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Conditions for recovery of sparse signals correlated by local transforms

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***Abstract*—This paper addresses the problem of correct recovery of multiple sparse correlated signals using distributed thresholding. We consider the scenario where multiple sensors capture the same event, but observe different signals that are correlated by local transforms of their sparse components. In this context, the signals do not necessarily have the same sparse support, but instead the support of one signal is built on local transforms of the atoms in the sparse support of another signal. We establish the sufficient condition for the correct recovery of such correlated signals using independent thresholding of the multiple signals. The condition is relevant in scenarios where low complexity processing such as thresholding is needed, for example in sensor networks. The validity of the derived recovery condition is confirmed by experimental results in noiseless and noisy scenarios.**

***Index Terms*—sparse approximations, thresholding, local transforms**

I. Introduction

Sparse signal approximation refers to a particular signal representation as a linear combination of a few functions (atoms) chosen from an overcomplete dictionary. In the last decade sparse approximations have attracted a lot of attention, partly due to the fact that they offer convenient solutions for signal coding and compression. With the design of flexible dictionaries, sparse representations have also found applications in signal analysis and distributed signal processing.

This paper deals with sparse approximations of correlated signals, where sparse components of different signals are linked with local transforms. A variety of correlated signal sets can be modeled this way, like videos and multi-view images at low bit rates, seismic signals, etc. In particular, we establish the sufficient condition for the recovery of sparse components of such correlated signals using thresholding. Thresholding is a fast algorithm where sparse components are simply chosen as those that have the highest inner product with the signal. However, in case of redundant dictionaries, thresholding does not guarantee to find the correct signal elements. The sufficient condition for the correct signal recovery by thresholding has been given in [1]. When a given signal satisfies the recovery condition, thresholding becomes an interesting approach for sparse recovery, due to its very low complexity compared to other sparse approximation algorithms (Matching Pursuit, Basis Pursuit Denoising). Moreover, thresholding can be seen as a part of the complexity-adaptive signal approximation strategy [2], where the signal is approximated using thresholding if the recovery condition is satisfied, or using more complex algorithms otherwise. Therefore, it is crucial to perform the worst case analysis and derive the sufficient condition under which a sparse signal could be recovered by thresholding.

Recovery of correlated signals by thresholding has been considered in [3], where the correlated signals share a common sparse support, but are observed under different noisy conditions. Sparse representation of correlated signals has been also presented within the concept of distributed compressed sensing for two correlation models: common sparse component with sparse innovations, and common sparse supports [4]. The recovery algorithms in [4] are based on greedy and convex optimizations algorithms. The thresholding recovery analysis for correlated signals that we present here differs from the previous work [3], [4] in one major assumption: we do not require the signals to share the same support (i.e., to have exactly the same atoms in the representation). Instead, we allow each atom in one signal to have its corresponding atom in another signal, which is obtained by a local transform such as shift, scaling, or any combination of those. These signals can be, for example, obtained by a set of sensors that look at the same event, but record different observations, as shown on the Fig. 1. We derive the sufficient recovery condition for this correlation model, validate it on randomly generated 1D signals and illustrate its usage on seismic signals.

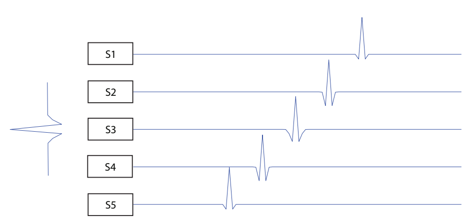

Fig. 1. Sensors observe the same event and record correlated observations.

