \tilde{c}^b_{F,t})$$

$$\pi_{H,t} = \lambda\left[\varphi\tilde{y}^b_{H,t} + \sigma\left(\tilde{c}^b_{H,t} + (1-\gamma_b)\left(\tilde{c}^b_{F,t} - \tilde{c}^b_{H,t}\right)\right)\right] + \lambda\upsilon^b_{H,t} + \beta E_t\{\pi_{H,t+1}\}$$

$$\pi_{F,t} = \lambda\left[\varphi\tilde{y}^b_{F,t} + \sigma\left(\tilde{c}^b_{F,t} + (1-\gamma_b)\left(\tilde{c}^b_{H,t} - \tilde{c}^b_{F,t}\right)\right)\right] + \lambda\upsilon^b_{F,t} + \beta E_t\{\pi_{F,t+1}\}$$

for all t where

$$\tilde{\upsilon}^b_t = \varphi\hat{y}^b_{H,t} + \sigma\hat{c}^b_{H,t} + \sigma\left(1-\gamma_b\right)\left(\hat{c}^b_{F,t} - \hat{c}^b_{H,t}\right) - (1+\varphi)\hat{a}_{H,t} + \hat{\mu}_{H,t}$$
$$\tilde{\upsilon}^b_{F,t} = \varphi\hat{y}^b_{F,t} + \sigma\hat{c}^b_{F,t} + \sigma\left(1-\gamma_b\right)\left(\hat{c}^b_{H,t} - \hat{c}^b_{F,t}\right) - (1+\varphi)\hat{a}_{F,t} + \hat{\mu}_{F,t}$$

[32] Notice that in order to reach the efficient level of foreign output, the labour subsidy should be set equal to $\tilde{\tau} = -\frac{(1-\delta_b)(\sigma+\varphi)}{((1-\delta_b)\varphi+(1-\gamma_b)\sigma)}$, a level such that foreign labour is over-subsidized!

The solution to this problem allows to determine the average inflation in area H and all the other area variables, given a state contingent path of the average inflation in area F. A symmetric problem can be stated for the foreign area. Notice that once the average union variables are determined, the region specific variables can be recovered directly from the equilibrium conditions namely (32), (33), (45), (46), (59) and (60). Moreover, under this formulation, the optimal monetary policy problem is independent of whether there is either monetary autonomy among countries or a monetary union in the other area.

5 Optimal monetary policies

The solution to the optimal policy problems of both the small open policy maker and the central bank of the monetary union enable us to simulate the impulse responses to a one percent decrease in home and foreign mark ups under regimes A and B. These impulses responses are plotted in figures 1-2. The baseline calibration is listed in the appendix and is in line with the literature[33].

5.1 Dynamic Simulation

The impulse responses to a global negative mark up shock can be interpreted as follows.

As shown in figures 1 and 2, under optimal policies, given the fall in their marginal costs, both home and foreign firms cut prices and expand output supply. Workers increase consumption and reduce leisure. Monetary policies have then to trade off between output and inflation stabilization. These patterns are common to both areas and regimes. However, under regime A, consumption and output in area H increase by less than in regime B, while deflation in area H and output in area F increase by more. These differences are explained *exclusively* by the diverging conduct of policy makers under the two policy regimes.

Regime B. Under regime B, when there are two currency unions, impulse responses are symmetric across areas. Under the baseline calibration both domestic and foreign per-period output are perceived as too low. However, the distortion in the foreign production is considered relatively stronger (i.e. the desired steady state output ratio $\frac{Y_H}{Y_F} < 1.$). As consequence under global mark up shocks, monetary union policy makers would like foreign output to fluctuate relatively more. In other words they attempt to generate a positive covariance between mark up shocks and their terms of trade in order to induce foreign consumers to rise their production by more than domestic households. Obviously, in equilibrium none of the policy makers in area H and F reaches her goal. Indeed given symmetry, home and foreign output perfectly commove in such a way that their relative average per-period ratio is always equal to one.

Regime A. Under regime A, the conduct of the monetary policy makers in area H is dissimilar from that in regime B in two respects. On the one hand, under the baseline calibration, the per-period domestic output is too high from the small open

[33] See in particular Galí and Monacelli (2007), Galí and Monacelli (2005) and Pappa (2004).

economy perspective. As a result, when there is monetary independence, they are more focused on output stabilization than the central bank of the monetary union. Indeed, in response to a negative mark up shock, they seek to restrain output expansion and allow for a higher deflation by increasing on average the nominal interest rate by more than what the single central bank of the monetary union does in regime B. In this way they push the economy in the direction of an improvement of the terms of trade in their area. In fact being their economies small, these monetary authorities consider what happens in the world economy as exogenous. Thus, they do not take into account (as the monetary authority of a currency area does) how their joint action affects per-period foreign production. Therefore they do not realize that boosting the negative covariance between foreign production and mark up shocks can be beneficial: it induces foreign workers to produce additional output that can be consumed even by domestic households thanks to the consumption risk sharing.

Given the restrictive monetary policy in area H, the monetary authority of area F restrains monetary policy as well, but not as much as the central banks of area H, allowing for a terms of trade worsening. By doing so, she wants to oppose the restrictive policies of the other area, because she finds an expansion of foreign output beneficial. Nevertheless, she also wants to stabilize domestic price dynamics. Deflation response in area F is similar across regimes, whereas output and consumption are influenced by the restrictive policy of the policy makers in area H.

There is a crucial question that is still left open. When are the consumers of area H better off? In regime A or in regime B? This question is addressed in the next section.

6 Welfare evaluation

The analysis of the previous section reveals that, in the presence of mark up shocks, there are potential welfare benefits from the adoption of a common currency. Moreover, it makes clear which are the sources of these benefits: on the one hand the internalization of the spillover effects generated within area H; on the other hand the gains in monopoly power in controlling the terms of trade across areas. The household welfare based criterion derived in (81) allows to quantify the welfare gains of being in a currency area as average per-period losses expressed as a fraction of the steady state consumption. The results are quite robust: under mark up shocks, even for relatively low levels of the elasticity of substitution between home and foreign bundles, there are welfare benefits of forming a monetary union. In the next sections we analyze how these benefits vary according to the key parameters of the model.

The intertemporal and the intratemporal elasticities of substitution. Both the intratemporal elasticity of substitution between home and foreign bundles, η, and the relative risk adversion coefficient (the inverse of intertemporal elasticity of substitution of consumption), σ, are crucial to determine the size of the welfare gains (or losses) of abandoning monetary autonomy[34]. Indeed they influence directly the effects that

[34]This finding is actually consistent with the literature. See in particular Benigno and Benigno (2003),

movements in the terms of trade produce on the demand of foreign goods. The higher are η and σ[35], the larger is the switching effect from domestic towards foreign goods, the stronger is the increase in foreign production due to a terms of trade improvement and the more domestic production (and leisure) decreases allowing home households to reach a higher level of utility. Summing up, these parameters govern the real effects of the beggar-thy-neighbour policies and therefore the benefits of policy coordination that arise from being in a monetary union. Figure 3 plots how welfare benefits increase in area H relatively to an increase of η and σ. η varies from 1 to 3, while σ varies from 1 to 2.5. Within this range, these gains reach a maximum of 0.3 percentage of the steady state consumption. However, for low levels η the adoption of a common currency brings about welfare losses up to 0.1 percentage of steady state consumption.

The degree of home bias. The welfare benefits of a monetary union are due to two main channels: the internalization of all the external effects produced within the monetary union by the national authorities; the gains of monopoly power (due to the bigger size of the area) on the terms of trade (and thus average output differentials) across areas. A relevant question is which of these channels contributes more to explain the welfare benefits themselves. For this reason, we investigate to what extent these gains depend on the degree of home bias of area H, α_b.

Figure 4 plots the welfare gains of being in the regime B for the consumers of area H relatively to different degree of η (from 1 to 3) and α_b (from 0.6 to 1) and shows the following result. For low degree of η the welfare gains - which are actually losses - are lower in a closed economy (i.e. $\alpha_b = 1$), whereas for high degree of η the converse is true. This finding can be explained as follows. If η and α_b are high, the main sources of welfare gains is due to the elimination of the spillover effects within the union in area H. Indeed if the area is very closed, the welfare benefits due to an increase in control on the terms of trade across area and on average area output differential are not important. However if η and α_b are low the main gain in adopting the same currency is due to the internalization of both the impact of its actions on the foreign area and related feedback effects on the same area H.

In order to better disentangle these two sources of welfare gains, it would be useful to allow for different elasticities of substitution between bundles produced in different regions and in different areas. In this way, in fact, it would be possible to understand how the welfare gains of forming a monetary union vary in response to a variation of a parameter, the elasticity of substitution between bundles produced in different areas, that affects *exclusively* exeternalities generated by the big economy on the terms of trade across areas.

The correlation between region specific shocks. For the purpose of this paper, it is important to check how welfare gains depend on the correlation between region specific shocks. Indeed, this correlation is the key determinant of the costs due to the loss of an

Benigno and Benigno (2006) and Pappa (2004).

[35]The lower is the intertemporal elasticity of consumption, the higher is the incentive to smooth consumption across periods. Thus, when there is a terms of trade improvement, consumers are more inclined to keep the same level of overall consumption, buying more foreign goods or working more to substitute between the present and future consumption.

independent instrument of policy that can suit specific country economic conditions.

Figure 5 plots the welfare gains of the consumers in area H relative to the elasticity of substitution between home and foreign bundles η and to that ς_1. Not surprisingly according to that figure the lower is the correlation between regional shocks the lower are the welfare benefits of adopting a common currency. In fact for small levels of η and ς_1 there are significant welfare losses across policy regimes up to 0.15 of the steady state consumption. However for high level of η independently of the degree of correlation between region specific shocks, the welfare gains of having the same currency are always greater than 0.1 percent of the steady state consumption.

7 Conclusion

This paper has shown that, in the presence of mark up shocks, under plausible calibration there are welfare gains due to the adoption of a common currency. This finding is obtained in a New Keynesian open economy framework in which forming a monetary union entails a meaningful trade-off: on the one hand, because of nominal rigidities, losing monetary independence implies the welfare costs of renouncing to a policy instrument that can stabilize country-specific shocks; on the other hand, delegating the monetary policy to the monetary union's central bank generates welfare gains by improving the conduct of the national authorities. In a world constituted by two economic areas as the one laid out in our basic setup, two are the main sources of this improvement. The first is due to the internalization of the spillover effects produced by autonomous authorities within the monetary union. The second is due by the gain in monopoly power in controlling the terms of trade across areas and the feedback effects of the policy maker decision.

References

Benigno, Gianluca and Pierpaolo Benigno, "Price Stability in Open Economies," *Review of Economic Studies*, 2003, *70* (4), 743–764.

— **and** — , "Designing Targeting Rules for International Monetary Policy Cooperation," *Journal of Monetary Economics*, 2006, *53*, 473–506.

Benigno, Pierpaolo and Michael Woodford, "Inflation Stabilization and Welfare: The Case of a Distorted Steady State," *Journal of the European Economics Association*, 2005, *3*, 1185–1236.

— **and** — , "Linear-Quadratic Approximation of Optimal Policy Problems," *NBER Working Paper*, 2006, *12672.*

Clarida, Richard, Jordi Galí, and Mark Gertler, "A Simple Framework for International Monetary Policy Analysis," *Journal of Monetary Economics*, 2002, *49* (5), 879–904.

Corsetti, Giancarlo, "A Micro-founded Reconsideration of the Theory of the Optimal Currency Areas," 2007. Mimeo.

— **and Paolo Pesenti**, "Welfare and Macroeconomic Interdependence," *Quarterly Journal of Economics*, 2001, *16* (2), 421–446.

Epifani, Paolo and Gino Gancia, "Globalization and the Size of Government," 2005. Universitat Pompeu Fabra, mimeo.

Galí, Jordi, *Monetary Policy, Inflation, and the Business Cycle*, Princeton University Press, 2008.

— **and Tommaso Monacelli**, "Monetary Policy and Exchange Rate Volatility in a Small Open Economy," *Review of Economic Studies*, 2005, *72* (3), 707–734.

