FREDHOLM AND PROPERNESS PROPERTIES OF QUASILINEAR ELLIPTIC OPERATORS ON $\mathbb{R}^N$

BY

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1. Introduction.

Properness is a crucial ingredient in many aspects of the existence theory for nonlinear equations. For instance, it is the properness of the compact perturbations of identity that makes it possible to define the Leray-Schauder degree, and indeed properness pervades the topological degree literature from the origins to the latest developments of the subject. Proper maps are closed, and the properness assumption is essentially the only known one that guarantees that a mapping has closed range, a key feature in the approach to existence via "normal solvability". Properness also plays a role in uniqueness questions, as testified by the famous "properness criterion" ensuring that a local homeomorphism is globally invertible.

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In problems arising from nonlinear PDEs over bounded domains, the routine procedure to prove properness is to rely upon the compactness of various embeddings. When the domain is unbounded, e.g. $\mathbb{R}^N$, the compactness property vanishes, and establishing properness becomes a much more serious challenge. No clearly delineated strategy has yet emerged to tackle this issue, although the maximum principle, when available, has been shown to provide an adequate tool in recent work by Jeanjean, Lucia and Stuart [9] devoted to second order semilinear elliptic equations.

In this paper, we consider second order quasilinear elliptic operators

$$F(u) := -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x,u,\nabla u)\partial^2_{\alpha\beta}u + b(x,u,\nabla u), \tag{1.1}$$

viewed as mappings from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for some $N < p < \infty$, and we investigate conditions for the properness of F on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$. Of course, these conditions also ensure that F does map into $L^p(\mathbb{R}^N)$, but they do not place any a priori growth limitation with respect to u or ∇u in (1.1). The idea behind our work is that properness on closed bounded subsets ought to be related to the Fredholm properties of the mapping F. Indeed, if X and Y are Banach spaces and $\Phi_+(X,Y) \subset \mathcal{L}(X,Y)$ denotes the subset of operators with closed range and finite dimensional null-space, it is well-known that $L \in \Phi_+(X,Y)$ if and only if L is proper on the closed bounded subsets of X (Yood's criterion). Thus, in the linear case, "proper on closed bounded subsets" is the same as "semi-Fredholm with index $\nu \in \mathbb{Z} \cup \{-\infty\}$".

Another compelling reason to study the Fredholm and properness properties of F simultaneously is that both are involved in recent (and less recent) degree theories for Fredholm mappings. Such theories are now as complete as Leray-Schauder's, at least in the index 0 case (see [5], [13] and the references therein). They are useful in the investigation of various existence or bifurcation questions in problems over unbounded domains, where the Leray-Schauder theory does not apply. Yet, we are not aware of any prior work exploring the Fredholm or properness properties of the operator F in (1.1) in a broad setting. In Corollaries 6.2 and 7.1, we formulate sets of conditions which are necessary and sufficient to ensure that (1.1) defines an operator satisfying all the requirements for the use of the degree theory in [13].

Our contribution splits into four different but complementary main results, which fortunately are also very easy to describe. First, we show under general assumptions about the coefficients $a_{\alpha\beta}$ and b that F in (1.1) is semi-Fredholm of index $\nu \in \mathbb{Z} \cup \{-\infty\}$, i.e. that $DF(u) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for every $u \in W^{2,p}(\mathbb{R}^N)$, if and only if this is true for *one* value u^0 (Theorem 3.1). The local constancy of the index then implies that the index of $DF(u)$ is independent of u. This essentially settles the Fredholm question for F in the simplest possible way. The proof relies heavily upon the concept of a bounded linear operator being compact modulo another bounded linear operator (Definition 3.1), which differs markedly from other somewhat related ideas in the literature (relative compactness, strictly singular operators, etc...).

Our second main result, which for its most part follows from the first one plus a subtle feature of Nemytskii operators (Theorem 2.5; the subtlety is explained in the subsequent Remark 2.4), reduces the properness property to a seemingly weaker one: Clearly, F is proper on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ if and only if every bounded sequence (u_n) from $W^{2,p}(\mathbb{R}^N)$ such that $(F(u_n))$ converges in $L^p(\mathbb{R}^N)$ has a subsequence converging in $W^{2,p}(\mathbb{R}^N)$. We prove in Theorem 4.1 that it suffices to find a subsequence converging in the weaker C^1 sense, provided that $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for some, and hence every, $u^0 \in W^{2,p}(\mathbb{R}^N)$. A convenient formulation based on the concept of "sequence vanishing uniformly at infinity" in the C^1 sense, introduced in Definition 4.1, is given in Corollary 4.1.

The usefulness of the criteria in Theorem 4.1 or Corollary 4.1 is not self-evident: To find a subsequence converging in the C^1 sense is indeed weaker in principle, but since the embedding $W^{2,p}(\mathbb{R}^N) \hookrightarrow C^1(\mathbb{R}^N)$ is not compact, there is no obvious reason why such a subsequence should exist. This is where our third result comes into play, showing that it is far from being unusual for bounded sequences of $W^{2,p}(\mathbb{R}^N)$ to possess subsequences that are convergent in C^1 norm (for the purpose of this casual discussion, we deliberately ignore the obvious fact that $C^1(\mathbb{R}^N)$ is not a normed space since this difficulty can be disposed of in a straightforward way): In the "shifted subsequence lemma" (Theorem 4.3), we prove that every sequence (u_n) in $W^{2,p}(\mathbb{R}^N)(N < p < \infty)$ such that $u_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ has the following property: Either it tends to 0 in C^1 norm, or it contains a subsequence (u_{n_k})

which, after suitable shifts of the variable (each shift depending upon k), produces a new sequence $(\tilde{u}_{n_k})$ having a *nonzero* weak limit in $W^{2,p}(\mathbb{R}^N)$. Hence, $u_n \to 0$ in the C^1 sense if and only if no such subsequence exists. Our statement of the shifted subsequence lemma is different, but equivalent, to the formulation given here. One salient feature of the shifted subsequence lemma that sets it apart from other results in a similar spirit (see e.g. Willem [20] and the original work by P.L. Lions [12]) is that it shows that the shift can mostly be controlled, i.e. chosen to leave invariant any given lattice in $\mathbb{R}^N$.

The properness criteria in Theorem 4.1 and Corollary 4.1 require the existence of $u^0 \in W^{2,p}(\mathbb{R}^N)$ such that $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$. In our fourth and last main result (Corollary 5.1) we prove that this condition is *necessary* for the properness of F on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$. The proof hinges upon the existence of a stronger form of Yood's criterion for linear elliptic differential operators (Theorem 5.1), which is of independent interest and complements the results of Section 3.

As an application of all these results, we prove in Theorem 6.1 of Section 6 that surprisingly simple necessary and sufficient criteria for the properness of F on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ can be given when $a_{\alpha\beta}, 1 \leq \alpha, \beta \leq N$ and b in (1.1) are N-periodic in x (in particular, x-independent) and $b(\cdot, 0) = 0$: It is so if and only if $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for some $u^0 \in W^{2,p}(\mathbb{R}^N)$ and the equation $F(u) = 0$ possesses no nonzero solution. Alternatively, F is proper on the closed bounded subsets of $\mathbb{R}^N$ if and only if it is proper "at 0": Every sequence (u_n) from $W^{2,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ and $F(u_n) \to F(0)$ in $L^p(\mathbb{R}^N)$ contains a convergent subsequence. These results have generalizations when $a_{\alpha\beta}, 1 \leq \alpha, \beta \leq N$, and b are "asymptotically" N-periodic in x (in particular, asymptotically x-independent) and it is this more general case that we actually investigate. It is interesting that in the setting of Section 6, "semi-Fredholm of index $\nu \in \mathbb{Z} \cup \{-\infty\}$" implies "Fredholm of index 0" (i.e. $\nu = 0$) under additional assumptions frequently met in concrete problems (Theorem 6.2).

More general quasilinear operators are considered in Section 7, the coefficients of which no longer have to satisfy any periodicity condition, even asymptotically. In Theorem 7.1, we prove a natural variant of the properness criteria given in Theorem 6.1 by assuming only that the coefficients are "well-behaved" at infinity in each direction through the origin.

When $N \geq 2$, this hypothesis is fulfilled by scores of coefficients that fail to have the asymptotic N-periodicity property. This clarifies the connection between the behavior of the coefficients at infinity and the properness properties of the corresponding quasilinear elliptic operator. An even more general result that encompasses both Theorems 6.1 and 7.1 is discussed in Remark 7.1.

The arguments of this paper do not rely upon special features of scalar second order operators, and they could be extended to higher order problems or systems modulo appropriate but still general conditions. In fact, some ideas from Sections 3, 4 and 5 have successfully been used in Galdi and Rabier [6] to study bifurcation phenomena in the Navier-Stokes problem on planar exterior domains. To avoid further tedious technicalities, we have not attempted to formulate the weakest possible assumptions about the functions $a_{\alpha\beta}$ and b in (1.1): Our hypotheses make it unnecessary to discuss measurability questions, yet are general enough to be relevant in many concrete applications.

Due to space limitation, such applications are discussed elsewhere. The sufficiency of the conditions given here is exploited in [16] to obtain new global bifurcation theorems, and in [17] to resolve existence questions for the equation $F(u) = f$ with general $f \in L^p(\mathbb{R}^N)$. The "necessity" part can be used in conjunction with abstract results in [14] to prove that the lack of properness of the operator F on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ results in $F^{-1}(f)$ being either empty or noncompact for almost every $f \in L^p(\mathbb{R}^N)$, (see [15]).

2. Continuity and differentiability of some Nemytskii operators.

Let $f(= f(x,\xi)) : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}$ be a function. Whenever we need to display the components of x and ξ, we shall always write $x = (x_1, \cdots, x_N)$ and $\xi = (\xi_0, \cdots, \xi_N)$. By viewing $\mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N)$ as a bundle over $\mathbb{R}^N$, f can be identified with the bundle map

$$(x,\xi) \in \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \longmapsto (x, f(x,\xi)) \in \mathbb{R}^N \times \mathbb{R}.$$

The terminology "bundle map" is convenient to refer to properties of f in which the "base" variable x and the "fiber" variable ξ play markedly different roles, as in

Definition 2.1. *We shall say that f is an equicontinuous C^0 bundle map if f is continuous and the collection $(f(x,\cdot))_{x\in\mathbb{R}^N}$ is equicontinuous at every point of $\mathbb{R} \times \mathbb{R}^N$. If $k \geq 0$ is an* *integer, we shall say that f is an equicontinuous C_ξ^k bundle map if the partial derivatives $D_\xi^\kappa f, |\kappa| \leq k$, exist and are equicontinuous C^0 bundle maps.*

Lemma 2.1. *Let f be an equicontinuous C^0 bundle map. Then:*
(i) The collection $(f(x,\cdot))_{x\in\mathbb{R}^N}$ is uniformly equicontinuous on the compact subsets of $\mathbb{R}\times\mathbb{R}^N$.
(ii) If $\omega \subset \mathbb{R}^N$ is an open subset and $f(\cdot,0) \in L^\infty(\omega)$, the collection $(f(x,\cdot))_{x\in\bar{\omega}}$ is equibounded on the compact subsets of $\mathbb{R}\times\mathbb{R}^N$.

Proof. The proof of part (i) follows by a straightforward modification of the classical proof that continuous functions are uniformly continuous on compact subsets. For the proof of part (ii), it suffices to show that the result is true when the compact subset K is the closed ball with center 0 and radius $R > 0$. By part (i), there is $\delta > 0$ and a finite covering $\mathcal{U}$ of K by open balls with radius δ such that $|f(x,\xi) - f(x,\eta)| < 1$ for every $x \in \mathbb{R}^N$ and every $\xi, \eta \in B$ whenever B is a ball from the covering $\mathcal{U}$.

Let $\rho > 0$ be a Lebesgue number for the covering $\mathcal{U}$. Given $\xi \in K$, the subdivision of the ray through 0 and ξ into $[R/\rho]+1$ intervals produces intervals with length less than ρ. Therefore, $|f(x,\xi) - f(x,0)| < [R/\rho]+1$, so that $|f(x,\xi)| < [R/\rho]+1+|f(x,0)|$ for every $(x,\xi) \in \mathbb{R}^N \times K$. The conclusion follows from the assumption that $f(\cdot,0) \in L^\infty(\omega)$. $\square$

Naturally, the condition $f(\cdot,0) \in L^\infty(\omega)$ in Lemma 2.1 (ii) may be replaced by $f(\cdot,\xi^0) \in L^\infty(\omega)$ for any $\xi^0 \in \mathbb{R}\times\mathbb{R}^N$. Also, it is obvious that the sum and product of two equicontinuous C_ξ^k bundle maps are equicontinuous C_ξ^k bundle maps.

Remark 2.1: If f is of class C^k and $f(x,\xi)$ is N-periodic in x, with period $T = (T_1,\cdots,T_N)$, for every $\xi \in \mathbb{R}\times\mathbb{R}^N$, it follows from the uniform continuity of $D_\xi^\kappa f$ on $[0,T_1]\times\cdots\times[0,T_N]\times K$ for every compact subset K of $\mathbb{R}\times\mathbb{R}^N$ and $|\kappa| \leq k$, that f is an equicontinuous C_ξ^k bundle map. Other (nonperiodic) examples are easily found. $\square$

Given an equicontinuous C_ξ^k bundle map $f : \mathbb{R}^N \times (\mathbb{R}\times\mathbb{R}^N) \to \mathbb{R}, k = 0$, or 1, we shall be interested in various continuity and differentiablity properties of the Nemytskii operator $u \longmapsto f(\cdot, u, \nabla u)$. Let $\omega \subset \mathbb{R}^N$ be an open subset (bounded or unbounded) and let $C^1(\overline{\omega})$ denote the subspace of $C^0(\overline{\omega}) \cap C^1(\omega)$ of those (real-valued) functions v such that ∇v extends as a continuous function on $\overline{\omega}$. We introduce the space

$$C^1_d(\overline{\omega}) := \{v \in C^1(\overline{\omega}) : \lim_{\substack{|x|\to\infty \\ x\in\omega}} |v(x)| = \lim_{\substack{|x|\to\infty \\ x\in\omega}} |\nabla v(x)| = 0\}, \tag{2.1}$$

where "d" stands fro "decay", so that

$$C^1_d(\overline{\omega}) = C^1(\overline{\omega}) \text{ if } \omega \text{ is bounded.} \tag{2.2}$$

The space $C^1_d(\overline{\omega})$ is a Banach space for the norm

$$||v||_{C^1_d(\overline{\omega})} := \max_{x\in\overline{\omega}} |v(x)| + \max_{x\in\overline{\omega}} |\nabla v(x)|. \tag{2.3}$$

Of course, $C^1_d(\overline{\omega}) \hookrightarrow W^{1,\infty}(\omega)$ and the norm (2.3) is just the norm induced by $W^{1,\infty}(\omega)$.

Throughout the paper, $\mathcal{D}(\omega)$ denotes the space of smooth real-valued functions with support in some compact subset of ω.

Theorem 2.1. *Let $f : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}$ be an equicontinuous C^0 bundle map, and let $\omega \subset \mathbb{R}^N$ be an open subset (bounded or unbounded). Suppose that $f(\cdot, 0) \in L^\infty(\omega)$. Then, the Nemytskii operator $u \longmapsto f(\cdot, u, \nabla u)$ has the following properties.*
(i) It is well defined and continuous from $C^1_d(\overline{\omega})$ into $L^\infty(\omega)$ and maps bounded subsets onto bounded subsets.
(ii) If $\partial\omega$ is Lipschitz continuous (possibly $\emptyset$) and $N < p < \infty$, it is well defined and continuous from $W^{2,p}(\omega)$ into $L^\infty(\omega)$ and maps bounded subsets onto bounded subsets.
(iii) If ω is bounded with $\partial\omega$ Lipschitz continuous and $N < p < \infty$, it is completely continuous, i.e. transforms weakly convergent sequences into strongly convergent ones, from $W^{2,p}(\omega)$ into $L^\infty(\omega)$ (hence also into $L^q(\omega), 1 \le q \le \infty$).
(iv) If $\partial\omega$ is Lipschitz continuous (possibly $\emptyset$) and $N < p < \infty$, the "multiplication" $(u, v) \in W^{2,p}(\omega) \times L^p(\omega) \longmapsto f(\cdot, u, \nabla u)v \in L^p(\omega)$ is well defined and weakly sequentially continuous (i.e. sequentially continuous when $W^{2,p}(\omega)$ and $L^p(\omega)$ are equipped with their weak topologies).

Proof. (i) If $u \in C^1_d(\overline{\omega})$, the function $x \in \overline{\omega} \longmapsto f(x, u(x), \nabla u(x))$ is continuous, hence measurable. Also, there is a compact subset $K \subset \mathbb{R} \times \mathbb{R}^N$ such that $(u(x), \nabla u(x)) \in K$ for every $x \in \overline{\omega}$. By Lemma 2.1 (ii) there is a constant $M_K > 0$ such that $|f(x, u(x), \nabla u(x))| \leq M_K$ for every $x \in \overline{\omega}$. This shows that $f(\cdot, u, \nabla u) \in L^\infty(\omega)$ and hence the Nemytskii operator is well defined. If now $u_n, u \in C^1_d(\overline{\omega})$ and $u_n \to u$ in $C^1_d(\overline{\omega})$, then there is a compact subset $K \subset \mathbb{R} \times \mathbb{R}^N$ containing $(u(x), \nabla u(x))$ and $(u_n(x), \nabla u_n(x))$ for every $x \in \overline{\omega}$ and every $n \in \mathbb{N}$. Since $|(u_n(x), \nabla u_n(x)) - (u(x), \nabla u(x))|$ can be made arbitrarily small uniformly in $x \in \overline{\omega}$ for n large enough, it follows from Lemma 2.1 (i) that given $\epsilon > 0$, we have $|f(x, u_n(x), \nabla u_n(x)) - f(x, u(x), \nabla u(x))| < \epsilon$ for every $x \in \overline{\omega}$. This proves the desired continuity property. Lastly, if $B \subset C^1_d(\overline{\omega})$ is a bounded subset, then there is a compact subset $K \subset \mathbb{R} \times \mathbb{R}^N$ such that $(u(x), \nabla u(x)) \in K$ for every $x \in \overline{\omega}$ and every $u \in B$, whence the boundedness of the set $\{f(\cdot, u, \nabla u) : u \in B\}$ in $L^\infty(\omega)$ follows once again from Lemma 2.1 (ii).

(ii) This is due to (i) and the continuity of the embedding $W^{2,p}(\omega) \hookrightarrow C^1_d(\overline{\omega})$. Since this embedding is compact when ω is bounded (see (2.2)), this also proves part (iii).

(iv) Let $u_n \in W^{2,p}(\omega)$ and $v_n \in L^p(\omega)$ be sequences such that $u_n \rightharpoonup u$ in $W^{2,p}(\omega)$ and $v_n \rightharpoonup v$ in $L^p(\omega)$. From part (ii), the sequence $(f(\cdot, u_n, \nabla u_n))$ is bounded in $L^\infty(\omega)$, and hence the sequence $(f(\cdot, u_n, \nabla u_n)v_n)$ is bounded in $L^p(\omega)$. As a result, there is a subsequence $(f(\cdot, u_{n_k}, \nabla u_{n_k})v_{n_k})$ and there is $w \in L^p(\omega)$ such that $f(\cdot, u_{n_k}, \nabla u_{n_k})v_{n_k} \rightharpoonup w$ in $L^p(\omega)$. Let $\omega' \subset \omega$ be any open ball. By part (iii) we have $f(\cdot, u_n, \nabla u_n)_{|\omega'} \to f(\cdot, u, \nabla u)_{|\omega'}$ in $L^\infty(\omega')$, whence $[f(\cdot, u_n, \nabla u_n)v_n]_{|\omega'} \rightharpoonup [f(\cdot, u, \nabla u)v]_{|\omega'}$ in $L^p(\omega')$. This implies $w_{|\omega'} = [f(\cdot, u, \nabla u)v]_{|\omega'}$, hence $w = f(\cdot, u, \nabla u)v$ since the ball ω' is arbitrary. Thus, every subsequence of the bounded sequence $(f(\cdot, u_n, \nabla v_n)u_n)$ that is weakly convergent in $L^p(\omega)$ has weak limit $f(\cdot, u, \nabla u)v$. This yields $f(\cdot, u_n, \nabla u_n)v_n \rightharpoonup f(\cdot, u, \nabla u)v$ in $L^p(\omega)$ by the usual argument. $\square$

When ω is bounded (and $\partial\omega$ is Lipschitz continuous) and $N < p < \infty$, Theorem 2.1 (iii) implies that the Nemytskii operator $u \to f(\cdot, u, \nabla u)$ is completely continuous from $W^{2,p}(\omega)$ into $L^p(\omega)$, but the hypotheses made in Theorem 2.1 do not even ensure that $f(\cdot, u, \nabla u) \in L^p(\omega)$ for $u \in W^{2,p}(\omega)$ when ω is unbounded, even if $f(\cdot, 0) = 0$. This issue is clarified in our next theorem.

Theorem 2.2. *Let $f : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}$ have the form*

$$f(x,\xi) = f_0(x) + \sum_{i=0}^{N} g_i(x,\xi)\xi_i \tag{2.4}$$

with $g_i(\cdot,0) \in L^\infty(\mathbb{R}^N)$ and g_i equicontinuous C^0 bundle map, $0 \le i \le N$. Let $N < p < \infty$ and suppose that $f_0 \in L^p(\mathbb{R}^N)$. In particular, it is so if f is a C^1_ξ equicontinuous bundle map with $f(\cdot,0) \in L^p(\mathbb{R}^N)$ and $\partial_{\xi_i} f(\cdot,0) \in L^\infty(\mathbb{R}^N), 0 \le i \le N$. Then, the Nemytskii operator $u \to f(\cdot,u,\nabla u)$ has the following properties:
(i) It is well defined and continuous from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ and maps bounded subsets onto bounded subsets.
(ii) It is weakly sequentially continuous from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ (i.e. sequentially continuous when $W^{2,p}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ are equipped with their weak topologies).