II. Sparse correlation model

We consider two signals y_1 and y_2 that have sparse representations in dictionaries Φ and Ψ, respectively. We assume that the signals are not exactly sparse, but they can be approximated by sparse decompositions of m atoms up to an approximation error, i.e.:

$$\begin{aligned} y_1 &= \Phi_I a_1 + e_1 = \sum_{k=1}^{m} a_{1,k}\phi_{i_k} + e_1 \\ y_2 &= \Psi_J a_2 + e_2 = \sum_{k=1}^{m} a_{2,k}\psi_{j_k} + e_2, \end{aligned} \quad (1)$$

where $I = \{i_k\}, J = \{j_k\}, k = 1, ...m$ label the sets of atoms that participate in the sparse decompositions of y_1 and y_2, respectively. Matrices Φ_I and Ψ_J denote respectively sub-matrices of Φ and Ψ with respect to I and J. We are particularly interested in signals that are correlated by local transforms of the atoms in different signals. The correlation model is described in the rest of this section through two assumptions.

Assumption 1. We assume that signals y_1 and y_2 are correlated in the following way:

$$y_2 = \sum_{k=1}^{m} a_{2,k}\psi_{j_k} + e_2 = \sum_{k=1}^{m} a_{2,k}F_k(\phi_{i_k}) + e_2, \quad (2)$$

where $F_k(\cdot)$ denotes a transform of an atom ϕ_{i_k} in y_1 to an atom ψ_{j_k}, and it differs for each $k = 1, ..., m$.

In particular, we consider a special class of transforms F, which result from a linear transform of the coordinate system in the space $\mathcal{H}$ where the signal and the dictionaries are defined. Let $\mathbf{v}$ denote the vector of coordinates in $\mathcal{H}$ and $\mathbf{u}$ denote the vector of coordinates obtained by transforming $\mathbf{v}$ with an arbitrary linear transform T, i.e., $\mathbf{u} = T(\mathbf{v})$. Let further atom ϕ be defined as a continuous function on $\mathcal{H}$ normalized to have the l_2 norm equal to one, i.e., $\phi = g(\mathbf{v})/\|g\|$. Equivalently, we define $\psi = h(\mathbf{v})/\|h\|$. We consider the atom transforms F which satisfy:

$$h(\mathbf{v}) = F(g(\mathbf{v})) = g(T(\mathbf{v})) = g(\mathbf{u}). \quad (3)$$

The second equality in Eq.(3) directly defines the class of transforms F considered in this work, which result from a linear transform T. These types of transforms have been shown to be of great practical use, for example in dictionary design for sparse image approximation [5]. The considered transforms can be illustrated by the following example.

Example 1: Consider a function $g(x) \in \mathbb{R}$ and an overcomplete dictionary obtained by various local transforms of $g(x)$, like shifts and scaling. These transforms can be realized by a single transform of the x coordinate, which includes translation b and anisotropic scaling s; i.e., $x' = (x-b)/s$. This class of transforms obviously satisfies Eq.(3), where $\mathbf{v} = x$ and $\mathbf{u} = x'$.

We assume further that the signals satisfy the local transforms applied to each atom, within the local support of that atom:

Assumption 2. For all k and $\mathbf{v}$ such that $\phi_{i_k}(\mathbf{v}) \neq 0$, it holds:

$$y_2(\mathbf{v}) = y_1(T_k(\mathbf{v})) = y_1(\mathbf{u}). \quad (4)$$

The type of signal correlation under different local transforms given by Assumptions 1 and 2 can be found in many practical cases where the same signal is observed by sensors at different positions. Locality of the transforms is highly important in practical cases as different parts of the signal can be captured under different transforms, like for example at different distances to the sensor. Since the signal correlation model includes a noise component, slight deviations from the assumed model (e.g., occlusions, interference) can be considered as noise components and hence the signal correlation model is not very restrictive.

We are now interested in establishing the conditions under which thresholding, performed independently on each signal, recovers the correct sparse representations of signals y_1 and y_2. This can be stated as follows:

Problem 1: Assume that we are given two correlated signals y_1 and y_2 in Eq.(1), and the assumptions 1 and 2 hold. Suppose that thresholding recovers the correct sparsity pattern I of the signal y_1. We want to derive the sufficient condition for the correct sparse recovery of the sparsity pattern J of the signal y_2 using thresholding, without having all the information about the atom transforms.

