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Polynomial identities for partitions

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For any partition λ of an integer n, we write

$$\lambda = < 1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots, n^{m_n(\lambda)} >$$

where $m_i(\lambda)$ is the number of parts equal to i. We denote by $r(\lambda)$ the number of parts of λ (i.e. $r(\lambda) = \sum_{i=1}^{n} m_i(\lambda)$). Recall that the notation $\lambda \vdash n$ means that λ is a partition of n.

For $1 \leq k \leq N$, let e_k be the k-th elementary symmetric function in the variables $x_1, \dots, x_N$, let h_k be the sum of all monomials of total degree k and let $p_k = \sum_i x_i^k$. Each of the sets $\{e_k\}$, $\{h_k\}$ or $\{p_k\}$ is a set of algebraically independent generators of the ring of symmetric functions over the field $\mathbb{Q}$. The expressions of h_n and e_n in terms of the p_k are the following [Mac, page 17]:

$$h_n = \sum_{\lambda \vdash n} \prod_{k=1}^{n} \frac{p_k^{m_k(\lambda)}}{k^{m_k(\lambda)} \cdot m_k(\lambda)!} \quad ,$$

$$(-1)^n e_n = \sum_{\lambda \vdash n} (-1)^{r(\lambda)} \prod_{k=1}^{n} \frac{p_k^{m_k(\lambda)}}{k^{m_k(\lambda)} \cdot m_k(\lambda)!} \quad .$$

By specialization of the variables, one then obtains various interesting identities, see [Mac].

In this note we present in a special case two identities which are similar in nature to the above two, except that the multiplicative behaviour is slightly different: instead of raising p_k to the m-th power (where $m = m_k(\lambda)$), we take one of the following products

$$p_k(p_k + k)(p_k + 2k) \dots (p_k + (m-1)k) ,$$

$$\text{or} \quad p_k(p_k - k)(p_k - 2k) \dots (p_k - (m-1)k) .$$

In the special case we consider, the e_n, h_n and p_k are replaced by certain polynomials in a variable q, which we now define.

Let q be an indeterminate and for every positive integer k, consider the polynomial $P_k(q)$ defined inductively by the equations $\sum_{d|k} P_d(q) = q^k - 1$. By Möbius inversion, this is equivalent to the formula $P_k(q) = \sum_{d|k} \mu(k/d)(q^d - 1)$, where μ denotes the ordinary Möbius function of number theory. Since $\sum_{d|k} \mu(k/d) = 0$ if $k \neq 1$ we obtain

$$P_k(q) = \begin{cases} \sum_{d|k} \mu(k/d) q^d & \text{if } k \neq 1, \\ q - 1 & \text{if } k = 1. \end{cases}$$

When q is a power of a prime, $P_k(q)$ has a natural interpretation in terms of finite fields (see Lemma 1.2). For every positive integer m we define

$$P_{k,m}^{+}(q) = P_k(q)\,(P_k(q) + k)\,(P_k(q) + 2k) \dots (P_k(q) + (m-1)k) ,$$

$$P_{k,m}^{-}(q) = P_k(q)\,(P_k(q) - k)\,(P_k(q) - 2k) \dots (P_k(q) - (m-1)k) .$$

Thus in particular $P_{k,1}^{+}(q) = P_{k,1}^{-}(q) = P_k(q)$. For $m = 0$, we extend this definition by setting $P_{k,0}^{+}(q) = P_{k,0}^{-}(q) = 1$.

THEOREM A. *Let n be a positive integer. Then*

$$\sum_{\lambda\vdash n}\prod_{k=1}^{n}\frac{P^{+}_{k,m_k(\lambda)}(q)}{k^{m_k(\lambda)}\cdot m_k(\lambda)!} = q^n - q^{n-1}\,.$$

THEOREM B. *Let n be a positive integer. Then*

$$\sum_{\lambda\vdash n}(-1)^{r(\lambda)}\prod_{k=1}^{n}\frac{P^{-}_{k,m_k(\lambda)}(q)}{k^{m_k(\lambda)}\cdot m_k(\lambda)!} = -(q-1)\,.$$

For $n=1$, both results reduce to the tautology $q-1=q-1$. For $n=2$, we obtain respectively

$$\begin{aligned}\frac{(q-1)q}{2}+\frac{q^2-q}{2}&=q^2-q\,,\\ \frac{(q-1)(q-2)}{2}-\frac{q^2-q}{2}&=-(q-1)\,,\end{aligned}$$

and for $n=3$, we have

$$\begin{aligned}\frac{(q-1)q(q+1)}{6}+(q-1)\frac{(q^2-q)}{2}+\frac{(q^3-q)}{3}&=q^3-q^2\,,\\ -\frac{(q-1)(q-2)(q-3)}{6}+(q-1)\frac{(q^2-q)}{2}-\frac{(q^3-q)}{3}&=-(q-1)\,.\end{aligned}$$

For larger n, the complexity of the formulas grows rapidly. Note that if $k=p^a$ is a prime power, then $P_{p^a}(q)=q^{p^a}-q^{p^{a-1}}$. Thus the first occurence of a polynomial $P_k(q)$ having more than two terms appears for $n=6$ where $P_6(q)=q^6-q^3-q^2+q$.

It is sometimes convenient to define $\tilde{P}_k(q)=P_k(q)/k$ and then

$$\begin{aligned}\tilde{P}^{+}_{k,m}(q)&=\tilde{P}_k(q)\,(\tilde{P}_k(q)+1)\,(\tilde{P}_k(q)+2)\ldots(\tilde{P}_k(q)+(m-1))\,,\\ \tilde{P}^{-}_{k,m}(q)&=\tilde{P}_k(q)\,(\tilde{P}_k(q)-1)\,(\tilde{P}_k(q)-2)\ldots(\tilde{P}_k(q)-(m-1))\,.\end{aligned}$$

With this notation, each term in the product in either theorem has the form

$$\frac{\tilde{P}^{+}_{k,m}(q)}{m!}\quad\text{or}\quad\frac{\tilde{P}^{-}_{k,m}(q)}{m!}\,,$$

and this is simply a binomial coeeficient.