Proof. To prove the "in particular" part, notice that for $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R} \times \mathbb{R}^N$ we have $f(x,\xi) = f(x,0) + \sum_{i=0}^{N} g_i(x,\xi)\xi_i$, where $g_i(x,\xi) := \int_0^1 \partial_{\xi_i} f(x,t\xi)dt, 0 \le i \le N$. It is straightforward to check with Lemma 2.1 (i) that g_i is an equicontinuous C^0 bundle map. Furthermore, $g_i(\cdot,0) = \partial_{\xi_i} f(\cdot,0) \in L^\infty(\mathbb{R}^N), 0 \le i \le N$.

(i) From Theorem 2.1 (ii), the operators $u \in W^{2,p}(\mathbb{R}^N) \to g_i(\cdot,u,\nabla u) \in L^\infty(\mathbb{R}^N)$, $0 \le i \le N$, are continuous and map bounded subsets onto bounded subsets. As a result, the operator

$$u \in W^{2,p}(\mathbb{R}^N) \to g_0(\cdot,u,\nabla u)u + \sum_{i=1}^{N} g_i(\cdot,u,\nabla u)\partial_i u \in L^p(\mathbb{R}^N)$$

(where of course $\partial_i u := \partial_{x_i} u$) is continuous and maps bounded subsets onto bounded subsets. By (2.4) this operator is just $u \to f(\cdot,u,\nabla u) - f_0$, and the conclusion follows from the assumption $f_0 \in L^p(\mathbb{R}^N)$.

(ii) Let $u_n \in W^{2,p}(\mathbb{R}^N)$ be a sequence such that $u_n \rightharpoonup u$ in $W^{2,p}(\mathbb{R}^N)$. By part (i), the sequence $(f(\cdot,u_n,\nabla u_n))$ is bounded in $L^p(\mathbb{R}^N)$ and hence there is a subsequence $(f(\cdot,u_{n_k},\nabla u_{n_k}))$ and there is $w \in L^p(\mathbb{R}^N)$ such that $f(\cdot,u_{n_k},\nabla u_{n_k}) \rightharpoonup w$ in $L^p(\mathbb{R}^N)$. Let $\omega \subset \mathbb{R}^N$ be an open ball. Since ω is bounded and $f - f_0$ is continuous and vanishes when $\xi = 0$, Theorem 2.1 (iii) applies. Accordingly, we have $f(\cdot,u_n,\nabla u_n)_{|\omega} \to f(\cdot,u,\nabla u)_{|\omega}$

in $L^p(\omega)$. This implies $w_{|\omega} = f(\cdot, u, \nabla u)_{|\omega}$, whence $w = f(\cdot, u, \nabla u)$ since the ball ω is arbitrary. Thus, the only weak cluster point of the sequence $(f(\cdot, u_n, \nabla u_n))$ in $L^p(\mathbb{R}^N)$ is $f(\cdot, u, \nabla u)$, so that $f(\cdot, u_n, \nabla u_n) \rightharpoonup f(\cdot, u, \nabla u)$ in $L^p(\mathbb{R}^N)$. $\square$

Remark 2.2: Theorem 2.2 remains valid, with the same proof, if $\mathbb{R}^N$ is replaced by an open subset $\omega \subset \mathbb{R}^N$ with $\partial\omega$ Lipschitz continuous. $\square$

A more subtle continuity property will be proved in Theorem 2.4. Before that, we turn to differentiability questions. The derivative of the Nemytskii operator $f(\cdot, u, \nabla u)$ must be carefully distinguished from the derivative of the mapping f. For that reason, we shall use the notation

$$\mathbf{f}(u) := f(\cdot, u, \nabla u), \tag{2.5}$$

so that $D\mathbf{f}(u)$ will unambiguously refer to the derivative of the Nemytskii operator.

Theorem 2.3. *Let $f : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}$ be an equicontinuous C^1_ξ bundle map. Let $N < p < \infty$ and suppose that $f(\cdot, 0) \in L^\infty(\mathbb{R}^N)$ (resp. $f(\cdot, 0) \in L^p(\mathbb{R}^N)$) and that $\partial_{\xi_i} f(\cdot, 0) \in L^\infty(\mathbb{R}^N), 0 \leq i \leq N$. Then, the Nemytskii operator $\mathbf{f}(u) := f(\cdot, u, \nabla u)$ is of class C^1 from $W^{2,p}(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$ (resp. $L^p(\mathbb{R}^N)$) with derivative given by*

$$D\mathbf{f}(u)v := \partial_{\xi_0} f(\cdot, u, \nabla u)v + \sum_{i=1}^{N} \partial_{\xi_i} f(\cdot, u, \nabla u)\partial_i v, \ \forall v \in W^{2,p}(\mathbb{R}^N). \tag{2.6}$$

In particular, $D\mathbf{f}$ is bounded and $\mathbf{f}$ is uniformly continuous on the bounded subsets of $W^{2,p}(\mathbb{R}^N)$.

Proof. For breviety, we denote by T the operator $Tv := \partial_{\xi_0} f(\cdot, u, \nabla u)v + \sum_{i=1}^{N} \partial_{\xi_i} f(\cdot, u, \nabla u)\partial_i v$. Note that $T \in \mathcal{L}(W^{2,p}(\mathbb{R}^N), L^\infty(\mathbb{R}^N))$ and $T \in \mathcal{L}(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$. This follows from Theorem 2.1 (ii) with $\partial_{\xi_i} f$ replacing f and, for the former relation, from the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. Next,

$$f(\cdot, u + v, \nabla(u + v)) - f(\cdot, u, \nabla u) - Tv = k_{u,0}(\cdot, v, \nabla v)v + \sum_{i=1}^{N} k_{u,i}(\cdot, v, \nabla v)\partial_i v \tag{2.7}$$

where, for $0 \leq i \leq N$, $k_{u,i}(\cdot, v, \nabla v)$ is the Nemytskii operator associated with

$$k_{u,i}(x,\xi) := \int_0^1 [\partial_{\xi_i} f(x, u(x) + t\xi_0, \nabla u(x) + t\xi') - \partial_{\xi_i} f(\cdot, u(x), \nabla u(x))]dt.$$

Since $u \in W^{2,p}(\mathbb{R}^N) \hookrightarrow C_0^1(\mathbb{R}^N)$, it is easily seen by Lemma 2.1 (i) that $k_{u,i}$ is an equicontinuous C^0 bundle map. Since also $k_{u,i}(\cdot, 0) = 0 \in L^\infty(\mathbb{R}^N)$, it follows from Theorem 2.1 (ii) that $v \longmapsto k_{u,i}(\cdot, v, \nabla v)$ is continuous from $W^{2,p}(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$. Thus, for every $\epsilon > 0$, we have $|k_{u,i}(\cdot, v, \nabla v)|_{0,\infty,\mathbb{R}^N} < \epsilon, 0 \leq i \leq N$, provided that $||v||_{2,p,\mathbb{R}^N}$ is small enough. By (2.7), we obtain $|\mathbf{f}(u+v) - \mathbf{f}(u) - Tv|_{0,\infty,\mathbb{R}^N} \leq c\epsilon||v||_{1,\infty,\mathbb{R}^N} \leq C\epsilon||v||_{2,p,\mathbb{R}^N}$ and $|\mathbf{f}(u+v) - \mathbf{f}(u) - Tv||_{0,p,\mathbb{R}^N} \leq c\epsilon||v|_{1,p,\mathbb{R}^N} \leq C\epsilon||v||_{2,p,\mathbb{R}^N}$, where $c, C > 0$ are constants independent of $v \in W^{2,p}(\mathbb{R}^N)$. Thus, if $f(\cdot, 0) \in L^\infty(\mathbb{R}^N)$ (resp. $f(\cdot, 0) \in L^p(\mathbb{R}^N)$) so that $\mathbf{f}$ maps $W^{2,p}(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$ by Theorem 2.1 (ii) (resp. into $L^p(\mathbb{R}^N)$ by Theorem 2.2 (i)) this shows that $D\mathbf{f}(u) = T$. In both cases when $\mathbf{f}$ maps into $L^\infty(\mathbb{R}^N)$ or into $L^p(\mathbb{R}^N)$, the continuity of $D\mathbf{f}$ follows from (2.6) and Theorem 2.1 (ii) with f replaced by $\partial_{\xi_i} f, 0 \leq i \leq N$. This also shows that $D\mathbf{f}$ is bounded on the bounded subsets of $W^{2,p}(\mathbb{R}^N)$, which in turn implies the uniform continuity of $\mathbf{f}$ on such subsets by the mean value theorem. $\square$

Remark 2.3: More generally, let $k \geq 1$ be an integer and let f in Theorem 2.3 be an equicontinuous C_ξ^k bundle map such that $f(\cdot, 0) \in L^\infty(\mathbb{R}^N)$ (resp. $f(\cdot, 0) \in L^p(\mathbb{R}^N)$) and $D_\xi^\kappa f(\cdot, 0) \in L^\infty(\mathbb{R}^N)$ for every $\kappa := (\kappa_0, \cdots, \kappa_N) \in \mathbb{N}^{N+1}$ with $1 \leq |\kappa| \leq k$. Then, the Nemytskii operator $\mathbf{f}(u) := f(\cdot, u, \nabla u)$ is of class C^k from $W^{2,p}(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$ (resp. $L^p(\mathbb{R}^N)$) if $N < p < \infty$. The most convenient way to see this is to use the so-called "converse of Taylor's theorem" (see e.g. [1, pp. 93 and 97-99]) and argue as in the proof of the "omega lemma" ([1, p. 101]) with appropriate modifications. These modifications are mostly suggested by the proof of Theorem 2.3 when $k = 1$, but the fact that $W^{1,p}(\mathbb{R}^N)$ is a Banach algebra when $p > N$ is also important when $k \geq 2$. $\square$

Part of the argument needed in the proof of our next theorem is better couched in abstract terms:

Lemma 2.2. *Let X, Y and Z be normed spaces with $X \hookrightarrow Y$ (continuous embedding) and let $\mathbf{f} : X \to Z$ be uniformly continuous on the bounded subsets of X. Suppose that there is a dense subset $D \subset X$ such that whenever $u \in D$ and (u_n) is a bounded sequence from X with $u_n \to u$ in Y, we have $\mathbf{f}(u_n) \to \mathbf{f}(u)$ in Z. Then, the restriction of $\mathbf{f}$ to the bounded subsets of X remains continuous for the topology induced by Y.*

Proof. It suffices to show that if $\bar{B} \subset X$ is any closed ball, then $\mathbf{f}_{|\bar{B}}$ is continuous for the Y-topology. In what follows, we denote by B' a fixed open ball in X with $\bar{B} \subset B'$. Let $\epsilon > 0$ be given. By the uniform continuity of $\mathbf{f}$ on B', there is $\delta > 0$ such that

$$\{v, w \in B', ||v - w||_X < \delta\} \Rightarrow ||\mathbf{f}(v) - \mathbf{f}(w)||_Z < \epsilon/3, \tag{2.8}$$

and it may be assumed with no loss of generality that $\delta > 0$ in (2.8) is such that $\bar{B} + \delta B_1 \subset B'$ where B_1 is the unit ball of X.

Let $u \in \bar{B}$ and let (u_n) be a sequence from $\bar{B}$ such that $u_n \to u$ in Y. Choose $v \in D$ such that $||v - u||_X < \delta$, so that $v \in B'$ and $v_n := v - u + u_n \in B'$ for every $n \in \mathbb{N}$. We have $||\mathbf{f}(u_n) - \mathbf{f}(u)||_Z \leq ||\mathbf{f}(u_n) - \mathbf{f}(v_n)||_Z + ||\mathbf{f}(v_n) - \mathbf{f}(v)||_Z + ||\mathbf{f}(v) - \mathbf{f}(u)||_Z$. Since $||u_n - v_n||_X = ||u - v||_X < \delta$, it follows from (2.8) that $||\mathbf{f}(u_n) - \mathbf{f}(v_n)||_Z < \epsilon/3$ for every $n \in \mathbb{N}$ and that $||\mathbf{f}(u) - \mathbf{f}(v)||_Z < \epsilon/3$. Hence, $||\mathbf{f}(u_n) - \mathbf{f}(u)||_Z < ||\mathbf{f}(v_n) - \mathbf{f}(v)||_Z + 2\epsilon/3$. Now, $v_n \to v$ in Y and, since $v \in D$, we know by hypothesis that $\mathbf{f}(v_n) \to \mathbf{f}(v)$ in Z. Thus, $||\mathbf{f}(v_n) - \mathbf{f}(v)||_Z < \epsilon/3$, whence $||\mathbf{f}(u_n) - \mathbf{f}(u)||_Z < \epsilon$, for n large enough. □

Theorem 2.4. *Let $f : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}$ be an equicontinuous C^1_ξ bundle map. Let $N < p < \infty$ and suppose that $f(\cdot, 0) \in L^p(\mathbb{R}^N)$ and $\partial_{\xi_i} f(\cdot, 0) \in L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), 0 \leq i \leq N$. Then, the restriction of the Nemytskii operator $u \longmapsto f(\cdot, u, \nabla u)$ to the bounded subsets of $W^{2,p}(\mathbb{R}^N)$ is continuous into $L^p(\mathbb{R}^N)$ for the topology of $C^1_d(\mathbb{R}^N)$. (In other words, if (u_n) is a bounded sequence from $W^{2,p}(\mathbb{R}^N)$ and there is $u \in W^{2,p}(\mathbb{R}^N)$ such that $u_n \to u$ in $C^1_d(\mathbb{R}^N)$, then $f(\cdot, u_n, \nabla u_n) \to f(\cdot, u, \nabla u)$ in $L^p(\mathbb{R}^N)$.)*

Proof. This will follow from Lemma 2.2 with $X = W^{2,p}(\mathbb{R}^N), Y = C^1_d(\mathbb{R}^N), Z = L^p(\mathbb{R}^N)$ and $D = \mathcal{D}(\mathbb{R}^N)$. Since we already know that $\mathbf{f}$ in (2.5) is uniformly continuous on the bounded subsets of $W^{2,p}(\mathbb{R}^N)$ (Theorem 2.3), it suffices to show that if $u \in \mathcal{D}(\mathbb{R}^N)$ and (u_n) is a bounded sequence from $W^{2,p}(\mathbb{R}^N)$ such that $u_n \to u$ in $C^1_d(\mathbb{R}^N)$, then $\mathbf{f}(u_n) \to \mathbf{f}(u)$ in $L^p(\mathbb{R}^N)$.

In the proof of Theorem 2.2, we have already established that $\mathbf{f}(v) = f(\cdot, 0) + \mathbf{g}_0(v)v + \sum_{i=1}^{N} \mathbf{g}_i(v)\partial_i v$ for every $v \in W^{2,p}(\mathbb{R}^N)$, where $\mathbf{g}_i(v) := g_i(\cdot, v, \nabla v)$ and g_i is given by $g_i(x, \xi) = \int_0^1 \partial_{\xi_i} f(x, t\xi)dt, 0 \leq i \leq N$. We also noticed that each mapping g_i is an equicontinuous C^0 bundle map. Hence, by Theorem 2.1 (i), $\mathbf{g}_i : C^1_d(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$ is continuous, $0 \leq i \leq N$.

Clearly, the problem reduces to showing that $\mathbf{g}_0(u_n)u_n \to \mathbf{g}_0(u)u$ in $L^p(\mathbb{R}^N)$ and $\mathbf{g}_i(u_n)\partial_i u_n \to \mathbf{g}_i(u)\partial_i u$ in $L^p(\mathbb{R}^N), 1 \leq i \leq N$. To see this, write

$$\mathbf{g}_0(u_n)u_n - \mathbf{g}_0(u)u = (\mathbf{g}_0(u_n) - \mathbf{g}_0(u))u_n + \mathbf{g}_0(u)(u_n - u). \tag{2.9}$$

The first term in the right-hand side of (2.9) tends to 0 in $L^p(\mathbb{R}^N)$ because $\mathbf{g}_0(u_n) \to \mathbf{g}_0(u)$ in $L^\infty(\mathbb{R}^N)$ and (u_n) is bounded in $L^p(\mathbb{R}^N)$. Since $u_n - u \to 0$ in $C^1_d(\mathbb{R}^N)$, hence in $L^\infty(\mathbb{R}^N)$, the second term in the right-hand side of (2.9) also tends to 0 in $L^p(\mathbb{R}^N)$ because $\mathbf{g}_0(u) \in L^p(\mathbb{R}^N)$. This follows from the assumption $u \in \mathcal{D}(\mathbb{R}^N)$. Indeed, let $\omega \subset \mathbb{R}^N$ be a bounded open subset such that Supp $u \subset \omega$. We have $\mathbf{g}_0(u) \in L^\infty(\mathbb{R}^N) \subset L^\infty(\omega) \subset L^p(\omega)$ since ω is bounded. On the other hand, $\mathbf{g}_0(u)(x) = g_0(x, u(x), \nabla u(x)) = g_0(x, 0) = \partial_{\xi_0} f(x, 0)$ for $x \in \mathbb{R}^N \backslash \omega$. Since $\partial_{\xi_0} f(\cdot, 0) \in L^p(\mathbb{R}^N)$, this implies $\mathbf{g}_0(u) \in L^p(\mathbb{R}^N \backslash \omega)$. Hence, altogether, $\mathbf{g}_0(u) \in L^p(\mathbb{R}^N)$.

At this stage, we have shown that $\mathbf{g}_0(u_n)u_n \to \mathbf{g}_0(u)u$ in $L^p(\mathbb{R}^N)$. Similar arguments yield $\mathbf{g}_i(u_n)\partial_i u_n \to \mathbf{g}_i(u)\partial_i u$ in $L^p(\mathbb{R}^N), 1 \leq i \leq N$, and the proof is complete. □

The following somewhat surprising property (see Remark 2.4 below) is obtained as a simple corollary to Theorems 2.3 and 2.4. It is one of the keys to the proof of the properness results in Section 4.

Theorem 2.5. *Let $f : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N)$ be an equicontinuous C^1_ξ bundle map. Let $N < p < \infty$ and suppose that $f(\cdot, 0) \in L^p(\mathbb{R}^N), \partial_{\xi_i} f(\cdot, 0) \in L^\infty(\mathbb{R}^N), 0 \leq i \leq N$ (so that the Nemytskii operator $\mathbf{f}$ is of class C^1 from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ by Theorem 2.3). If (u_n) is a bounded sequence from $W^{2,p}(\mathbb{R}^N)$ and $u \in W^{2,p}(\mathbb{R}^N)$ is such that $u_n \to u$ in $C^1_d(\mathbb{R}^N)$*

(hence, $u_n \rightharpoonup u$ in $W^{2,p}(\mathbb{R}^N)$) we have

$$\mathbf{f}(u_n) - \mathbf{f}(u) - D\mathbf{f}(u)(u_n - u) \longrightarrow 0 \text{ in } L^p(\mathbb{R}^N). \tag{2.10}$$

Proof. Set $v_n := u_n - u$, so that $v_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ and $v_n \to 0$ in $C^1_d(\mathbb{R}^N)$. The left-hand side of (2.10) is $\mathbf{f}(u+v_n) - \mathbf{f}(u) - D\mathbf{f}(u)v_n$, i.e. $\mathbf{g}(v_n) - \mathbf{g}(0)$ where $\mathbf{g}(v) := \mathbf{f}(u+v) - D\mathbf{f}(u)v$ for $v \in W^{2,p}(\mathbb{R}^N)$. Note that $\mathbf{g}$ is the Nemytskii operator associated with (see (2.6))

$$g(x,\xi) := f(x, u(x) + \xi_0, \nabla u(x) + \xi') - \sum_{i=0}^{N} \partial_{\xi_i} f(x, u(x), \nabla u(x))\xi_i.$$

Since $\partial_{\xi_i} f(\cdot, u, \nabla u) \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ by Theorem 2.1 (ii), it is straightforward to check with Lemma 2.1 (i) that g is an equicontinuous C^1_ξ bundle map with $\partial_{\xi_i} g(x,\xi) = \partial_{\xi_i} f(x, u(x) + \xi_0, \nabla u(x) + \xi') - \partial_{\xi_i} f(x, u(x), \nabla u(x)), 0 \leq i \leq N$. In particular, $g(\cdot, 0) = f(\cdot, u, \nabla u) \in L^p(\mathbb{R}^N)$ by Theorem 2.2 (i). Also, $\partial_{\xi_i} g(\cdot, 0) = 0 \in L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), 0 \leq i \leq N$. It thus follows from Theorem 2.4 that $\mathbf{g}(v_n) - \mathbf{g}(0) \to 0$ in $L^p(\mathbb{R}^N)$, which is exactly what is claimed in (2.10). □

Remark 2.4: Relation (2.10) is very much unexpected in light of the fact that the hypotheses of Theorem 2.5 do *not* even imply that $\mathbf{f}(u_n) - \mathbf{f}(u) \to 0$ in $L^p(\mathbb{R}^N)$ (Theorem 2.4 does not apply because $\partial_{\xi_i} f(\cdot, 0) \in L^\infty(\mathbb{R}^N)$, not $L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$), nor do they imply that $D\mathbf{f}(u)(u_n - u) \to 0$ in $L^p(\mathbb{R}^N)$ (only $D\mathbf{f}(u)(u_n - u) \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$ and, by (2.6), $D\mathbf{f}(u)(u_n - u) \to 0$ in $L^\infty(\mathbb{R}^N)$). Strong convergence in $L^p(\mathbb{R}^N)$ holds only for the *difference* of these two terms. Note also that (2.10) does *not* say that $\mathbf{f}$ is differentiable at u when the source space $W^{2,p}(\mathbb{R}^N)$ is equipped with the C^1_d-norm. □

3. Fredhom properties of quasilinear elliptic operators.

Let $a_{\alpha\beta} : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}, 1 \leq \alpha, \beta \leq N$, and $b : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}$ be given functions. We now begin the investigation of the Fredholm properties of the second order differential operator

$$F(u) := -\sum_{\alpha\beta=1}^{N} a_{\alpha\beta}(\cdot, u, \nabla u)\partial^2_{\alpha\beta} u + b(\cdot, u, \nabla u). \tag{3.1}$$

We shall make additional assumptions (listed below) ensuring that the operator F is elliptic and maps $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for some $N < p < \infty$. Naturally, the suitable hypotheses for the latter will be derived from the results in Section 2.