— **and** — , "Optimal Monetary and Fiscal Policy in a Currency Union," 2007. Universitat Pompeu Fabra, mimeo.

Mundell, Robert, "A Theory of Optimum Currency Areas," *American Economic Review*, 1961, *51*, 657–675.

Obstfeld, Maurice and Kenneth Rogoff, "Do We Need a New International Monetary Policy Compact?," *NBER Working Paper*, 2000, *7864.*

Pappa, Evi, "Do the ECB and the Fed Really Need to Cooperate? Optimal Monetary Policy in a Two-Country World," *Journal of Monetary Economics*, 2004, *51* (4), 753–779.

Rotemberg, Julio J. and Michael Woodford, "An Optimization-Based Econometric Framework for the Evaluation of Monetary Policy," *NBER Macroeconomic Annual*, 1997.

Woodford, Michael, *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, 2003.

Baseline Calibration

$\sigma^{-1} = 1/2$	Intertemporal elasticity of substitution of the private goods;
$\eta = 2$	Elasticity of substitution between home and foreign private goods;
$\varphi^{-1} = 1/3$	Intertemporal elasticity of substitution of labor;
$\alpha_s = 0.6$	Degree of home bias for the bundle of the region;
$\alpha_b = 0.8$	Degree of home bias for the bundle of the area;
$\varepsilon = 6$	Elasticity of substitution among goods produced in the same region;
$\beta = 0.99$	Preferences discount factor;
$SDva = 0.0071$	Standard deviation of the white noise of the aggregate technological shocks;
$SDv\mu = 0.03$	Standard deviation of the white noise of the aggregate markup shocks;
$ac = 0.9$	Autocorrelation of shocks;
$\tilde{\tau} = -1$	Steady state labour subsidy;
$\varsigma = 2$	Ratio between the variance of the idiosyncratic innovation and the variance of the aggregate innovation.

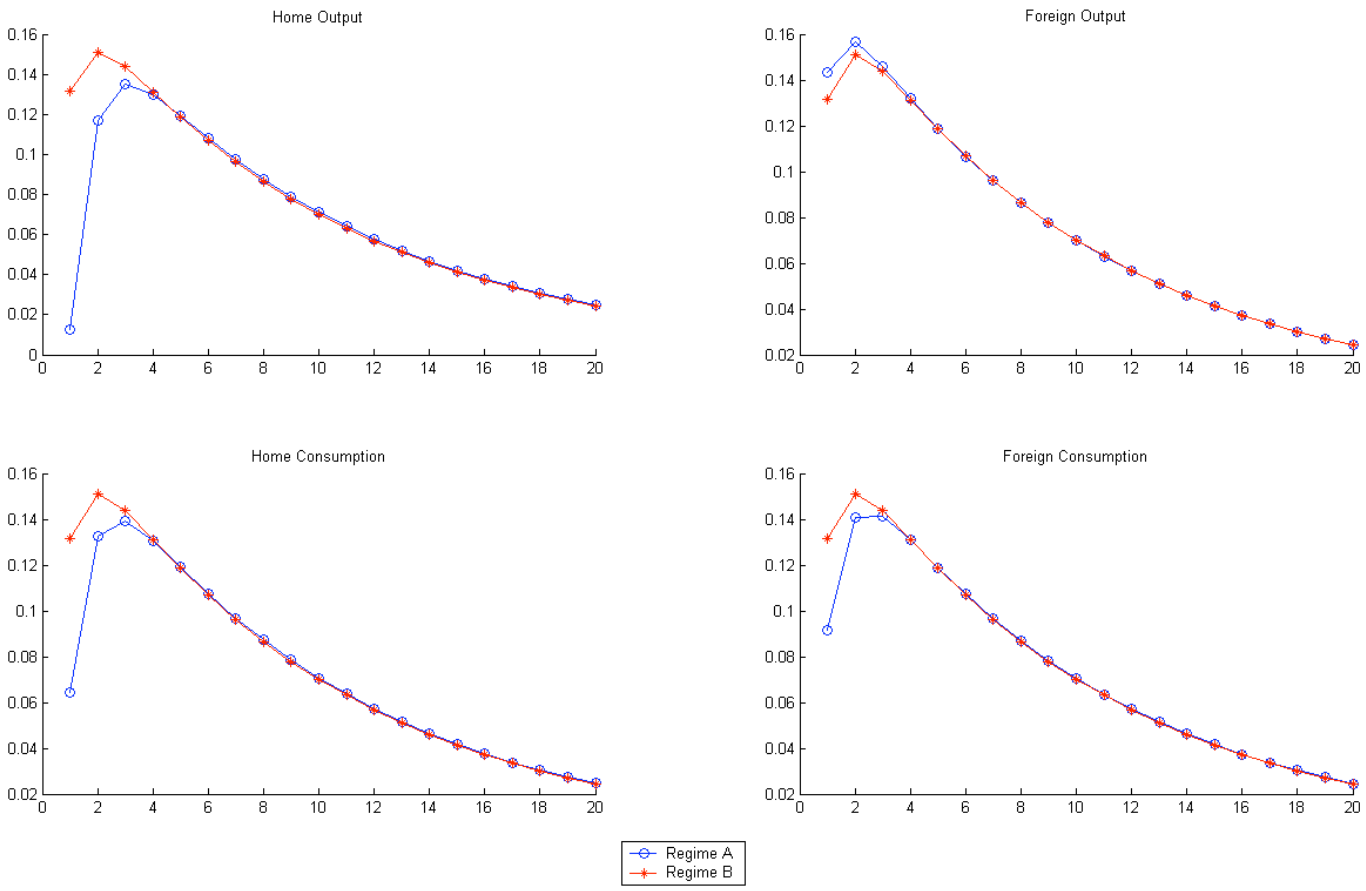

Figure 1: Impulse responses to a negative aggregate markup shock.

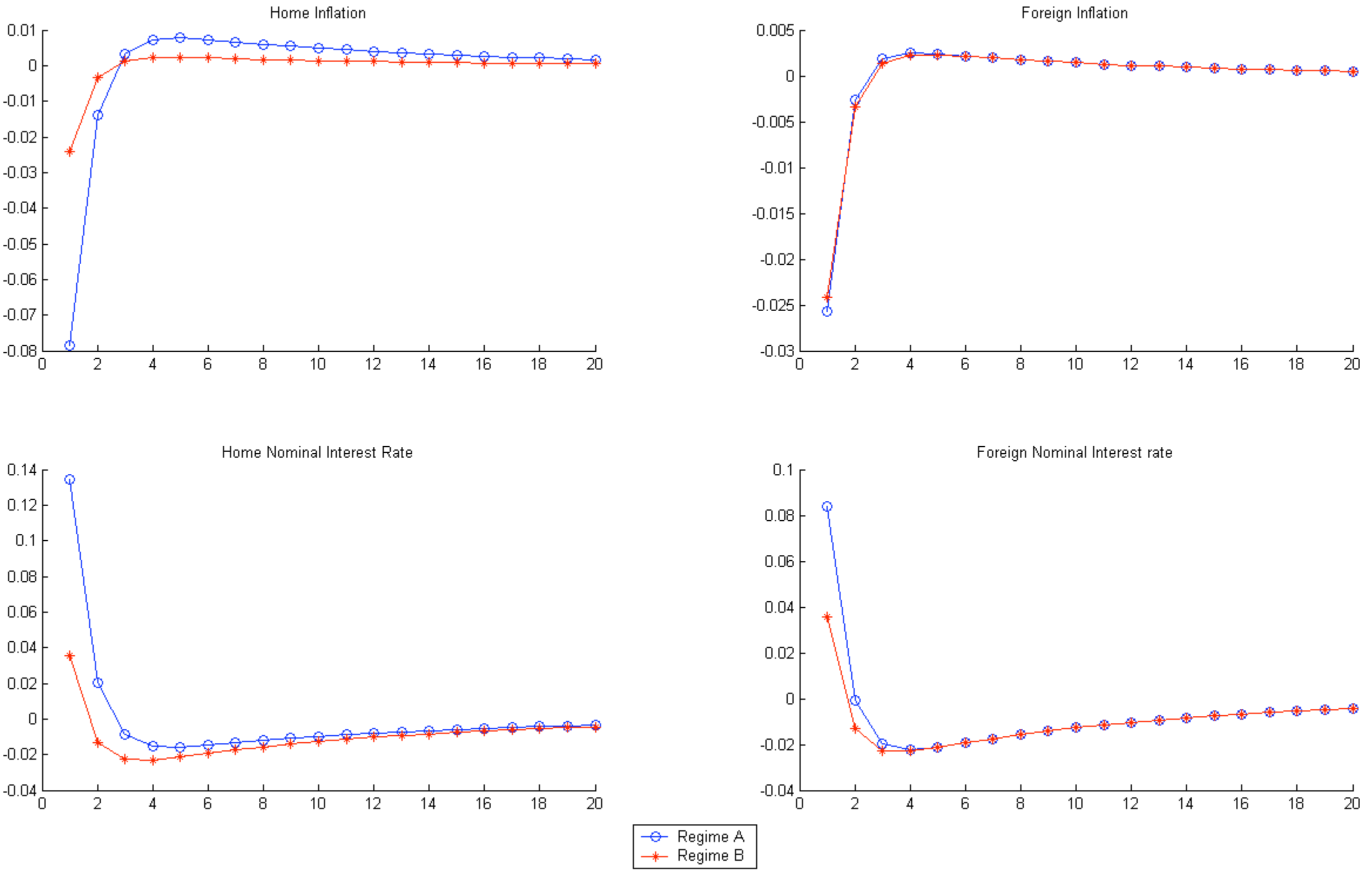

Figure 2: Impulse responses to a negative aggregate markup shock.

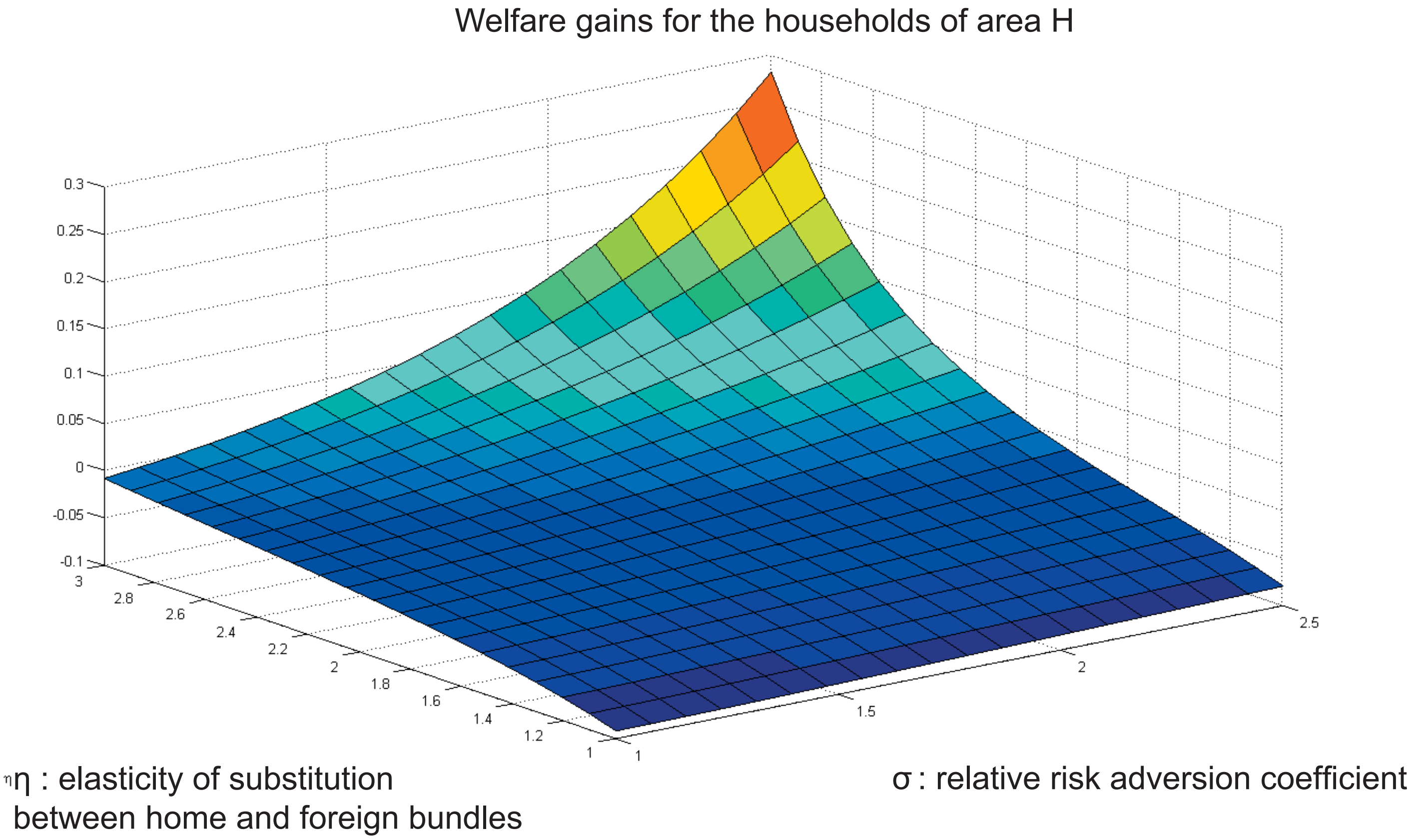

Figure 3: Welfare gains for area H expressed as percentage of the steady state consumption.