Since a polynomial is determined by its values for infinitely many choices of the variable, it is clear that it suffices to prove the theorems when $q=p^a$ is a prime power. The proof in that case is based on a known (but non trivial) result about the general linear group $GL_n(\mathbb{F}_q)$ over the finite field $\mathbb{F}_q$ with q elements. This result asserts that the number of semi-simple conjugacy classes of $GL_n(\mathbb{F}_q)$ is equal to q^n-q^{n-1}. For this reason, we make no claim to originality as far as Theorem A is concerned: the proof simply consists in a combinatorial count of the number of semi-simple classes of $GL_n(\mathbb{F}_q)$ (Section 1). The proof of the second theorem (Section 4) is more elaborate. It involves a Möbius inversion in the poset of parabolic subgroups of $GL_n(\mathbb{F}_q)$ (Section 2), followed by an analysis of the fixed points in the building of $GL_n(\mathbb{F}_q)$ under the action of a semi-simple element of the group (Section 3). As several formulas appearing in this paper find their origin in the modular representation theory of finite groups (in relationship with a recent conjecture of Alperin), we explain the connections in Section 5.

We wish to raise two questions suggested by the results above:

(1) Is there a combinatorial proof of the theorems? One can hope for a more general combinatorial setting which would specialize to our two statements. Suitable generating functions are expected to play some role. A positive answer to this question would lead to a new proof of the above mentioned fact that the number of semi-simple classes of $GL_n(\mathbb{F}_q)$ is equal to $q^n - q^{n-1}$.

(2) In the case mentioned at the beginning of this introduction, there is an involution which interchanges the role of e_n and h_n, see [Mac]. Is there some similar phenomenon in our situation? A positive answer would allow to prove only one of the results and deduce the other one by applying the involution. Note in this direction that if one changes the sign of $P_k(q)$, then $P^+_{k,m}(q)$ is replaced by $(-1)^m P^-_{k,m}(q)$.

1. Semi-simple classes of the general linear group

Let G denote the general linear group $GL_n(\mathbb{F}_q)$ of invertible $n \times n$ matrices over the finite field $\mathbb{F}_q$ with q elements, where q is a power of a prime p. Each $g \in G$ acts on the vector space $V = \mathbb{F}_q^n$ and therefore defines a structure of $\mathbb{F}_q[t]$-module on V, such that $t \cdot v = gv$ for every $v \in V$. Two elements of G are conjugate if and only if the corresponding $\mathbb{F}_q[t]$-modules are isomorphic. Since the polynomial ring $\mathbb{F}_q[t]$ is a principal ideal domain, every $\mathbb{F}_q[t]$-module decomposes uniquely (up to isomorphism) as a direct sum of indecomposable cyclic modules, that is, of the form $\mathbb{F}_q[t]/(f^r)$ where f is an irreducible polynomial and $r \geq 1$. Since we are interested in the action of $g \in G$, which is invertible, the possibility $f = t$ has to be excluded. A Jordan canonical form for g can be deduced from this: g is conjugate to a matrix which decomposes with diagonal blocks, each block being the companion matrix of some polynomial f^r as above.

Now $g \in G$ is called *semi-simple* if the corresponding $\mathbb{F}_q[t]$-module is semi-simple, that is, if each integer r above is equal to 1. In other words the minimal polynomial of g must be square-free. Since an irreducible polynomial f over a finite field has no multiple root, the simple module $\mathbb{F}_q[t]/(f)$ decomposes over an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ as a direct sum of one-dimensional $\overline{\mathbb{F}}_q[t]$-modules of the form $\overline{\mathbb{F}}_q[t]/(t-a)$. Thus $g \in G$ is semi-simple if and only if it is conjugate in $GL_n(\overline{\mathbb{F}}_q)$ to a diagonal matrix.

For later use we also recall that $g \in G$ is semi-simple if and only if g is p-regular, that is, the order of g is prime to p (where p is the characteristic of the field $\mathbb{F}_q$). Indeed the minimal polynomial of g divides $t^h - 1$ where h is the order of g, and the cyclotomic polynomial $t^h - 1$ has no multiple root when p does not divide h, hence is square-free. Conversely if p divides h, then it suffices to show that $g' = g^{h/p}$ is not semi-simple. But since g' has order p, its minimal polynomial f is a divisor of $t^p - 1 = (t-1)^p$, but is not $t - 1$. Thus $f = (t-1)^k$ is not square-free.

By a *semi-simple class* of G, we mean a conjugacy class of semi-simple elements of G, and this is the same as a p-regular class. We first record a crucial result for the present paper.

(1.1) THEOREM (Steinberg). *The number $\ell(G)$ of semi-simple classes of the group $G = GL_n(\mathbb{F}_q)$ is equal to $q^n - q^{n-1}$.*

Proof. We briefly indicate two different proofs. The first consists in applying Theorem 3.7.6 of [Car] which asserts that $\ell(G) = |Z|\, q^r$ where Z is the centre of G and r is the semi-simple rank of the algebraic group $GL_n(\overline{\mathbb{F}}_q)$, that is, $r = n-1$. Note that the assumption of that theorem is that the derived group of the algebraic group should be simply-connected (in the sense of the theory of algebraic groups), and this is indeed the case here because the derived group is $SL_n(\overline{\mathbb{F}}_q)$. Since Z is the group of scalar matrices, we have $|Z| = q-1$ and we obtain $\ell(G) = (q-1)q^{n-1}$ as required.