As before, we shall use the notation $x = (x_1, \cdots, x_N)$ and $\xi = (\xi_0, \cdots, \xi_N)$. The assumptions about the "coefficients" $a_{\alpha\beta}, 1 \leq \alpha, \beta \leq N$, and b are as follows (recall $N < p < \infty$):

$$a_{\alpha\beta} \text{ is an equicontinuous } C^1_\xi \text{ bundle map, } 1 \leq \alpha, \beta \leq N, \tag{3.2}$$

$$a_{\alpha\beta}(\cdot, 0) \in L^\infty(\mathbb{R}^N), \partial_{\xi_i} a_{\alpha\beta}(\cdot, 0) \in L^\infty(\mathbb{R}^N),\ 1 \leq \alpha, \beta \leq N,\ 0 \leq i \leq N. \tag{3.3}$$

$$\begin{cases} \sum\limits_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x,\xi)\eta_\alpha\eta_\beta \geq \gamma(x,\xi)|\eta|^2, \\ \forall \eta = (\eta_1, \cdots, \eta_N) \in \mathbb{R}^N,\ \forall (x,\xi) \in \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N), \end{cases} \tag{3.4}$$

where $\gamma : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to (0, \infty)$ is bounded from below by a positive constant $\gamma_{\tilde{K}}$ on every compact subset $\tilde{K}$ of $\mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N)$ (e.g. γ lower semicontinuous).

$$b \text{ is an equicontinuous } C^1_\xi \text{ bundle map.} \tag{3.5}$$

$$b(\cdot, 0) \in L^p(\mathbb{R}^N), \quad \partial_{\xi_i} b(\cdot, 0) \in L^\infty(\mathbb{R}^N),\ 0 \leq i \leq N. \tag{3.6}$$

Our first task will be to establish the continuity and differentiability of the operator F in (3.1).

Lemma 3.1. *The operator F in (3.1) is both continuous and weakly sequentially continuous from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, and it maps bounded subsets onto bounded subsets.*

Proof. By Theorem 2.1 (ii) with $f = a_{\alpha\beta}$, the Nemytskii operators $u \longmapsto a_{\alpha\beta}(\cdot, u, \nabla u), 1 \leq \alpha, \beta \leq N$, are continuous from $W^{2,p}(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$ and they map bounded subsets onto bounded subsets. By Theorem 2.2 (i) with $f = b$, the Nemytskii operator $u \longmapsto b(\cdot, u, \nabla u)$ is continuous from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ and it maps bounded subsets onto bounded subsets. This proves the continuity of the operator F as well as its boundedness on bounded subsets.

If (u_n) is a sequence from $W^{2,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $W^{2,p}(\mathbb{R}^N)$, we have $\partial^2_{\alpha\beta} u_n \rightharpoonup \partial^2_{\alpha\beta} u$ in $L^p(\mathbb{R}^N)$. By Theorem 2.1 (iv) with $f = a_{\alpha\beta}$, we find that $a_{\alpha\beta}(\cdot, u_n, \nabla u_n)\partial^2_{\alpha\beta} u_n \rightharpoonup a_{\alpha\beta}(\cdot, u, \nabla u)\partial^2_{\alpha\beta} u$ in $L^p(\mathbb{R}^N)$. Next, by Theorem 2.2 (ii) with $f = b$, we have $b(\cdot, u_n, \nabla u_n) \rightharpoonup b(\cdot, u, \nabla u)$ in $L^p(\mathbb{R}^N)$. The weak sequential continuity of F follows at once from these properties. □

Remark 3.1: The proof of Lemma 3.1 requires only the assumption that $a_{\alpha\beta}$ is a C^0 bundle map with $a_{\alpha\beta}(\cdot, 0) \in L^\infty(\mathbb{R}^N), 1 \leq \alpha, \beta \leq N$ and that $b(x, \xi) = b_0(x) + \sum_{i=1}^{N} c_i(x, \xi)\xi_i$ with $b_0 \in L^p(\mathbb{R}^N), c_i(\cdot, 0) \in L^\infty(\mathbb{R}^N)$ and c_i equicontinuous C^0 bundle maps. In addition, by Remark 2.2, this generalization of Lemma 3.1 remains valid when $\mathbb{R}^N$ is replaced by an arbitrary open subset $\omega \subset \mathbb{R}^N$ with Lipschitz-continuous boundary. This will be used in Section 6. □

Lemma 3.2. *The operator F in (3.1) is of class C^1 from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, with derivative (using the notation (2.5) for simplicity)*

$$DF(u)v := -\sum_{\alpha,\beta=1}^{N} \mathbf{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta} v + D\mathbf{b}(u)v - \sum_{\alpha,\beta=1}^{N} (D\mathbf{a}_{\alpha\beta}(u)v)\partial^2_{\alpha\beta} u, \tag{3.7}$$

where $D\mathbf{b}(u)$ and $D\mathbf{a}_{\alpha\beta}(u)$ are given by (2.6).

Proof. A routine application of Theorem 2.3. □

Remark 3.2: More generally, given an integer $k \geq 1$, there are simple conditions ensuring that the operator F in (3.1) is of class C^k from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ $(N < p < \infty)$. By Remark 2.3 and straightforward arguments, it suffices that the coefficients $a_{\alpha\beta}$ and b are equicontinuous C^k_ξ bundle maps with $D^\kappa_\xi a_{\alpha\beta} \in L^\infty(\mathbb{R}^N),\ 1 \leq \alpha, \beta \leq N,\ 0 \leq |\kappa| \leq k$, and $b(\cdot, 0) \in L^p(\mathbb{R}^N), D^\kappa_\xi b \in L^\infty(\mathbb{R}^N),\ 1 \leq |\kappa| \leq k$. The $C^{\nu+1}$ regularity of Fredholm mappings of index $\nu \geq 0$ is needed e.g. to use the Sard-Smale theorem, and hence this remark may be important when $\nu \geq 1$. However, we shall not make use of it in this paper. $\square$

Our investigation of the Fredholm properties of the operator F makes crucial use of the concept of semi-Fredholm operator. Since the standard literature does not fully agree upon a definition and since most texts provide only an incomplete description of their properties, we now include a brief summary for the convenience of the reader. (A full treatment of semi-Fredholm operators can however be found in Kato [10, pp. 229-244], but the setting there is the more general one of unbounded operators, substantially more technical.)

Given real Banach spaces X and Y, an operator $L \in \mathcal{L}(X,Y)$ is said to be *semi-Fredholm* if rge L is closed in Y and at least one among dim ker L and codim rge L is finite. Some authors, e.g., Schechter [19], consider only the case when dim ker $L < \infty$, which indeed will be the more relevant one here. A semi-Fredholm operator has a well defined index $\nu \in \mathbb{Z} \cup \{\pm\infty\}$ given by the formula $\nu :=$ dim ker $L-$ codim rge L. We shall denote by $\Phi_\nu(X,Y)$ the set of semi-Fredholm operators of index ν and (as is customary) by $\Phi_+(X,Y)$ the set of semi-Fredholm operators of index $\nu \in \mathbb{Z} \cup \{-\infty\}$. In other words, $L \in \Phi_+(X,Y)$ if and only if rge L is closed in Y and dim ker $L < \infty$. As recalled in the Introduction, Yood's criterion characterizes $\Phi_+(X,Y) : L \in \Phi_+(X,Y)$ if and only if $L \in \mathcal{L}(X,Y)$ and L is proper on the closed bounded subsets of X. For a proof, see e.g. Deimling [2, p. 78].

Semi-Fredholm operators enjoy two fundamental stability properties, well-known for finite index (Fredholm case) but less familiar in general, although those results go back to the work of Dieudonné [3]: For $\nu \in \mathbb{Z} \cup \{\pm\infty\}, \Phi_\nu(X,Y)$ is stable under small enough perturbations (hence $\Phi_\nu(X,Y)$ and $\Phi_+(X,Y)$ are open subsets of $\mathcal{L}(X,Y)$) and stable under compact perturbations of arbitrary magnitude (hence the same thing is true of $\Phi_+(X,Y)$). A full proof of these properties can be found in Lindenstrauss and Tzafriri [11, pp. 78-79]. Note that the openness of $\Phi_\nu(X,Y)$ for $\nu \in \mathbb{Z} \cup \{\pm\infty\}$ implies the *local constancy* of the index of semi-Fredholm operators. Before proceeding, we need a straightforward variant of Yood's criterion when the space X is reflexive.

Lemma 3.3. *Let X and Y be real Banach spaces with X reflexive and let $L \in \mathcal{L}(X,Y)$. Then, $L \in \Phi_+(X,Y)$ if and only if the following property holds: If (u_n) is a sequence from*

X such that $u_n \rightharpoonup 0$ in X and $Lu_n \to 0$ in Y, then $u_n \to 0$ in X.

Proof. It is trivial to check that when X is reflexive the given condition is equivalent to the properness of L on the closed bounded subsets of X, so that the conclusion follows from Yood's criterion. $\square$

The variant of Yood's criterion in Lemma 3.3 will be used together with the following (apparently new) concept.

Definition 3.1. *Let X and Y be real Banach spaces with X reflexive and let $T, L \in \mathcal{L}(X, Y)$. We shall say that T is compact modulo L if, for every sequence (u_n) from X we have*

$$\{u_n \rightharpoonup 0 \text{ in } X, Lu_n \to 0 \text{ in } Y\} \Rightarrow Tu_n \to 0 \text{ in } Y.$$

Compactness modulo L in Definition 3.1 should be carefully distinguished from Kato's "compactness relative L" [10], which coincides with compactness for bounded operators, and from other properties, e.g. completely singular operators, also introduced in connection with semi-Fredholm operators. ¿From Definition 3.1 with $L = 0$, it follows that "T compact modulo 0" (or, more generally, modulo a compact L) is the same as "T compact" because X is reflexive (this is the only reason why the reflexivity of X was assumed in Definition 3.1). In contrast, if $L \in \Phi_+(X, Y)$, it follows from Lemma 3.3 that *every* operator $T \in \mathcal{L}(X, Y)$ is compact modulo L. The importance of Definition 3.1 is due to the following simple result:

Lemma 3.4. *Let X and Y be real Banach spaces with X reflexive and let $L_0, L_1 \in \mathcal{L}(X, Y)$. Suppose that $L_1 - L_0$ is compact modulo both L_0 and L_1. Then:*

(i) If (u_n) is a sequence from X such that $u_n \rightharpoonup 0$ in X, we have $L_0 u_n \to 0$ in Y if and only if $L_1 u_n \to 0$ in Y.

(ii) $L_0 \in \Phi_+(X, Y)$ if and only if $L_1 \in \Phi_+(X, Y)$.

For $t \in [0, 1]$, set $L_t := tL_1 + (1 - t)L_0$. If $L_t - L_0$ is compact modulo L_0 and L_t for every $t \in [0, 1]$, then

(iii) $L_t \in \Phi_+(X, Y)$ for every $t \in [0, 1]$ if and only if this holds for one such $t_0 \in [0, 1]$, and in that case index L_t is independent of t.

Proof. (i) follows from Definition 3.1 and the remark that if T is compact modulo L and $u_n \rightharpoonup 0$ in $X, Lu_n \to 0$ in Y, then $(L \pm T)u_n \to 0$ in Y. (ii) holds because of (i) and Lemma 3.3. For the proof of (iii), note that for every $t \in [0,1], t(L_1 - L_0) = L_t - L_0$ is compact modulo L_t and L_0. Hence, $L_1 - L_0$ is compact modulo L_t and L_0, so that $(t - t_0)(L_1 - L_0) = L_t - L_{t_0}$ is compact modulo L_t (and L_0). By exchanging the roles of t and t_0 in this statement, it follows that $L_t - L_{t_0}$ is also compact modulo L_{t_0}. Thus, $L_t - L_{t_0}$ is compact modulo L_t and L_{t_0} for every $t \in [0,1]$. That $L_t \in \Phi_+(X,Y)$ if and only if $L_{t_0} \in \Phi_+(X,Y)$ now follows from (ii), and index $L_t =$ index L_{t_0} by the local constancy of the index and the connectedness of $[0,1]$. □

Part (iii) of Lemma 3.4 will be useful only later (Lemma 6.5).

Remark 3.3: In the proof of part (i) of Lemma 3.4, we used the fact that if T is compact modulo L and if $T + L \in \Phi_+(X,Y)$, then $L \in \Phi_+(X,Y)$. But if $L \in \Phi_+(X,Y)$, nothing can be said about $T + L$ (since every operator is compact modulo L when $L \in \Phi_+(X,Y)$). Also, when T is compact modulo L and $T + L \in \Phi_+(X,Y)$, so that $L \in \Phi_+(X,Y)$, it cannot be ascertained that the indices of $T + L$ and L are the same. To see this, let $L, M \in \Phi_+(X,Y)$ with index $L \neq$ index M, and set $T = M - L$. If so, T is compact modulo L (since $L \in \Phi_+(X,Y)$) and $T + L = M$ and L have different indices. □

For $N < p < \infty$ and $u \in W^{2,p}(\mathbb{R}^N)$, we define the operator $L(u) \in \mathcal{L}(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ by

$$(3.8) \qquad L(u)v := -\sum_{\alpha,\beta=1}^{N} \mathbf{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta}v + D\mathbf{b}(u)v \in L^p(\mathbb{R}^N), \ \forall v \in W^{2,p}(\mathbb{R}^N),$$

where the notation (2.5) was used. That $L(u)$ is well defined follows from Theorem 2.1 (ii) and Theorem 2.3.

Lemma 3.5. *Suppose $N < p < \infty$ and let $u \in W^{2,p}(\mathbb{R}^N)$ and $\nu \in \mathbb{Z} \cup \{\pm\infty\}$ be given. Then, $DF(u) \in \Phi_\nu(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ if and only if $L(u) \in \Phi_\nu(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$. In particular, $DF(u) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ if and only if $L(u) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$.*

Proof. From (3.7) and (3.8), we have $DF(u)v - L(u)v = -\sum_{\alpha,\beta=1}^{N}(D\mathbf{a}_{\alpha\beta}(u)v)\partial^2_{\alpha\beta}u$. From

Theorem 2.3, this is just

$$(3.9) \qquad -\sum_{\alpha,\beta=1}^{N}(\partial_{\xi_0}a_{\alpha\beta}(\cdot,u,\nabla u)\partial^2_{\alpha\beta}u)v - \sum_{\alpha,\beta=1}^{N}\sum_{i=1}^{N}(\partial_{\xi_i}a_{\alpha\beta}(\cdot,u,\nabla u)\partial^2_{\alpha\beta}u)\partial_i v.$$

By Theorem 2.1 (ii), we have $\partial_{\xi_i}a_{\alpha\beta}(\cdot,u,\nabla u) \in L^\infty(\mathbb{R}^N)$, hence $\partial_{\xi_i}a_{\alpha\beta}(\cdot,u,\nabla u)\partial^2_{\alpha\beta}u \in L^p(\mathbb{R}^N), 1 \leq \alpha,\beta \leq N, 0 \leq i \leq N$. But since $p > N$, multiplication by a function $\varphi \in L^p(\mathbb{R}^N)$ is a compact operator from $W^{1,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ (if $B \subset \mathbb{R}^N$ is an open ball and supp $\varphi \subset \bar{B}$, use the compactness of the embedding $W^{1,p}(B) \hookrightarrow C^0(\bar{B})$; the general case obtains by truncation and a limiting process because $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$). It follows from this remark that (3.9) defines a compact operator from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$. Thus, $DF(u) - L(u)$ is compact, and the conclusion follows from the stability of $\Phi_\nu(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ by compact perturbations. □

Our next result is about linear elliptic differential operators on $\mathbb{R}^N$.

Lemma 3.6. *Let* $L := -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\partial^2_{\alpha\beta} + \sum_{\alpha=1}^{N} B_\alpha(x)\partial_\alpha + C(x)$ *be an operator strictly elliptic on the compact subsets of* $\mathbb{R}^N$*, with coefficients* $A_{\alpha\beta} \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $B_\alpha, C \in L^\infty(\mathbb{R}^N), 1 \leq \alpha,\beta \leq N$*. Let* $1 < q < \infty$ *and let* (u_n) *be a sequence from* $W^{2,q}(\mathbb{R}^N)$ *such that* $u_n \rightharpoonup 0$ *in* $W^{2,q}(\mathbb{R}^N)$ *and* $Lu_n \to 0$ *in* $L^q(\mathbb{R}^N)$*. Denote by* $B_R \subset \mathbb{R}^N$ *the open ball with center* 0 *and radius* $R > 0$*. Then,* $u_n \to 0$ *in* $W^{2,q}(B_R)$*.*

Proof. From elliptic regularity on bounded domains, e.g. Gilbarg and Trudinger [7, Theorem 9.11], there is a constant $C > 0$ such that

$$(3.10) \qquad ||u||_{2,q,B_R} \leq C(|u|_{0,q,B_{2R}} + |Lu|_{0,q,B_{2R}}), \ \forall u \in W^{2,q}(\mathbb{R}^N).$$

Since $u_n \rightharpoonup 0$ in $W^{2,q}(\mathbb{R}^N)$, we have $u_n \rightharpoonup 0$ in $W^{2,q}(B_{2R})$ and hence $u_n \to 0$ in $L^q(B_{2R})$. Also, $Lu_n \to 0$ in $L^q(\mathbb{R}^N)$ implies $Lu_n \to 0$ in $L^q(B_{2R})$. By letting $u = u_n$ in (3.10), we thus get $u_n \to 0$ in $W^{2,q}(B_R)$. □

Lemma 3.7. *Let $N < p < \infty$. The relation $L(u) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ holds for every $u \in W^{2,p}(\mathbb{R}^N)$ if and only if it holds for some $u^0 \in W^{2,p}(\mathbb{R}^N)$ (see (3.8) for the definition of $L(u)$).*

Proof. In what follows u is fixed once and for all. We shall prove that if (v_n) is a sequence from $W^{2,p}(\mathbb{R}^N)$ such that $v_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ and $L(u)v_n \to 0$ in $L^p(\mathbb{R}^N)$, then $(L(u) - L(u^0))v_n \to 0$ in $L^p(\mathbb{R}^N)$. By exchanging the roles of u and u^0, the same conclusion holds if we assume $L(u^0)v_n \to 0$ in $L^p(\mathbb{R}^N)$ instead of $L(u)v_n \to 0$ in $L^p(\mathbb{R}^N)$. Thus, $L(u) - L(u^0)$ is compact modulo both $L(u)$ and $L(u^0)$ (Definition 3.1), and the lemma follows from Lemma 3.4 (ii).

Our assumptions about the coefficients $a_{\alpha\beta}$ and b ensure that the collections of mappings $(a_{\alpha\beta}(x,\cdot))_{x\in\mathbb{R}^N}, (\partial_{\xi_i} b(x,\cdot))_{x\in\mathbb{R}^N}$ are all equicontinuous at $\xi = 0$. As a result, given $\epsilon > 0$, there is $\delta > 0$ such that $|a_{\alpha\beta}(x,\xi) - a_{\alpha\beta}(x,0)| < \epsilon/2$ and $|\partial_{\xi_i} b(x,\xi) - \partial_{\xi_i} b(x,0)| < \epsilon/2$ for every $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{R} \times \mathbb{R}^N$ with $|\xi| < \delta$. Evidently, δ can be chosen independent of $1 \leq \alpha, \beta \leq N$ and $0 \leq i \leq N$. On the other hand, owing to the embedding $W^{2,p}(\mathbb{R}^N) \hookrightarrow C^1_d(\mathbb{R}^N)$, there is $R > 0$ such that $|(u(x), \nabla u(x))| < \delta$ and $|(u^0(x), \nabla u^0(x))| < \delta$ for $|x| \geq R$, i.e. for $x \in \tilde{B}_R$ where $\tilde{B}_R$ is the complement of the open ball $B_R \subset \mathbb{R}^N$ with center 0 and radius R. Therefore, we have $|a_{\alpha\beta}(x, u(x), \nabla u(x)) - a_{\alpha\beta}(x, u^0(x), \nabla u^0(x))| < \epsilon$ and $|\partial_{\xi_i} b(x, u(x), \nabla u(x)) - \partial_{\xi_i} b(x, u^0(x), \nabla u^0(x))| < \epsilon$ for $x \in \tilde{B}_R$ and $1 \leq \alpha, \beta \leq N$, $0 \leq i \leq N$. This implies at once that $|L(u)v_n - L(u_0)v_n|_{0,p,\tilde{B}_R} \leq \epsilon(N+1)^{2-\frac{2}{p}} ||v_n||_{2,p,\mathring{\tilde{B}}_R}$ and hence that

$$
(3.11) \qquad |L(u)v_n - L(u^0)v_n|_{0,p,\tilde{B}_R} \leq M(N+1)^{2-\frac{2}{p}}\epsilon, \ \forall\, n \in \mathbb{N},
$$

where $M > 0$ is a constant such that $||v_n||_{2,p,\mathbb{R}^N} \leq M$ for every $n \in \mathbb{N}$.