Figure 4: Welfare gains for area H expressed as percentage of the steady state consumption.

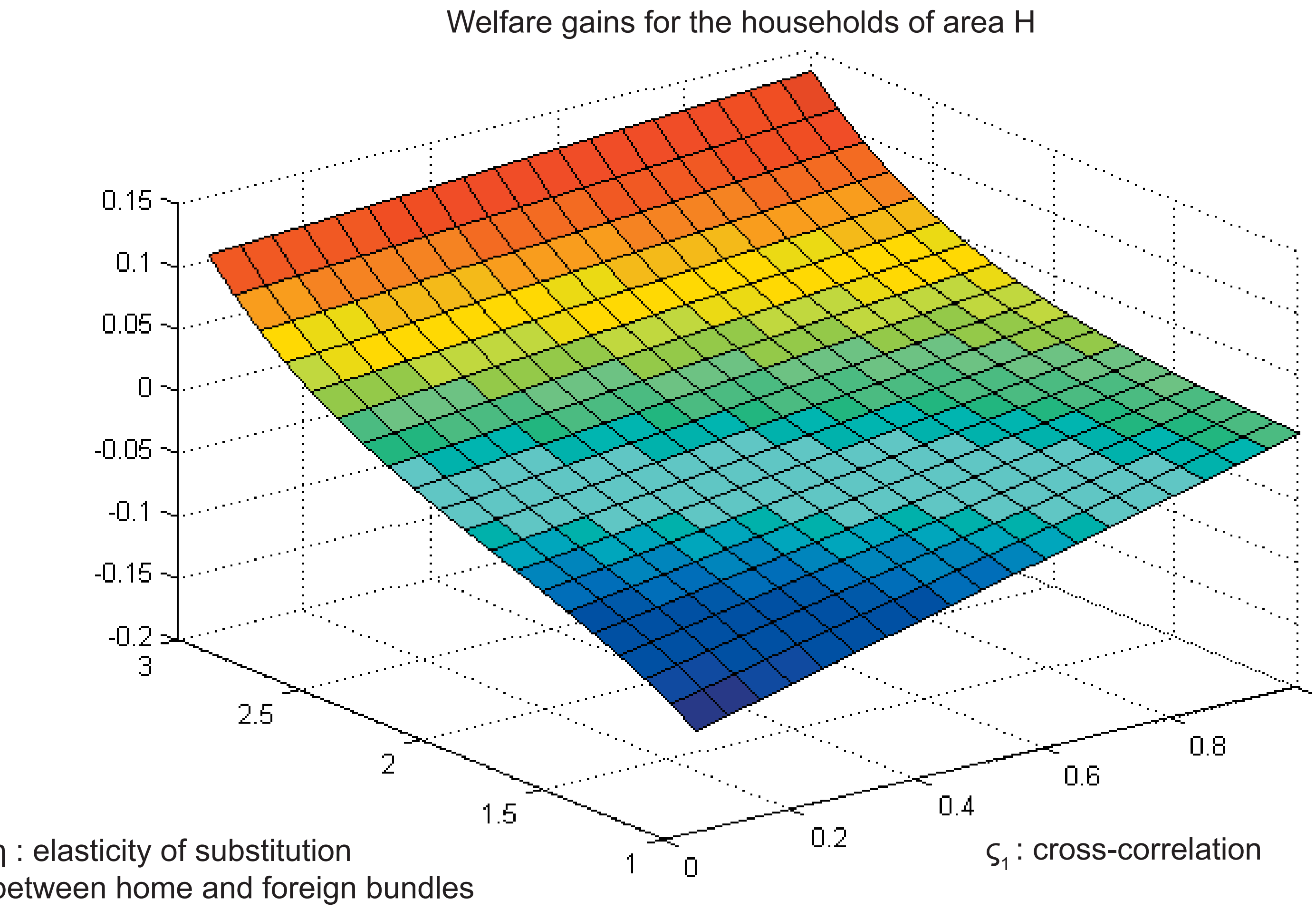

Figure 5: Welfare gains for area H expressed as percentage of the steady state consumption.

A Retrieving condition (26)

Given the definitions of $P^*_{H,t}$ and $P^*_{F,t}$ it is easy to show that:

$$\mathcal{E}_{iH,t}P^*_{H,t} = [\alpha_b P_{H,t}{}^{1-\eta} + (1-\alpha_b)P_{F,t}{}^{1-\eta}]^{\frac{1}{1-\eta}} \qquad \mathcal{E}_{iF,t}P^*_{F,t} = [\alpha_b P_{F,t}{}^{1-\eta} + (1-\alpha_b)P_{H,t}{}^{1-\eta}]^{\frac{1}{1-\eta}} \tag{83}$$

By (83):

$$\frac{\mathcal{E}_{iH,t}P^*_{H,t}}{P_{H,t}} = \left[\alpha_b + (1-\alpha_b)\left(\frac{P_{F,t}}{P_{H,t}}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \qquad \frac{\mathcal{E}_{iF,t}P^*_{F,t}}{P_{F,t}} = \left[\alpha_b + (1-\alpha_b)\left(\frac{P_{H,t}}{P_{F,t}}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \tag{84}$$

which jointly with (25) leads to:

$$\left(\frac{C^*_{F,t}}{C^*_{H,t}}\right) = \left[\frac{\alpha_b\left(\frac{P_{F,t}}{P_{H,t}}\right)^{1-\eta} + (1-\alpha_b)}{(1-\alpha_b)\left(\frac{P_{F,t}}{P_{H,t}}\right)^{1-\eta} + \alpha_b}\right]^{-\frac{1}{\sigma(1-\eta)}} \tag{85}$$

Moreover thanks to (6):

$$\frac{P_{i,t}}{P_{C^i,t}} = \left[\frac{1}{\alpha_s} - \frac{\alpha_b - \alpha_s}{\alpha_s}\left(\frac{P_{H,t}}{P_{C^i,t}}\right)^{1-\eta} - \frac{(1-\alpha_b)}{\alpha_s}\left(\frac{P_{F,t}}{P_{C^i,t}}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \qquad i \in \left[0, \frac{1}{2}\right) \tag{86}$$

which can be read as:

$$\frac{P_{i,t}}{P_{C^i,t}} = \left[\frac{1}{\alpha_s} - \frac{\alpha_b - \alpha_s}{\alpha_s}\left(\frac{P_{H,t}}{\mathcal{E}_{i,H}P^*_{H,t}}\right)^{(1-\eta)}\left(\frac{C^*_{H,t}}{C^i_t}\right)^{-\sigma(1-\eta)} - \frac{(1-\alpha_b)}{\alpha_s}\left(\frac{P_{F,t}}{\mathcal{E}_{i,F}P^*_{F,t}}\right)^{(1-\eta)}\left(\frac{C^*_{F,t}}{C^i_t}\right)^{-\sigma(1-\eta)}\right]^{\frac{1}{1-\eta}} \tag{87}$$

Finally by using (84) and (85) we can rewrite (86) as (26).

B Zero Inflation Deterministic Steady State

In this section we show that, given appropriate initial conditions, zero inflation is a Nash equilibrium policy at the deterministic steady state under both regimes A and B.

In the regime A the *timelessly* optimal policy problem of a monetary authority of country i in the area H can be formulated as the maximization of the following

Lagragian:

$$
\begin{aligned}
L^i = & \sum_{t=0}^{\infty} \beta^t E_0 \Big\{ \frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1} \left(\frac{Y_t^i Z_t^i}{A_t^i} \right)^{\varphi+1} \\
& + \zeta_{1,t}^{s,i} \left[Y_t^i - \left(\frac{P_{i,t}}{P_{C^i,t}} \right)^{-\eta} \left(\alpha_s C_t^i + (\alpha_b - \alpha_s) C_t^{i\sigma\eta} \mathcal{C}_{H,t} + (1-\alpha_b) C_t^{i\sigma\eta} \mathcal{C}_{F,t} \right) \right] \\
& + \zeta_{2,t}^{s,i} \left[K_t^i - \left(\frac{Y_t^i}{A_t^i} \right)^{\varphi+1} Z_t^{i\varphi} (1+\mu_t^i)(1-\tau^i) \frac{\varepsilon}{\varepsilon-1} \right] - \zeta_{2,t-1}^{s,i} \theta \Pi_{i,t}^{\varepsilon} K_t^i \\
& + \zeta_{3,t}^{s,i} \left[F_t^i - Y_t^i C_t^{i\,-\sigma} \frac{P_t^i}{P_{C^i,t}} \right] - \zeta_{3,t-1}^{s,i} \theta \Pi_{i,t}^{(\varepsilon-1)} F_t^i \\
& + \zeta_{4,t}^{s,i} \left[F_t^i - K_t^i \left(\frac{1-\theta \Pi_{i,t}^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{\varepsilon-1}} \right] \\
& + \zeta_{5,t}^{s,i} \left[Z_t^i - \theta Z_{t-1}^i \Pi_{i,t}^{\varepsilon} - (1-\theta) \left(\frac{1-\theta \Pi_{i,t}^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} \right] \Big\}
\end{aligned}
$$

with respect to C_t^i, Y_t^i, Z_t^i, K_t^i, F_t^i and $\Pi_{i,t}$ and where $P_{i,t}/P_{C^i,t}$ are determined consistently with (26), while $C_{H,t}^*$, $C_{F,t}^*$, $\mathcal{C}_{H,t}$ and $\mathcal{C}_{F,t}$ are taken as given. Assume that $\mu_t^j = \mu$, $A_t^j = A$, $\tau^j = \tau$ and $Z_t^j = \Pi_{j,t} = 1$ for all $j \in [0,1]$ and t. Assume in addition that $Z_{-1}^i = 1$. Recalling that $\tilde{\tau} = 1 - (1-\tau)\frac{(1+\mu)\varepsilon}{\varepsilon-1}$ it can be shown that according to the first order conditions at the symmetric deterministic steady state:

$$C^{-\sigma} = \zeta_1^s \delta_s - \zeta_3^s \sigma \gamma_s Y C^{-\sigma-1} \tag{88}$$
$$Y^{\varphi} = \zeta_1^s - \zeta_2^s (\varphi+1) Y^{\varphi} (1-\tilde{\tau}) - \zeta_3^s C^{-\sigma} \tag{89}$$
$$Y^{\varphi+1} = -\zeta_2^s \varphi Y^{\varphi+1} + \zeta_5^s (1-\theta) \tag{90}$$
$$\zeta_2^s (1-\theta) = \zeta_4^s \tag{91}$$
$$\zeta_3^s (1-\theta) = -\zeta_4^s \tag{92}$$
$$\zeta_2^s \theta \varepsilon K = -\zeta_3^s \theta (\varepsilon-1) F + \zeta_4^s \frac{\theta}{1-\theta} K \tag{93}$$

with $\delta_s = \alpha_s(1-\sigma\eta) + \gamma_s \eta \sigma$. Then

$$
\begin{aligned}
& Y = (1-\tilde{\tau})^{-\frac{1}{\sigma+\varphi}} \\
& C = Y \\
& F = K = \frac{Y C^{-\sigma}}{1-\theta} = \frac{Y^{\varphi+1}(1-\tilde{\tau})}{1-\theta} \\
& \Pi_H = \Pi_F = 1 \qquad Z = 1 \\
& \zeta_1^s = \frac{\gamma_s \sigma + (1-\tilde{\tau}\varphi)}{\delta_s \varphi + \gamma_s \sigma} \\
& \zeta_2^s = \frac{\zeta_4^s}{1-\theta} = -\zeta_3^s = -\frac{\delta_s - (1-\tilde{\tau})\varphi}{(\delta_s \varphi + \gamma_s \sigma)(1-\tilde{\tau})} \qquad \zeta_5^s = \frac{Y^{\varphi+1}(1-\varphi\zeta_3^s)}{1-\theta}
\end{aligned}
$$

is a steady state symmetric solution of the optimal policy problem just stated[36].