III. Single signal thresholding

We review here the conditions under which simple thresholding algorithm recovers correctly the sparse representation of the signals [1]. Before describing their result, let us define some functions that will be used throughout the analysis. *Setwise cumulative coherence function*, defined in [6], is given as:

$$\mu_1(\Phi, I) := \sup_{k \notin I} \sum_{i \in I} |\langle \phi_k, \phi_i \rangle|.$$

Next, the *Dictionary Inter Symbol Interference* is given as:

$$ISI(\Phi, I) := \mu_1(\Phi, I) + \sup_{l \in I} \mu_1(\Phi_I, I \setminus \{l\}).$$

This function measures the interference of atoms in the sparse decomposition that can lead to incorrect recovery. We will denote the second term in the expression for $ISI(\Phi, I)$ as $\chi(\Phi_I, I)$, i.e.,

$$\chi(\Phi_I, I) := \sup_{l \in I} \mu_1(\Phi_I, I \setminus \{l\}).$$

The recovery condition is given by the following theorem:

Theorem 1 (Gribonval, Nielsen, Vandergheynst [2]): Let $y = \Phi x + e$ be a noisy sparse representation of the data. Moreover, assume $x_{l_k}, k = 1, ...|I|$ are the $|I|$ nonzero components of x in decreasing order of magnitude, i.e., $|x_{l_1}| \geq |x_{l_2}| \geq ... \geq |x_{l_{|I|}}|$. If for a certain m, $1 \leq m \leq |I|$, the following condition is satisfied:

$$\frac{|x|_{l_m}}{||x||_\infty} > \frac{||\Phi_I^* e||_\infty + ||\Phi_{\bar{I}}^* e||_\infty}{||x||_\infty} + ISI(\Phi, I) \quad (5)$$

then each inner product of the observed data y with the atoms $\{\phi_{l_i}\}_{1 \leq i \leq m}$ exceeds all the inner products with the atoms $\{\phi_i\}_{i \in I \setminus \{l_1,...,l_m\}}$ indexed by the complementary set

$\bar{I}$. In particular the $m = |I|$ largest inner products correspond exactly to the support I of x.

For the general proof, please see the generalization of the theorem to the multi-channel case [1].

IV. Thresholding of correlated signals

We will first assume that the sparsity pattern I of the signal y_1 can be recovered by thresholding, i.e., that signal y_1 satisfies:

$$\frac{|a_{1,m}|}{||a_1||_\infty} > \frac{||\Phi_I^* e_1||_\infty + ||\Phi_{\bar{I}}^* e_1||_\infty}{||a_1||_\infty} + ISI(\Phi, I). \quad (6)$$

Before establishing the recovery conditions for the signal y_2, we prove the following lemma:

Lemma 1: Let two correlated signals y_1 and y_2 in Eq.(1) be correlated by the model in Eq.(2), with transforms defined by Eq.(3). Let further assume that the condition in Eq.(4) holds. Then, for all $k = 1, ..., m$, it holds:

$$\langle y_2, \psi_{j_k} \rangle = C_k \langle y_1, \phi_{i_k} \rangle,$$

where:

$$C_k = \frac{1}{\sqrt{|\frac{\partial T_k(\mathbf{v})}{\partial \mathbf{v}}|}} = \frac{1}{\sqrt{|\frac{\partial \mathbf{u}}{\partial \mathbf{v}}|}} = \frac{1}{\sqrt{J_k}},$$

and $J_k = |\frac{\partial \mathbf{u}}{\partial \mathbf{v}}|$ is the Jacobian of the linear transform T_k.