The second proof is based on the property that the number $\ell(G)$, which is the number of p-regular classesof G, is also (by a well-known theorem of Brauer [CR, 17.11]) the number of isomorphism classes of irreducible representations of G over an algebraically closed field k of characteristic p (e.g. $k = \overline{\mathbb{F}}_q$). This number is known for finite groups of Lie type, see [CR, 72.29]. Explicitly let T be the group of all diagonal matrices and let T_i be the subgroup of matrices of the form $\mathrm{diag}(1,\ldots,1,a,a^{-1},1,\ldots,1)$ where a and a^{-1} occur at the i-th and $(i+1)$-th position ($1 \le i \le n-1$). Finally for every subset I of $S = \{1,\ldots,n-1\}$, let T_I be the subgroup generated by all T_i for $i \in I$. Then by [CR, 72.29], we have $\ell(G) = \sum_{I\subseteq S} |T/T_I|$. Now $|T| = (q-1)^n$ and $|T_I| = (q-1)^{|I|}$, from which it follows that

$$\sum_{I\subseteq S} |T/T_I| = \sum_{k=0}^{n-1} \binom{n-1}{k} (q-1)^{n-k} = (q-1)\left((q-1)+1\right)^{n-1} = (q-1)q^{n-1},$$

as was to be shown. □

Each semi-simple class of G contains a matrix g in canonical form, that is, with blocks on the diagonal such that each block is the companion matrix g_f of an irreducible polynomial f (and with zeros outside the blocks). A permutation of the blocks does not change the conjugacy class, so that one can always arrange the sizes of the blocks in decreasing order. Grouping together all blocks of the same size k (i.e. all irreducible polynomials f_i of the same degree k), we define m_k to be the number of blocks of size k. Then it is clear that $\lambda = < 1^{m_1}, \ldots, n^{m_n} >$ is a partition of n. We call it *the partition associated* with the semi-simple class.

Conversely with any partition λ of n are associated several semi-simple classes of G: we must have $m_k = m_k(\lambda)$ blocks of size k and it suffices to choose m_k irreducible polynomial of degree k and fill in the blocks with their companion matrices, in any order.

These observations immediately lead to a proof of Theorem A.

Proof of Theorem A. We count the number of semi-simple clases of $G = GL_n(\mathbb{F}_q)$. It is equal to $q^n - q^{n-1}$ by Theorem 1.1 and this provides the right hand side of the formula in Theorem A. The left hand side is obtained by summing over all partitions λ of n the number of ways of choosing $m_k(\lambda)$ irreducible polynomials of degree k (regardless of their order), for each $1 \le k \le n$. This is the number of *weak compositions* of $m_k = m_k(\lambda)$ into r parts, where r is the number of irreducible polynomials of degree k over $\mathbb{F}_q$ (different from the polynomial t if $k = 1$). Here a weak composition of m_k is a decomposition $m_k = x_1 + \ldots + x_r$ in integers $x_i \ge 0$, and the number of such weak compositions is equal to the binomial coefficient

$$\binom{m_k + r - 1}{m_k},$$

see [St, page 15]. Now $r = \tilde{P}_k(q)$ by Lemma 1.2 below, and therefore the binomial coefficient above is equal to

$$\frac{\tilde{P}^{+}_{k,m_k}(q)}{m_k!},$$

so that the total number of semi-simple classes of G is

$$\sum_{\lambda \vdash n} \prod_{k=1}^{n} \frac{\tilde{P}^{+}_{k,m_k(\lambda)}(q)}{m_k(\lambda)!} \,.$$

This completes the proof. $\square$

(1.2) LEMMA. *Let $\mathbb{F}_q$ be the finite field with q elements and let $k \geq 1$ be an integer.*

(a) The number of non-zero elements in $\mathbb{F}_{q^k}$ which do not belong to an extension of $\mathbb{F}_q$ properly contained in $\mathbb{F}_{q^k}$ is equal to $P_k(q)$.

(b) The number of irreducible polynomials of degree k over $\mathbb{F}_q$, distinct from the polynomial t (if $k = 1$), is equal to $\tilde{P}_k(q) = P_k(q)/k$.

Proof. (a) Since any intermediate field extension has the form F_{q^d} where $d|k$, we have

$$q^k - 1 = |\mathbb{F}^*_{q^k}| = \sum_{d|k} P_d(q) \,.$$

(b) Every element of $\mathbb{F}_{q^k}$, not lying in any proper subextension of $\mathbb{F}_q$ is a root of an irreducible polynomial of degree k over $\mathbb{F}_q$, and every irreducible polynomial of degree k over $\mathbb{F}_q$ has k distinct roots in $\mathbb{F}_{q^k}$. All roots are non-zero except in the case of the polynomial t. Thus we obtain that the number of irreducible polynomials of degree k, distinct from t, is equal to $P_k(q)/k = \tilde{P}_k(q)$. $\square$