Consider now the monetary union in the area F[37]. Suppose that for all $i \in [0\frac{1}{2})$ $\Pi_t^i = 1$ at all times[38]. Then we want to show that given other policymakers strategy, $\Pi_t^i = 1$ for all $i \in [\frac{1}{2}, 1]$ and t is optimal.

If for all $i \in [0\frac{1}{2})$ $\Pi_t^i = 1$ at all times, the optimal policy problem of the monetary authority in the area F can be written as maximizing:

$$
\begin{aligned}
L = \sum_{t=0}^{\infty}\beta^t E_0 \Bigg\{ & \int_{\frac{1}{2}}^{1} \left[\frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1}\left(\frac{Y_t^i Z_t^i}{A_t^i}\right)^{\varphi+1} \right] \\
& + \zeta_{1,t}^{b,i}\left[Y_t^i - \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta}\left(\alpha_s C_t^i + 2(\alpha_b-\alpha_s) C_t^{i\sigma\eta}\int_{\frac{1}{2}}^{1} C_t^{j\,1-\sigma\eta} dj + 2(1-\alpha_b) C_t^{i\sigma\eta}\int_0^{\frac{1}{2}} C_t^{j\,1-\sigma\eta} dj\right)\right] \\
& + \zeta_{2,t}^{b,i}\left[K_t^i - \left(\frac{Y_t^i}{A_t^i}\right)^{\varphi+1} Z_t^{i\varphi}(1+\mu_t^i)(1-\tau^i)\frac{\varepsilon}{\varepsilon-1}\right] - \zeta_{2,t-1}^{b,i}\theta\Pi_{i,t}^{\varepsilon}K_t^i \\
& + \zeta_{3,t}^{b,i}\left[F_t^i - Y_t^i C_t^{i\,-\sigma}\frac{P_t^i}{P_{C^i,t}}\right] - \zeta_{3,t-1}^{s}\theta\Pi_{i,t}^{(\varepsilon-1)}F_t^i \\
& + \zeta_{4,t}^{b,i}\left[F_t^i - K_t^i\left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta}\right)^{\frac{1}{\varepsilon-1}}\right] \\
& + \zeta_{5,t}^{b,i}\left[Z_t^i - \theta Z_{t-1}^i\Pi_{i,t}^{\varepsilon} - (1-\theta)\left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right] \\
& + \zeta_{6,t}^{b,i}\left[\left(\frac{C_{F,t}^*}{C_{F,t-1}^*}\right)^{-\sigma}\frac{P_{F,t}}{P_{F,t}^*}\frac{P_{F,t-1}^*}{P_{F,t-1}}\Pi_{F,t}^{-1} - \left(\frac{C_t^i}{C_{t-1}^i}\right)^{-\sigma}\frac{P_{i,t}}{P_{C^i,t}}\frac{P_{C^i,t-1}}{P_{i,t-1}}\Pi_{i,t}^{-1}\right]\Bigg\} di \\
& + \int_0^{\frac{1}{2}}\Bigg\{ \zeta_{7,t}^{b,i}\left[Y_t^i - \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta}\left(\alpha_s C_t^i + 2(\alpha_b-\alpha_s)C_t^{i\sigma\eta}\int_0^{\frac{1}{2}} C_t^{j\,1-\sigma\eta}dj + 2(1-\alpha_b)C_t^{i\sigma\eta}\int_{\frac{1}{2}}^{1} C_t^{j\,1-\sigma\eta}dj\right)\right] \\
& + \zeta_{8,t}^{b,i}\left[\left((1+\mu_t^i)(1-\tau)\frac{\varepsilon}{\varepsilon-1}\left(\frac{Y_t^i}{A_t^i}\right)^{\varphi+1}\right) - \frac{P_{i,t}}{P_{C^i,t}}Y_t^i C_t^{i\,-\sigma}\right] di \Bigg\}
\end{aligned}
$$

with respect to C_t^i, Y_t^i for all i and Z_t^i, K_t^i, F_t^i and $\Pi_{i,t}$ all $i \in [\frac{1}{2}, 1]$ and where $P_{i,t}/P_{C^i,t}$ and $P_{F,t}^*/P_{F,t}$ are determined consistently with (26) , (27), (84) and (85) . Assume that $\mu_t^i = \mu$, $A_t^j = A$, $\tau^j = \tau$ and $Z_t^j = \Pi_{j,t} = 1$ for all $j \in [0,1]$ and t. Moreover assume that $Z_{-1}^i = 1$ for all $i \in \left[\frac{1}{2}, 1\right]$ Given that $\tilde{\tau} = 1 - (1-\tau)(1+\mu)\frac{\varepsilon}{\varepsilon-1}$. Then according to the first order conditions at the symmetric deterministic steady

[36]In other words, given a zero inflation policy of the other central banks, zero inflation is a best response of the central bank of the country i.

[37]We follow closely Benigno and Benigno (2006).

[38]..which implies that $F_t^i = F$, $K_t^i = K$ and $\frac{F_t^i}{K_t^i} = 1$ for all i and t

state:

$$C^{-\sigma} = \zeta_1^b \delta_b + \zeta_7^b (1-\delta_b) - \zeta_3^b \sigma \gamma_b Y C^{-\sigma-1} - \zeta_8^b \sigma (1-\gamma_b) Y C^{-\sigma-1} \tag{94}$$

$$Y^{\varphi} = \zeta_1^b - \zeta_2^b (\varphi+1) Y^{\varphi} (1-\tilde{\tau}) - \zeta_3^b C^{-\sigma} \tag{95}$$

$$Y^{\varphi+1} = -\zeta_2^b \varphi Y^{\varphi+1} + \zeta_5^b (1-\theta) \tag{96}$$

$$\zeta_2^b (1-\theta) = \zeta_4^b \tag{97}$$

$$\zeta_3^b (1-\theta) = -\zeta_4^b \tag{98}$$

$$\zeta_2^b \theta \varepsilon K = -\zeta_3^b \theta (\varepsilon - 1) F + \zeta_4^b \frac{\theta}{1-\theta} K \tag{99}$$

$$0 = \zeta_1^b (1-\delta_b) + \zeta_7^b \delta_b - \zeta_3^b \sigma (1-\gamma_b) Y C^{-\sigma-1} - \zeta_8^b \sigma \gamma_b Y C^{-\sigma-1} \tag{100}$$

$$0 = \zeta_7^b + \zeta_8^b \left[(\varphi+1) Y^{\varphi} (1-\tilde{\tau}) - C^{-\sigma} \right] \tag{101}$$

where $\delta_b \equiv (1-\sigma\eta)\alpha_b + \eta\sigma\gamma_b$.

Then it is easy to show:

$$Y = (1-\tilde{\tau})^{-\frac{1}{\sigma+\varphi}} \tag{102}$$

$$C = Y \tag{103}$$

$$F = K = \frac{Y C^{-\sigma}}{1-\theta} = \frac{Y^{\varphi+1}(1-\tilde{\tau})}{1-\theta} \tag{104}$$

$$\Pi_H = \Pi_F = 1 \qquad Z = 1 \tag{105}$$

$$\zeta_1^b = Y^{\varphi} \qquad \zeta_2^b = \frac{\zeta_4^b}{1-\theta} = -\zeta_3^b = 0 \qquad \zeta_5^b = \frac{Y^{\varphi+1}}{1-\theta} \tag{106}$$

$$\zeta_6^b = 0 \qquad \zeta_7^b = \frac{\tilde{\tau}}{(1-\tilde{\tau})(\sigma+\varphi)} \qquad \zeta_8^b = \frac{-Y^{\varphi}\tilde{\tau}}{(\sigma+\varphi)} \tag{107}$$

Hence being the best response of both monetary union and the small open economy policymakers, zero inflation is a Nash equilibrium solution in regime A.

Consider now the case of regime B and suppose that the central bank of area H set $\Pi_{H,t}^{-1}=1$ for all t. The central bank of the monetary union in area F maximizes:

$$
\begin{aligned}
L = \sum_{t=0}^{\infty}\beta^t E_0 \Bigg\{ \int_{\frac{1}{2}}^{1} & \left[\frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1}\left(\frac{Y_t^i Z_t^i}{A_t^i}\right)^{\varphi+1} \right] \\
& + \zeta_{1,t}^{b,i}\left[Y_t^i - \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta} \left(\alpha_s C_t^i + 2(\alpha_b-\alpha_s) C_t^{i\,\sigma\eta} \int_{\frac{1}{2}}^{1} C_t^{j\,1-\sigma\eta} dj + 2(1-\alpha_b) C_t^{i\,\sigma\eta} \int_0^{\frac{1}{2}} C_t^{j\,1-\sigma\eta} dj \right) \right] \\
& + \zeta_{2,t}^{b,i}\left[K_t^i - \left(\frac{Y_t^i}{A_t^i}\right)^{\varphi+1} Z_t^{i\varphi}(1+\mu_t^i)(1-\tau^i)\frac{\varepsilon}{\varepsilon-1} \right] - \zeta_{2,t-1}^{b,i}\theta \Pi_{i,t}^{\varepsilon} K_t^i \\
& + \zeta_{3,t}^{b,i}\left[F_t^i - Y_t^i C_t^{i\,-\sigma}\frac{P_t^i}{P_{C^i,t}} \right] - \zeta_{3,t-1}^{s}\theta \Pi_{i,t}^{(\varepsilon-1)} F_t^i \\
& + \zeta_{4,t}^{b,i}\left[F_t^i - K_t^i \left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta}\right)^{\frac{1}{\varepsilon-1}} \right] \\
& + \zeta_{5,t}^{b,i}\left[Z_t^i - \theta Z_{t-1}^i \Pi_{i,t}^{\varepsilon} - (1-\theta)\left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}} \right] \\
& + \zeta_{6,t}^{b,i}\left[\left(\frac{C_{F,t}^*}{C_{F,t-1}^*}\right)^{-\sigma} \frac{P_{F,t}}{P_{F,t}^*}\frac{P_{F,t-1}^*}{P_{F,t-1}}\Pi_{F,t}^{-1} - \left(\frac{C_t^i}{C_{t-1}^i}\right)^{-\sigma}\frac{P_{i,t}}{P_{C^i,t}}\frac{P_{C^i,t-1}}{P_{i,t-1}}\Pi_{i,t}^{-1} \right] \Bigg\} di \\
+ \int_0^{\frac{1}{2}} \Bigg\{ & \zeta_{7,t}^{b,i}\left[Y_t^i - \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta} \left(\alpha_s C_t^i + 2(\alpha_b-\alpha_s) C_t^{i\,\sigma\eta} \int_0^{\frac{1}{2}} C_t^{j\,1-\sigma\eta} dj + 2(1-\alpha_b) C_t^{i\,\sigma\eta} \int_{\frac{1}{2}}^{1} C_t^{j\,1-\sigma\eta} dj \right) \right] \\
& + \zeta_{8,t}^{b,i}\left[K_t^i - \left(\frac{Y_t^i}{A_t^i}\right)^{\varphi+1} Z_t^{i\varphi}(1+\mu_t^i)(1-\tau^i)\frac{\varepsilon}{\varepsilon-1} \right] - \zeta_{8,t-1}^{b,i}\theta \Pi_{i,t}^{\varepsilon} K_t^i \\
& + \zeta_{9,t}^{b,i}\left[F_t^i - Y_t^i C_t^{i\,-\sigma}\frac{P_t^i}{P_{C^i,t}} \right] - \zeta_{9,t-1}^{s}\theta \Pi_{i,t}^{(\varepsilon-1)} F_t^i \\
& + \zeta_{10,t}^{b,i}\left[F_t^i - K_t^i \left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta}\right)^{\frac{1}{\varepsilon-1}} \right] \\
& + \zeta_{11,t}^{b,i}\left[Z_t^i - \theta Z_{t-1}^i \Pi_{i,t}^{\varepsilon} - (1-\theta)\left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}} \right] \\
& + \zeta_{12,t}^{b,i}\left[\left(\frac{C_{H,t}^*}{C_{H,t-1}^*}\right)^{-\sigma} \frac{P_{H,t}}{P_{H,t}^*}\frac{P_{H,t-1}^*}{P_{H,t-1}} - \left(\frac{C_t^i}{C_{t-1}^i}\right)^{-\sigma}\frac{P_{i,t}}{P_{C^i,t}}\frac{P_{C^i,t-1}}{P_{i,t-1}}\Pi_{i,t}^{-1} \right] \Bigg\} di
\end{aligned}
$$