Proof: From the definition of the inner product, we have:

$$\langle y_2, \psi_{j_k} \rangle = \int_{\widetilde{S}_k} y_2(\mathbf{v}) \psi_{j_k}(\mathbf{v}) d\mathbf{v},$$

where $\widetilde{S}_k$ represents the subspace where $\psi_{j_k}(\mathbf{v}) \neq 0$. Substituting $\psi_{j_k} = h(\mathbf{v})/\|h\|$, we get:

$$\langle y_2, \psi_{j_k} \rangle = \int_{\widetilde{S}_k} y_2(\mathbf{v}) \frac{h(\mathbf{v})}{\|h\|} d\mathbf{v}. \quad (7)$$

The l_2 norm $\|h\|$ can be evaluated as follows:

$$\|h\| = \sqrt{\langle h, h \rangle} = \sqrt{\int_{\widetilde{S}_k} h^2(\mathbf{v}) d\mathbf{v}} = \sqrt{\int_{S_k} g^2(\mathbf{u}) |\frac{\partial \mathbf{v}}{\partial \mathbf{u}}| d\mathbf{u}},$$

where S_k represents the subspace where $g(\mathbf{u}) \neq 0$ (or equivalently, where $\phi_{i_k}(\mathbf{u}) \neq 0$). The last equality is obtained by applying a change of variables using the Eq.(3) and Eq.(4) for $\mathbf{u} = T_k(\mathbf{v})$. Since T_k is defined as a linear transform of coordinates, the mapping $S_k \to \widetilde{S}_k$ is smooth, and the change of variables holds. Furthermore, we have that $|\partial \mathbf{v}/\partial \mathbf{u}| = 1/J_k$ does not depend on $\mathbf{v}$, and it can go in front of the integral:

$$\|h\| = \frac{1}{\sqrt{J_k}} \|g\|. \quad (8)$$

We can now go back to the Eq.(7) and similarly apply a change of variables, using Eq.(3), (4) and (8) and obtain:

$$\langle y_2, \psi_{j_k} \rangle = \int_{S_k} y_1(\mathbf{u}) \sqrt{J_k} \frac{g(\mathbf{u})}{\|g\|} \frac{1}{J_k} d\mathbf{u}.$$

Finally, $1/\sqrt{J_k}$ can go in front of the integral as a constant C_k and we get:

$$\langle y_2, \psi_{j_k} \rangle = 1/\sqrt{J_k} \int_{S_k} y_1(\mathbf{u}) \phi_{j_k}(\mathbf{u}) d\mathbf{u} = C_k \langle y_1, \phi_{i_k} \rangle. \blacksquare$$

If we go back now to our Example 1, we have the Jacobian that is equal to: $|\frac{\partial(x')}{\partial(x)}| = 1/s_k$ and hence $C_k = \sqrt{s_k}$, where s_k is the scale transform for each pair of correlated atoms (ϕ_{i_k}, ψ_{j_k}).

We can now give the recovery condition for the signal y_2:

Theorem 2: Suppose we are given two correlated signals $y_1 = \Phi_I a_1 + e_1$ and $y_2 = \Psi_J a_2 + e_2$, which satisfy assumptions 1 and 2. Furthermore, suppose that thresholding recovers the correct sparsity pattern I of the signal y_1, i.e., the condition given by Eq.(6) is satisfied. If for all $k = 1, ..., m$ the following sufficient condition is satisfied:

$$\begin{aligned} C_k |a_{1,m}| &> C_k \|a_1\|_\infty \chi(\Phi, I) + C_k \|\Phi_I^* e_1\|_\infty \\ &+ \|\Psi_{\bar{J}}^* e_2\|_\infty + \|a_2\|_\infty \mu_1(\Psi, J) \end{aligned} \quad (9)$$

where $C_k = 1/\sqrt{|\frac{\partial T_k(\mathbf{v})}{\partial \mathbf{v}}|}$, then thresholding recovers all sparse components of the signal y_2, i.e., each inner product of the signal y_2 with the atoms $\{\psi_{j_k}\}_{1 \le k \le m}$ exceeds all the inner products with the atoms $\{\psi_i\}_{i \in J \setminus \{j_1, ..., j_m\}}$ indexed by the complementary set $\bar{J}$.