¿From Theorem 2.1 (ii) (and the expression for the derivative $D\mathbf{b}(u)v$ given in Theorem 2.3), the operator $L = L(u)$ satisfies the conditions required in Lemma 3.6, and $v_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N), Lv_n \to 0$ in $L^p(\mathbb{R}^N)$. By Lemma 3.6, we infer that $v_n \to 0$ in $W^{2,p}(B_R)$, which in turn implies that $L(u^0)v_n \to 0$ in $L^p(B_R)$ and $L(u)v_n \to 0$ in $L^p(B_R)$ (of course, the latter also follows from the assumption $L(u)v_n \to 0$ in $L^p(\mathbb{R}^N)$). As a result, we have

$L(u)v_n - L(u^0)v_n \to 0$ in $L^p(B_R)$, so that

$$|L(u)v_n - L(u^0)v_n|_{0,p,B_R} \leq \epsilon, \tag{3.12}$$

for n large enough. Altogether, (3.11) and (3.12) yield $|L(u)v_n - L(u^0)v_n|_{0,p,\mathbb{R}^N} \leq \epsilon(M^p(N+1)^{2p-2}+1)^{1/p}$ for n large enough. Since $\epsilon > 0$ is arbitrary, this is just the desired relation $(L(u) - L(u^0))v_n \to 0$ in $L^p(\mathbb{R}^N)$, and the proof is complete. □

We are now in a position to show that the question: "When is the operator F in (3.1) semi-Fredholm of index $\nu \in \mathbb{Z} \cup \{-\infty\}$?" has the simplest possible answer.

Theorem 3.1. *Let $N < p < \infty$ and suppose that the coefficients of the operator F in (3.1) satisfy the conditions (3.2) to (3.6). Then, F is semi-Fredholm of index $\nu \in \mathbb{Z} \cup \{-\infty\}$ (i.e. $DF(u) \in \Phi_\nu(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for every $u \in W^{2,p}(\mathbb{R}^N)$) if and only if there is $u^0 \in W^{2,p}(\mathbb{R}^N)$ such that $DF(u^0) \in \Phi_\nu(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$.*

Proof. The necessity is obvious. Conversely, suppose $DF(u^0) \in \Phi_\nu(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$, so that $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$. By Lemma 3.5, this implies that $L(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$, and hence $L(u) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for every $u \in W^{2,p}(\mathbb{R}^N)$ by Lemma 3.7. Once again by Lemma 3.5, we find that $DF(u) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$. By the local constancy of the index and the continuity of DF (Lemma 3.2), it follows that the index of $DF(u)$ is independent of u, hence equals the index ν of $DF(u^0)$. □

Even in the case of finite index (Fredholm case) our proof of Theorem 3.1, based upon Yood's criterion, requires making use of the semi-Fredholm setting.

4. The properness of quasilinear elliptic operators on $\mathbb{R}^N$: A new criterion.

The operator F in (3.1) is proper on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ if and only if every bounded sequence (u_n) from $W^{2,p}(\mathbb{R}^N)$ such that $F(u_n)$ converges in $L^p(\mathbb{R}^N)$ contains a subsequence converging in $W^{2,p}(\mathbb{R}^N)$. We begin this section by showing in Theorem 4.1 that when $N < p < \infty$ and $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for some $u^0 \in W^{2,p}(\mathbb{R}^N)$ a *weaker* property of the above sequence (u_n) suffices to obtain the desired properness result. We shall see later (Theorem 5.1) that the existence of u^0 above is necessary for properness and hence is not a restrictive assumption.

Lemma 4.1. *Let $N < p < \infty$ and let (u_n) be a bounded sequence from $W^{2,p}(\mathbb{R}^N)$. Suppose that there is $u \in W^{2,p}(\mathbb{R}^N)$ such that $u_n \to u$ in $C^1_d(\mathbb{R}^N)$. Then, $F(u_n) - F(u) - DF(u)(u_n - u) \to 0$ in $L^p(\mathbb{R}^N)$.*

Proof. By (3.1) and (3.7), we have

$$(4.1) \quad F(u_n) - F(u) - DF(u)(u_n - u) = -\sum_{\alpha,\beta=1}^{N} (\mathbf{a}_{\alpha\beta}(u_n) - \mathbf{a}_{\alpha\beta}(u))\partial^2_{\alpha\beta} u_n \; + \\ \mathbf{b}(u_n) - \mathbf{b}(u) - D\mathbf{b}(u)(u_n - u) + \sum_{\alpha,\beta=1}^{N} [D\mathbf{a}_{\alpha\beta}(u)(u_n - u)]\partial^2_{\alpha\beta} u,$$

where the notation (2.5) was used. In the proof of Lemma 3.5, we already observed that the linear operator $v \in W^{2,p}(\mathbb{R}^N) \to \sum_{\alpha,\beta=1}^{N} (D\mathbf{a}_{\alpha\beta}(u)v)\partial^2_{\alpha\beta} u \in L^p(\mathbb{R}^N)$ is compact. Hence, since our assumptions ensure that $u_n \rightharpoonup u$ in $W^{2,p}(\mathbb{R}^N)$, we have

$$(4.2) \qquad \sum_{\alpha,\beta=1}^{N} [D\mathbf{a}_{\alpha\beta}(u)(u_n - u)]\partial^2_{\alpha\beta} u \to 0 \text{ in } L^p(\mathbb{R}^N).$$

Next,

$$(4.3) \qquad \mathbf{b}(u_n) - \mathbf{b}(u) - D\mathbf{b}(u)(u_n - u) \to 0 \text{ in } L^p(\mathbb{R}^N)$$

by Theorem 2.5. It thus follows from (4.1), (4.2) and (4.3) that $F(u_n) - F(u) - DF(u)(u_n - u) \to 0$ in $L^p(\mathbb{R}^N)$ if and only if

$$(4.4) \qquad \sum_{\alpha,\beta=1}^{N} (\mathbf{a}_{\alpha\beta}(u_n) - \mathbf{a}_{\alpha\beta}(u))\partial^2_{\alpha\beta} u_n \to 0 \text{ in } L^p(\mathbb{R}^N).$$

That (4.4) holds follows at once from Theorem 2.1 (i), the hypothesis $u_n \to u$ in $C^1_d(\mathbb{R}^N)$ and the boundedness of the derivatives $\partial^2_{\alpha\beta} u_n$ in $L^p(\mathbb{R}^N)$. □

The comments made in Remark 2.4 may be repeated here: The hypotheses of Lemma 4.1 do not ensure that $F(u_n) - F(u) \to 0$ in $L^p(\mathbb{R}^N)$ or that $DF(u)(u_n - u) \to 0$ in $L^p(\mathbb{R}^N)$. Yet, the difference of these two terms does tend strongly to 0 in $L^p(\mathbb{R}^N)$.

Theorem 4.1. *Let $N < p < \infty$ and suppose that the coefficients of the operator F in (3.1) satisfy the conditions (3.2) to (3.6). Suppose also that there is $u^0 \in W^{2,p}(\mathbb{R}^N)$ such that $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$. The following properties are equivalent:*

(i) The restriction of F to the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ is proper.

(ii) Every bounded sequence (u_n) from $W^{2,p}(\mathbb{R}^N)$ such that $(F(u_n))$ converges in $L^p(\mathbb{R}^N)$ contains a subsequence converging in $C^1_d(\mathbb{R}^N)$.

Proof. That (i) $\Rightarrow$ (ii) is obvious. Conversely, let (u_n) be a bounded sequence from $W^{2,p}(\mathbb{R}^N)$ such that $(F(u_n))$ converges in $L^p(\mathbb{R}^N)$. With no loss of generality assume that there is $u \in W^{2,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $W^{2,p}(\mathbb{R}^N)$. By Lemma 3.1, we have $F(u_n) \rightharpoonup F(u)$ in $L^p(\mathbb{R}^N)$, hence $F(u_n) \to F(u)$ in $L^p(\mathbb{R}^N)$ since the sequence $(F(u_n))$ is strongly convergent by hypothesis. It follows from property (ii) that after replacing (u_n) by a subsequence, we may assume that (u_n) converges in $C^1_d(\mathbb{R}^N)$. Evidently, the limit of (u_n) in $C^1_d(\mathbb{R}^N)$ must be u, i.e. $u_n \to u$ in $C^1_d(\mathbb{R}^N)$. By Lemma 4.1, we thus have $F(u_n) - F(u) - DF(u)(u_n - u) \to 0$ in $L^p(\mathbb{R}^N)$, and hence $DF(u)(u_n - u) \to 0$ in $L^p(\mathbb{R}^N)$ from the above. From Theorem 3.1, we have $DF(u) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$, so that $DF(u)$ is proper on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ by Yood's criterion. Therefore, $(u_n - u)$ and hence (u_n) contains a subsequence converging in $W^{2,p}(\mathbb{R}^N)$, as desired. □

Several of our previous results, and in particular Theorem 4.1 (ii), involve the convergence in $C^1_d(\mathbb{R}^N)$ of a sequence bounded in $W^{2,p}(\mathbb{R}^N)$. We now give a convenient equivalent formulation of this condition.

Definition 4.1. *We shall say that the sequence (u_n) from $C^1_d\mathbb{R}^N)$ vanishes uniformly at infinity in the sense of $C^1_d(\mathbb{R}^N)$ if the following condition holds: For every $\epsilon > 0$, there are $R > 0$ and $n_0 \in \mathbb{N}$ such that $|u_n(x)| + |\nabla u_n(x)| \leq \epsilon$ for every $|x| \geq R$ and every $n \geq n_0$.*

Theorem 4.2. *Let $N < p < \infty$ and let (u_n) be a bounded sequence from $W^{2,p}(\mathbb{R}^N)$. If $u \in W^{2,p}(\mathbb{R}^N)$, the following conditions are equivalent:*

(i) $u_n \to u$ in $C^1_d(\mathbb{R}^N)$.

(ii) $u_n \rightharpoonup u$ *in* $W^{2,p}(\mathbb{R}^N)$ *and* (u_n) *vanishes uniformly at infinity in the sense of* $C^1_d(\mathbb{R}^N)$.

Proof. (i) $\Rightarrow$ (ii): That $u_n \rightharpoonup u$ in $W^{2,p}(\mathbb{R}^N)$ is clear. Next, let $\epsilon > 0$ be given. Let $R > 0$ be such that $|u(x)| + |\nabla u(x)| \leq \epsilon/2$ for $|x| \geq R$, and let $n_0 \in \mathbb{N}$ be such that $||u_n - u||_{1,\infty,\mathbb{R}^N} \leq \epsilon/2$ for $n \geq n_0$. Then, for $|x| \geq R$ and $n \geq n_0$, we have $|u_n(x)| + |\nabla u_n(x)| \leq \epsilon$.

(ii) $\Rightarrow$ (i): Let $\epsilon > 0$ be given and let $R > 0$ and $n_0 \in \mathbb{N}$ be such that $|u_n(x)| + |\nabla u_n(x)| \leq \epsilon/2$ for $|x| \geq R$ and $n \geq n_0$. After increasing R if necessary, we may also assume that $|u(x)| + |\nabla u(x)| \leq \epsilon/2$ for $|x| \geq R$. Hence,

$$|u_n(x) - u(x)| + |\nabla u_n(x) - \nabla u(x)| \leq \epsilon, \ \forall |x| \geq R, \ \forall n \geq n_0. \tag{4.5}$$

¿From the compactness of the embedding $W^{2,p}(B_R) \hookrightarrow C^1(\bar{B}_R)$ where B_R denotes the open ball with radius R centered at the origin, there is $n_1 \in \mathbb{N}$ such that

$$|u_n(x) - u(x)| + |\nabla u_n(x) - \nabla u(x)| \leq \epsilon, \ \forall x \in \bar{B}_R, \ \forall n \geq n_1. \tag{4.6}$$

By (4.5) and (4.6), we have $||u_n - u||_{1,\infty,\mathbb{R}^N} \leq \epsilon$ for $n \geq \max(n_0, n_1)$, which shows that $u_n \to u$ in $C^1_d(\mathbb{R}^N)$ since $\epsilon > 0$ is arbitrary. $\square$

As a corollary to both Theorems 4.1 and 4.2, we obtain:

Corollary 4.1. *Let* $N < p < \infty$ *and suppose that the coefficients of the operator* F *in (3.1) satisfy the conditions (3.2) to (3.6). Suppose also that there is* $u^0 \in W^{2,p}(\mathbb{R}^N)$ *such that* $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$. *The following properties are equivalent:*
(i) F *is proper on the closed bounded subsets of* $W^{2,p}(\mathbb{R}^N)$.
(ii) Every bounded sequence (u_n) *from* $W^{2,p}(\mathbb{R}^N)$ *such that* $(F(u_n))$ *converges in* $L^p(\mathbb{R}^N)$ *contains a subsequence vanishing uniformly at infinity in the sense of* $C^1_d(\mathbb{R}^N)$.
(iii) Every bounded sequence (u_n) *from* $W^{2,p}(\mathbb{R}^N)$ *such that* $(F(u_n))$ *converges in* $L^p(\mathbb{R}^N)$ *vanishes uniformly at infinity in the sense of* $C^1_d(\mathbb{R}^N)$.

Proof. Straightforward. $\square$

Corollary 4.1 leaves us with the following question: Given that the embedding $W^{2,p}(\mathbb{R}^N)$

$\hookrightarrow C_d^1(\mathbb{R}^N)$ is not compact, what does it take for a bounded sequence from $W^{2,p}(\mathbb{R}^N)$ to vanish uniformly at infinity in the sense of $C_d^1(\mathbb{R}^N)$? We answer this question in the next theorem, by showing that this property is part of an alternative satisfied by *every* bounded sequence from $W^{2,p}(\mathbb{R}^N)$.

Let $T_1 > 0, \cdots, T_N > 0$ be real numbers fixed once and for all. We shall set $T := (T_1, \cdots, T_N) \in \mathbb{R}^N$, and, for every multi-index $\ell = (\ell_1, \cdots, \ell_N) \in \mathbb{Z}^N$,

$$\ell T := (\ell_1 T_1, \cdots, \ell_N T_N) \in \mathbb{R}^N. \tag{4.7}$$

Theorem 4.3. *(shifted subsequence lemma): Let $T \in \mathbb{R}^N$ be as above and let $N < p < \infty$. If (u_n) is a bounded sequence from $W^{2,p}(\mathbb{R}^N)$, then either*

(i) (u_n) vanishes uniformly at infinity in the sense of $C_d^1(\mathbb{R}^N)$ (see Definition 4.1), or

(ii) there are a sequence $\ell_k \in \mathbb{Z}^N$ with $\lim\limits_{k\to\infty} |\ell_k| = \infty$ and a subsequence (u_{n_k}) such that the sequence $(\tilde{u}_{n_k})$ defined by $\tilde{u}_{n_k}(x) := u_{n_k}(x + \ell_k T)$ is weakly convergent (in $W^{2,p}(\mathbb{R}^N)$) to $\tilde{u} \in W^{2,p}(\mathbb{R}^N), \tilde{u} \neq 0$.

Proof. In this proof, we set $Q_0 := (0, T_1) \times \cdots \times (0, T_N)$. Suppose that (i) does not hold, so that there is $\epsilon > 0$ such that for every $k \in \mathbb{N}$, there is $x_k \in \mathbb{R}^N$ with $|x_k| \geq k$ and there is $n_k \in \mathbb{N}, n_k > n_{k-1} (n_0 := 0)$ such that $|u_{n_k}(x_k)| + |\nabla u_{n_k}(x_k)| \geq \epsilon$. Since $\mathbb{R}^N = \bigcup\limits_{\ell \in \mathbb{Z}^N} (\bar{Q}_0 + \ell T)$ (see (4.7)), there is $\ell_k \in \mathbb{Z}^N$ such that $y_k := x_k - \ell_k T \in \bar{Q}_0$. Clearly, $\lim\limits_{k\to\infty} |\ell_k| = \infty$. Let $\tilde{u}_{n_k}(x) := u_{n_k}(x + \ell_k T)$. Note that $||\tilde{u}_{n_k}||_{2,p,\mathbb{R}^N} = ||u_{n_k}||_{2,p,\mathbb{R}^N}$ and hence the sequence $(\tilde{u}_{n_k})$ is bounded in $W^{2,p}(\mathbb{R}^N)$.

By passing to a subsequence, we may assume that there is $\tilde{u} \in W^{2,p}(\mathbb{R}^N)$ such that $\tilde{u}_{n_k} \rightharpoonup \tilde{u}$ in $W^{2,p}(\mathbb{R}^N)$. We now show that $\tilde{u} \neq 0$: By the compactness of the embedding $W^{2,p}(Q_0) \hookrightarrow C^1(\bar{Q}_0)$, we have $\tilde{u}_{n_k} \to \tilde{u}$ in $C^1(\bar{Q}_0)$. In particular, $||\tilde{u}_{n_k}||_{1,\infty,Q_0} \to ||\tilde{u}||_{1,\infty,Q_0}$, and since $||\tilde{u}_{n_k}||_{1,\infty,Q_0} \geq |\tilde{u}_{n_k}(y_k)| + |\nabla \tilde{u}_{n_k}(y_k)| = |u_{n_k}(x_k)| + |\nabla u_{n_k}(x_k)| \geq \epsilon$, it follows that $||\tilde{u}||_{1,\infty,Q_0} \geq \epsilon$, whence $\tilde{u} \neq 0$. $\square$

Remark 4.1: Properties (i) and (ii) of Theorem 4.3 are mutually exclusive, for if (i) holds and $(\tilde{u}_{n_k})$ is a subsequence as in (ii), then $\tilde{u}_{n_k}$ tends uniformly to 0 on every compact subset of $\mathbb{R}^N$, whence $\tilde{u}_{n_k} \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$. Hence, $(\tilde{u}_{n_k})$ cannot tend weakly to $\tilde{u} \neq 0$ in $W^{2,p}(\mathbb{R}^N)$. $\square$

In conjunction with Theorem 4.3, we shall later need the following intuitively clear technical result.

Lemma 4.2. *Let $1 < q < \infty$ and let $k \geq 0$ be an integer. Given $u \in W^{k,q}(\mathbb{R}^N)$ and a sequence $h_n \in \mathbb{R}^N$ such that $\lim |h_n| = \infty$, set $\tilde{u}_n(x) := u(x + h_n)$. Then, $\tilde{u}_n \to 0$ in $W^{k,q}(\omega)$ for every bounded open subset $\omega \subset \mathbb{R}^N$. In particular, $\tilde{u}_n \rightharpoonup 0$ in $W^{k,q}(\mathbb{R}^N)$.*

Proof. Suppose first that $k = 0$. Let $\epsilon > 0$ be given. Choose $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $|u-\varphi|_{0,q,\mathbb{R}^N} \leq \epsilon$, so that $|\tilde{u}_n-\tilde{\varphi}_n|_{0,q,\mathbb{R}^N} = |u-\varphi|_{0,q,\mathbb{R}^N} \leq \epsilon$ for every $n \in \mathbb{N}$, where of course $\tilde{\varphi}_n(x) := \varphi(x+h_n)$. If $\omega \subset \mathbb{R}^N$ is any bounded open subset, we thus have $|\tilde{u}_n-\tilde{\varphi}_n|_{0,q,\omega} \leq \epsilon$ for every $n \in \mathbb{N}$. Now, $|\tilde{u}_n|_{0,q,\omega} \leq |\tilde{u}_n - \tilde{\varphi}_n|_{0,q,\omega} + |\tilde{\varphi}_n|_{0,q,\omega} \leq \epsilon + |\tilde{\varphi}_n|_{0,q,\omega}$, and Supp $\tilde{\varphi}_n =$ (Supp $\varphi) - h_n$. Thus, (Supp $\tilde{\varphi}_n) \cap \omega = \emptyset$ for n large enough, and hence $|\tilde{u}_n|_{0,q,\omega} \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, this yields $\tilde{u}_{n_{|\omega}} \to 0$ in $L^q(\omega)$.

If now $u \in W^{k,q}(\mathbb{R}^N)$, the above shows that $D^\kappa \tilde{u}_n \to 0$ in $L^q(\omega)$ for every $\kappa \in \mathbb{N}^N, |\kappa| \leq k$, so that $\tilde{u}_n \to 0$ in $W^{k,q}(\omega)$. The sequence $(\tilde{u}_n)$ is bounded in $W^{k,q}(\mathbb{R}^N)$(because $||\tilde{u}_n||_{k,q,\mathbb{R}^N} = ||u||_{k,q,\mathbb{R}^N}$). If $\tilde{u} \in W^{2,q}(\mathbb{R}^N)$ is a weak cluster point of $(\tilde{u}_n)$, it follows from the above that $\tilde{u}_{|\omega} = 0$ for every open bounded subset $\omega \subset \mathbb{R}^N$, so that $\tilde{u} = 0$. Thus, $\tilde{u} = 0$, i.e. 0 is the only cluster point of the sequence $(\tilde{u}_n)$, and this implies $\tilde{u}_n \rightharpoonup 0$ in $W^{k,q}(\mathbb{R}^N)$. $\square$

5. A strong form of Yood's criterion for linear elliptic differential operators.

In this section, we prove that if the operator F in (3.1) is proper on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)(N < p < \infty)$, then F is semi-Fredholm of index $\nu \in \mathbb{Z} \cup \{-\infty\}$. This result will be obtained as a corollary to a strong form of Lemma 3.3, valid for linear elliptic differential operators (Theorem 5.1), which is also useful to establish the condition $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ required in Theorem 4.1 or Corollary 4.1.