with respect to C_t^i, Y_t^i, Z_t^i, K_t^i, F_t^i and $\Pi_{i,t}$ all i and where$P_{i,t}/P_{C^i,t}$, $P_{F,t}^*/P_{F,t}$ and $P_{H,t}^*/P_{H,t}$ are determined consistently with (26) , (27), (84) and (85). Assume that $\mu_t^i=\mu$, $A_t^j=A$, $\tau^j=\tau$ and $Z_t^j=\Pi_{j,t}=1$ for all $j\in[0,1]$ and t. Moreover assume that $Z_{-1}^i=1$ for all i Given that $\tilde{\tau}=1-(1-\tau)(1+\mu)\frac{\varepsilon}{\varepsilon-1}$. Then according to the first order conditions at the symmetric deterministic steady state it can be shown:

$$Y = (1-\tilde{\tau})^{-\frac{1}{\sigma+\varphi}} \tag{108}$$
$$C = Y \tag{109}$$
$$F = K = \frac{YC^{-\sigma}}{1-\theta} = \frac{Y^{\varphi+1}(1-\tilde{\tau})}{1-\theta} \tag{110}$$
$$\Pi_H = \Pi_F = 1 \qquad Z = 1 \tag{111}$$

Therefore for the policymaker of the area F zero inflation is a best response to a zero inflation policy of the policymaker in the area H. A symmetric problem can be stated for the policymaker of the monetary union of the area H Thus zero inflation is a Nash equilibrium policy.

C The purely quadratic approximation to the welfare

In order to recover the optimal policies we need to approximate up to the second order single country representative agent utility given by (1) in the following way.

First we can approximate the utility derived from private consumption for generic region i as:

$$\frac{C_t^{i\,1-\sigma}}{1-\sigma} \simeq \frac{C^{1-\sigma}}{1-\sigma} + C^{1-\sigma}(\hat{c}_t^i + \frac{1}{2}(\hat{c}_t^i)^2) - \frac{\sigma}{2}C^{1-\sigma}(\hat{c}_t^i)^2 + t.i.p. \tag{112}$$

where $\hat{c}_t^i$ stands for the log-deviations of private consumption from the non-stochastic symmetric steady state[39].

Similarly the labor disutility can be approximated by taking into account that $N_t^i = \frac{Y_t^i Z_t^i}{A_t^i}$ and, as showed by Galí and Monacelli (2005), being $Z_t^i = \int_0^1 \left(\frac{p_t(h^i)}{P_{i,t}}\right)^{-\varepsilon} dh^i$:

$$\hat{z}_t^i \simeq \frac{\varepsilon}{2} Var_{h^i}(p_t(h^i)) \tag{113}$$

In words the approximation of Z_t^i around the symmetric steady state is purely quadratic. Moreover following Woodford (2001, NBER WP8071) it is possible to show that $\sum\limits_{t=0}^{\infty}\beta^t Var_{h^i}(p_t(h^i)) = \frac{1}{\lambda}\sum\limits_{t=0}^{\infty}\beta^t\pi_{i,t}^2$ with $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$. Thus:

$$\frac{1}{\varphi+1}\left(\frac{Y_t^i Z_t^i}{A_t^i}\right)^{\varphi+1} \simeq \frac{1}{\varphi+1}Y^{\varphi+1} + Y^{\varphi+1}(\hat{y}_t^i + \frac{1}{2}(\hat{y}_t^i)^2) + Y^{\varphi+1}\frac{\varepsilon}{2\lambda}(\pi_{i,t})^2 + \frac{\varphi}{2}Y^{\varphi+1}(\hat{y}_t^i)^2$$
$$-(\varphi+1)Y^{\varphi+1}\hat{y}_t^i a_t^i + t.i.p. \tag{114}$$

[39]From now this convention will be used: $\hat{x}_t$ represents the log-deviation of X_t from the steady state.

C.1 The case of the small open economy

By combining (112) and (114) and taking into account that at the steady state $C^{-\sigma} = (1-\tilde{\tau})Y^{\varphi}$, the second order approximation of welfare of the region i households can be written as:

$$\sum_{t=0}^{\infty} \beta^t Y^{\varphi+1} E_0 \left[\hat{s}_t^{i'} w_s - \frac{1}{2} \hat{s}_t^{i'} W_{s,s} \hat{s}_t^i + \hat{s}_t^{i'} W_{s,e} \hat{e}_t^i \right] + t.i.p. \tag{115}$$

where

$$\hat{s}_t^{i'} \equiv \left[\hat{y}_t^i,\ \hat{c}_t^i,\ \pi_{i,t} \right] \qquad w_s^{'} \equiv [-1,\ (1-\tilde{\tau}),\ \ 0] \qquad \hat{e}_t^{i'} \equiv \left[\hat{c}_{H,t},\ \hat{c}_{F,t},\ a_t^i,\ \ \mu_t^i \right]$$

$$W_{s,s} \equiv \begin{bmatrix} (\varphi+1) & 0 & 0 \\ 0 & (1-\tilde{\tau})(\sigma-1) & 0 \\ 0 & 0 & \frac{\varepsilon}{\lambda} \end{bmatrix} \qquad W_{s,e} \equiv \begin{bmatrix} 0 & 0 & (\varphi+1) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and with $i \in [0, \frac{1}{2})$ $\hat{c}_{H,t} \equiv 2 \int_0^{\frac{1}{2}} \hat{c}_t^j dj$ and $\hat{c}_{F,t} \equiv 2 \int_{\frac{1}{2}}^1 \hat{c}_t^j dj$. In order to recover a purely quadratic approximation to the welfare of the small open economy, we have to use the second order approximations to both the aggregate demand and the Phillips curves.

The second order approximation to the demand curve can be written as:

$$0 \simeq \left[\hat{s}_t^{i'} g_s - \hat{e}_t^{i'} g_e + \tfrac{1}{2} \hat{s}_t^{i'} G_{s,s} \hat{s}_t^{i'} - \hat{s}_t^{i'} G_{s,e} e_t^{i'} \right] + s.o.t.i.p. \tag{116}$$

where

$$g_s^{'} \equiv [-1,\ \delta_s,\ 0] \qquad g_e^{'} \equiv [-(\delta_b - \delta_s),\ -(1-\delta_b),\ 0,\ 0]$$

$$G_{s,s} \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & \delta_s + \omega_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{s,e} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2\omega_1 + 2\omega_2 & -2\omega_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\delta_s \equiv \alpha_s(1-\eta\sigma) + \eta\sigma/\alpha_s$, $\delta_b \equiv \alpha_b(1-\eta\sigma) - \alpha_b\eta\sigma/(1-2\alpha_b)$ and:

$$\omega_1 \equiv \frac{(1-\alpha_s)\eta\sigma(\sigma - (1-\alpha_s)\alpha_s(1-\eta\sigma))}{\alpha_s^2} \qquad \omega_2 \equiv \frac{(1-\alpha_b)\eta\sigma\left(\sigma + \left(\alpha_s^2 + (1-2\alpha_b)\right)(1-\eta\sigma)\right)}{\alpha_s(1-2\alpha_b)}$$

As in Benigno and Woodford (2005) the second order approximation to the (54) and be combined with (52) and (53) to obtain:

$$V_0 = \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty} \beta^t E_0 \left[\hat{s}_t^{i'} v_s - \hat{e}_t^{i'} v_e + \frac{1}{2} \hat{s}_t^{i'} V_{s,s} \hat{s}_t^i - \hat{s}_t^{i'} V_{s,e} \hat{e}_t^i \right] + s.o.t.i.p. \tag{117}$$

where

$$v_s' \equiv [\varphi,\ \sigma\gamma_s,\ 0] \qquad v_e' \equiv [\sigma(\gamma_s - \gamma_b),\ -\sigma(1-\gamma_b),\ -(\varphi+1),\ 1]$$

$$
V_{s,s} \equiv \begin{bmatrix} \varphi(\varphi+2) & \sigma\gamma_s & 0 \\ \sigma\gamma_s & -\eta\sigma^2\left(\gamma_s-1\right)\gamma_s-\sigma^2\gamma_s & 0 \\ 0 & 0 & \frac{\varepsilon(\varphi+1)}{\lambda} \end{bmatrix}
$$

$$
V_{s,e} \equiv \begin{bmatrix} -2\sigma(\gamma_b-\gamma_s) & -2\sigma(1-\gamma_b) & (\varphi+1)^2 & -(\varphi+1) \\ 2\eta\sigma^2\gamma_s\left(\gamma_b-\gamma_s\right)) & 2\eta\sigma\gamma_s^2\left(1-\gamma_b\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

Given (116) and (117), it is possible to rewrite (115) in a purely quadratic way. Indeed it is easy to show that:

$$
w_s = (1-\varphi\zeta_s)g_s - \zeta_s v_s \tag{118}
$$

where $\zeta_s = (\delta_s - (1-\tilde{\tau}))/(\delta_s\varphi + \gamma_s\sigma)$[40]:

Then we can write the second order approximation of union welfare as:

$$
Y^{\varphi+1}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\hat{s}_t^{i'}\Omega_{s,s}\hat{s}_t^i - \hat{s}_t^{i'}\Omega_{s,e}\hat{e}_t^i\right] + t.i.p. \tag{119}
$$

where

$$
\Omega_{s,s} \equiv W_{s,s} + (1-\varphi\zeta_s)G_{s,s} - \zeta_s V_{s,s} \qquad \Omega_{s,e} \equiv W_{s,e} + (1-\varphi\zeta_s)G_{s,e} - \zeta_s V_{s,e} \tag{120}
$$

and $\Omega_{s,s}$ and $\Omega'_{s,e}$ are respectively equal to:

$$
\begin{bmatrix} (1-\zeta_s(\varphi+1))\varphi & -\zeta_s\gamma_s\sigma & 0 \\ -\zeta_s\gamma_s\sigma & (1-\tilde{\tau})(\sigma-1)+\zeta_s\sigma^2\gamma_s\left(\eta\left(1-\gamma_s\right)+1\right)+(1-\zeta_s\varphi)(\delta_s+\omega_1) & 0 \\ 0 & 0 & \frac{(1-\zeta_s(\varphi+1))\varepsilon}{\lambda} \end{bmatrix}
$$

$$
\begin{bmatrix} 2\zeta_s\sigma(\gamma_b-\gamma_s) & -2\zeta_s\eta\sigma^2\gamma_s\left(\gamma_b-\gamma_s\right)+2\left(1-\zeta_s\varphi\right)\left(\omega_1+\omega_2\right) & 0 \\ 2\zeta_s\sigma(1-\gamma_b) & -2\zeta_s\eta\sigma^2\gamma_s(1-\gamma_b)-2\left(1-\zeta_s\varphi\right)\omega_2 & 0 \\ (1-\zeta_s(\varphi+1))(\varphi+1) & 0 & 0 \\ \zeta_s(\varphi+1) & 0 & 0 \end{bmatrix}
$$

Now we would like to rewrite this approximation in terms of deviations from the target of the small open economy policymaker. This target can be determined by maximizing (119) with respect to s_t^i subject to (45):

The Lagrangian associated to this problem can be written as:

$$
L = Y^{\varphi+1}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\hat{s}_t^{i'}\Omega_{s,s}\hat{s}_t^i - \hat{s}_t^{i'}\Omega_{s,e}\hat{e}_t^i - \phi_t^i\left(\hat{s}_t^{i'}g_s + \hat{e}_t^{i'}g_e\right)\right] \tag{121}
$$

The first order condition of L with respect to $\hat{s}_t^{i'}$ can be read as:

$$
\Omega_{s,s}\hat{s}_t^i - \Omega_{s,e}\hat{e}_t^i = \phi_t^i g_s \tag{122}
$$

[40]Notice that $\zeta_3^s = \zeta_s(1-\tilde{\tau})$ and $\zeta_1^s = (1-\varphi\zeta_s)$ with ζ_1^s and ζ_3^s being the lagrange multipliers previously recovered for the optimal policy problem of the small economy policymaker[41].