2. Parabolic subgroups and inversion

In this section we review some properties of parabolic subgroups and inversion. Let $V = \mathbb{F}_q^n$ be the n-th dimensional vector space over $\mathbb{F}_q$ with basis $(e_1, \ldots, e_n)$. Consider the maximal flag of subspaces

$$\mathcal{F} = (V_1 < V_2 < \ldots < V_{n-1})$$

where $V_i =< e_1, \ldots, e_i >$. Let $S = \{1, \ldots, n-1\}$ and for each subset $J \subseteq S$, consider the subflag $\mathcal{F}_J$ consisting of the subspaces V_j for $j \in J$. The parabolic subgroup P_J is the stabilizer of $\mathcal{F}_J$ in the group $GL_n(\mathbb{F}_q)$. In particular $P_\emptyset = GL_n(\mathbb{F}_q)$, and $P_S = B$ is the Borel subgroup of upper triangular matrices. We have clearly

$$I \subseteq J \iff P_I \supseteq P_J \,. \tag{2.1}$$

For each $J \subseteq S$, the unipotent subgroup U_J is the normal subgroup of P_J consisting of those elements which induce the identity on each successive quotient of the flag $\mathcal{F}_J$ (including the smallest subspace in $\mathcal{F}_J$ and the quotient of V by the largest subspace). In particular $U_\emptyset = \{1\}$ and U_S is the full unipotent subgroup of upper triangular matrices with entries equal to 1 on the diagonal, and this a Sylow p-subgroup of $GL_n(\mathbb{F}_q)$ (where p is the characteristic of $\mathbb{F}_q$). The passage from P_J to U_J is order-reversing. So in particular every U_J is a subgroup of U_S, hence is a p-group.

If $J = \{j_1, j_2, \ldots, j_k\}$ (where $1 \leq j_1 < j_2 < \ldots j_k \leq n-1$), we consider the action of P_J on the successive quotients $V_{j_1} \oplus V_{j_2}/V_{j_1} \oplus \ldots \oplus V_{j_k}/V_{j_{k-1}} \oplus V/V_{j_k}$. Then we have by definition

$$P_J/U_J \cong GL_{j_1}(\mathbb{F}_q) \times GL_{j_2-j_1}(\mathbb{F}_q) \times \ldots \times GL_{j_k-j_{k-1}}(\mathbb{F}_q) \times GL_{n-j_k}(\mathbb{F}_q),$$

and so P_J/U_J is a direct product of $|J|+1$ general linear groups.

In order to use induction arguments (in the form of a Möbius inversion, see Proposition 2.4), we generalize our setting to direct products of general linear groups. Thus for the group $GL_{n_1}(\mathbb{F}_q) \times \ldots \times GL_{n_k}(\mathbb{F}_q)$, we consider the sets $S_i = \{1, \ldots, n_i - 1\}$ and their disjoint union

$$S = S_1 \times \{1\} \bigcup \ldots \bigcup S_k \times \{k\}. \tag{2.2}$$

Any subset J of S is a disjoint union

$$J = J_1 \times \{1\} \bigcup \ldots \bigcup J_k \times \{k\},$$

where $J_i \subseteq S_i$. The parabolic subgroup P_J is the direct product $P_J = P_{J_1} \times \ldots \times P_{J_k}$ and similarly for the unipotent group U_J and for the quotient P_J/U_J.

Now define

$$c(GL_{n_1}(\mathbb{F}_q) \times \ldots \times GL_{n_k}(\mathbb{F}_q)) = k \qquad \text{(the number of factors)},$$

$$r(GL_{n_1}(\mathbb{F}_q) \times \ldots \times GL_{n_k}(\mathbb{F}_q)) = \sum_{i=1}^{k}(n_i - 1) = |S| \qquad \text{(the rank)}.$$

Then whenever G is a finite direct product of general linear groups over $\mathbb{F}_q$, we define

$$\ell(G) = (q-1)^{c(G)}\, q^{r(G)},$$
$$z(G) = (q-1)^{c(G)}.$$

By Theorem 1.1, $\ell(G)$ is the number of semi-simple classes of G, that is also the number of classes of elements of order prime to p. Indeed this number behaves multiplicatively with respect to direct products of groups (since clearly $c(G)$ and $r(G)$ behave additively). The number $z(G)$ is the order of the centre of G, but it has another important characterization in the modular representation theory (see Section 5).

(2.3) LEMMA. *If G is a finite direct product of general linear groups and P_J a parabolic subgroup of G, then*

$$c(P_J/U_J) = c(G) + |J|,$$
$$r(P_J/U_J) = r(G) - |J|.$$

Proof. Since all these numbers behave additively, it suffices to check the formulas when $G = GL_n(\mathbb{F}_q)$, that is, $c(G) = 1$. From a remark above we already know that $c(P_J/U_J) = 1 + |J|$. On the other hand if $J = \{j_1, \ldots, j_k\}$, then

$$r(P_J/U_J) = (j_1 - 1) + (j_2 - j_1 - 1) + \ldots + (j_k - j_{k-1} - 1) + (n - j_k - 1) = n - 1 - k = r(G) - |J|,$$

which completes the proof. □

Now we can state the main result of this section.

(2.4) PROPOSITION. *Let* G *be a finite direct product of general linear groups over* $\mathbb{F}_q$ *and let* S *be the corresponding subset as in (2.2). Then*

(a) $\ell(G) = \sum_{J\subseteq S} z(P_J/U_J)$,

(b) $z(G) = \sum_{J\subseteq S} (-1)^{|J|}\,\ell(P_J/U_J)$.