Proof: We start the proof by bounding the inner products of the signal y_1 with $\{\phi_{i_k}\}, k = 1, ..., m$, similarly to the proof of the Theorem 9 in [1]:

$$\begin{aligned} \{\langle y_1, \phi_{i_k} \rangle\}_{i_k \in I} &= \Phi_I^* y_1 = \Phi_I^* \Phi_I a_1 + \Phi_I^* e_1 \\ &= a_1 + (\Phi_I^* \Phi_I - I_d) a_1 + \Phi_I^* e_1, \end{aligned}$$

where the i_k-th term $1 \le k \le m$ can be bounded as follows:

$$\begin{aligned} &|\langle y_1, \phi_{i_k} \rangle| \ge |a_{1,k}| - \|(\Phi_I^* \Phi_I - I_d) a_1\|_\infty - \|\Phi_I^* e_1\|_\infty \\ &\ge |a_{1,m}| - \|(\Phi_I^* \Phi_I - I_d) a_1\|_\infty - \|\Phi_I^* e_1\|_\infty. \end{aligned} \quad (10)$$

From Lemma 1, we have $\langle y_2, \psi_{j_k} \rangle = C_k \langle y_1, \phi_{i_k} \rangle$. When combined with Eq.(10) it gives the following inequality:

$$|\langle y_2, \psi_{j_k} \rangle| \ge C_k (|a_{1,m}| - \|(\Phi_I^* \Phi_I - I_d) a_1\|_\infty - \|\Phi_I^* e_1\|_\infty).$$

In order for $|\langle y_2, \psi_{j_k} \rangle|$ to be recovered by thresholding, the following condition has to be satisfied for all $1 \le k \le m$:

$$|\langle y_2, \psi_{j_k} \rangle| > \sup_{l \notin J} |\langle y_2, \psi_l \rangle|.$$

We have that:

$$\sup_{l \notin J} |\langle y_2, \psi_l \rangle| = \|\Psi_{\bar{J}}^* y_2\|_\infty \le \|\Psi_{\bar{J}}^* e_2\|_\infty + \|\Psi_{\bar{J}}^* \Psi_J a_2\|_\infty,$$

so we have to show that the condition in Eq.(9) implies the following inequality:

$$\begin{aligned} &C_k (|a_{1,m}| - \|(\Phi_I^* \Phi_I - I_d) a_1\|_\infty - \|\Phi_I^* e_1\|_\infty) > \\ &\|\Psi_{\bar{J}}^* e_2\|_\infty + \|\Psi_{\bar{J}}^* \Psi_J a_2\|_\infty, \end{aligned}$$

or equivalently:

$$\begin{aligned} C_k |a_{1,m}| &> C_k \|(\Phi_I^* \Phi_I - I_d) a_1\|_\infty + C_k \|\Phi_I^* e_1\|_\infty \\ &+ \|\Psi_{\bar{J}}^* e_2\|_\infty + \|\Psi_{\bar{J}}^* \Psi_J a_2\|_\infty, \end{aligned} \quad (11)$$

Tropp has shown in [7] that the following inequalities hold:

$$\begin{aligned} &\frac{\|(\Phi_I^* \Phi_I - I_d) x\|_\infty}{\|x\|_\infty} \le \||(\Phi_I^* \Phi_I - I_d)\||_{\infty,\infty} = \\ &= \||(\Phi_I^* \Phi_I - I_d)\||_{1,1} = \sup_{l \in I} \mu_1(\Phi_I, I \setminus \{l\}), \end{aligned} \quad (12)$$

$$\frac{\|\Phi_{\bar{I}}^*\Phi_I x\|_\infty}{\|x\|_\infty} \leq |||\Phi_{\bar{I}}^*\Phi_I|||_{\infty,\infty} = |||\Phi_{\bar{I}}^*\Phi_I|||_{1,1} = \mu_1(\Phi, I). \quad (13)$$

Therefore, we have:

$$\begin{aligned} \|(\Phi_I^*\Phi_I - I_d)a_1\|_\infty &\leq \|a_1\|_\infty \sup_{l \in I} \mu_1(\Phi_I, I \setminus \{l\}) = \\ &= \|a_1\|_\infty \chi(\Phi, I), \\ \|\Psi_{\bar{J}}^*\Psi_J a_2\|_\infty &\leq \|a_2\|_\infty \mu_1(\Psi, J). \end{aligned}$$