Alternatively:

$$(1-\zeta_s(\varphi+1))\varphi\hat{y}_t^{i,s} - \zeta_s\gamma_s\sigma\hat{c}_t^{i,s} - \zeta_s\sigma(\gamma_b-\gamma_s)\hat{c}_{H,t} - \zeta_s\sigma(1-\gamma_b)\hat{c}_{F,t} - (1-\zeta_s(\varphi+1))(\varphi+1)\hat{a}_t^i$$
$$-\zeta_s(\varphi+1)\hat{\mu}_t^i = \phi_{1,t}^i$$

$$[(1-\tilde{\tau})(\sigma-1) + \zeta_s\left(\eta\sigma^2\gamma_s(1-\gamma_s) + \sigma^2\gamma_s\right) + (1-\zeta_s\varphi)(\delta_s+\omega_1)]\hat{c}_t^{i,s} - \zeta_s\gamma_s\sigma\hat{y}_t^{i,s}$$
$$+[\zeta_s\eta\sigma^2\gamma_s(\gamma_b-\gamma_s) - (1-\zeta_s\varphi)(\omega_1+\omega_2)]\hat{c}_{H,t} + [\zeta_s\eta\sigma^2\gamma_s(1-\gamma_b) + (1-\zeta_s\varphi)\omega_2]\hat{c}_{F,t} = -\delta_s\phi_{1,t}^i$$

$$(1-\zeta_s(\varphi+1))\frac{\varepsilon}{\lambda}\pi_{i,t} = 0$$

$$\hat{y}_t^{i,s} = \delta_s\hat{c}_t^{i,s} + (\delta_b-\delta_s)\hat{c}_{H,t} + (1-\delta_b)\hat{c}_{F,t}$$

for all $i \in [0, \frac{1}{2})$ and where ϕ_t^i is the lagrange multiplier of (45). Notice that in the perspective of the small open monetary authority $\hat{c}_{H,t}$ and $\hat{c}_{F,t}$ are taken as exogenous.

Then it is easy to show that (119) can be rewritten as (76). Indeed it is sufficient to add and subtract the corresponding target in each terms of (119) and then use the fist order conditions just listed.

C.2 The case of the Monetary Union

If in the area H there is a Monetary Union, then the second order approximation of average welfare of the union household can be read as:

$$\sum_{t=0}^{\infty}\beta^t Y^{\varphi+1}\int_0^{\frac{1}{2}} E_0\left[\hat{s}_t^{i'}w_s - \frac{1}{2}\hat{s}_t^{i'}W_{s,s}\hat{s}_t^i + \hat{s}_t^{i'}W_{s,u}\hat{u}_t^i\right]di + t.i.p. \qquad (123)$$

$$\hat{s}_t^{i'} \equiv \left[\hat{y}_t^i,\ \hat{c}_t^i,\ \pi_{i,t}\right] \qquad w_s^{'} \equiv [-1,\ (1-\tilde{\tau}),\ \ 0] \qquad \hat{u}_t^{i'} \equiv \left[a_t^i,\ \mu_t^i\right]$$

$$W_{s,s} \equiv \begin{bmatrix} (\varphi+1) & 0 & 0 \\ 0 & (1-\tilde{\tau})(\sigma-1) & 0 \\ 0 & 0 & \frac{\varepsilon}{\lambda} \end{bmatrix} \qquad W_{s,u} \equiv \begin{bmatrix} (\varphi+1) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A purely quadratic approximation to the welfare of the union households can be retrieved thanks to the second order approximations of the demand supply curves.

The second order approximation to the demand curve of a generic region i in the area H can be read as:

$$0 \simeq \hat{s}_t^{i'} g_s + \int_0^{\frac{1}{2}} \hat{s}_t^i di' g_{S_H} di + \int_{\frac{1}{2}}^1 \hat{s}_t^i di' g_{S_F} di + \frac{1}{2} \hat{s}_t^{i'} G_{s,s} \hat{s}_t^i di + \frac{1}{2} \int_0^{\frac{1}{2}} \hat{s}_t^{i'} G_{s_H,s_H} \hat{s}_t^i di$$

$$+\frac{1}{2} \int_{\frac{1}{2}}^1 \hat{s}_t^{i'} G_{s_F,s_F} \hat{s}_t^i di + \frac{1}{2} \int_0^{\frac{1}{2}} \hat{s}_t^i di' G_{S_H,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di + \frac{1}{2} \int_{\frac{1}{2}}^1 \hat{s}_t^i di' G_{S_F,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di$$

$$+\hat{s}_t^{i'} G_{s,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di + \hat{s}_t^{i'} G_{s,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di + \int_0^{\frac{1}{2}} \hat{s}_t^i di' G_{S_H,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di + s.o.t.i.p. \quad (124)$$

where

$$g_s^{'} \equiv [-1,\ \delta_s,\ 0] \qquad g_{S_H}^{'} \equiv [0,\ 2(\delta_b - \delta_s),\ 0] \qquad g_{S_F}^{'} \equiv [0,\ 2(1-\delta_b),\ 0,]$$

$$G_{s,s} \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & \delta_s + \omega_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{s_F,s_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(1-\delta_b) + 2\omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{s_H,s_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\eta\sigma^2(1-\gamma_s^2) + 2(\delta_b - \delta_s) - 2(\omega_1 + \omega_3) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4\left(\eta\sigma^2(1-\gamma_s^2) - \eta\sigma^2\gamma_b(1-\gamma_b) + 2\omega_1 + 2\omega_2 + \omega_3\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{S_F,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\left(\eta\sigma^2\gamma_b(1-\gamma_b) + \omega_3\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{s,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\left(\omega_1 + \omega_2\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{s,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\omega_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{S_H,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4\left(\eta\sigma^2\gamma_b(1-\gamma_b) - \omega_2\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and where

$$\omega_3 \equiv \frac{(1-\alpha_b)\eta\sigma(\sigma + 2(1-\alpha_b)(1-\eta\sigma))}{1-2\alpha_b}$$

By integrating (124):

$$0 \simeq \int_0^{\frac{1}{2}} \hat{s}_t^i di' h_{S_H} di + \int_{\frac{1}{2}}^1 \hat{s}_t^i di' h_{S_F} di + \frac{1}{2} \int_0^{\frac{1}{2}} \hat{s}_t^{i'} H_{s_H,s_H} \hat{s}_t^i di + \frac{1}{2} \int_{\frac{1}{2}}^1 \hat{s}_t^{i'} H_{s_F,s_F} \hat{s}_t^i di$$

$$+\frac{1}{2} \int_0^{\frac{1}{2}} \hat{s}_t^i di' H_{S_H,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di + \frac{1}{2} \int_{\frac{1}{2}}^1 \hat{s}_t^i di' H_{S_F,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di + \int_0^{\frac{1}{2}} \hat{s}_t^i di' H_{S_H,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di$$

$$+s.o.t.i.p.$$

with

$$h_{S_H}^{'} \equiv [-1,\ \delta_b,\ 0] \qquad h_{S_F}^{'} \equiv [0,\ (1-\delta_b),\ 0,]$$

$$H_{s_H,s_H} \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\eta\sigma^2(1-\gamma_s^2)+\delta_b-\omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H_{s_F,s_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1-\delta_b)+\omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\eta\sigma^2(1-\gamma_s^2)-2\eta\sigma^2\gamma_b(1-\gamma_b)+2\omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_{S_F,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\left(\eta\sigma^2(1-\gamma_b)\gamma_b+2\omega_3\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H_{S_H,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\eta\sigma^2\gamma_b(1-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A symmetric approximation can be stated for the resource constraints of the regions in the area F namely:

$$\begin{aligned}
0 \simeq & \int_{\frac{1}{2}}^{1} \hat{s}_t^i di' f_{S_F} di + \int_0^{\frac{1}{2}} \hat{s}_t^i di' f_{S_H} di + \frac{1}{2}\int_{\frac{1}{2}}^{1} \hat{s}_t^{i\prime} F_{s_F,s_F} \hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^{i\prime} F_{s_H,s_H} \hat{s}_t^i di \\
& + \frac{1}{2}\int_{\frac{1}{2}}^{1} \hat{s}_t^i di' F_{S_F,S_F} \int_{\frac{1}{2}}^{1} \hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^i di' F_{S_H,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di + \int_{\frac{1}{2}}^{1} \hat{s}_t^i di' F_{S_F,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di \\
& + s.o.t.i.p.
\end{aligned}$$

with $f_{S_F} \equiv h_{S_H}$, $f_{S_H} \equiv h_{S_F}$, $F_{s_F,s_F} \equiv H_{s_H,s_H}$, $F_{s_H,s_H} \equiv H_{s_F,s_F}$, $F_{S_F,S_F} \equiv H_{S_H,S_H}$, $F_{S_H,S_H} \equiv H_{S_F,S_F}$ and $F_{S_F,S_H} \equiv H_{S_H,S_F}$.

Conversely the second order approximation of the (54) for the area F can be obtained by combining (52) and (53):

$$\begin{aligned}
V_0 = & \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty}\beta^t E_0\Big[\hat{s}_t^{i\prime} v_s + \int_{\frac{1}{2}}^{1} \hat{s}_t^i di' v_{S_F} + \int_0^{\frac{1}{2}} \hat{s}_t^i di' v_{S_H} - \hat{u}_t^{i\prime} v_u + \frac{1}{2}\hat{s}_t^i V_{s,s}\hat{s}_t^i \\
& + \hat{s}_t^{i\prime} V_{s,S_F} \int_{\frac{1}{2}}^{1} \hat{s}_t^i di + \hat{s}_t^{i\prime} V_{s,s_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di + \frac{1}{2}\int_{\frac{1}{2}}^{1} \hat{s}_t^i V_{s_F,s_F} \hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^i V_{s_H,s_H} \hat{s}_t^i di \\
& + \frac{1}{2}\int_{\frac{1}{2}}^{1} \hat{s}_t^i di' V_{S_F,S_F} \int_{\frac{1}{2}}^{1} \hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^i di' V_{S_H,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di + \int_0^{\frac{1}{2}} \hat{s}_t^i di' V_{S_H,S_F} \int_{\frac{1}{2}}^{1} \hat{s}_t^i di \\
& - \hat{s}_t^{i\prime} V_{s,u} \hat{u}_t^i\Big] + s.o.t.i.p. \qquad (125)
\end{aligned}$$

where

$$v_s' \equiv [\varphi,\ \sigma\gamma_s,\ 0] \quad v_{S_F}' \equiv [0,\ 2\sigma(\gamma_b-\gamma_s),\ 0] \quad v_{S_H}' \equiv [0,\ 2\sigma(1-\gamma_b),\ 0] \quad v_u' \equiv [\ (\varphi+1),\ -1]$$

