C^1–Fredholm maps and bifurcation for quasilinear elliptic equations on $\mathbb{R}^N$

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Dedicated to Alfonso Vignoli on the occasion of his 60th anniversary

Summary : We discuss a broad class of second order quasilinear elliptic operators on $\mathbb{R}^N$ acting from the Sobolev space $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for $p \in (N, \infty)$. Conditions are given which ensure that such operators are C^1–Fredholm maps of index zero. Then we give additional assumptions which imply that they are proper on the closed bounded subsets of $W^{2,p}(\mathbb{R}^N)$. For operators with these properties the topological degrees developed by Fitzpatrick, Pejsachowicz and Rabier are available. We illustrate their use by deriving results about the bifurcation of connected components of solutions of quasilinear elliptic equations on $\mathbb{R}^N$.

Keywords : bifurcation, topological degree, Fredholm map, global branch, quasilinear elliptic equation, proper map

Classification : 35B32, 35J60, 55M25, 55C40

1 Introduction

In this paper we survey some of our recent work on second order quasilinear elliptic operators defined on $\mathbb{R}^N$. In particular, we discuss the global behaviour of some connected sets of solutions (λ, u) of a second order quasilinear elliptic equation

$$(1.1) \quad -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, u(x), \nabla u(x))\partial_\alpha\partial_\beta u(x) + b(x, u(x), \nabla u(x), \lambda) = 0$$

for $x \in \mathbb{R}^N$. Here λ is a real parameter and the function u is required to satisfy the condition

$$(1.2) \qquad \lim_{|x|\to\infty} u(x) = 0.$$

In addition to the ellipticity of the matrix $[a_{\alpha\beta}]$ of coefficients, we suppose that $b(x,0,\lambda) = 0$ for all $(x,\lambda) \in I\!R^{N+1}$. Thus $u \equiv 0$ is a solution of the problem (1.1), (1.2) for every $\lambda \in I\!R$ and our results deal with components of non-trivial solutions bifurcating from this line of trivial solutions.

As is well known, [4, 5, 6, 7], topological degree theory is the primary tool for establishing this kind of result and we have shown that the degree for proper Fredholm maps in the form developed by Fitzpatrick, Pejsachowicz and Rabier can be applied to the problem (1.1), (1.2) under very natural and rather general assumptions concerning the coefficients.

The first step is to express the problem (1.1), (1.2) as the set of zeros of a function $F \in C^1(I\!R \times X, Y)$ where X and Y are real Banach spaces. We use the standard Sobolev spaces $X_p = W^{2,p}(I\!R^N)$ and $Y_p = L^p(I\!R^N)$ where $p \in (N, \infty)$. There are two reasons for choosing these spaces :
(i) all elements of X_p vanish as $|x| \to \infty$, thus ensuring that (1.2) is satisfied, and
(ii) we can ensure that $F(\lambda, u) \in Y_p$ for all $u \in X_p$ without imposing restrictions on the growth of the functions $a_{\alpha\beta}(x,\xi)$ and $b(x,\xi,\lambda)$ as $|\xi| \to \infty$.
Our results in Section 5, giving explicit conditions for global bifurcation, do not depend on the choice of p within the range (N, ∞).

In this setting we have derived conditions on the functions $a_{\alpha\beta}(x,\xi)$ and $b(x,\xi,\lambda)$ which imply that $F : I\!R \times X \to Y$ is a C^1-Fredholm operator of index zero which is proper on closed bounded subsets of $I\!R \times X$. We summarize these criteria in Sections 2 and 3. The remainder of the paper concerns the conclusions about the bifurcation of solutions of (1.1), (1.2) which can be obtained within this framework. We recall in Section 4 the abstract bifurcation theorems found in [10] and [9]. The Fredholm property alone is sufficient to guarantee the bifurcation of a branch of solutions from a point across which the parity is equal to -1, so we begin with a result of this kind for (1.1),(1.2). The sense in which a branch can fail to be compact can be made more precise if $F : I\!R \times X \to Y$ is also proper on closed bounded subsets of $I\!R \times X$. This is a more delicate question and most of our work has been devoted to resolving this issue. The main conclusions about global bifurcation for (1.1),(1.2) are presented at the end of Sction 5.

2 C^1-Fredholm operators of index zero

Let X and Y be real Banach spaces. A function $F \in C^1(I\!R \times X, Y)$ is said to be a Fredholm map of index zero on an interval J if the partial derivative $D_u F(\lambda, u) : X \to Y$ is a bounded linear Fredholm operator of index zero for all points $(\lambda, u) \in J \times X$.