We can thus conclude that the condition given in Eq.(9) that has the same right hand side term as Eq.(11), but lower bounded by Eq.(12) and Eq.(13) implies also the condition in Eq.(11). ■ The derived condition in Eq.(9) represents the worst case analysis solution and in general case, it is not tight. The novel condition does not include the value of $\chi(\Psi, J)$ as the condition in Eq.(5) for signal y_2 would include when the correlation model is not considered. Furthermore, in order to test the condition in Eq.(9) we do not need the value of the smallest coefficient $|a_{2,m}|$ in the sparse decomposition of y_2, which is needed in order to test the condition in Eq.(5) for signal y_2. On the other hand, in Eq.(9) we need to have the values of the constant C_k. Note however that we do not need to know the local transforms between sparse components, but only the values C_k. If we use the dictionary in Example 1, we would just need to know the transform of scales between sparse components, or the bound on these transforms. Therefore, the new condition can be tested using less information about the signal y_2 than one would need in order to test the Eq.(5).

V. Experimental results

A. Randomly generated 1D signals

The sufficient condition given by the novel theorem 2 has been verified on pairs of one-dimensional synthetic signals. We have generated a dictionary of size M=1000, for signals of length N=700. We have constructed a parametric dictionary, where a generating function undergoes random shift and scaling operations to generate the different atoms in the dictionary. We have used the second derivative of the Gaussian as the generating function, i.e., $g(x) = (4x^2 - 2)\exp(-(x^2))$. The dictionary has been constructed by applying the coordinate transform $x' = (x - b)/s$. The shifts b have been selected randomly from 1 to N, while the scales s have been chosen randomly from a uniform distribution on the logarithmic scale from -1 to 3. All atoms have been normalized to have the unit norm. The Jacobian of this transform is $1/s$, and hence we have the constant $C = \sqrt{s}$. The same dictionary has been used for both signals, hence $\Phi = \Psi$.

We have performed experiments in noiseless and noisy scenarios. In both cases the signal y_1 has been chosen such that condition in Eq.(6) is fulfilled, for different values of the sparsity m. The sparsity pattern I and the coefficients a_1 have been chosen randomly. To construct the correlated signal y_2 we have randomly chosen different transforms $T_k, k = 1, ..., m$ defined by b_k and s_k, for each atom in the sparse support of the signal, from the range $(-2, 2)$ for b and $(1, s_{max})$ for the scale s. The transformed atoms from I then yield the sparse support for the signal y_2, denoted as J, and also give the values of C_k for each pair of atoms $(\phi_{i,k}, \psi_{j,k}), k = 1, ..., m$. The two signals have been constructed so that they verify the Lemma 1.

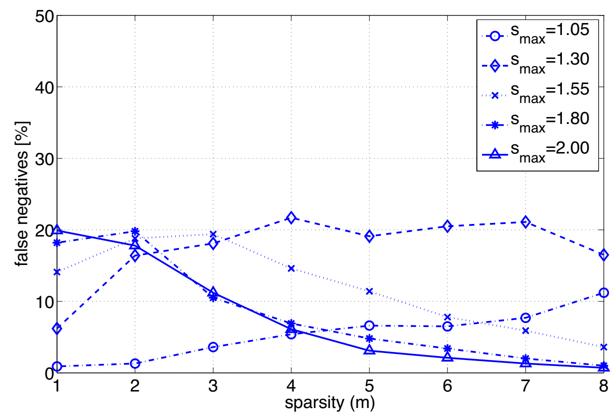

Fig. 2. Number of false negatives versus the sparsity m, for different values of the maximal scaling parameters.