$$V_{s,s} \equiv \begin{bmatrix} \varphi(\varphi+2) & \sigma\gamma_s & 0 \\ \sigma\gamma_s & \eta\sigma^2(1-\gamma_s)\gamma_s - \sigma^2\gamma_s & 0 \\ 0 & 0 & \frac{\varepsilon(\varphi+1)}{\lambda} \end{bmatrix} \quad V_{s,S_F} \equiv \begin{bmatrix} 0 & \sigma(1-\gamma_b) & 0 \\ \sigma(1-\gamma_b) & -2\eta\sigma^2(1-\gamma_b)\gamma_s & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{s,S_H} \equiv \begin{bmatrix} 0 & \sigma(\gamma_b-\gamma_s) & 0 \\ \sigma(\gamma_b-\gamma_s) & 2\eta\sigma^2\gamma_s(\gamma_s-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad V_{s_F,s_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(\eta-1)\sigma^2(1-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{s_H,s_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(\eta-1)\sigma^2(\gamma_b-\gamma_s) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad V_{S_F,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\eta\sigma^2(1-\gamma_b)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\eta\sigma^2(\gamma_b-\gamma_s)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad V_{S_F,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\eta\sigma^2(1-\gamma_b)(\gamma_b-\gamma_s) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{s,u} \equiv \begin{bmatrix} (\varphi+1)^2 & -(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By integrating (127) over $\left[\frac{1}{2},1\right]$

$$\begin{aligned}\frac{1}{2}V_0 \quad = \quad & \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty}\beta^t E_0\Big[\int_{\frac{1}{2}}^{1}\hat{s}_t^i di' r_{S_F} + \int_0^{\frac{1}{2}}\hat{s}_t^i di' r_{S_H} - \int_{\frac{1}{2}}^{1}\hat{u}_t^i di' r_u + \frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^{i'} R_{s_F,s_F}\hat{s}_t^i di \\ & + \frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^{i'} R_{s_H,s_H}\hat{s}_t^i di + \frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di' R_{S_F,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}}\hat{s}_t^i di' R_{S_H,S_H}\int_0^{\frac{1}{2}}\hat{s}_t^i di \\ & + \int_0^{\frac{1}{2}}\hat{s}_t^i di' R_{S_F,S_H}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di - \int_{\frac{1}{2}}^{1}\hat{s}_t^{i'} R_{s_F,u}\hat{u}_t^i di\Big] + s.o.t.i.p. \end{aligned} \tag{126}$$

where:

$$r'_{S_F} \equiv [\varphi,\ \sigma\gamma_b,\ 0] \qquad r'_{S_H} \equiv [0,\ \sigma(1-\gamma_b),\ 0] \qquad r'_u \equiv [(\varphi+1),\ -1]$$

$$R_{s_F,s_F} \equiv \begin{bmatrix} \varphi(\varphi+2) & \sigma\gamma_s & 0 \\ \sigma\gamma_s & -\eta\gamma_s^2\sigma^2 + \eta\gamma_b\sigma^2 - \gamma_b\sigma^2 & 0 \\ 0 & 0 & \frac{\varepsilon(\varphi+1)}{\lambda} \end{bmatrix} R_{s_H,s_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & (\eta-1)\sigma^2(1-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{S_F,S_F} \equiv \begin{bmatrix} 0 & 2\sigma(\gamma_b-\gamma_s) & 0 \\ 2\sigma(\gamma_b-\gamma_s) & 2\eta\sigma^2\left(\gamma_s^2-\gamma_b^2\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} R_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\eta\sigma^2(1-\gamma_b)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{S_F,S_H} \equiv \begin{bmatrix} 0 & \sigma(1-\gamma_b) & 0 \\ \sigma(1-\gamma_b) & -2\eta\sigma^2(1-\gamma_b)\gamma_b & 0 \\ 0 & 0 & 0 \end{bmatrix} R_{s_F,u} \equiv \begin{bmatrix} (\varphi+1)^2 & -(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Again a symmetric condition can be stated for the regions of the area H namely:

$$\begin{aligned}\frac{1}{2}V_0 &= \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty}\beta^t E_0\Big[\int_{\frac{1}{2}}^{1}\hat{s}_t^i di' k_{S_H} + \int_0^{\frac{1}{2}}\hat{s}_t^i di' k_{S_F} - \int_{\frac{1}{2}}^{1}\hat{u}_t^i di' k_u + \frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^i K_{s_H,s_H}\hat{s}_t^i di \\ &\quad +\frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di' K_{S_H,S_H}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}}\hat{s}_t^i di' K_{S_F,S_F}\int_0^{\frac{1}{2}}\hat{s}_t^i di + \int_0^{\frac{1}{2}}\hat{s}_t^i di' K_{S_H,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di \\ &\quad -\int_{\frac{1}{2}}^{1}\hat{s}_t^{i'} K_{s_H,u}\hat{u}_t^i di\Big] + s.o.t.i.p. \end{aligned} \tag{127}$$

where $k_{S_H} = r_{S_F}$, $k_{S_F} = r_{S_H}$, $k_u = r_u$, $K_{s_H,s_H} = R_{s_F,s_F}$, $K_{S_H,S_H} = R_{S_F,S_F}$, $K_{S_F,S_F} = R_{S_H,S_H}$, $K_{S_H,S_F} = R_{S_F,S_H}$ and $K_{s_H,u} = R_{s_F,u}$.

Then it can be shown that:

$$\begin{aligned} w_s &= (1-\varphi\zeta_b)h_{S_H} - (\xi-\zeta_b)\varphi f_{S_H} - \zeta_b k_{S_H} - (\xi-\zeta_b)r_{S_H} \\ 0 &= (1-\varphi\zeta_b)h_{S_F} - (\xi-\zeta_b)\varphi f_{S_F} - \zeta_b k_{S_F} - (\xi-\zeta_b)r_{S_F} \end{aligned} \tag{128}$$

where $\zeta_b = \frac{1}{2}\frac{\tilde{\tau}}{\sigma+\varphi} - \frac{\delta_b - 1 + (1/2)\tilde{\tau}}{(1-2\gamma_b)\sigma + (1-2\delta_b)\varphi}$ and $\xi = \frac{\tilde{\tau}}{\sigma+\varphi}$ Hence we can write the second order approximation of union welfare as:

$$\begin{aligned} &Y^{\varphi+1}\sum_{t=0}^{\infty}\beta^t E_0\Big[\frac{1}{2}\int_0^{\frac{1}{2}}\hat{s}_t^{i'}\Omega_{s_H,s_H}\hat{s}_t^i di + \frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^{i'}\Omega_{s_F,s_F}\hat{s}_t^i di + \int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_H}\int_0^{\frac{1}{2}}\hat{s}_t^i di \\ &+\int_{\frac{1}{2}}^{1}\hat{s}_t^i di'\Omega_{S_F,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di + \int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di - \int_0^{\frac{1}{2}}\hat{s}_t^{i'}\Omega_{s_H,u}\hat{u}_t^i di - \int_{\frac{1}{2}}^{1}\hat{s}_t^{i'}\Omega_{s_F,u}\hat{u}_t^i di\Big] \\ &+t.i.p. \end{aligned} \tag{129}$$

where

$$\begin{aligned} \Omega_{s_H,s_H} &\equiv W_{s,s} + (1-\varphi\zeta_b)H_{s_H,s_H} - (\xi-\zeta_b)\varphi F_{s_H,s_H} - \zeta_b K_{s_H,s_H} \\ \Omega_{s_F,s_F} &\equiv (1-\zeta_b\varphi)H_{s_F,s_F} - (\xi-\zeta_b)\varphi F_{s_F,s_F} - (\xi-\zeta_b)R_{s_F,s_F} \\ \Omega_{S_H,S_H} &\equiv \tfrac{1}{2}(1-\zeta_b\varphi)H_{S_H,S_H} - \tfrac{1}{2}(\xi-\zeta_b)\varphi F_{S_H,S_H} - \tfrac{1}{2}\zeta_b K_{S_H,S_H} - \tfrac{1}{2}(\xi-\zeta_b)R_{S_H,S_H} \\ \Omega_{S_F,S_F} &\equiv \tfrac{1}{2}(1-\zeta_b\varphi)H_{S_F,S_F} - \tfrac{1}{2}(\xi-\zeta_b)\varphi F_{S_F,S_F} - \tfrac{1}{2}\zeta_b K_{S_F,S_F} - \tfrac{1}{2}(\xi-\zeta_b)R_{S_F,S_F} \\ \Omega_{S_H,S_F} &\equiv (1-\zeta_b\varphi)H_{S_H,S_F} - (\xi-\zeta_b)\varphi F'_{S_F,S_H} - \zeta_b K_{S_H,S_F} - (\xi-\zeta_b)R'_{S_F,S_H} \\ \Omega_{s_H,u} &\equiv W_{s,u} - \zeta_b K_{s_H,u} \qquad \Omega_{s_F,u} \equiv -(\xi-\zeta_b)R_{s_F,u} \end{aligned} \tag{130}$$

and Ω_{s_H,s_H}, Ω_{s_F,s_F}, Ω_{S_H,S_H}, Ω_{S_F,S_F}, Ω_{S_H,S_F}, $\Omega_{s_H,u}$ and $\Omega_{s_F,u}$ are respectively equal to:

$$\begin{bmatrix} (1-\zeta_b(\varphi+1))\varphi & -\zeta_b\sigma\gamma_s & 0 \\ -\zeta_b\sigma\gamma_s & \omega_{sHsH} & 0 \\ 0 & 0 & \frac{(1-\zeta_b(\varphi+1))\varepsilon}{\lambda} \end{bmatrix}$$

$$\begin{bmatrix} -(\xi-\zeta_b)(\varphi+1)\varphi & -(\xi-\zeta_b)\sigma\gamma_s & 0 \\ -(\xi-\zeta_b)\sigma\gamma_s & \omega_{sFsF} & 0 \\ 0 & 0 & -\frac{((\xi-\zeta_b)(\varphi+1))\varepsilon}{\lambda} \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\zeta_b\sigma\left(\gamma_b-\gamma_s\right) & 0 \\ -\zeta_b\sigma\left(\gamma_b-\gamma_s\right) & \omega_{SHSH} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -(\xi-\zeta_b)\sigma\left(\gamma_b-\gamma_s\right) & 0 \\ -(\xi-\zeta_b)\sigma\left(\gamma_b-\gamma_s\right) & \omega_{SFSF} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\zeta_b\sigma\left(1-\gamma_b\right)-(\xi-\zeta_b)\sigma\left(1-\gamma_b\right) & 0 \\ -\zeta_b\sigma\left(1-\gamma_b\right)-(\xi-\zeta_b)\sigma\left(1-\gamma_b\right) & 2\omega_{SHSF} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} (1-\zeta_b(\varphi+1))(\varphi+1) & \zeta_b(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -(\xi-\zeta_b)(\varphi+1)^2 & (\xi-\zeta_b)(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with

$$\begin{aligned}
\omega_{sHsH} \equiv \quad & (\sigma-1)(1-\tilde{\tau}) \\
& +(1-\zeta_b\varphi)\left(-\eta\sigma^2\left(1-\gamma_s^2\right)+\delta_b-\omega_3\right) \\
& -(\xi-\zeta_b)\varphi(1-\delta_b+\omega_3) \\
& -\zeta_b\left(-\eta\sigma^2\gamma_s^2+\eta\sigma^2\gamma_b-\sigma^2\gamma_b\right) \\
& -(\xi-\zeta_b)\left(\eta\sigma^2(1-\gamma_b)-\sigma^2(1-\gamma_b)\right)
\end{aligned} \tag{131}$$

$$\tag{132}$$

$$\begin{aligned}
\omega_{sFsF} \equiv \quad & (1-\zeta_b\varphi)(1-\delta_b+\omega_3) \\
& -(\xi-\zeta_b)\varphi(-\eta\sigma^2\left(1-\gamma_s^2\right)+\delta_b-\omega_3) \\
& -\zeta_b\left(\eta\sigma^2(1-\gamma_b)-\sigma^2(1-\gamma_b)\right) \\
& +(\xi-\zeta_b)\left(\eta\sigma^2\gamma_s^2-\eta\sigma^2\gamma_b+\sigma^2\gamma_b\right)
\end{aligned} \tag{133}$$

$$\begin{aligned}
\omega_{SHSH} \equiv \quad & (1-\zeta_b\varphi)(\eta\sigma^2\left(1-\gamma_s^2-\gamma_b(1-\gamma_b)\right)+\omega_3) \\
& +(\xi-\zeta_b)\varphi(\eta\sigma^2\gamma_b(1-\gamma_b)+\omega_3) \\
& +\zeta_b\eta\sigma^2\left(\gamma_b^2-\gamma_s^2\right) \\
& +(\xi-\zeta_b)\eta\sigma^2(1-\gamma_b)^2
\end{aligned} \tag{134}$$

$$\begin{aligned}
\omega_{SFSF} \equiv \quad & -(1-\zeta_b\varphi)(\eta\sigma^2\gamma_b(1-\gamma_b)+\omega_3) \\
& -(\xi-\zeta_b)\varphi(\eta\sigma^2\left(1-\gamma_s^2-\gamma_b(1-\gamma_b)\right)+\omega_3) \\
& +\zeta_b\eta\sigma^2(1-\gamma_b)^2 \\
& +(\xi-\zeta_b)\eta\sigma^2\left(\gamma_b^2-\gamma_s^2\right)
\end{aligned} \tag{135}$$

$$\begin{aligned}
\omega_{SHSF} \equiv \quad & (1-\zeta_b\varphi)\eta\sigma^2\gamma_b(1-\gamma_b) \\
& -(\xi-\zeta_b)\varphi\eta\sigma^2\gamma_b(1-\gamma_b) \\
& +\zeta_b\eta\sigma^2(1-\gamma_b)\gamma_b \\
& +(\xi-\zeta_b)\eta\sigma^2(1-\gamma_b)\gamma_b
\end{aligned} \tag{136}$$