Lemma 5.1. *Let $L := -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\partial^2_{\alpha\beta} + \sum_{\alpha,\beta=1}^{N} B_\alpha(x)\partial_\alpha + C(x)$ be an operator strictly elliptic on the compact subsets of $\mathbb{R}^N$, with coefficients $A_{\alpha\beta} \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and*

$B_\alpha, C \in L^\infty(\mathbb{R}^N), 1 \le \alpha, \beta \le N$. Let $N < p < \infty$ and let (u_n) be a sequence from $W^{2,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ and $Lu_n \to 0$ in $L^p(\mathbb{R}^N)$. Let $v \in W^{2,p}(\mathbb{R}^N), 0 < \epsilon \le 1$ and $n_0 \in \mathbb{N}$ be given. Then, there is $n \in \mathbb{N}, n \ge n_0$, such that

(i) $||v||^p_{2,p,\mathbb{R}^N} + ||u_n||^p_{2,p,\mathbb{R}^N} - \epsilon \le ||v + u_n||^p_{2,p,\mathbb{R}^N} \le ||v||^p_{2,p,\mathbb{R}^N} + ||u_n||^p_{2,p,\mathbb{R}^N} + \epsilon$.

(ii) $||v + u_n||_{1,\infty,\mathbb{R}^N} \le \max(||v||_{1,\infty,\mathbb{R}^N}, ||u_n||_{1,\infty,\mathbb{R}^N}) + \epsilon$.

(iii) $|Lu_n|_{0,p,\mathbb{R}^N} \le \epsilon$.

Note: Parts (i) and (ii) of the lemma stress the fact that $||v + u_n||^p_{2,p,\mathbb{R}^N}$ is nearly the *sum* of $||v||^p_{2,p,\mathbb{R}^N}$ and $||u_n||^p_{2,p,\mathbb{R}^N}$ while $||v + u_n||_{1,\infty,\mathbb{R}^N}$ remains bounded by any constant bounding $||v||_{1,\infty,\mathbb{R}^N}$ and $||u_n||_{1,\infty\mathbb{R}^N}$.

Proof. Since $Lu_n \to 0$ in $L^p(\mathbb{R}^N)$, we may assume with no loss of generality that (iii) holds for every $n \ge n_0$. Hence, it suffices to show that (i) and (ii) hold for some $n \ge n_0$. Since $v \in W^{2,p}(\mathbb{R}^N)$, there is $R > 0$ such that

$$||v||_{2,p,\overset{\circ}{\tilde{B}}_R} \le \epsilon, \quad ||v||_{1,\infty,\overset{\circ}{\tilde{B}}_R} \le \epsilon, \tag{5.1}$$

where $\tilde{B}_R := \mathbb{R}^N \setminus B_R$ and $B_R \subset \mathbb{R}^N$ is the open ball with center 0 and radius R. Also, by Lemma 3.6, we have $u_n \to 0$ in $W^{2,p}(B_R)$ and hence $u_n \to 0$ in $C^1(\bar{B}_R)$. Thus,

$$||u_n||_{2,p,B_R} \le \epsilon, \quad ||u_n||_{1,\infty,B_R} \le \epsilon, \tag{5.2}$$

for n large enough. We shall need the (elementary) inequalities

$$(a+b)^p \le a^p + p(a+b)^{p-1}b, \quad |a-b|^p \ge a^p - p(a+b)^{p-1}b, \tag{5.3}$$

valid for $a \ge 0$ and $b \ge 0$ and $p \ge 1$.

Since $||v+u_n||^p_{2,p,\mathbb{R}^N} = ||v+u_n||^p_{2,p,B_R} + ||v+u_n||^p_{2,p,\overset{\circ}{\tilde{B}}_R}$, relations (5.1) and (5.2) together with the triangle inequality yield $||v + u_n||^p_{2,p,\mathbb{R}^N} \le (||v||_{2,p,B_R} + \epsilon)^p + (||u_n||_{2,p,\overset{\circ}{\tilde{B}}_R} + \epsilon)^p \le (||v||_{2,p,\mathbb{R}^N} + \epsilon)^p + (||u_n||_{2,p,\mathbb{R}^N} + \epsilon)^p$. From the first inequality in (5.3), we obtain (using $\epsilon \le 1$)

$$||v + u_n||^p_{2,p,\mathbb{R}^N} \le ||v||^p_{2,p,\mathbb{R}^N} + ||u_n||^p_{2,p,\mathbb{R}^N} + 2p(M+1)^{p-1}\epsilon, \tag{5.4}$$

where $M > 0$ is a constant such that $||v||_{2,p,\mathbb{R}^N} \leq M$ and $||u_n||_{2,p,\mathbb{R}^N} \leq M$ for every $n \in \mathbb{N}$.

The other form of the triangle inequality along with (5.1) and (5.2) also yields $||v + u_n||^p_{2,p,\mathbb{R}^N} \geq |\, ||v||_{2,p,B_R} - ||u_n||_{2,p,B_R}|^p + |\, ||u_n||_{2,p,\overset{\circ}{\tilde{B}}_R} - ||v||_{2,p\overset{\circ}{\tilde{B}}_R}|^p \geq ||v||^p_{2,p,B_R} + ||u_n||^p_{2,p,\overset{\circ}{\tilde{B}}_R} - 2p(M+1)^{p-1}\epsilon$, where the second inequality in (5.3) and $\epsilon \leq 1$ were used. Since (5.1) and (5.2) also yield $||v||^p_{2,p,\mathbb{R}^N} \leq ||v||^p_{2,p,B_R} + \epsilon^p$ and $||u_n||^p_{2,p,\mathbb{R}^N} \leq ||u_n||^p_{2,p,\overset{\circ}{\tilde{B}}_R} + \epsilon^p$ and since $\epsilon^p \leq \epsilon$, it follows that

$$(5.5) \qquad ||v||^p_{2,p,\mathbb{R}^N} + ||u_n||^p_{2,p,\mathbb{R}^N} - 2[p(M+1)^{p-1}+1]\epsilon \leq ||v+u_n||^p_{2,p,\mathbb{R}^N}.$$

¿From (5.4) and (5.5), part (i) of the lemma holds with ϵ replaced by $2[p(M+1)^{p-1} + 1]\epsilon$. Hence, it holds as stated after rescaling ϵ (which merely amounts to increasing n if necessary).

The proof of (ii) is straightforward since $||v + u_n||_{1,\infty,\mathbb{R}^N} = \max(||v+u_n||_{1,\infty,B_R}, ||v + u_n||_{1,\infty,\overset{\circ}{\tilde{B}}_R}) \leq \max(||v||_{1,\infty,B_R} + \epsilon, ||u_n||_{1,\infty,\overset{\circ}{\tilde{B}}_R} + \epsilon) = \max(||v||_{1,\infty,B_R}, ||u_n||_{1,\infty,\overset{\circ}{\tilde{B}}_R}) + \epsilon \leq \max(||v||_{1,\infty,\mathbb{R}^N}, ||u_n||_{1,\infty,\mathbb{R}^N}) + \epsilon.$ □

Lemma 5.2. *Let L be as in Lemma 5.1 and let $N < p < \infty$. Suppose that there is a sequence (u_n) from $W^{2,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$, $Lu_n \to 0$ in $L^p(\mathbb{R}^N)$ and (u_n) contains no subsequence converging to 0 in $W^{2,p}(\mathbb{R}^N)$. Then, there is a sequence (w_n) from $W^{2,p}(\mathbb{R}^N)$ such that $w_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N), w_n \to 0$ in $C^1_d(\mathbb{R}^N), Lw_n \to 0$ in $L^p(\mathbb{R}^N)$ and (w_n) contains no subsequence converging to 0 in $W^{2,p}(\mathbb{R}^N)$.*

Note: The whole point in this lemma is that the sequence (w_n) has all the properties of the sequence (u_n) *plus* that of tending to 0 in $C^1_d(\mathbb{R}^N)$, which neither (u_n) nor any of its subsequences need have.

Proof. The hypothesis that (u_n) contains no subsequence converging to 0 in $W^{2,p}(\mathbb{R}^N)$ implies that $||u_n||_{2,p,\mathbb{R}^N}$ is bounded away from 0 in $W^{2,p}(\mathbb{R}^N)$ for n large enough. As a result, we may assume after rescaling (u_n) if necessary that

$$(5.6) \qquad ||u_n||_{2,p,\mathbb{R}^N} \geq 1, \quad \forall n \in \mathbb{N}$$

(in particular, this does not affect the conditions $u_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ or $Lu_n \to 0$ in $L^p(\mathbb{R}^N)$).

Let (ϵ_k) be a sequence from $(0,1]$ such that

$$\sum_{k=1}^{\infty} \epsilon_k = 1. \tag{5.7}$$

In Lemma 5.1, let $v = u_1, \epsilon = \epsilon_1$ and $n_0 = 1$. This produces an integer $n_1 \in \mathbb{N}$, and hence an element u_{n_1} from the sequence (u_n). With $v_1 := u_1$, set $v_2 := v_1 + u_{n_1} = u_1 + u_{n_1}$. More generally, with $v_k \in W^{2,p}(\mathbb{R}^N)$ and $n_k \in \mathbb{N}$ already constructed, let $v = v_k, \epsilon = \epsilon_k$ and $n_0 = n_k$ in Lemma 5.1. This produces an integer $n_{k+1} \geq n_k$ and a corresponding element $u_{n_{k+1}}$ from the sequence (u_n). We then set $v_{k+1} := v_k + u_{n_{k+1}}$.

Property (i) of Lemma 5.1 yields $||v_k||^p_{2,p,\mathbb{R}^N} + ||u_{n_{k+1}}||^p_{2,p,\mathbb{R}^N} - \epsilon_k \leq ||v_{k+1}||^p_{2,p,\mathbb{R}^N} \leq ||v_k||^p_{2,p,\mathbb{R}^N} + ||u_{n_{k+1}}||^p_{2,p,\mathbb{R}^N} + \epsilon_k$, which, by induction, shows at once that with $n_0 := 1$, we have (recall $v_1 = u_1$)

$$\sum_{j=0}^{k+1} ||u_{n_j}||^p_{2,p,\mathbb{R}^N} - \sum_{j=1}^{k} \epsilon_j \leq ||v_{k+1}||^p_{2,p,\mathbb{R}^N} \leq \sum_{j=0}^{k+1} ||u_{n_j}||^p_{2,p,\mathbb{R}^N} + \sum_{j=1}^{k} \epsilon_j,$$

for every $k \geq 0$. By changing k into $k-1$ and using (5.6) and (5.7), it follows that

$$k^{1/p} \leq ||v_k||_{2,p,\mathbb{R}^N} \leq M(k+2)^{1/p}, \ \forall\, k \geq 1, \tag{5.8}$$

where $M(\geq 1)$ is a constant such that $||u_n||_{2,p,\mathbb{R}^N} \leq M$ for every $n \in \mathbb{N}$.

¿From Lemma 5.1 (ii), we find $||v_{k+1}||_{1,\infty,\mathbb{R}^N} \leq \max(||v_k||_{1,\infty,\mathbb{R}^N}, ||u_{n_{k+1}}||_{1,\infty,\mathbb{R}^N}) + \epsilon_k$ and hence, by induction, $||v_{k+1}||_{1,\infty,\mathbb{R}^N} \leq \max\{||u_{n_j}||_{1,\infty,\mathbb{R}^N} : 0 \leq j \leq k+1\} + \sum_{j=1}^{k} \epsilon_j$ for $k \geq 0$, where, once again, $n_0 := 1$. By (5.7), we get

$$||v_k||_{1,\infty,\mathbb{R}^N} \leq CM + 1, \quad \forall\, k \geq 1, \tag{5.9}$$

with M as above and $C > 0$ depending only upon the embedding $W^{2,p}(\mathbb{R}^N) \hookrightarrow C^1_0(\mathbb{R}^N)$.

Next, by Lemma 5.1 (iii), we have $|Lv_{k+1}|_{0,p,\mathbb{R}^N} \leq |Lv_k|_{0,p,\mathbb{R}^N} + \epsilon_k$, whence, by induction, $|Lv_{k+1}|_{0,p,\mathbb{R}^N} \leq |Lu_1|_{0,p,\mathbb{R}^N} + \sum_{j=1}^{k} \epsilon_j$ for every $k \geq 0$. By (5.7),

$$|Lv_k|_{0,p,\mathbb{R}^N} \leq |Lu_1|_{0,p,\mathbb{R}^N} + 1. \tag{5.10}$$

Now, set $w_k := v_k/k^{1/p}$. It follows from (5.8) that (w_k) is bounded in $W^{2,p}(\mathbb{R}^N)$ and that $||w_k||_{2,p,\mathbb{R}^N} \geq 1$ for every $k \in \mathbb{N}$ (so that (w_k) contains no subsequence converging to 0 in $W^{2,p}(\mathbb{R}^N)$). By (5.9), $||w_k||_{1,\infty,\mathbb{R}^N} \leq (CM+1)/k^{1/p}$, whence $w_k \to 0$ in $C^1_d(\mathbb{R}^N)$. Together with the boundedness of (w_k) in $W^{2,p}(\mathbb{R}^N)$, this also implies $w_k \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$. Lastly, $|Lw_k|_{0,p,\mathbb{R}^N} \leq (|Lu_1|_{0,p,\mathbb{R}^N} + 1)/k^{1/p}$ by (5.10), which shows that $Lw_k \to 0$ in $L^p(\mathbb{R}^N)$. □

Theorem 5.1. *Let* $L := -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\partial^2_{\alpha\beta} + \sum_{\alpha=1}^{N} B_\alpha(x)\partial_\alpha + C(x)$ *be an operator strictly elliptic on the compact subsets of* $\mathbb{R}^N$*, with coefficients* $A_{\alpha\beta} \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $B_\alpha, C \in L^\infty(\mathbb{R}^N)$*. Given* $N < p < \infty$*, we have* $L \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ *if and only if every bounded sequence* (u_n) *from* $W^{2,p}(\mathbb{R}^N)$ *such that* $u_n \to 0$ *in* $C^1_d(\mathbb{R}^N)$ *and* $Lu_n \to 0$ *in* $L^p(\mathbb{R}^N)$ *contains a subsequence converging in* $W^{2,p}(\mathbb{R}^N)$ *(and hence tends to* 0 *in* $W^{2,p}(\mathbb{R}^N)$*).*

Proof. The necessity is obvious. For the converse, we use Lemma 3.3: We must show that if (u_n) is a sequence from $W^{2,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ and $Lu_n \to 0$ in $L^p(\mathbb{R}^N)$, then $u_n \to 0$ in $W^{2,p}(\mathbb{R}^N)$. By contradiction, suppose that it is not so, and hence that (u_n) contains a subsequence (u_{n_k}) which is bounded away from 0 in $W^{2,p}(\mathbb{R}^N)$. For simplicity of notation, replace (u_{n_k}) by (u_n). Then, (u_n) satisfies the conditions required in Lemma 5.2, and hence after replacing (u_n) by the sequence (w_n) of that lemma, we may assume that $u_n \to 0$ in $C^1_d(\mathbb{R}^N)$. But then, from the hypothesis made in the theorem, (u_n) must contain a subsequence converging to 0 in $W^{2,p}(\mathbb{R}^N)$, which is the desired contradiction. □

Theorem 5.1 is a strengthening of Lemma 3.3, hence of Yood's criterion, for elliptic differential operators, showing that attention may be confined to bounded sequences tending to 0 in $C^1_d(\mathbb{R}^N)$ (instead of considering all the sequences tending weakly to 0 in $W^{2,p}(\mathbb{R}^N)$).

Corollary 5.1. *Let* $N < p < \infty$ *and suppose that the coefficients of the operator* F *in (3.1) satisfy the conditions (3.2) to (3.6). Suppose also that every bounded sequence* (u_n) *from* $W^{2,p}(\mathbb{R}^N)$ *such that* $u_n \to 0$ *in* $C^1_d(\mathbb{R}^N)$ *and* $F(u_n) \to F(0)$ *in* $L^p(\mathbb{R}^N)$ *contains a convergent subsequence*[(1)] *(In particular, it is so if* F *is proper on the closed bounded subsets*

[(1)] Hence $u_n \to 0$ in $W^{2,p}(\mathbb{R}^N)$.

of $W^{2,p}(\mathbb{R}^N)$.) Then, there is $\nu \in \mathbb{Z} \cup \{-\infty\}$ such that $DF(u) \in \Phi_\nu(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for every $u \in W^{2,p}(\mathbb{R}^N)$.

Proof. By Theorem 3.1, it suffices to show that $DF(0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$. According to Theorem 5.1 with $L = DF(0)$, this amounts to proving that if (u_n) is a sequence from $W^{2,p}(\mathbb{R}^N)$ such that $u_n \to 0$ in $C^1_d(\mathbb{R}^N)$ and $DF(0)u_n \to 0$ in $L^p(\mathbb{R}^N)$, then (u_n) contains a subsequence converging (to 0) in $W^{2,p}(\mathbb{R}^N)$. By Lemma 4.1, we have $F(u_n) - F(0) - DF(0)u_n \to 0$ in $L^p(\mathbb{R}^N)$. Since $DF(0)u_n \to 0$ in $L^p(\mathbb{R}^N)$, this implies $F(u_n) \to F(0)$ in $L^p(\mathbb{R}^N)$, and now the hypothesis about F made in the corollary requires (u_n) to have a subsequence converging (to 0) in $W^{2,p}(\mathbb{R}^N)$. □

We omit the (obvious) explicit statements of the necessary and sufficient criteria for the properness of F on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ that follow from Corollary 5.1 and either Theorem 4.1 or Corollary 4.1.

6. Operators with asymptotically N-periodic coefficients.

Let $T = (T_1, \cdots, T_N) \in \mathbb{R}^N$ with $T_i > 0, 1 \leq i \leq N$. A mapping f defined on $\mathbb{R}^N$ is said to be N-periodic with period T if $f(x_1, \cdots, x_i + T_i, \cdots, x_N) = f(x_1, \cdots, x_N)$ for every $x \in \mathbb{R}^N$ and every $1 \leq i \leq N$. In this section, we shall assume that the coefficients $a_{\alpha\beta}, 1 \leq \alpha, \beta \leq N$ and b are "asymptotically" N-periodic (in addition to the properties required earlier and listed in Section 3). Precisely, this means that we consider continuous mappings $a^\infty_{\alpha\beta} : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}, 1 \leq \alpha, \beta \leq N$, all N-periodic in x with period $T = (T_1, \cdots, T_N)$, and assume that

$$\lim_{|x|\to\infty} a_{\alpha\beta}(x, \cdot) - a^\infty_{\alpha\beta}(x, \cdot) = 0, \ 1 \leq \alpha, \beta \leq N, \tag{6.1}$$

where the limits are taken uniformly over the compact subsets of $\mathbb{R} \times \mathbb{R}^N$. Recall that $b(x, \xi)$ may be written as

$$b(x, \xi) = b(x, 0) + \sum_{i=0}^{N} c_i(x, \xi)\xi_i \tag{6.2}$$

with $c_i(x, \xi) = \int_0^1 \partial_{\xi_i} b(x, t\xi)dt$, so that $c_i(\cdot, 0) = \partial_{\xi_i} b(\cdot, 0) \in L^\infty(\mathbb{R}^N)$and c_i is an equicontinuous C^0 bundle map (see Theorem 2.2), $0 \leq i \leq N$. We also assume the existence of continuous mappings $c_i^\infty : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}, N$-periodic in x with period T as above such that

$$\lim_{|x|\to\infty} c_i(x,\cdot) - c_i^\infty(x,\cdot) = 0,\ 0 \leq i \leq N, \tag{6.3}$$

where once again the limits are taken uniformly over the compact subsets of $\mathbb{R} \times \mathbb{R}^N$. We shall set

$$b^\infty(x,\xi) := \sum_{i=0}^{N} c_i^\infty(x,\xi)\xi_i, \tag{6.4}$$

so that $b^\infty(\cdot,0) = 0$.

Note that (6.3) and (6.4) hold e.g. when $b^\infty : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}$ is a C_ξ^1 bundle map with $b^\infty(\cdot,0) = 0$, which is N-periodic in x with period T and such that

$$\lim_{|x|\to\infty} \partial_{\xi_i} b(x,\cdot) - \partial_{\xi_i} b^\infty(x,\cdot) = 0, \quad 0 \leq i \leq N, \tag{6.3'}$$

uniformly over the compact subsets of $\mathbb{R} \times \mathbb{R}^N$. It suffices to define $c_i^\infty(x,\xi) := \int_0^1 \partial_{\xi_i} b^\infty(x, t\xi)dt$.

By continuity and N-periodicity, we have $a_{\alpha\beta}^\infty(\cdot,0) \in L^\infty(\mathbb{R}^N)$, $\quad 1 \leq \alpha,\beta \leq N$, and $c_i^\infty(\cdot,0) \in L^\infty(\mathbb{R}^N)$, $0 \leq i \leq N$. Also, by Remark 2.1, the coefficients $a_{\alpha\beta}^\infty$ and c_i^∞ are all equicontinuous C^0 bundle maps. No ellipticity condition is required of the coefficients $a_{\alpha\beta}^\infty$ at this stage. We define the operator F^∞ in the obvious way, namely

$$F^\infty(u) := -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}^\infty(\cdot,u,\nabla u)\partial_{\alpha\beta}^2 u + b^\infty(\cdot,u,\nabla u). \tag{6.5}$$

The assumptions made above ensure that when $N < p < \infty$, F^∞ is both continuous and weakly sequentially continuous from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, and maps bounded subsets onto bounded subsets. This follows from Lemma 3.1 and Remark 3.1.

Under the assumptions listed above, we shall prove that two very simple necessary and sufficient criteria for the properness of F on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ exist (Theorem 6.1).