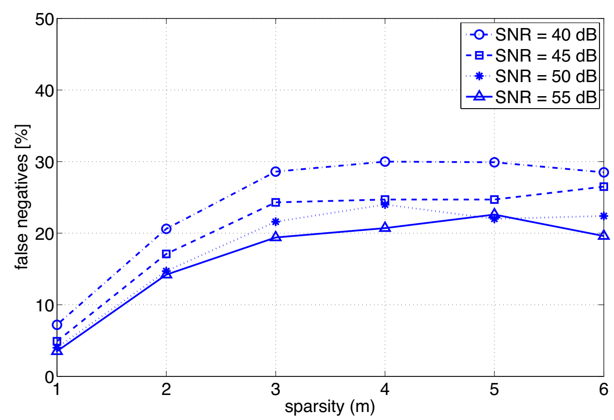

Fig. 3. Number of false negatives versus the the sparsity m for different SNR, the maximum scale is set constant $s = 1.5$.

We have verified the sufficient condition in Eq.(9) by running experiments over 10 different realizations of the dictionary. For each dictionary we have performed 100 trials on y_1 and y_2 constructed as explained above. No false positive has been recorded, and all components where Eq.(9) holds have been recovered. This is because the condition in Eq.(9) is sufficient. The condition of Eq.(9) is however not necessary, as it is based on a worst case analysis. In order to evaluate the quality of the bound, we count the number of false negatives (when the condition is not fulfilled but thresholding still recovers the correct J). On Fig. 2 we plot the percentage of false negatives (FN) depending on the number of sparse components m, for different maximal values of the transform scale s_{max} between all pairs of signals y_1 and y_2. The maximum number of false negatives appears towards smaller sparsity values m when s_{max} increases. For larger m, the smaller number of

FN appears for large s_{max}. Although this might look counter-intuitive, it can be easily explained. In order for the signal y_1 to be recovered by thresholding, it needs to have a low value of $\chi(\Phi, I)$, which then reduces the first term on the rhs of Eq.(9), thus the condition holds in more cases. Finally, we can see that the number of FN is small for most cases.

Finally, Fig.3 shows the number of FN as a function of m, in the case where signals are distorted by additive white Gaussian noise. We can see that FN is now higher for smaller SNR value but then tends to the values obtained in the noiseless case when SNR increases. The influence of noise is smaller for small m or equivalently, for higher sparsity.

B. Seismic 1D signals

Seismic signals captured at neighboring locations exhibit the type of correlation assumed by Assumption 1. Two seismic signals shown on the Fig. 4 are obviously correlated and the second signal is shifted towards the front with respect to the first signal. This shift is important to detect in seismic signals as it represents the propagation of the seismic wave. In the following, we approximate these signals with Gabor atoms:

$$g(x) = \frac{1}{K} \exp\left(-\pi\left(\frac{x-b}{s}\right)^2\right) \sin\left(2\pi \frac{w}{N}(x-b)\right), \qquad (14)$$

where K is a normalization constant. Atoms are chosen from a dictionary, which is constructed by the discretization of parameters (s, b, w) that respectively represent scale, translation and frequency. The scales are discretized in a dyadic manner, i.e., $s = 2^j, j = 1, ..., log_2(N)$, where N is the signal length. Translation (shift) parameter b is chosen uniformly from 1 to N with step 2, such that the dictionary is overcomplete and its ISI is not too high. Finally, to construct the dictionary that is invariant to shifts and scales as given in the Eq.(3), frequency has to be linked to the scale as: $w = w_0/s$, where w_0 is the basic modulating frequency and it is constant. We have chosen it to be $5N$, which is the approximate frequency of the given seismic signals. Seismic signals y_1 and y_2 are approximated by one Gabor atom per signal ($m = 1$), and the approximated signals are p_1 and p_2, respectively (see Fig. 4). The Gabor atoms recovered by independent thresholding on two signals have the following parameters: $s_1 = 128, s_2 = 128, b_1 = 571, b_2 = 553$, and the recovery conditions in Eq.(5) for the signal y_1 and Eq.(9) for the signal y_2 are shown to be satisfied. Therefore, the observed seismic signals are sparse in the chosen dictionary, correlated by the proposed model, and the derived recovery condition holds. Interestingly, the condition in Eq.(5) evaluated for the signal y_2 does not hold (false negative), thus giving evidence that our new condition in Eq.(9) is less conservative than the condition in Eq.(5). Moreover, the recovered atoms directly give us the shift between signals. The recovered shift is equal to the shift evaluated by the cross-correlation of original signals, thus it is correctly recovered.