Now we rewrite the welfare approximation in (129) as:

$$Y^{\varphi+1}\sum_{t=0}^{\infty}\beta^t E_0\Big[\frac{1}{2}\int_0^{\frac{1}{2}}\left(\hat{s}_t^i-2\int_{\frac{1}{2}}^1\hat{s}_t^i di\right)'\Omega_{s_H,s_H}\left(\hat{s}_t^i-2\int_{\frac{1}{2}}^1\hat{s}_t^i di\right)di$$

$$+\frac{1}{2}\int_{\frac{1}{2}}^1\left(\hat{s}_t^i-2\int_{\frac{1}{2}}^1\hat{s}_t^i di\right)'\Omega_{s_F,s_F}\left(\hat{s}_t^i-2\int_{\frac{1}{2}}^1\hat{s}_t^i di\right)'di$$

$$-\int_0^{\frac{1}{2}}\left(\hat{s}_t^i-2\int_0^{\frac{1}{2}}\hat{s}_t^i di\right)'\Omega_{s_H,u}\left(\hat{u}_t^i-2\int_0^{\frac{1}{2}}\hat{u}_t^i di\right)di$$

$$-\int_{\frac{1}{2}}^1\left(\hat{s}_t^i-2\int_{\frac{1}{2}}^1\hat{s}_t^i di\right)'\Omega_{s_F,u}\left(\hat{u}_t^i-2\int_{\frac{1}{2}}^1\hat{u}_t^i di\right)di$$

$$+\int_0^{\frac{1}{2}}\hat{s}_t^i di'(\Omega_{s_H,s_H}+\Omega_{S_H,S_H})\int_0^{\frac{1}{2}}\hat{s}_t^i di$$

$$+\int_{\frac{1}{2}}^1\hat{s}_t^i di'(\Omega_{s_F,s_F}+\Omega_{S_F,S_F})\int_{\frac{1}{2}}^1\hat{s}_t^i di$$

$$+\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^i di$$

$$-2\int_0^{\frac{1}{2}}\hat{s}_t^{i'}di\Omega_{s_H,u}\int_0^{\frac{1}{2}}\hat{u}_t^i di$$

$$-2\int_{\frac{1}{2}}^1\hat{s}_t^{i'}di\Omega_{s_F,u}\int_{\frac{1}{2}}^1\hat{u}_t^i di\Big]+t.i.p. \tag{137}$$

Notice that the components expressed as the difference between specific country and average union variables can be considered terms independent of policy (even if they should be taken into account for welfare evaluation). Indeed movements in the common nominal interest rate can just influence the average union economic performance. Thus (137) can be read as:

$$Y^{\varphi+1}\sum_{t=0}^{\infty}\beta^t E_0\Big[\int_0^{\frac{1}{2}}\hat{s}_t^i di'(\Omega_{s_H,s_H}+\Omega_{S_H,S_H})\int_0^{\frac{1}{2}}\hat{s}_t^i di+\int_{\frac{1}{2}}^1\hat{s}_t^i di'(\Omega_{s_F,s_F}+\Omega_{S_F,S_F})\int_{\frac{1}{2}}^1\hat{s}_t^i di$$

$$+\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^i di-2\int_0^{\frac{1}{2}}\hat{s}_t^{i'}di\Omega_{s_H,u}\int_0^{\frac{1}{2}}\hat{u}_t^i di-2\int_{\frac{1}{2}}^1\hat{s}_t^{i'}di\Omega_{s_F,u}\int_{\frac{1}{2}}^1\hat{u}_t^i di\Big]+t.i.p.$$

The last step consists in rewriting (138) in terms of gaps with respect to the target of the policymaker of the monetary union. In order to do so first consider that by (138):

$$\frac{1}{2}Y^{\varphi+1}\sum_{t=0}^{\infty}\beta^t E_0\Big[\frac{1}{2}\hat{s}_{H,t}(\Omega_{s_H,s_H}+\Omega_{S_H,S_H})\hat{s}_{H,t}+\frac{1}{2}\hat{s}'_{F,t}(\Omega_{s_F,s_F}+\Omega_{S_F,S_F})\hat{s}_{F,t}$$

$$+\hat{s}_{H,t}\Omega_{S_H,S_F}\hat{s}_{F,t}-\hat{s}_{H,t}\Omega_{s_H,u}\hat{u}_{H,t}-\hat{s}_{F,t}\Omega_{s_F,u}\hat{u}_{F,t}\Big]+t.i.p.$$

Them it is easy to show that target is determined by maximizing (138) with respect $\hat{y}_{H,t}$, $\hat{y}_{F,t}$, $\hat{c}_{H,t}$, $\hat{c}_{F,t}$ and $\pi_{H,t}$ subject to:

$$\begin{aligned}\hat{y}_{H,t} &= \delta_b\hat{c}_{H,t} + (1-\delta_b)\hat{c}_{F,t} \qquad i \in \left[0, \frac{1}{2}\right) \\ \hat{y}_{F,t} &= \delta_b\hat{c}_{F,t} + (1-\delta_b)\hat{c}_{H,t} \qquad i \in \left[\frac{1}{2}, 0\right]\end{aligned} \tag{138}$$

In other words the target of the benevolent central bank of the monetary union coincides with the constrained efficient allocation (namely the allocation that a planner would choose having as objective (138)). According to the first order conditions with respect to $\hat{y}^b_{H,t}$, $\hat{y}^b_{F,t}$, $\hat{c}^b_{H,t}$, $\hat{c}^b_{F,t}$ and $\pi_{H,t}$:

$$\begin{aligned}&(1-\zeta_b(\varphi+1))\varphi\hat{y}^b_{H,t} - \zeta_b\sigma\left(\gamma_b\hat{c}^b_{H,t} + (1-\gamma_b)\,\hat{c}^b_{F,t}\right) - (1-\zeta_b(\varphi+1))(\varphi+1)\hat{a}_{H,t} \\ &-\zeta_b(\varphi+1)\hat{\mu}_{H,t} = \phi^H_{1,t}\end{aligned} \tag{139}$$

$$\begin{aligned}&-(\xi-\zeta_b)(\varphi+1)\varphi\hat{y}^b_{F,t} - (\xi-\zeta_b)\sigma\left(\gamma_b\hat{c}^b_{F,t} + (1-\gamma_b)\,\hat{c}^b_{H,t}\right) + (\xi-\zeta_b)(\varphi+1)^2\hat{a}_{F,t} \\ &-(\xi-\zeta_b)(\varphi+1)\hat{\mu}_{F,t} = \phi^F_{1,t}\end{aligned} \tag{140}$$

$$\begin{aligned}&[(\sigma-1)(1-\tilde{\tau}) + (1-\zeta_b\varphi)\delta_b - (\xi-\zeta_b)\varphi(1-\delta_b)]\,\hat{c}^b_{H,t} - \zeta_b\sigma\gamma_b\hat{y}^b_{H,t} - (\xi-\zeta_b)\sigma\,(1-\gamma_b)\,\hat{y}^b_{F,t} \\ &+\zeta_b\sigma^2\gamma_b\left(\gamma_b\hat{c}^b_{H,t} + (1-\gamma_b)\,\hat{c}^b_{F,t}\right) - (\xi-\zeta_b)\sigma^2\,(1-\gamma_b)\left(\gamma_b\hat{c}^b_{F,t} + (1-\gamma_b)\,\hat{c}^b_{H,t}\right) \\ &+(1-\xi\varphi)\eta\sigma^2\gamma_b(1-\gamma_b)(\hat{c}^b_{H,t} - \hat{c}^b_{F,t}) = -(\delta_b\phi^H_{1,t} + (1-\delta_b)\phi^F_{1,t})\end{aligned} \tag{141}$$

$$\begin{aligned}&[(1-\zeta_b\varphi)(1-\delta_b) - (\xi-\zeta_b)\varphi\delta_b]\,\hat{c}^b_{F,t} - (\xi-\zeta_b)\sigma\gamma_b\hat{y}^b_{F,t} - \zeta_b\sigma(1-\gamma_b)\hat{y}^b_{H,t} \\ &+(\xi-\zeta_b)\sigma^2\gamma_b\left(\gamma_b\hat{c}^b_{F,t} + (1-\gamma_b)\,\hat{c}^b_{H,t}\right) + \zeta_b\sigma^2(1-\gamma_b)\left(\gamma_b\hat{c}^b_{H,t} + (1-\gamma_b)\,\hat{c}^b_{F,t}\right) \\ &-(1-\xi\varphi)\eta\sigma^2\gamma_b(1-\gamma_b)(\hat{c}^b_{F,t} - \hat{c}^b_{H,t}) = -(\delta_b\phi^F_{1,t} + (1-\delta_b)\phi^H_{1,t})\end{aligned} \tag{142}$$

$$(1-\zeta_b(\varphi+1))\frac{\varepsilon}{\lambda}\pi_{H,t} = 0 \tag{143}$$

where $\phi^H_{1,t}$ and $\phi^F_{1,t}$ are the lagrange multipliers of constraints (47) and (48). Then it can be shown that (129) corresponds to (81) (again by adding and subtracting the target in each term of (129) and then using the conditions just listed) where:

$$\begin{aligned}\varrho_H \equiv\ &\big[(\sigma-1)(1-\tilde{\tau}) + (1-\zeta_b(\varphi+1))\delta_b - (\xi-\zeta_b)\varphi(1-\delta_b) - (1-\zeta_b\varphi)\eta\sigma^2\gamma_b(1-\gamma_b) \\ &+\zeta_b\sigma^2\gamma_b^2 + (\xi-\zeta_b)\sigma^2\,(1-\gamma_b)^2\,\big] \\ \varrho_F \equiv\ &\big[(1-\zeta_b(\varphi+1))(1-\delta_b) - (\xi-\zeta_b)\varphi\delta_b + \zeta_b\sigma^2(1-\gamma_b)^2 + (1-\zeta_b\varphi)\eta\sigma^2\gamma_b(1-\gamma_b) \\ &+(\xi-\zeta_b)\sigma^2\gamma_b^2\big] \\ \varrho_{H,F} \equiv\ &(1-\zeta_b\varphi)\eta\sigma^2\gamma_b(1-\gamma_b) + \zeta_b\sigma^2\gamma_b(1-\gamma_b) + (\xi-\zeta_b)\sigma^2\,(1-\gamma_b)\,\gamma_b\end{aligned}$$