Lemma 6.1. *For $R > 0$, denote by $\tilde{B}_R$ the complement of the open ball with center 0 and radius R in $\mathbb{R}^N$. Suppose $N < p < \infty$ and let $\mathcal{B} \subset W^{2,p}(\mathbb{R}^N)$ be a bounded subset. For every $\epsilon > 0$, there is $R > 0$ such that $|F(u) - F(0) - F^\infty(u)|_{0,p,\tilde{B}_R} \leq \epsilon$ for every $u \in \mathcal{B}$.*

Proof. Since $\mathcal{B}$ is bounded in $W^{2,p}(\mathbb{R}^N)$, hence in $C^1_d(\mathbb{R}^N)$, there is a compact subset $K \subset \mathbb{R} \times \mathbb{R}^N$ such that $(u(x), \nabla u(x)) \in K$ for every $x \in \mathbb{R}^N$ and every $u \in \mathcal{B}$. It thus follows from the limit in (6.1) being uniform in $\xi \in K$ that $|a_{\alpha\beta}(x,\xi) - a^\infty_{\alpha\beta}(x,\xi)| \leq \epsilon$ for every $x \in \tilde{B}_R$) and every $\xi \in K$ if $R > 0$ is large enough. Hence

$$|\mathbf{a}_{\alpha\beta}(u) - \mathbf{a}^\infty_{\alpha\beta}(u)|_{0,\infty,\tilde{B}_R} \leq \epsilon, \ \forall\, u \in \mathcal{B}, \ 1 \leq \alpha, \beta \leq N, \tag{6.6}$$

where the notation (2.6) was used.

A similar argument, based on (6.3) instead of (6.1), yields

$$|c_i(x,\xi) - c^\infty_i(x,\xi)| \leq \epsilon, \ \forall x \in \tilde{B}_R, \ \forall \xi \in K, \ 0 \leq i \leq N, \tag{6.7}$$

after increasing $R > 0$ if necessary. Since K above is arbitrary, we may also assume that K is convex and contains 0. By (6.2), (6.4) and (6.7), we find $|b(x,\xi) - b(x,0) - b^\infty(x,\xi)| \leq \epsilon \sum_{i=0}^{N} |\xi_i|$ for $x \in \tilde{B}_R$ and $\xi \in K$. As a result, $|\mathbf{b}(u) - \mathbf{b}(0) - \mathbf{b}^\infty(u)|_{0,p\tilde{B}_R} \leq \epsilon(N+1)||u||_{2,p,\mathbb{R}^N}$ for every $u \in \mathcal{B}$. Since $b(\cdot, 0) \in L^p(\mathbb{R}^N)$, we have $|b(\cdot,0)|_{0,p,\tilde{B}_R} \leq \epsilon$ for $R > 0$ large enough, whence

$$|\mathbf{b}(u) - \mathbf{b}^\infty(u)|_{0,p,\tilde{B}_R} \leq \epsilon(N+1)||u||_{2,p\mathbb{R}^N}, \ \forall u \in \mathcal{B}.$$

Together with (6.6), this shows that $|F(u) - F(0) - F^\infty(u)|_{0,p,\tilde{B}_R} \leq \epsilon(N^2 + N + 1)||u||_{2,p,\mathbb{R}^N}$ for every $u \in \mathcal{B}$. Since $\mathcal{B}$ is bounded, the conclusion follows after a mere rescaling of ϵ. □

Given $h \in \mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}$ any function, we denote by $\tau_h(f) : \mathbb{R}^N \to \mathbb{R}$ the function

$$\tau_h(f)(x) := f(x+h), \ \forall x \in \mathbb{R}^N. \tag{6.8}$$

Lemma 6.2. *Suppose $N < p < \infty$ and let $\mathcal{B} \subset W^{2,p}(\mathbb{R}^N)$ be a bounded subset. Let $\omega \subset \mathbb{R}^N$ be a bounded open subset. For every $\epsilon > 0$, we have $|\tau_h(F(u) - F(0)) - \tau_h(F^\infty(u))|_{0,p,\omega} \leq \epsilon$ for every $u \in \mathcal{B}$ provided that $|h|$ is large enough.*

Proof. By translation invariance of the Lebesgue measure, $|\tau_h(F(u)-F(0))-\tau_h(F^\infty(u))|_{0,p,\omega} = |F(u) - F(0) - F^\infty(u)|_{0,p,\omega+h}$ for every $u \in \mathcal{B}$. Let then $R > 0$ be as in Lemma 6.1. For $|h|$ large enough, we have $\omega + h \subset \tilde{B}_R$ since ω is bounded, so that $|F(u) - F(0) - F^\infty(u)|_{0,p,\omega+h} \leq |F(u) - F(0) - F^\infty(u)|_{0,p,\tilde{B}_R} \leq \epsilon$ for every $u \in \mathcal{B}$. □

Lemma 6.3. *Suppose that $N < p < \infty$ and*
(i) there is $u^0 \in W^{2,p}(\mathbb{R}^N)$ such that $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$,
(ii) $\{u \in W^{2,p}(\mathbb{R}^N), F^\infty(u) = 0\} \Leftrightarrow u = 0$.

Then, the restriction of the operator F to the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ is proper.

Proof. By Corollary 4.1, we must show that if (u_n) is a bounded sequence from $W^{2,p}(\mathbb{R}^N)$ and if there is $y \in L^p(\mathbb{R}^N)$ such that $F(u_n) \to y$ in $L^p(\mathbb{R}^N)$, then there is a subsequence (u_{n_k}) vanishing uniformly at infinity in the sense of $C^1_d(\mathbb{R}^N)$. After replacing F by $F - F(0)$ and y by $y - F(0)$, we may assume $F(0) = 0$.

Since the sequence (u_n) is bounded, we may also assume that $u_n \rightharpoonup u$ in $W^{2,p}(\mathbb{R}^N)$ after replacing (u_n) by a subsequence. The problem reduces to showing that (u_n) vanishes uniformly at infinity in the sense of $C^1_d(\mathbb{R}^N)$. To do this, it suffices to prove that the case (ii) of Theorem 4.3 cannot occur, i.e. that there are no sequence $\ell_k \in \mathbb{Z}^N$ with $\lim |\ell_k| = \infty$ and subsequence (u_{n_k}) of (u_n) such that $\tilde{u}_{n_k}(x) := u_{n_k}(x + \ell_k T)$ has a nonzero weak limit $\tilde{u}$ in $W^{2,p}(\mathbb{R}^N)$. See (4.7) for the notation $\ell_k T$. By contradiction, suppose that such sequences exist. For simplicity of notation, we replace (u_{n_k}) by (u_n), thereby assuming that there is a sequence $\ell_n \in \mathbb{Z}^N$ with $\lim_{n\to\infty} |\ell_n| = \infty$ such that $\tilde{u}_n \rightharpoonup \tilde{u} \neq 0$ in $W^{2,p}(\mathbb{R}^N)$, where $\tilde{u}_n(x) := u_n(x + \ell_n T)$. The translation invariance of the Lebesgue measure implies $F(u_n)^\sim - \tilde{y}_n \to 0$ in $L^p(\mathbb{R}^N)$, where $F(u_n)^\sim(x) := F(u_n)(x + \ell_n T)$ and $\tilde{y}_n(x) := y(x + \ell_n T)$. In particular, for every bounded subset $\omega \subset \mathbb{R}^N$ we have

$$F(u_n)^\sim - \tilde{y}_n \to 0 \text{ in } L^p(\omega). \tag{6.9}$$

The definition of $F(u_n)^\sim$ given above coincides with $\tau_{\ell_n T}(F(u_n))$ if the notation (6.8) is used. Since $|\ell_n| \to \infty$, hence $|\ell_n T| \to \infty$, it follows from (6.9) (with $F(0) = 0$) and Lemma 6.2 that $\tau_{\ell_n T}(F^\infty(u_n)) - \tilde{y}_n \to 0$ in $L^p(\omega)$. But because the coefficients of F^∞ are N-periodic in x, this is just

$$F^\infty(\tilde{u}_n) - \tilde{y}_n \to 0 \text{ in } L^p(\omega). \tag{6.10}$$

Since the sequence $(\tilde{u}_n)$ is bounded in $W^{2,p}(\mathbb{R}^N)$ (indeed, $||\tilde{u}_n||_{2,p\mathbb{R}^N} = ||u_n||_{2,p,\mathbb{R}^N}$ and (u_n) is a bounded sequence by hypothesis) and F^∞ maps bounded subsets to bounded subsets, as observed earlier, the sequence $(F^\infty(\tilde{u}_n) - \tilde{y}_n)$ is bounded in $L^p(\mathbb{R}^N)$ and hence has weakly convergent subsequences in $L^p(\mathbb{R}^N)$. By (6.10), every subsequence of $(F^\infty(\tilde{u}_n) - \tilde{y}_n)$ converging weakly to some limit in $L^p(\mathbb{R}^N)$ converges to 0 in $L^p(\omega)$ for every bounded open subset $\omega \subset \mathbb{R}^N$, and hence converges weakly to 0 in $L^p(\mathbb{R}^N)$. As a result, $F^\infty(\tilde{u}_n) - \tilde{y}_n \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$, and since $\tilde{y}_n \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$ by Lemma 4.2, it follows that $F^\infty(\tilde{u}_n) \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$. Thus, $F^\infty(\tilde{u}) = 0$ by the weak sequential continuity of F^∞ (also observed earlier). This contradicts condition (ii) of the lemma since $\tilde{u} \neq 0$. □

Our assumptions do not ensure that the operator F^∞ in (6.5) is differentiable, but it is easily checked that they suffice for F^∞ to be differentiable at $0 \in W^{2,p}(\mathbb{R}^N)$, with derivative

$$DF^\infty(0)v = -\sum_{\alpha,\beta=1}^{N} a^\infty_{\alpha\beta}(\cdot,0)\partial^2_{\alpha\beta}v + \sum_{\alpha=1}^{N} c^\infty_\alpha(\cdot,0)\partial_\alpha v + c^\infty_0(\cdot,0)v, \tag{6.11}$$

for every $v \in W^{2,p}(\mathbb{R}^N)$.

Lemma 6.4. *Suppose $N < p < \infty$ and let (u_n) be a bounded sequence from $W^{2,p}(\mathbb{R}^N)$ such that $u_n \to 0$ in $W^{2,p}(\omega)$ for every bounded open subset $\omega \subset \mathbb{R}^N$. Then, we have*
(i) $F(u_n) - F(0) - F^\infty(u_n) \to 0$ in $W^{2,p}(\mathbb{R}^N)$.
(ii) $(DF(0) - DF^\infty(0))u_n \to 0$ in $W^{2,p}(\mathbb{R}^N)$.

Proof. (i) Let $\epsilon > 0$ be given. Since (u_n) is bounded, it follows from Lemma 6.1 that for $R > 0$ large enough, we have $|F(u_n) - F(0) - F^\infty(u_n)|_{0,p,\tilde{B}_R} \leq \epsilon$ for every $n \in \mathbb{N}$.

Next, since $u_n \to 0$ in $W^{2,p}(B_R)$ by hypothesis, and since F and F^∞ are continuous from $W^{2,p}(B_R)$ into $L^p(B_R)$ (see Remark 3.1), we have $F(u_n) - F(0) \to 0$ in $L^p(B_R)$ and $F^\infty(u_n) \to 0$ in $L^p(B_R)$. Hence, $|F(u_n) - F(0) - F^\infty(u_n)|_{0,p,B_R} \leq \epsilon$ for n large enough. As a result, $|F(u_n) - F(0) - F^\infty(u_n)|_{0,p,\mathbb{R}^N} \leq 2^{1/p}\epsilon$ for n large enough. This proves (i).

(ii) follows from (i) after replacing F and F^∞ by $DF(0)$ and $DF^\infty(0)$, respectively. Indeed, because smoothness in the variable ξ alone is required in the definition of C^k_ξ bundle maps (Definition 2.1), it is readily checked that $DF(0)$ (resp. $DF^\infty(0)$) satisfies all the conditions required of F (resp. F^∞). Furthermore, conditions (6.1) to (6.4) are unaffected when replacing F (resp. F^∞) by $DF(0)$ (resp. $DF^\infty(0)$) (see (3.7) and (6.11) for the expression of $DF(0)$ and $DF^\infty(0)$). □

Theorem 6.1. *Suppose $N < p < \infty$ and that the coefficients of the operator F in (3.1) satisfy the conditions (3.2) to (3.6) and (6.1) and (6.3). The following statements are equivalent:*

(i) The restriction of F to the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ is proper.

(ii) Every sequence (u_n) from $W^{2,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ and $F(u_n) \to F(0)$ in $L^p(\mathbb{R}^N)$ contains a convergent subsequence (hence tends to 0 in $W^{2,p}(\mathbb{R}^N)$).

(iii) There is $u^0 \in W^{2,p}(\mathbb{R}^N)$ such that $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ and the equation $F^\infty(u) = 0$ (see (6.5)) has no nonzero solution $u \in W^{2,p}(\mathbb{R}^N)$.

Proof. (i) $\Rightarrow$ (ii) is obvious and (iii) $\Rightarrow$ (i) is Lemma 6.3. Thus, it suffices to show that (ii) $\Rightarrow$ (iii). That $DF(u) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for every $u \in W^{2,p}(\mathbb{R}^N)$ when (ii) holds follows from Corollary 5.1 since $u_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ whenever (u_n) is bounded in $W^{2,p}(\mathbb{R}^N)$ and $u_n \to 0$ in $C^1_d(\mathbb{R}^N)$. To show that $F^\infty(u) \neq 0$ if $u \neq 0$, assume by contradiction that $u \in W^{2,p}(\mathbb{R}^N), u \neq 0$ and $F^\infty(u) = 0$. For $n \in \mathbb{N}$, set $\tilde{u}_n(x) := u(x + nT)$, so that $F^\infty(\tilde{u}_n) = 0$ by the N-periodicity in x of the coefficients of F^∞, and $\tilde{u}_n \to 0$ in $W^{2,p}(\omega)$ for every bounded open subset $\omega \subset \mathbb{R}^N$ by Lemma 4.2. By Lemma 6.4 (i), we thus obtain $F(\tilde{u}_n) - F(0) \to 0$ in $L^p(\mathbb{R}^N)$, i.e. $F(\tilde{u}_n) \to F(0)$ in $L^p(\mathbb{R}^N)$. Also, $\tilde{u}_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ (e.g. by Lemma 4.2 once again). It now follows from part (ii) of the theorem that $(\tilde{u}_n)$ contains a convergent subsequence. Necessarily, this subsequence tends to 0, which is impossible since $||\tilde{u}_n||_{2,p,\mathbb{R}^N} = ||u||_{2,p,\mathbb{R}^N}$ for every $n \in \mathbb{N}$ and $u \neq 0$. □

Both the implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) in Theorem 6.1 are very simple criteria for the properness of F. Note that condition (ii) is much weaker than properness, and actually a genuine (nonlinear) generalization of Lemma 3.3. The nonexistence of nonzero solutions to the equation $F^\infty(u) = 0$ can often be checked, at least when F^∞ has x-independent coefficients, via general Pohozaev identities (see [18]).

Remark 6.1: Due to the continuity and N-periodicity of the coefficients $a^\infty_{\alpha\beta}(\cdot, 0)$, the ellipticity and strict ellipticity on $\mathbb{R}^N$ are identical properties for these coefficients. Also (assuming (3.4)) it follows easily from (6.1) that the (strict) ellipticity condition for the coefficients $a^\infty_{\alpha\beta}(\cdot, 0)$ is equivalent to the strict ellipticity condition for the coefficients $a_{\alpha\beta}(\cdot, 0)$ (i.e. the constant $\gamma(x, 0)$ in (3.4) can be chosen independent of $x \in \mathbb{R}^N$). $\square$

Our next task will be to show that under a mild additional assumption, $DF(0)$ and $DF^\infty(0)$ are in $\Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ simultaneously. This will lead to a sharpening of Theorem 6.1 in Theorem 6.2.

Lemma 6.5. *Suppose $N < p < \infty$ and suppose that the coefficients $a_{\alpha\beta}(\cdot, 0)$ satisfy a strict ellipticity condition on $\mathbb{R}^N$. Then, $DF(0) - DF^\infty(0)$ is compact modulo both $DF(0)$ and $DF^\infty(0)$ (see Definition 3.1). Furthermore, $DF(0) \in \Phi_\nu(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ if and only if $DF^\infty(0) \in \Phi_\nu(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$.*

Proof. We must show that if (v_n) is a sequence from $W^{2,p}(\mathbb{R}^N)$ such that $v_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ and either $DF(0)v_n \to 0$ in $L^p(\mathbb{R}^N)$ or $DF^\infty(0)v_n \to 0$ in $L^p(\mathbb{R}^N)$, then $(DF(0) - DF^\infty(0))v_n \to 0$ in $L^p(\mathbb{R}^N)$. By Lemma 6.4 (ii), it suffices to show that $v_n \to 0$ in $W^{2,p}(\omega)$ for every bounded open subset $\omega \subset \mathbb{R}^N$. This follows from Lemma 3.6 with either $L = DF(0)$ or $L = DF^\infty(0)$. That this lemma does apply with both choices is due to the strict ellipticity assumption and Remark 6.1. This shows that $DF(0) - DF^\infty(0)$ is compact modulo both $DF(0)$ and $DF^\infty(0)$.

More generally, set $L_t := tDF(0) + (1-t)DF^\infty(0)$ for $t \in [0, 1]$. By Remark 6.1, the second order coefficients of L_t, i.e. $ta_{\alpha\beta}(\cdot, 0) + (1-t)a^\infty_{\alpha\beta}(\cdot, 0)$ satisfy a strict ellipticity condition on $\mathbb{R}^N$. As a result, we may repeat the above arguments with $L_1 = DF(0)$ replaced by L_t to prove that $L_t - L_0$ is compact modulo both L_t and L_0 for every $t \in [0, 1]$.

By Lemma 3.4 (iii), $L_t \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ if and only if this is true of either $L_1 = DF(0)$ or $L_0 = DF^\infty(0)$, and if so the index of L_t is independent of $t \in [0,1]$. This completes the proof. $\square$

The following lemma complements Lemma 6.5 by showing that in some cases of practical importance, the index ν can only be 0.

Lemma 6.6. *Let $1 < q < \infty$ and let $L \in \mathcal{L}(W^{2,q}(\mathbb{R}^N), L^q(\mathbb{R}^N))$ be a second order elliptic differential operator with continuous N-periodic coefficients. We have*
(i) $\dim\ker L = 0$ *or* ∞ *and* $\dim\ker L^* = 0$ *or* ∞.
(ii) If $N = 1$, then $\dim\ker L = 0$, *and if in addition* rge L *is closed then L is an isomorphism (hence Fredholm of index* 0*)*(2).
(iii) If $q \geq 2$, L has constant coefficients and $L \in \Phi_+(W^{2,q}(\mathbb{R}^N), L^q(\mathbb{R}^N))$, then L is an isomorphism (hence Fredholm of index 0*).*

Proof. (i) Suppose that $\ker L \neq \{0\}$ and let $u \in \ker L, u \neq 0$. For $n \in \mathbb{N}$, set $\tilde{u}_n(x) := u(x + nT)$ where T is the period of the coefficients of L. By N-periodicity, we have $L\tilde{u}_n = 0$ for every $n \in \mathbb{N}$, and $\tilde{u}_n \rightharpoonup 0$ in $W^{2,q}(\mathbb{R}^N)$ by Lemma 4.2. If $\ker L$ were finite dimensional, this would imply $\tilde{u}_n \to 0$ in $W^{2,q}(\mathbb{R}^N)$, which is impossible since $||\tilde{u}_n||_{2,q,\mathbb{R}^N} = ||u||_{2,q,\mathbb{R}^N} (\neq 0)$ for every $n \in \mathbb{N}$. Thus, $\dim\ker L = \infty$. The proof that $\dim\ker L^* = 0$ or ∞ follows the same lines after noticing that $\ker L^* \subset L^{q'}(\mathbb{R}^N)(q' =$ Hölder conjugate of $q)$ is also invariant by T-translation (i.e. $u^* \in \ker L^* \Rightarrow \tilde{u}^* \in \ker L^*$ where $\tilde{u}^*(x) := u^*(x+T)$). This follows easily from the T-translation invariance of rge L and of the Lebesgue measure.

For the proof of (ii), note first that by the continuity and periodicity of its coefficients, L is strictly elliptic. The solutions $u \in W^{2,q}(\mathbb{R})$ of $Lu = 0$ are in $C^1(\mathbb{R})$ since $q > 1$, hence in $C^2(\mathbb{R})$ by an immediate bootstrapping argument. The theory of second order linear ODEs with bounded continuous coefficients ensures that there are only two linearly independent solutions of $Lu = 0$ in $C^2(\mathbb{R})$, whence $\dim\ker L \leq 2$, and $\ker L = \{0\}$ by (i).

The same procedure does not directly apply to obtain $\ker L^* = \{0\}$ because L^* need not be a differential operator. However, the above shows that when rge L is closed, then

(2)This result can be improved; see Remark 6.2.

L is semi-Fredholm. Since the index is unaffected by small enough perturbations of the coefficients, we may assume that L has C^∞ (still periodic) coefficients. If so, L^* remains an elliptic second-order differential operator with C^∞ periodic coefficients. Note that L^* acts from $L^{q'}(\mathbb{R})$ into $W^{-2,q'}(\mathbb{R})$, but by (local) elliptic regularity, the members of $\ker L^*$ are in $W^{2,q'}_{\text{loc}}(\mathbb{R})$, hence in $C^1(\mathbb{R})$ since $q' > 1$, and also in $C^2(\mathbb{R})$ by boostrapping. Thus, $\dim\ker L^* \leq 2$ by the arguments used above with L, whence $\ker L^* = \{0\}$ by (i). This shows that index $L = 0$. Since also $\ker L = \{0\}, L$ is an isomorphism.