Proof. (a) This is straightforward by Lemma 2.3:

$$\begin{aligned}\sum_{J\subseteq S} z(P_J/U_J) &= \sum_{J\subseteq S}(q-1)^{c(G)+|J|}\\ &= (q-1)^{c(G)}\sum_{k=0}^{|S|}\binom{|S|}{k}(q-1)^k\\ &= (q-1)^{c(G)}\,((q-1)+1)^{|S|}\\ &= (q-1)^{c(G)}\,q^{r(G)} = \ell(G)\,.\end{aligned}$$

(b) If $I\subseteq J$, then $P_I\supseteq P_J\supseteq U_J\supseteq U_I$ by (2.1). Then P_J/U_I is a parabolic subgroup of P_I/U_I with corresponding unipotent normal subgroup U_J/U_I and quotient $(P_J/U_I)/(U_J/U_I)\cong P_J/U_J$. Moreover any parabolic subgroup of P_I/U_I arises in this way. Therefore the formula (a) applied to the group P_I/U_I yields

$$\ell(P_I/U_I) = \sum_{I\subseteq J\subseteq S} z(P_J/U_J)\,.$$

By Möbius inversion in the poset of all subsets of S , we have

$$z(P_I/U_I) = \sum_{I\subseteq J\subseteq S} \mu(I,J)\,\ell(P_J/U_J)\,.$$

Taking $I=\emptyset$, we have $\mu(\emptyset,J)=(-1)^{|J|}$ and therefore

$$z(G) = \sum_{J\subseteq S}(-1)^{|J|}\,\ell(P_J/U_J)\,,$$

as required. □

3. Fixed points of the building

Let $V=\mathbb{F}_q^n$ be the n-th dimensional vector space over the finite field $\mathbb{F}_q$ with basis $(e_1,\ldots,e_n)$. Let $\overline{\mathcal{P}}(V)$ be the poset of all subspaces of V and let $\mathcal{P}(V)$ be the proper part of $\overline{\mathcal{P}}(V)$, that is $\mathcal{P}(V)=\overline{\mathcal{P}}(V)-\{0,V\}$. The *building* of the general linear group $GL_n(\mathbb{F}_q)$ is the simplicial complex $\Delta=\Delta(\mathcal{P}(V))$ associated with $\mathcal{P}(V)$ (i.e. the order complex of $\mathcal{P}(V)$). By definition a k-simplex in Δ is a flag of subspaces $W_0<W_1<\ldots<W_k$ with $W_i\in\mathcal{P}(V)$. If $V_i=<e_1,\ldots,e_i>$, the $(n-2)$-simplex $V_1<\ldots<V_{n-1}$ is called the *standard maximal flag*, and its subflags are called *standard flags*.

The set of dimensions of the subspaces in a flag σ is a subset of $\{1,\ldots,n-1\}$, called the *type* of σ . From elementary linear algebra, all flags of the same type are in the same orbit under the action of the group $G=GL_n(\mathbb{F}_q)$. In particular every orbit contains a unique standard flag and thus Δ/G can be identified with the set of standard flags. Moreover the stabilizer G_σ of a simplex σ of Δ is conjugate to the stabilizer of a standard flag, that is, a parabolic subgroup. (Note that throughout this paper a parabolic subgroup refers only to what is usually called a standard parabolic subgroup, namely the stabilizer of a *standard* flag.)

Before we proceed any further, we recall that if a group H acts on a poset X by order-preserving maps, then the set of orbits X/H is again a poset: for $x, y \in X$, then by definition the relation $Hx \leq Hy$ between the orbits of x and y holds if there exists $h \in H$ such that $hx \leq y$. Now H also acts on the simplicial complex $\Delta(X)$ and on its geometric realization $|\Delta(X)|$ and we can consider the orbit space $|\Delta(X)|/H$. Under a strong condition on the action of H, the order complex $\Delta(X/H)$ associated with the poset X/H has a geometric realization which is homeomorphic to $|\Delta(X)|/H$. The condition is that X should be a regular H-poset in the sense of Bredon [Br, page 116], that is: whenever $\sigma = (x_0 < \ldots < x_k)$ and $\tau = (y_0 < \ldots < y_k)$ are k-simplices such that for all i there exists $h_i \in H$ with $h_i x_i = y_i$, then there exists $h \in H$ such that $h\sigma = \tau$. (We warn the reader that this notion of regular poset does not coincide with the one defined by Curtis-Reiner [CR, 66.4].) All the posets we shall consider will turn out to be regular.

We return now to the group $G = GL_n(\mathbb{F}_q)$, the G-poset $\mathcal{P}(V)$ and the building $\Delta = \Delta(\mathcal{P}(V))$. We note that $\mathcal{P}(V)$ is a regular G-poset, because G acts transitively on the set of flags of a given type. For $g \in G$, we denote by $\mathcal{P}(V)^g$ the poset of g-fixed points in $\mathcal{P}(V)$, and by $\Delta^g = \Delta(\mathcal{P}(V)^g)$ the corresponding simplicial complex. The centralizer $C_G(g)$ acts on $\mathcal{P}(V)^g$ and we shall prove below that $\mathcal{P}(V)^g$ is a regular $C_G(g)$-poset.

We are interested in the orbit complex $\Delta^g/C_G(g)$, which appears in our next result where the reduced Euler characteristic $\tilde{\chi}(\Delta^g/C_G(g)) = \chi(\Delta^g/C_G(g)) - 1$ comes into play. We denote by $[g]$ the conjugacy class of $g \in G$.

(3.1) PROPOSITION. *Let* Δ *be the building of* $G = GL_n(\mathbb{F}_q)$. *Then*

$$\sum_{\mathrm{semi-simple}\ [g]} \tilde{\chi}(\Delta^g/C_G(g)) = -z(G),$$

where $[g]$ *runs over the set of all semi-simple classes of* G.