VI. Conclusion

We have derived the sufficient condition for recovery of sparse correlated signals by thresholding. The obtained solution is novel with respect to the state of the art work due to the particular correlation model based on the local transforms of the sparse signal components. The new model and the recovery condition is important for practical cases of correlated signals where local geometric transforms are usually present. We show that in both noisy and noiseless cases the number of false negatives stays reasonably small. In the future work, we would like to apply the obtained conditions for correlated images. Moreover, we would like to extend the worst case analysis presented in this work to the average case analysis.

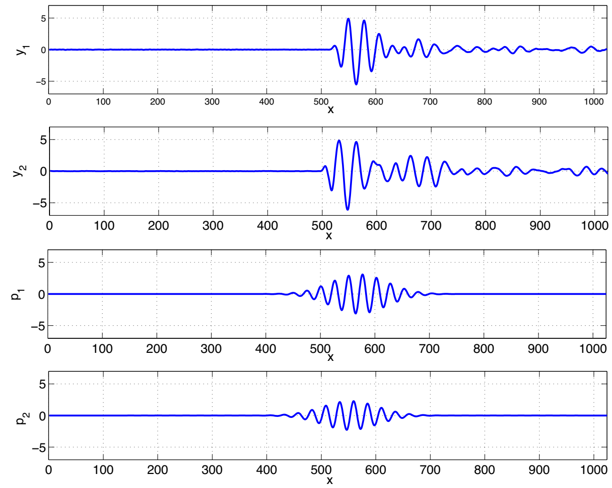

Fig. 4. Seismic signals y_1 and y_2 captured at two neighboring locations and their respective approximations p_1 and p_2 with one Gabor atom per signal. The signals y_1 and y_2 are correlated by a shift on the x axis, which is correctly captured by shifted Gabor atoms.

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References

[1] R. Gribonval, H. Rauhut, K. Schnass, and P. Vandergheynst, "Atoms of all channels, unite! average case analysis of multi-channel sparse recovery using greedy algorithms," *Journal of Fourier Analysis and Applications*, vol. 14, no. 5, pp. 655–687, 2008.

[2] M. Nielsen R. Gribonval and P. Vandergheynst, "Towards an adaptive computational strategy for sparse signal approximation," *preprint of the Institut de Recherche en Informatique et Systèmes Aléatoires (IRISA)*, 2006.

[3] P. Vandergheynst K. Schnass and P. Frossard, "Distributed sensing of noisy signals by thresholding of redundant expansions," *IEEE Workshop on Multimedia Signal Processing*, pp. 1–4, Oct. 2006.

[4] M.F. Duarte, S. Sarvotham, D. Baron, M.B. Wakin, and R.G. Baraniuk, "Distributed compressed sensing of jointly sparse signals," *Thirty-Ninth Asilomar Conference on Signals, Systems and Computers*, pp. 1537–1541, 28 - November 1, 2005.

[5] R.M. Figueras i Ventura, P. Vandergheynst and P. Frossard, "Low rate and flexible image coding with redundant representations," *IEEE Transactions on Image Processing*, vol. 15, no. 3, pp. 726–739, March 2006.

[6] Gribonval R. and Nielsen M., "Beyond sparsity: Recovering structured representations by l1 minimization and greedy algorithms," *Advances in computational mathematics*, vol. 28, no. 1, pp. 23–41, 2008.

[7] J. Tropp, "Greed is good: Algorithmic results for sparse approximation," *IEEE Transactions on Information Theory*, vol. 50, no. 10, pp. 2231–2242, October 2004.