(iii) Since $q \geq 2$, we have $q' \leq 2$. Since L, hence L^*, has constant coefficients and Fourier transform maps $L^{q'}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ (because $q' \leq 2$; see [4]), it follows that $\ker L^* = \{0\}$. Also, $\dim\ker L < \infty$ by hypothesis, so that $\ker L = \{0\}$ by (i). Thus, index $L = 0$ and L is one-to-one, hence an isomorphism. □

From Lemmas 6.5 and 6.6, we obtain the following sharpening of Theorem 3.1 (see also Remark 6.2):

Theorem 6.2. *Suppose $N < p < \infty$ and that the coefficients of the operator F in (3.1) satisfy the conditions (3.2) to (3.6) and (6.1) and (6.3). Suppose also that the coefficients $a_{\alpha\beta}(\cdot, 0)$ satisfy a strict ellipticity condition on $\mathbb{R}^N$. Then, if either $N = 1$ or $N \geq 2$ and the coefficients $a^\infty_{\alpha\beta}(\cdot, 0)$ and $c^\infty_i(\cdot, 0)$ are constant, $F : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is Fredholm of index 0 if and only if there is $u^0 \in W^{2,p}(\mathbb{R}^N)$ such that $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$.*

Proof. The necessity is obvious. Conversely, suppose that $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$. Then, by Theorem 3.1, we have $DF(0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ and , by Lemma 6.5 with $L = DF^\infty(0)$ and Remark 6.1, it follows that $DF^\infty(0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ with index $DF^\infty(0) =$ index $DF(0)$. We now apply Lemma 6.6 with $L = DF^\infty(0)$, which is possible once again by Remark 6.1. If $N = 1$, Lemma 6.6 (ii) with $q = p$ yields index $DF^\infty(0) = 0$. If $N \geq 2$, the condition $p > N$ ensures that $p > 2$, so that index $DF^\infty(0) = 0$ by Lemma 6.6 (iii) with $q = p$. Thus, $DF(0) \in \Phi_0(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$. That F is Fredholm of index 0 follows from another application of Theorem 3.1. □

Now, we can use Theorem 6.2 to complement Theorem 6.1.

Corollary 6.1. *If, in addition to the assumptions of Theorem 6.1, the coefficients* $a_{\alpha\beta}(\cdot,0)$ *satisfy a strict ellipticity condition on* $\mathbb{R}^N$, *and if either* $N=1$, *or* $N\geq 2$ *and the coefficients* $a^{\infty}_{\alpha\beta}(\cdot,0)$ *and* $c^{\infty}_i(\cdot,0)$ *are constant, then the statements (i), (ii) and (iii) of Theorem 6.1 are equivalent to:*

(iv) there is $u^0\in W^{2,p}(\mathbb{R}^N)$ *such that* $DF(u^0)\in\Phi_0(W^{2,p}(\mathbb{R}^N),L^p(\mathbb{R}^N))$ *and the equation* $F^{\infty}(u)=0$ *(see (6.5)) has no nonzero solution in* $W^{2,p}(\mathbb{R}^N)$.

Proof. Evidently, (iv) $\Rightarrow$ (iii), and the converse is due to Theorem 6.2. $\square$

Alternatively, we obtain simple necessary and sufficient conditions ensuring that the degree theory in [13] is available for the operator F.

Corollary 6.2. *Suppose* $N<p<\infty$ *and that the coefficients of the operator* F *in (3.1) satisfy the conditions (3.2) to (3.6) and (6.1) and (6.3). Then,* $F:W^{2,p}(\mathbb{R}^N)\to L^p(\mathbb{R}^N)$ *is of class* C^1, *Fredholm of index* 0 *and proper on the closed bounded subsets of* $W^{2,p}(\mathbb{R}^N)$ *if and only if*

(i) there is $u^0\in W^{2,p}(\mathbb{R}^N)$ *such that* $DF(u^0)\in\Phi_0(W^{2,p}(\mathbb{R}^N),L^p(\mathbb{R}^N))$

and

(ii) the equation $F^{\infty}(u)=0$ *(see (6.5)) has no nonzero solution* $u\in W^{2,p}(\mathbb{R}^N)$.

If, in addition, the coefficients $a_{\alpha\beta}(\cdot,0)$ *satisfy a strict ellipticity condition on* $\mathbb{R}^N$, *and if either* $N=1$ *or* $N\geq 2$ *and the coefficients* $a^{\infty}_{\alpha\beta}(\cdot,0)$ *and* $c^{\infty}_i(\cdot,0)$ *are constant, then condition (i) above may be replaced by*

(i') there is $u^0\in W^{2,p}(\mathbb{R}^N)$ *such that* $DF(u^0)\in\Phi_+(W^{2,p}(\mathbb{R}^N),L^p(\mathbb{R}^N))$.

Proof. The first part follows directly from Lemma 3.2, Theorem 3.1 and the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 6.1. By Theorem 6.2, (i) may be replaced by (i') under the stated additional assumptions. $\square$

In concrete problems, Remark 6.1 and Lemma 6.5 may be useful to check the strict ellipticity condition and conditions such as $DF(0)\in\Phi_\nu(W^{2,p}(\mathbb{R}^N),L^p(\mathbb{R}^N))$. For the record, we observe that there are conditions other than those given in Lemma 6.6 ensuring that $L \in \Phi_0(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ if and only if $L \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$. For instance, the assumption that L is *formally self adjoint* plays an important role in such questions.

Remark 6.2 Let $L := -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}\partial^2_{\alpha\beta} + \sum_{\alpha=1}^{N} B_\alpha \partial_\alpha + C$ be an elliptic differential operator with *constant* coefficients. It is mostly folklore, though an explicit statement is hard to find, that given $1 < q < \infty, L : W^{2,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ is semi-Fredholm if and only if it is an isomorphism, and that in turn this amounts to a condition about the coefficients of L. More precisely, this condition is that $C > 0$ if either $N \geq 2$ or $N = 1$ and $B_1 = 0$. If $N = 1$ and $B_1 \neq 0$ then $C \neq 0$ suffices. This result is stronger than Lemma 6.6 (iii). The sufficiency follows by Fourier transform and Mikhlin's multiplier theorem (see e.g. Grisvard [8, p.99]). A proof of the necessity can be based upon the special properties of Gaussians and Fourier transform. When the coefficients $a^\infty_{\alpha\beta}(\cdot, 0)$ and $c^\infty_i(\cdot, 0)$ are constant, this can be used with $L = DF^\infty(0)$, and hence $A_{\alpha\beta} = a^\infty_{\alpha\beta}(0), B_\alpha = c^\infty_\alpha(0)$ and $C = c^\infty_0(0), 1 \leq \alpha, \beta \leq N$. Together with Lemma 6.5 and Theorem 3.1, this gives a simple necessary and sufficient condition for F to be semi-Fredholm of index $\nu \in \mathbb{Z} \cup \{-\infty\}$ or, equivalently, Fredholm of index 0. □

Remark 6.3 (case $N = 1$): When $N = 1$, there is only one function $a_{11} := a$, but it is also possible to allow for different "limit" functions a^∞ and $a^{-\infty}$ (and also b^∞ and $b^{-\infty}$) with different periods T_+ and T_-, respectively. This leads to two "limit" operators F^∞ and $F^{-\infty}$. In this case, condition (iii) of Theorem 6.1 must incorporate the assumptions that $F^\infty(u) = 0 \Leftrightarrow u = 0$ *and* $F^{-\infty}(u) = 0 \Leftrightarrow u = 0$ for Theorem 6.1 to remain valid: (i) $\Rightarrow$ (ii) is still trivial, and (iii) $\Rightarrow$ (i) by a straightforward variant of the proof of Lemma 6.3 (after extracting a subsequence, it may be assumed that ℓ_k tends to ∞ or to $-\infty$). The proof of (ii) $\Rightarrow$ (iii) is a little more delicate. First, Lemma 6.1 can easily be modified in the form: For every $\epsilon > 0$, there is $R > 0$ such that $|F(u) - F^\infty(u)|_{0,p,(-\infty,-R)} \leq \epsilon$ and $|F(u) - F^\infty(u)|_{0,p,(R,\infty)} \leq \epsilon$ for every $u \in \mathcal{B}$. With this result, the following special case of Lemma 6.4 (i) can be proved: $F(\tilde{u}_n) - F(0) - F^{-\infty}(\tilde{u}_n) \to 0$ in $L^p(\mathbb{R})$ whenever $\tilde{u}_n(x) := u(x + nT_-), n \in \mathbb{N}$, and $F(\tilde{u}_n) - F(0) - F^\infty(\tilde{u}_n) \to 0$ in $L^p(\mathbb{R})$ whenever $\tilde{u}_n(x) := u(x - nT_+), n \in \mathbb{N}$. This uses the remark that $\tilde{u}_n(x) := u(x + nT_-)$ (resp. $u(x - nT_+)$) satisfies the condition $||\tilde{u}_n||_{2,p,(-R,\infty)} \to 0$ (resp. $||\tilde{u}_n||_{2,p,(-\infty,R)} \to 0$). This suffices to repeat the arguments of the proof of (ii) ⇒ (iii) in Theorem 6.1 with evident modifications. In contrast, there is no obvious generalization of Lemma 6.5 and, accordingly, it can no longer be ascertained that condition (iii) of Theorem 6.1 is equivalent to condition (iv) of Corollary 6.1. □

7. More general quasilinear elliptic operators.

Simple examples show that when $N \geq 2$, the asymptotic N-periodicity in x of the coefficients $a_{\alpha\beta}$ and b is rather atypical, although relevant in some important physical problems. What often happens is that these coefficients continue to have an orderly behavior at infinity, but their limit depends upon the direction in which the x-variable tends to infinity. We shall show in this section that if the directional behavior of the coefficients is sufficiently "stable", then a necessary and sufficient criterion for the properness of the operator F on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ can still be obtained. This criterion generalizes Theorem 6.1 in an obvious way (Theorem 7.1) and it was originally motivated by Remark 6.3, dealing with the case $N = 1$. However, Theorem 7.1 extends the result of Remark 6.3 to arbitrary N only when $a^{\pm\infty}$ and $b^{\pm\infty}$ in that remark are x-independent.

We continue to assume that the coefficients $a_{\alpha\beta}$ and b satisfy the conditions listed in Section 3. We denote by S^{N-1} the unit sphere of $\mathbb{R}^N$ and we assume that for each $s \in S^{N-1}$, there are continuous mappings $a_{\alpha\beta}^{\infty,s} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $c^{\infty,s} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ (hence all x-independent) such that for every sequence (λ_n, s_n) from $\mathbb{R}_+ \times S^{N-1}$ with $\lim \lambda_n = \infty, \lim s_n = s \in S^{N-1}$, we have

$$\lim a_{\alpha\beta}(\lambda_n s_n, \xi) = a_{\alpha\beta}^{\infty,s}(\xi),\ 1 \leq \alpha, \beta \leq N, \tag{7.1}$$

$$\lim\ c_i(\lambda_n s_n, \xi) = c_i^{\infty,s}(\xi), \quad 0 \leq i \leq N, \tag{7.2}$$

where in (7.1) and (7.2) the limits are uniform over the compact subsets of $\mathbb{R} \times \mathbb{R}^N$ and the coefficients c_i are as in (6.2). We shall set

$$b^{\infty,s}(\xi) := \sum_{i=1}^{N} c_i^{\infty,s}(\xi)\xi_i, \quad \forall\ s \in S^{N-1}. \tag{7.3}$$

Examples of coefficients satisfying conditions (7.1) and (7.2) are easily obtained starting with ξ-independent rational functions $R(x) := P(x)/Q(x)$ where P and Q are polynomials in N variables with $\deg P = \deg Q = m$: If P_m and Q_m denote the homogeneous parts of degree m of P and Q, respectively, and if Q (resp. Q_m) does not vanish on $\mathbb{R}^N$ (resp. S^{N-1}), then $\lim R(\lambda_n s_n) = P_m(s)/Q_m(s)$ for every sequence (λ_n, s_n) as in (7.1) and (7.2). Note that it is the rule rather than the exception that the limit $P_m(s)/Q_m(s)$ depends upon s. Other (nonrational) examples are easily found.

Conditions (7.1) and (7.2) imply an apparently stronger property:

Lemma 7.1. *Let (λ_n, s_n) be a sequence from $\mathbb{R}^+ \times S^{N-1}$ such that $\lim \lambda_n = \infty$ and $\lim s_n = s \in S^{N-1}$. Then,*

$$\lim a_{\alpha\beta}(x + \lambda_n s_n, \xi) = a_{\alpha\beta}^{\infty,s}(\xi), 1 \leq \alpha, \beta \leq N, \tag{7.4}$$

$$\lim c_i(x + \lambda_n s_n, \xi) = c_i^{\infty,s}(\xi),\ 0 \leq i \leq N \tag{7.5}$$

and the limits are uniform over the compact subsets of $\mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N)$.

Proof. Let $1 \leq \alpha, \beta \leq N$ be fixed. If (7.4) is not true for some pair $(x, \xi) \in \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N)$, or if the limit is not uniform over the compact subsets of $\mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N)$, there are $\epsilon > 0$ and a bounded sequence (x_n, ξ_n) from $\mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N)$ such that $|a_{\alpha\beta}(x_n + \lambda_n s_n, \xi_n) - a_{\alpha\beta}^{\infty,s}(\xi_n)| \geq \epsilon$ for arbitrarily large $n \in \mathbb{N}$. But since (x_n) is bounded and $\lim \lambda_n = \infty$, we may write $x_n + \lambda_n s_n = \mu_n s_n'$, where $s_n' = (x_n + \lambda_n s_n)/|x_n + \lambda_n s_n| \in S^{N-1}$ tends to s and $\mu_n = |x_n + \lambda_n s_n|$ tends to ∞. Hence, $|a_{\alpha\beta}(\mu_n s_n', \xi_n) - a_{\alpha\beta}^{\infty,s}(\xi_n)| \geq \epsilon$ for arbitrarily large n. On the other hand, since (ξ_n) is bounded and the limit in (7.1) is uniform over the compact subsets of $\mathbb{R} \times \mathbb{R}^N$, it follows that $|a_{\alpha\beta}(\mu_n s_n', \xi_n) - a_{\alpha\beta}^{\infty,s}(\xi_n)| < \epsilon$ for n large enough, a contradiction. The proof of (7.5) is similar. $\square$

For $s \in S^{N-1}$, we introduce the quasilinear operators

$$F^{\infty,s}(u) := -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}^{\infty,s}(u, \nabla u)\partial_{\alpha\beta}^2 u + b^{\infty,s}(u, \nabla u). \tag{7.6}$$

Since $b^{\infty,s}(0) = 0$ by (7.3) and $a_{\alpha\beta}^{\infty,s}(0)$ and $c_i^{\infty,s}(0)$ are constant, it follows from Lemma 3.1 and Remark 3.1 that $F^{\infty,s}$ is well defined, continuous and weakly sequentially continuous from $W^{2,p}(\omega)$ into $L^p(\omega)$ where $\omega \subset \mathbb{R}^N$ is an arbitrary open subset with Lipschitz continuous boundary. In what follows, we shall also need to extend the translation operator τ_h in (6.8) to bundle maps. This extension is defined in the obvious way, i.e.

$$\tau_h(f)(x,\xi) := f(x+h,\xi). \tag{7.7}$$

We may then define the operator $\tau_h(F)$ obtained by shifting all the coefficients of F by h:

$$\tau_h(F)(u) := -\sum_{\alpha,\beta=1}^{N} \tau_h(a_{\alpha\beta})(\cdot,u,\nabla u)\partial^2_{\alpha\beta}u + \tau_h(b)(\cdot,u,\nabla u). \tag{7.8}$$

Note that with the definition (7.8), we have the relation

$$\tau_h(F(u)) = \tau_h(F)(\tau_h(u)) \tag{7.9}$$

since differentiation commutes with h-translations. From (7.7) and (7.8) it is obvious that $\tau_h(F)$ enjoys the same properties as F.

Lemma 7.2. *Let $N < p < \infty$ and let (u_n) be a sequence from $W^{2,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $W^{2,p}(\mathbb{R}^N)$. On the other hand, let (λ_n, s_n) be a sequence from $\mathbb{R}_+ \times S^{N-1}$ such that $\lim \lambda_n = \infty$ and $\lim s_n = s \in S^{N-1}$. Then, $\tau_{\lambda_n s_n}(F)(u_n) \rightharpoonup F^{\infty,s}(u)$ in $L^p(\mathbb{R}^N)$.*

Proof. First, note that $\tau_{\lambda_n s_n}(F) = \tau_{\lambda_n s_n}(F - F(0)) + \tau_{\lambda_n s_n}(F(0))$ and $\tau_{\lambda_n s_n}(F(0)) = \tau_{\lambda_n s_n}(b(\cdot,0)) \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$ by Lemma 4.2. Thus, it suffices to show that $\tau_{\lambda_n s_n}(F - F(0))(u_n) \rightharpoonup F^{\infty,s}(u)$ in $L^p(\mathbb{R}^N)$. In other words, it is not restrictive to assume that $F(0) = b(\cdot,0) = 0$ in the first place.

Next, it must be observed that if $\mathcal{B}$ is a bounded subset of $W^{2,p}(\mathbb{R}^N)$, then the set $\{\tau_h(u) : u \in \mathcal{B}, h \in \mathbb{R}^N\}$ is bounded. From (7.9), $\tau_h(F)(u) = \tau_h(F(\tau_{-h}(u)))$, whence $|\tau_h(F)(u)|_{0,p,\mathbb{R}^N} = |F(\tau_{-h}(u))|_{0,p,\mathbb{R}^N}$. By Lemma 3.1, this implies that the set $\{\tau_h(F)(u) : u \in \mathcal{B}, h \in \mathbb{R}^N\}$ is bounded in $L^p(\mathbb{R}^N)$. In particular, the sequence $(\tau_{\lambda_n s_n}(F)(u_n))$ is bounded in $L^p(\mathbb{R}^N)$, so that $\tau_{\lambda_n s_n}(F)(u_n) \rightharpoonup F^{\infty,s}(u)$ in $L^p(\mathbb{R}^N)$ if and only if $\tau_{\lambda_n s_n}(F)(u_n) \rightharpoonup F^{\infty,s}(u)$ in $L^p(\omega)$, where $\omega \subset \mathbb{R}^N$ denotes an arbitrary open ball.

¿From the embedding $W^{2,p}(\mathbb{R}^N) \hookrightarrow C^1_d(\mathbb{R}^N)$, there is a compact subset $K \subset \mathbb{R} \times \mathbb{R}^N$ such that $(u_n(x), \nabla u_n(x)) \in K$ for every $x \in \mathbb{R}^N$ and every $n \in \mathbb{N}$. It thus follows from Lemma 7.1 that, given $1 \leq \alpha, \beta \leq N$ and $\epsilon > 0$, we have

$$(7.10) \qquad |a_{\alpha\beta}(x + \lambda_n s_n, u_n(x), \nabla u_n(x)) - a^{\infty,s}_{\alpha\beta}(u_n(x), \nabla u_n(x))| < \epsilon/2, \ \forall x \in \omega,$$

for $n \in \mathbb{N}$ large enough. Since $a^{\infty,s}_{\alpha\beta}(0)$ is constant, Theorem 2.1 (iii) ensures that $a^{\infty,s}_{\alpha\beta}(u_n, \nabla u_n) \to a^{\infty,s}_{\alpha\beta}(u, \nabla u)$ in $L^\infty(\omega)$, so that $|a^{\infty,s}_{\alpha\beta}(u_n(x), \nabla u_n(x)) - a^{\infty,s}_{\alpha\beta}(u(x), \nabla u(x))| < \epsilon/2$ for every $x \in \omega$ if n is large enough. Together with (7.10), we obtain $|a_{\alpha\beta}(x+\lambda_n s_n, u_n(x), \nabla u_n(x)) - a^{\infty,s}_{\alpha\beta}(u(x), \nabla u(x))| < \epsilon$ for every $x \in \omega$ and n large enough , i.e.

$$(7.11) \qquad \tau_{\lambda_n s_n}(a_{\alpha\beta})(\cdot, u_n, \nabla u_n) \to a^{\infty,s}_{\alpha\beta}(u, \nabla u) \text{ in } L^\infty(\omega).$$

Likewise, we find

$$(7.12) \qquad \tau_{\lambda_n s_n}(b)(\cdot, u_n, \nabla u_n) \to b^{\infty,s}(u, \nabla u) \text{ in } L^\infty(\omega) \hookrightarrow L^p(\omega)$$

because $b(\cdot, 0) = 0$ (as assumed earlier in the proof) and in that case (7.3) and (7.5) imply at once that $\lim b(x + \lambda_n s_n, \xi) = b^{\infty,s}(\xi)$, the limit being uniform over the compact subsets of $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$. Since the hypothesis $u_n \rightharpoonup u$ in $W^{2,p}(\mathbb{R}^N)$ implies $\partial^2_{\alpha\beta} u_n \rightharpoonup \partial^2_{\alpha\beta} u$ in $L^p(\mathbb{R}^N)$, it is clear that (7.11) and (7.12) yield the desired result that $\tau_{\lambda_n s_n}(F)(u_n) \rightharpoonup F^{\infty,s}(u)$ in $L^p(\omega)$. □

Lemma 7.3. *Suppose that $N < p < \infty$ and that*
(i) there is $u^0 \in W^{2,p}(\mathbb{R}^N)$ such that $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$,
(ii) for every $s \in S^{N-1}$, the equation $F^{\infty,s}(u) = 0$ has no nonzero solution $u \in W^{2,p}(\mathbb{R}^N)$.
Then, the restriction of F to the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ is proper.