Proof. From Proposition 2.4, we know that

$$z(G) = \sum_{J \subseteq S} (-1)^{|J|}\, \ell(P_J/U_J)$$

where $S = \{1, \ldots, n-1\}$. First we use the fact that the number $\ell(P_J/U_J)$ of p-regular clases of P_J/U_J is equal to $\ell(P_J)$, because U_J is a normal p-subgroup of P_J. (This is an easy group-theoretic exercise, using for instance the Schur-Zassenhaus theorem. Alternatively one can use the fact that $\ell(P_J)$ is the number of irreducible representations of P_J in characteristic p and that the normal p-subgroup U_J always acts trivially on an irreducible representation [CR, 17.16].) Now P_J is the stabilizer G_σ of a standard flag σ, and we can interpret the sum above as running over the set Δ/G of standard flags, including the empty flag (corresponding to $P_\emptyset = G$). The cardinality $|J|$ of the flag σ is equal to $\dim(\sigma) + 1$ (with the empty simplex of dimension -1). Therefore

$$\begin{aligned}
z(G) &= \sum_{\sigma\in[\Delta/G]} (-1)^{\dim(\sigma)+1}\,\ell(G_\sigma) = -\sum_{\sigma\in\Delta} \frac{(-1)^{\dim(\sigma)}}{|G:G_\sigma|}\,\ell(G_\sigma) \\
&= -\sum_{\sigma\in\Delta} \frac{(-1)^{\dim(\sigma)}}{|G:G_\sigma|} \sum_{p-\text{regular } g\in G_\sigma} \frac{1}{|G_\sigma : C_{G_\sigma}(g)|} \\
&= -\sum_{p-\text{regular } g\in G} \frac{1}{|G:C_G(g)|} \sum_{\sigma\in\Delta^g} \frac{(-1)^{\dim(\sigma)}}{|C_G(g):C_G(g)_\sigma|} \\
&= -\sum_{p-\text{regular } [g]} \ \sum_{\sigma\in[\Delta^g/C_G(g)]} (-1)^{\dim(\sigma)} \\
&= -\sum_{p-\text{regular } [g]} \tilde{\chi}(\Delta^g/C_G(g))\,.
\end{aligned}$$

Note that we end up with the *reduced* Euler characteristic because the empty simplex appears in the sum. □

Our strategy is to compute explicitly $\tilde{\chi}(\Delta^g/C_G(g))$ for each semi-simple class $[g]$ of G. As it does not require more effort to determine the homotopy type of $\Delta^g/C_G(g)$, we do so and single out in the process the result for the reduced Euler characteristic (which is also the Möbius number of the corresponding poset). We fix $g\in G$ and we have to work with the proper part $\mathcal{P}(V)^g/C_G(g)$ of the poset $\overline{\mathcal{P}}(V)^g/C_G(g)$.

An element W of $\overline{\mathcal{P}}(V)$ (that is, a subspace of V) is invariant under g if and only if it is an $\mathbb{F}_q[t]$-submodule of V, where t acts on V via g, as in Section 1. We decompose V as a direct sum of simple $\mathbb{F}_q[t]$-submodules, and then for each simple $\mathbb{F}_q[t]$-module X we group together all simples isomorphic to X into an isotypical component V_X of type X. Then V_X can be written $V_X = X\otimes Y$ where Y is a vector space whose dimension is the multiplicity of X in V_X. Therefore

$$V = \bigoplus_{i=1}^{s} X_i\otimes Y_i$$

where s is the number of isotypical components. Any $\mathbb{F}_q[t]$-submodule of V has the form

$$V = \bigoplus_{i=1}^{s} X_i\otimes Z_i$$

where Z_i is an arbitrary subspace of Y_i. It follows that the poset $\overline{\mathcal{P}}(V)^g$ is isomorphic to

$$\overline{\mathcal{P}}(V)^g \cong \overline{\mathcal{P}}(Y_1)\times\ldots\times\overline{\mathcal{P}}(Y_s)$$

where $\overline{\mathcal{P}}(Y_i)$ is the poset of subspaces of Y_i. Of course we consider the product order in this decomposition.

Now we introduce the action of $C_G(g)$. By definition of the centralizer, it is the group of $\mathbb{F}_q[t]$-automorphisms of V. Any $\mathbb{F}_q[t]$-endomorphism of V stabilizes every isotypical component and so $\mathrm{End}_{\mathbb{F}_q[t]}(V)$ is the direct product of the rings

$$\mathrm{End}_{\mathbb{F}_q[t]}(X_i\otimes Y_i)\cong K_i\otimes \mathrm{End}_{\mathbb{F}_q}(Y_i)\,,$$

where $K_i = \mathrm{End}_{\mathbb{F}_q[t]}(X_i)$ (a finite field extension of $\mathbb{F}_q$ by Schur's lemma). Now any $\alpha\otimes 1$ with $\alpha\in K_i$ acts trivially on the poset of $\mathbb{F}_q[t]$-submodules of $X_i\otimes Y_i$ (because it stabilizes each simple summand of $X_i\otimes Y_i$). Thus we only need to consider the action of $\mathrm{End}_{\mathbb{F}_q}(Y_i)$ on $\overline{\mathcal{P}}(Y_i)$, and in fact only $GL(Y_i)$ comes into play since we are considering automorphisms. It follows from this analysis that

$$\overline{\mathcal{P}}(V)^g/C_G(g)\cong \overline{\mathcal{P}}(Y_1)/GL(Y_1)\times\ldots\times\overline{\mathcal{P}}(Y_s)/GL(Y_s)\,. \tag{3.2}$$

Note that it is now clear that $\overline{\mathcal{P}}(V)^g$ is a regular $C_G(g)$-poset (hence also $\mathcal{P}(V)^g$), since we know that each $\overline{\mathcal{P}}(Y_i)$ is a regular $GL(Y_i)$-poset.

(3.3) PROPOSITION. *Let* Δ *be the building of the group* $G = GL_n(\mathbb{F}_q)$ *and let* $g \in G$ *.*

(a) If g *is semi-simple and if at least one isotypical component of* g *has multiplicity* ≥ 2 *, then* $\Delta^g/C_G(g)$ *is contractible. In particular* $\tilde{\chi}(\Delta^g/C_G(g)) = 0$ *.*

(b) If g *is semi-simple and if every isotypical component of* g *has multiplicity 1, then the geometric realization of* $\Delta^g/C_G(g)$ *is homeomorphic to a sphere* S^{s-2} *, where* s *is the number of isotypical components. In particular* $\tilde{\chi}(\Delta^g/C_G(g)) = (-1)^s$ *.*

(c) If g *is not semi-simple, then* Δ^g *is* $C_G(g)$*-contractible. In particular* $\Delta^g/C_G(g)$ *is contractible and* $\tilde{\chi}(\Delta^g/C_G(g)) = 0$ *.*

Proof. We first assume that g is semi-simple and use the decomposition 3.2. As before $\mathcal{P}(Y_i)$ denotes the proper part of $\overline{\mathcal{P}}(Y_i)$, and similarly for orbit posets. We first describe each factor in the decomposition 3.2.

Since $GL(Y_i)$ acts transitively on the subspaces of a given dimension,

$$\mathcal{P}(Y_i)/GL(Y_i) = [d_i - 1]$$

where $[d_i - 1] = \{1, \ldots, d_i - 1\}$ and $d_i = \dim(Y_i)$. This is contractible if $d_i \geq 2$ (because there is a maximal element in the poset) and empty if $d_i = 1$. (In terms of simplicial complexes, this corresponds to the fact that taking the type of flags allows to identify $\Delta(\mathcal{P}(Y_i))/GL(Y_i)$ with the poset of non-empty subsets of $[d_i - 1]$, and this is just a simplex of dimension $d_i - 2$, hence contractible if $d_i \geq 2$ and empty if $d_i = 1$.) Of course we also have $\overline{\mathcal{P}}(Y_i)/GL(Y_i) = \overline{[d_i - 1]} = \{0, \ldots, d_i\}$, and this is a lattice with proper part $[d_i - 1]$. Note that d_i is the multiplicity of the i-th isotypical component. Note also that the reduced Euler characteristic of $[d_i - 1]$ (which can be viewed as the Möbius function $\mu(0, d_i)$) is equal to 0 if $d_i \geq 2$ and -1 if $d_i = 1$. The decomposition 3.2 is the direct product of lattices

$$\overline{\mathcal{P}}(V)^g/C_G(g) \cong \overline{[d_1 - 1]} \times \ldots \times \overline{[d_s - 1]} \,.$$

Now we can prove (a). If $d_i \geq 2$ for some i, then $(0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 in i-th position) has no complement in the lattice (since 1 does not have a complement in $\overline{[d_i - 1]}$). It is well-known that this implies the contractibility of the proper part of the lattice. In fact we have conical contractibility in the sense of Quillen [Qu, 1.5], because

$$(a_1, \ldots, a_s) \leq (a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_s) \geq (0, \ldots, 0, 1, 0, \ldots, 0) \,.$$

In terms of Möbius functions, which behave multiplicatively with respect to direct products, we have

$$\tilde{\chi}\big(\mathcal{P}(V)^g/C_G(g)\big) = \prod_{j=1}^{s} \mu(0, d_j) = 0$$

since $\mu(0, d_i) = 0$.

Turning to the proof of (b), we assume that $d_i = 1$ for all i. Then the lattice $\overline{\mathcal{P}}(V)^g/C_G(g)$ is isomorphic to the direct product of s copies of the poset with 2 elements, that is, the lattice of all subsets of a set of cardinality s. Therefore the simplicial complex associated to the proper part of the lattice consists of all proper faces of a simplex of cardinality s (i.e. of dimension $s-1$). Thus one obtains the boundary of an $(s-1)$-simplex, which is homeomorphic to a sphere of dimension $s-2$. The reduced Euler characteristic is $(-1)^{s-2}$. This is obtained directly in terms of Möbius functions, since one has to multiply s times the value -1.

Finally we recall the proof of (c), which is standard. Some power h of g has order p by assumption. Since the minimal polynomial of h is a divisor of $t^p - 1 = (t-1)^p$ different from $t-1$, the subspace V^h of h-invariant points of V is a non-zero proper subspace, hence an element of $\mathcal{P}(V)^g$. It is invariant under g because g commutes with h. Then $\mathcal{P}(V)^g$ is conically contractible via

$$W \geq W \cap V^h \leq V^h ,$$

where $W \cap V^h \neq 0$ since it is the eigenspace of the action of h on W. The conical contraction above is $C_G(g)$-equivariant, so $\mathcal{P}(V)^g$ is $C_G(g)$-contractible (see [TW] for more details). This implies that the orbit poset is contractible. □

We have included statement (c) for completeness. Note that it holds more generally for the fixed points Δ^H under the action of a subgroup H having a non-trivial normal p-subgroup, using the fact that a p-group always has non-zero fixed points when acting on a vector space in characteristic p.

4. Proof of Theorem B

After having prepared the grounds in the previous sections, the proof of Theorem B is now easy. We use the formula of Proposition 3.1 for the group $G = GL_n(\mathbb{F}_q)$. The number $-z(G)$ is equal to $-(q-1)$ by definition, and this gives the right hand of the formula in Theorem B. Thus we have to show that the other side of the formula 3.1, namely

$$\sum_{\mathrm{semi-simple}\ [g]} \tilde{\chi}(\Delta^g / C_G(g)) ,$$

gives the sum over all partitions of n which appears in Theorem B.

By Proposition 3.3, the reduced Euler characteristic above vanishes, unless g has isotypical components with multiplicity 1. In that case we shall say that the class $[g]$ has defect zero, because the condition is equivalent to the requirement that $C_G(g)$ be of order prime to p (see the analysis in the last section), and this is the definition of a class of defect zero in representation theory.