Proof. To prove properness, we use the equivalence (i) $\Leftrightarrow$ (iii) in Corollary 4.1, which reduces the problem to showing that every bounded sequence (u_n) from $W^{2,p}(\mathbb{R}^N)$ such that $F(u_n) \to y$ in $L^p(\mathbb{R}^N)$ vanishes uniformly at infinity in the sense of $C^1_d(\mathbb{R}^N)$. To see this, we choose $T = (T_1, \cdots, T_N)$ with $T_i > 0, 1 \leq i \leq N$, and prove that case (ii) of Theorem 4.3 cannot occur.

By contradiction, suppose that there are a sequence $\ell_k \in \mathbb{Z}^N$ with $\lim |\ell_k| = \infty$ and a subsequence (u_{n_k}) such that the sequence $(\tilde{u}_{n_k})$ defined by $\tilde{u}_{n_k}(x) := u_{n_k}(x + \ell_k T)$ is weakly convergent (in $W^{2,p}(\mathbb{R}^N)$) to $\tilde{u} \in W^{2,p}(\mathbb{R}^N), \tilde{u} \neq 0$.

Set $\lambda_k := |\ell_k T|$ and $s_k := \ell_k T / |\ell_k T|$, so that $\lim \lambda_k = \infty$ and $s_k \in S^{N-1}$. After extracting a subsequence, we may assume that $\lim s_k = s \in S^{N-1}$. By Lemma 7.2, and since $\lambda_k s_k = \ell_k T$, we have $\tau_{\ell_k T}(F)(\tilde{u}_{n_k}) \rightharpoonup F^{\infty,s}(\tilde{u})$ in $L^p(\mathbb{R}^N)$. But $\tilde{u}_{n_k} = \tau_{\ell_k T}(u_{n_k})$, whence, by (7.9), $\tau_{\ell_k T}(F(u_{n_k})) \rightharpoonup F^{\infty,s}(\tilde{u})$. Since $\tilde{u} \neq 0$, it suffices to show that $\tau_{\ell_k T}(F(u_{n_k})) \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$, for then $F^{\infty,s}(\tilde{u}) = 0$ and a contradiction arises with condition (ii) of the theorem. That $\tau_{\ell_k T}(F(u_{n_k})) \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$ follows from $\tau_{\ell_k T}(F(u_{n_k}) - y) \to 0$ in $L^p(\mathbb{R}^N)$ by translation invariance of the Lebesgue measure and from $\tau_{\ell_k T}(y) \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$ by Lemma 4.2. This completes the proof. □

Lemma 7.3 is the analog of Lemma 6.3, i.e. the analog of the implication (iii) ⇒ (i) in Theorem 6.1 to prove the analog of the implication (ii) ⇒ (iii) of that theorem, we need a variant of a special case of Lemma 6.4 (i), given in Lemma 7.8 below. First, we have to go through some more technicalities.

For $\rho > 0$, let B_ρ denote the open ball in $\mathbb{R}^N$ with center 0 and radius ρ. Given $s \in S^{N-1}$, the set $B_\rho \cap s^\perp$ is the open ball with center 0 and radius ρ in $s^\perp$. We denote by $C_{s,\rho}$ the cylinder with axis $\mathbb{R}s$:

$$C_{s,\rho} := \mathbb{R}s \oplus (B_\rho \cap s^\perp), \tag{7.13}$$

and by $C^+_{s,\rho}$ the half-cylinder

$$C^+_{s,\rho} = \mathbb{R}^+_s \oplus (B_\rho \cap s^\perp). \tag{7.14}$$

Thus, for $R \in \mathbb{R}$, the set $Rs \pm C^+_{s,\rho}$ is the half-cylinder $C^+_{s,\rho}$ shifted by R in the direction $\pm s$ (hence along its axis).

Lemma 7.4. *Let $\rho > 0$ and $s \in S^{N-1}$ be given. Then, $\lim_{R\to\infty} a_{\alpha\beta}(x+Rs,\xi) = a^{\infty,s}_{\alpha\beta}(\xi)$, $1 \leq \alpha,\beta \leq N$, and $\lim_{R\to\infty} c_i(x+Rs,\xi) = c_i^{\infty,s}(\xi)$, $0 \leq i \leq N$, and the limits are uniform over $C^+_{s,\rho} \times K$ where K is any compact subset of $\mathbb{R} \times \mathbb{R}^N$.*

Proof. Since $x \in C^+_{s,\rho}$ has the form $x = ts + x'$ with $x' \in B_\rho \cap s^\perp$ (hence $|x'| < \rho$) and $t > 0$, we have $|x + Rs| \to \infty$ uniformly in $x \in C^+_{s,\rho}$ when $R \to \infty$, and the proof of Lemma 7.1 can be repeated with obvious modifications. □

Our next lemma is the appropriate substitute for Lemma 6.1.

Lemma 7.5. *Suppose $N < p < \infty$ and let $\mathcal{B} \subset W^{2,p}(\mathbb{R}^N)$ be a bounded subset. Given $\rho > 0$ and $s \in S^{N-1}$, there is $R > 0$ such that $|F(u) - F(0) - F^{\infty,s}(u)|_{0,p,Rs+C^+_{s,\rho}} < \epsilon$ for every $u \in \mathcal{B}$.*

Proof. Since $\mathcal{B}$ is bounded in $W^{2,p}(\mathbb{R}^N)$, hence in $C^1_d(\mathbb{R}^N)$, there is a compact subset $K \subset \mathbb{R} \times \mathbb{R}^N$ such that $(u(x), \nabla u(x)) \in K$ for every $x \in \mathbb{R}^N$ and every $u \in \mathcal{B}$. Lemma 7.4 implies that $|a_{\alpha\beta}(x,\xi) - a^{\infty,s}_{\alpha\beta}(\xi)| \leq \epsilon$ and $|c_i(x,\xi) - c_i^{\infty,s}(\xi)| \leq \epsilon$ for $1 \leq \alpha,\beta \leq N, 0 \leq i \leq N$ and every $(x,\xi) \in (Rs + C^+_{s,\rho}) \times K$. From this point on, the proof proceeds exactly as the proof of Lemma 6.1. □

Lemma 7.6. *Let $1 \leq q < \infty, \rho > 0$ and $s \in S^{N-1}$ be given. With $u \in W^{2,q}(C_{s,\rho})$ and $\theta > 0$ being fixed, set $\tilde{u}_n(x) := u(x - n\theta s), n \in \mathbb{N}$. Then, for every $R \in \mathbb{R}$, we have $\tilde{u}_n \to 0$ in $W^{2,q}(Rs - C^+_{s,\rho})$(3).*

Proof. By the translation invariance of the Lebesgue measure, we have $||\tilde{u}_n||_{2,p,Rs-C^+_{s,\rho}} = ||u||_{2,p,(R-n\theta)s-C^+_{s,\rho}}$. As $n \to \infty$, the set $(R - n\theta)s - C^+_{s,\rho}$ becomes contained in the complement of any fixed bounded subset of $\mathbb{R}^N$, whence $\lim_{n\to\infty} ||u||_{2,p,(R-n\theta)s-C^+_{s,\rho}} = 0$. □

Our last preliminary result (not needed in Section 6) is one in the spirit of equicontinuity.

(3)The choice $\theta \neq 1$ will be needed in Remark 7.1.

Lemma 7.7. *Suppose $N < p < \infty$ and let $\mathcal{B} \subset W^{2,p}(\mathbb{R}^N)$ be a bounded subset. Then, for every $\epsilon > 0$, there is $\delta > 0$ such that $u \in \mathcal{B}$ and $||u||_{2,p,\omega} < \delta$ implies $|F(u) - F(0)|_{0,p,\omega} < \epsilon$ for every open subset $\omega \subset \mathbb{R}^N$. Likewise, if $s \in S^{N-1}$ is fixed, $\delta > 0$ can also be chosen so that $|F^{\infty,s}(u)|_{0,p,\omega} < \epsilon$ for every open subset $\omega \subset \mathbb{R}^N$.*

Proof. Write $b(x,\xi) = b(x,0) + \sum_{i=0}^{N} c_i(x,\xi)\xi_i$ with $c_i(x,\xi) = \int_0^1 \partial_{\xi_i} b(x,t\xi)dt$. This shows that $F(u) - F(0) = \sum_{\alpha\beta=1}^{N} a_{\alpha\beta}(\cdot,u,\nabla u)\partial^2_{\alpha\beta}u + \sum_{\alpha=1}^{N} c_\alpha(\cdot,u,\nabla u)\partial_\alpha u + c_0(\cdot,u,\nabla u)u$. By Theorem 2.1 (ii), the terms $a_{\alpha\beta}(\cdot,u,\nabla u)$ and $c_i(\cdot,u,\nabla u)$ are all bounded by a constant $C_{\mathcal{B}} > 0$ in $L^\infty(\mathbb{R}^N)$, hence in $L^\infty(\omega)$ irrespective of ω, for every $u \in \mathcal{B}$. Thus, $|F(u) - F(0)|_{0,p,\omega} \leq C_{\mathcal{B}}\ ||u||_{2,p,\omega}$ for every $u \in \mathcal{B}$ after modifying $C_{\mathcal{B}}$ in a way depending only upon N and p, so that $\delta = \epsilon/C_{\mathcal{B}}$ works. By (7.3), the same argument gives the result with F replaced by $F^{\infty,s}$. $\square$

Lemma 7.8. *Suppose $N < p < \infty$. Given $u \in W^{2,p}(\mathbb{R}^N)$ and $s \in S^{N-1}$, set $\tilde{u}_n(x) := u(x - ns), n \in \mathbb{N}$. Then, $F(\tilde{u}_n) - F(0) - F^{\infty,s}(\tilde{u}_n) \to 0$ in $L^p(\mathbb{R}^N)$.*

Proof. Let $\epsilon > 0$ be fixed, and let $\mathcal{B} \subset W^{2,p}(\mathbb{R}^N)$ be the closed ball with center 0 and radius $||u||_{2,p,\mathbb{R}^N}$. By Lemma 7.7, there is $\delta > 0$ such that, for every open subset $\omega \subset \mathbb{R}^N$ we have that $v \in \mathcal{B}$ and $||v||_{2,p,\omega} < \delta$ implies $|F(v) - F(0)|_{0,p,\omega} < \epsilon/6$ and $|F^{\infty,s}(v)|_{0,p,\omega} < \epsilon/6$. Therefore,

$$\{v \in \mathcal{B}, ||v||_{2,p,\omega} < \delta\} \Rightarrow |F(v) - F(0) - F^\infty(v)|_{0,p,\omega} < \epsilon/3. \tag{7.15}$$

Choose $\rho > 0$ such that $||u||_{2,p,\overset{\circ}{\tilde{B}}_\rho} < \delta$, where $\tilde{B}_\rho$ denotes the complement of the open ball $B_\rho \subset \mathbb{R}^N$ with center 0 and radius ρ. Obviously, $B_\rho \subset C_{s,\rho}$ (see (7.13)) and hence $\tilde{C}_{s,\rho} := \mathbb{R}^N \setminus C_{s,\rho} \subset \tilde{B}_\rho$. It follows that $||u||_{2,p,\overset{\circ}{\tilde{C}}_{s,\rho}} < \delta$. Since $\overset{\circ}{\tilde{C}}_{s,\rho}$ is invariant by translation along $\mathbb{R}s$, we have $||\tilde{u}_n||_{2,p,\overset{\circ}{\tilde{C}}_{s,\rho}} = ||u||_{2,p,\overset{\circ}{\tilde{C}}_{s,\rho}} < \delta$ for every $n \in \mathbb{N}$. Also, $||\tilde{u}_n||_{2,p,\mathbb{R}^N} = ||u||_{2,p,\mathbb{R}^N}$, whence $\tilde{u}_n \in \mathcal{B}$ for every $n \in \mathbb{N}$. Since (7.15) holds for every open subet $\omega \subset \mathbb{R}^N$, we may choose $\omega = \overset{\circ}{\tilde{C}}_{s,\rho}$ and $v = \tilde{u}_n, n \in \mathbb{N}$, in (7.15). This yields

$$|F(\tilde{u}_n) - F(0) - F^\infty(\tilde{u}_n)|_{0,p,\overset{\circ}{\tilde{C}}_{s,\rho}} < \epsilon/3,\ \forall n \in \mathbb{N}. \tag{7.16}$$

Now, Lemma 7.5 provides $R > 0$ such that

$$|F(\tilde{u}_n) - F(0) - F^{\infty,s}(\tilde{u}_n)|_{0,p,Rs+C^+_{s,\rho}} < \epsilon/3, \ \forall n \in \mathbb{N}, \tag{7.17}$$

and, by Lemma 7.6 with $\theta = 1$ and Lemma 7.7 with $\omega = \mathbb{R}s - C^+_{s,\rho}$,

$$|F(\tilde{u}_n) - F(0) - F^{\infty,s}(\tilde{u}_n)|_{0,p,Rs-C^+_{s,\rho}} < \epsilon/3, \tag{7.18}$$

for n large enough. Since $\mathbb{R}^N = \tilde{C}_{s,\rho} \cup (Rs + C^+_{s,\rho}) \cup (Rs - C^+_{s,\rho})$ to within a set of measure 0, namely, $\partial C_{s,\rho} \cup (Rs + (B_\rho \cap s^\perp))$, it follows from (7.16), (7.17) and (7.18) that $|F(\tilde{u}_n) - F(0) - F^{\infty,s}(\tilde{u}_n)|_{0,p,\mathbb{R}^N} < \epsilon$ for n large enough. □

Here is now the "generalization" of Theorem 6.1.

Theorem 7.1. *Suppose $N < p < \infty$ and that the coefficients of the operator F in (3.1) satisfy the conditions (3.2) to (3.6) and (7.1) to (7.3). The following statements are equivalent:*

(i) The restriction of F to the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ is proper.

(ii) Every sequence (u_n) from $W^{2,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ and $F(u_n) \to F(0)$ in $L^p(\mathbb{R}^N)$ contains a convergent subsequence (hence tends to 0 in $W^{2,p}(\mathbb{R}^N)$).

(iii) There is $u^0 \in W^{2,p}(\mathbb{R}^N)$ such that $DF(u^0) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ and the equation $F^{\infty,s}(u) = 0$ has no nonzero solution $u \in W^{2,p}(\mathbb{R}^N)$ for any $s \in S^{N-1}$.

Proof. Once again, (i) ⇒ (ii) is obvious. That (iii) ⇒ (i) is Lemma 7.3, and that condition (ii) implies that $DF(u) \in \Phi_+(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for every $u \in W^{2,p}(\mathbb{R}^N)$ follows at once from Corollary 5.1. To show that $F^{\infty,s}(u) \neq 0$ when $u \neq 0$ for every given $s \in S^{N-1}$, suppose by contradiction that it is not so and let $u \in W^{2,p}(\mathbb{R}^N), u \neq 0$, be such that $F^{\infty,s}(u) = 0$. Then, with $\tilde{u}_n(x) := u(x - ns)$, we have $F^{\infty,s}(\tilde{u}_n) = 0$ since the coefficients of $F^{\infty,s}$ are x-independent. Thus, Lemma 7.8 yields $F(\tilde{u}_n) - F(0) \to 0$ in $L^p(\mathbb{R}^N)$. Since $\tilde{u}_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ by Lemma 4.2, condition (ii) of the theorem implies that $\tilde{u}_n \to 0$ in $W^{2,p}(\mathbb{R}^N)$, contradicting $||\tilde{u}_n||_{2,p,\mathbb{R}^N} = ||u||_{2,p,\mathbb{R}^N} \neq 0$. □

Since the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 7.1 does not make any explicit reference to the behavior of the coefficients of F as $|x|$ tends to ∞, it is likely to remain true in a more general setting. Yet, we have found no direct proof of the implication (ii) $\Rightarrow$ (i). However, when the coefficients $a_{\alpha\beta}$ are *independent* of ξ (semilinear case), then (ii) $\Rightarrow$ (i) (hence "$\Leftrightarrow$") holds irrespective of the behavior of the coefficients $a_{\alpha\beta}$ and b as $|x|$ tends to ∞. This follows from the remark that, in that case, the mapping $v \in W^{2,p}(\mathbb{R}^N) \longmapsto F(u+v) - F(v) \in L^p(\mathbb{R}^N)$ is compact for every fixed $u \in W^{2,p}(\mathbb{R}^N)$ (the case Supp u compact is easy; in general, use truncation and a limiting process) and from the weak sequential continuity of F. When the coefficients $a_{\alpha\beta}$ depend upon ξ, the mapping $v \longmapsto F(u+v) - F(v)$ is not compact and this argument fails.

¿From Theorem 7.1, it is clear that Corollary 6.1 can be generalized as follows:

Corollary 7.1. *Suppose $N < p < \infty$ and that the coefficients of the operator L in (3.1) satisfy the conditions (3.2) to (3.6) and (7.1) to (7.3). Then, $F : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is C^1, Fredholm of index 0 and proper on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$ if and only if*

(i) there is $u^0 \in W^{2,p}(\mathbb{R}^N)$ such that $DF(u^0) \in \Phi_0(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))$

and

(ii) the equation $F^{\infty,s}(u) = 0$ (see (7.6)) has no nonzero solution $u \in W^{2,p}(\mathbb{R}^N)$ for any $s \in S^{N-1}$.

Corollary 7.1 remains true if the index $\nu = 0$ is replaced by any index $\nu \in \mathbb{Z} \cup \{-\infty\}$. See also Remark 3.2 for conditions ensuring extra smoothness of F. Since $DF^{\infty,s}(0)$ has constant coefficients, it follows from Remark 6.2 that, when defined, the index of $DF^{\infty,s}(0)$ is 0. But unlike in Section 6, it is no longer possible to ascertain that F in Theorem 7.1 has index 0. Counterexamples exist even when $N = 1$ and F is linear. The simplest one is given by

$$F(u) := -u'' - b(x)u' - u, \tag{7.13}$$

where b is a smooth function such that $b(x) = 1$ for $x \geq 1$ and $b(x) = -1$ for $x \leq -1$. In this case, we have $F^{\infty}(u) := F^{\infty,1}(u) = -u'' - u' - u$ and $F^{-\infty}(u) := F^{\infty,-1}(u) =$

$-u'' + u' - u$. By elementary ODE arguments, $F : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}), 1 < p < \infty$, has a two-dimensional null-space (if $F(u) = 0$, then $u(x)$ has exponential decay as $|x| \to \infty$) and, with a little more work, F is surjective. Hence, F has index 2. The formal adjoint of F in (7.13), i.e.

$$F^*(u) = -u'' + b(x)u' + (b'(x) - 1)u,$$

gives an example with index -2. Yet, both (linear) operators $F^{\pm\infty}$ have index 0, e.g. by Remark 6.2.

Remark 7.1: There is no conceptual difficulty, only some notational inconvenience, in extending Theorem 7.1 to the case when F is replaced by $F^\natural + F$ where $F^\natural$ has asymptotically N-periodic coefficients, i.e. satisfies the conditions required in Section 6. In this setting, the ellipticity condition (3.4) should be required only of the coefficients $a^\natural_{\alpha\beta} + a_{\alpha\beta}$, where a self-explanatory notation was used. The technical modificiations needed to handle this case are minor. First, instead of choosing $T = (T_1, \cdots, T_N)$ arbitrarily in the proof of Lemma 7.3 with F replaced by $F^\natural + F, T$ should be chosen as a period of the operator $F^{\natural,\infty}$ with N-periodic coefficients, and Lemma 6.3 should be used to handle the part depending upon $F^\natural$. This provides the implication (iii) $\Rightarrow$ (i) for $F^\natural + F$ in Theorem 7.1, where of course $F^{\infty,s}$ should be replaced by $F^{\natural,\infty} + F^{\infty,s}$. That (i) $\Rightarrow$ (ii) remains obvious. Lemmas 7.4 to 7.8 are needed only for the proof of (ii) $\Rightarrow$ (iii). There is no need to change anything, not even F into $F^\natural + F$, in Lemmas 7.4 to 7.6. To obtain Lemma 7.5 with F replaced by $F^\natural + F$, it necessary to confine attention to those $s \in S^{N-1}$ of the form $s = \ell_{k_0}T/|\ell_{k_0}T|$ for some $k_0 \in \mathbb{Z}^N \setminus \{0\}$ and to $R = m|\ell_{k_0}T|s, m \in \mathbb{N}$, and to use Lemma 6.1 to handle the part depending upon $F^\natural$. Lemma 7.7 can be repeated verbatim with F replaced by $F^\natural + F$, and Lemma 7.8 for $F^\natural + F$ follows under the same restriction about s as above, provided that the sequence $(\tilde{u}_n)$ is now defined by $\tilde{u}_n(x) := u(x - n|\ell_{k_0}T|s)$. The only modifications to the proof consist in choosing $R = m|\ell_{k_0}T|s$ for some large enough $m \in \mathbb{N}$ and in using Lemma 7.6 with $\theta = |\ell_{k_0}T|$ instead of $\theta = 1$. These preliminaries give the implication (ii) $\Rightarrow$ (iii) in Theorem 7.1 with F replaced by $F^\natural + F$ and $F^{\infty,s}$ replaced by $F^{\natural,\infty} + F^{\infty,s}$ for all $s \in S^{N-1}$ of the form $s = \ell_{k_0}T/|\ell_{k_0}T|$ for some $k_0 \in \mathbb{Z}^N \setminus \{0\}$, hence for a dense subset of S^{N-1}. To obtain the full implication, it seems necessary to assume some continuity of $F^{\infty,s}$ with respect to s, not needed in Theorem 7.1. The pointwise condition that $F^{\infty,s_k}(u) \to F^{\infty,s}(u)$ in $L^p(\mathbb{R}^N)$ whenever $s_k \in S^{N-1}$ tends to s for every fixed $u \in W^{2,p}(\mathbb{R}^N)$ suffices. The metrizability of the weak topology of $W^{2,p}(\mathbb{R}^N)$ on closed bounded subsets is also helpful to obtain a sequence (u_n) tending weakly to 0. The corresponding (routine) details are left to the reader. □

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