We know from Section 1 that a partition is associated with every semi-simple class. Conversely with any partition λ of n are associated several semi-simple classes of G: we must have $m_k = m_k(\lambda)$ blocks of size k and it suffices to choose m_k irreducible polynomial of degree k (distinct from the polynomial t when $k=1$) and fill in the blocks with their companion matrices, in any order. In order to get a class of defect zero, all isotypical components must have multiplicity 1, and so the only new requirement is that the m_k irreducible polynomials of degree k must be all *distinct*, for each $1 \leq k \leq n$. Thus the number of semi-simple classes of defect zero associated with a partition λ is equal to

$$\prod_{k=1}^{n} \binom{\tilde{P}_k(q)}{m_k(\lambda)} ,$$

because by Lemma 1.2, $\tilde{P}_k(q)$ is the number of irreducible polynomials of degree k, distinct from t.

For any semi-simple class $[g]$ of defect zero associated with the partition λ, the number s of isotypical components for the action of g on V is the same as the number of irreducible summands of the action of g (because all multiplicities are 1), and this is the number of parts of the partition, i.e.

$$s = \sum_k m_k(\lambda) = r(\lambda)\,.$$

Therefore $\tilde{\chi}(\Delta^g/C_G(g)) = (-1)^s = (-1)^{r(\lambda)}$ by Proposition 3.3 and we obtain

$$\begin{aligned}\sum_{\mathrm{semi-simple}\ [g]} \tilde{\chi}(\Delta^g/C_G(g)) &= \sum_{\lambda\vdash n}\prod_{k=1}^{n}(-1)^{r(\lambda)}\binom{\tilde{P}_k(q)}{m_k(\lambda)}\\ &= \sum_{\lambda\vdash n}\prod_{k=1}^{n}(-1)^{r(\lambda)}\prod_{k=1}^{n}\frac{P^-_{k,m_k(\lambda)}(q)}{k^{m_k(\lambda)}\cdot m_k(\lambda)!}\,.\end{aligned}$$

This completes the proof. □

5. Connections with modular representation theory

Several formulas of the previous sections find their origin in the modular representation theory of finite groups, specialized to the case of the general linear group. In this section we explain how these formulas generalize to important conjectures for arbitrary finite groups.

For an arbitrary finite group G and a prime p, we consider the irreducible p-modular representations of G, namely the irreducible kG-modules, where k is an algebraically closed field of characteristic p. As before we write $\ell(G)$ for the number of isomorphism classes of irreducible kG-modules. By a well-known theorem of Brauer [CR, 17.11], it is also the number of p-regular conjugacy classes of G, and Steinberg's result 1.1 asserts that for $G = GL_n(\mathbb{F}_q)$ in natural characteristic p, we have $\ell(G) = q^n - q^{n-1}$.

Among the p-modular representations, we consider the irreducible kG-modules which are projective modules over the group algebra kG. The number of isomorphism classes of such modules is written $z(G)$. It is also the number of p-blocks of G of defect zero. When $G = GL_n(\mathbb{F}_q)$, it is well known that the Steinberg module St is a projective irreducible kG-module in characteristic p. Now $GL_n(\mathbb{F}_q)$ has an abelian quotient $GL_n(\mathbb{F}_q)/SL_n(\mathbb{F}_q)$ of order $q-1$ and the tensor product of St with any of the $q-1$ possible representations of dimension 1 is again a projective irreducible kG-module. It is also known that this gives the complete list of projective irreducible kG-modules, so that $z(G) = q-1$. Thus we recover the notation of the previous sections.

Alperin's conjecture [Al] asserts that for an arbitrary finite group G, the number $\ell(G)$ is equal to

$$\ell(G) = \sum_P z(N_G(P)/P)\,,$$

where P runs over the set of all p-subgroups of G up to conjugation (including $P = 1$). For $G = GL_n(\mathbb{F}_q)$ in natural characteristic p, this specializes to the statement (a) in Proposition 2.4, because only the p-subgroups U_J give a non-zero contribution. Thus Alperin's conjecture holds for $GL_n(\mathbb{F}_q)$ and the prime p (in fact also for the other primes). For groups of Lie type this was proved by Alperin in some cases and by Cabanes [Cab] in general.

The second statement in Proposition 2.4 is obtained from the first one by a Möbius inversion. The same idea can be applied to an arbitrary finite group G, using the poset $S_p(G)$ of non-trivial p-subgroups instead of the parabolic subgroups. This was first observed by Robinson [KR] who proved that Alperin's conjecture is equivalent to the following statement:

$$z(G) = \sum_{\sigma \in [\Delta(S_p(G))/G]} (-1)^{\dim(\sigma)} \ell(G_\sigma) \,,$$

where $\Delta(S_p(G))$ is the order complex of $S_p(G)$ and G_σ denotes the stabilizer of the simplex σ. In fact $\Delta(S_p(G))$ can be replaced by any G-complex which is G-homotopy equivalent to it (see [We, §6]). For a finite group of Lie type (such as $GL_n(\mathbb{F}_q)$), this fact applies with the building of G (see [TW] for the G-homotopy equivalence). Thus the second formula in Proposition 2.4 is a special case of Robinson's conjectural formula (using also the fact that $\ell(P_J) = \ell(P_J/U_J)$).

Finally the proof of Proposition 3.1 generalizes without change to an arbitrary finite group, using $\Delta(S_p(G))$ instead of the building. This appears in [Th, 6.3] and one gets the conjectural formula

$$-z(G) = \sum_{p-\text{regular } [g]} \tilde{\chi}\big(\Delta(S_p(G))^g / C_G(g)\big) \,.$$

Proposition 3.1 is a special case of this formula.

The crucial tool for the proof of Theorem B is Proposition 3.1. We could have proved this proposition by applying the results mentioned in this section, but it would not have been a very direct approach and it would not have shortened the proof substantially. This is why we have preferred a more streamlined approach.

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