In this section we consider the differential operator

$$F(\lambda,u)(x) = - \sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x,u(x),\nabla u(x))\partial_\alpha\partial_\beta u(x) + b(x,u(x),\nabla u(x),\lambda) \tag{2.1}$$

as a mapping between the spaces $X_p = W^{2,p}(I\!R^N)$ and $Y_p = L^p(I\!R^N)$ where $p \in (N, \infty)$. Our aim is to formulate conditions which ensure that it is a C^1-Fredholm map of index zero on an interval J. We must first deal with its smoothness and in this connection we have found the following definition to be convenient when discussing the assumptions on of the coefficients required to ensure that $F \in C^1(I\!R \times X_p, Y_p)$.

Using the notation

$$f : I\!R^N \times I\!R^{N+1} \to I\!R \text{ with } (x,\xi) = (x, \xi_0, ..\xi_N) \longmapsto f(x, \xi_0, ..\xi_N)$$

and

$$g : I\!R^N \times I\!R^{N+2} \to I\!R \text{ with } (x,\eta) = (x, \xi_0, ..\xi_N, \lambda) \longmapsto g(x, \xi_0, ..\xi_N, \lambda),$$

we see that the variable x plays a markedly different role from the variables ξ and η when deriving the smoothness properties of the associated Nemytskii operators $u \mapsto f(\cdot, u(\cdot), \nabla u(\cdot))$ and $(\lambda, u) \mapsto g(\cdot, u(\cdot), \nabla u(\cdot), \lambda)$. The terminology "bundle map" provides a convenient way of handling this distinction where x is the "base" variable and ξ and η are "fiber" variables. Note that since we require smoothness with respect to u and λ it is natural to treat λ as a fiber variable.

Definition 2.1 *A function $f : I\!R^N \times I\!R^M \to I\!R$ is called an equicontinuous C^0- bundle map if f is continuous and the collection $\left\{f(x,\cdot) : x \in I\!R^N\right\}$ is equicontinuous at ξ for every $\xi \in I\!R^M$. For a positive integer k, we say that $f = f(x,\xi)$ is an equicontinuous $C^k_\xi-$bundle map if the partial derivatives $D^\alpha_\xi f$ exist and are equicontinuous C^0-bundle maps for all multi-indices α with $|\alpha| \le k$.*

Remark 2.1 If $V \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $g \in C^k(\mathbb{R}^M)$ then the function $f(x,\eta) = V(x)g(\eta)$ is an equicontinuous $C^k_\eta-$bundle map, as are finite sums of such functions.

Remark 2.2 Equicontinuous C^0-bundle maps are uniformly equicontinuous on compact subsets of $\mathbb{R}^M$ in the following sense. Let $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ be an equicontinuous C^0-bundle map. Given a compact subset K of $\mathbb{R}^M$ and $\varepsilon > 0$, there exists $\delta(K,\varepsilon) > 0$ such that $|f(x,\xi) - f(x,\eta)| < \varepsilon$ for all $x \in \mathbb{R}^N$ and $\xi, \eta \in K$ with $|\xi - \eta| < \delta(K,\varepsilon)$. See Lemma 2.1 of [1].

We can now formulate the essential smoothness properties of the family of quasilinear second order differential operators defined by (2.1) where the functions $a_{\alpha\beta} : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ and $b : \mathbb{R}^N \times \mathbb{R}^{N+2} \to \mathbb{R}$ are bundle maps having the following properties.

(B) For $\alpha, \beta = 1,..,N$, the function $a_{\alpha\beta} = a_{\beta\alpha} : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ is an equicontinuous $C^1_\xi-$bundle map with

$$a_{\alpha\beta}(\cdot,0) \text{ and } \partial_{\xi_i} a_{\alpha\beta}(\cdot,0) \in L^\infty(\mathbb{R}^N) \text{ for } i = 0,1,..,N. \tag{2.2}$$

The function $b : \mathbb{R}^N \times \mathbb{R}^{N+2} \to \mathbb{R}$ is continuous and its partial derivatives $\partial_{\xi_i} b, \partial_\lambda b, \partial_\lambda \partial_{\xi_i} b$ and $\partial_{\xi_i} \partial_\lambda b$ exist and are continuous on $\mathbb{R}^N \times \mathbb{R}^{N+2}$ for $i = 0,1,..,N$. For each $\lambda \in \mathbb{R}, b(\cdot,\lambda) : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ is an equicontinuous $C^1_\xi-$bundle map and $\partial_\lambda \partial_{\xi_i} b : \mathbb{R}^N \times \mathbb{R}^{N+2} \to \mathbb{R}$ is an equicontinuous C^0-bundle map for $i = 0,1,..,N$. Furthermore, for all $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$,

$$b(x,0,\lambda) = 0 \tag{2.3}$$

and

$$\partial_{\xi_i} b(\cdot,0,\lambda) \text{ and } \partial_{\xi_i} \partial_\lambda b(\cdot,0,\lambda) \in L^\infty(\mathbb{R}^N) \text{ for } i = 0,..,N. \tag{2.4}$$

Remark 2.3 The hypothesis (B) ensures that $\partial_\lambda \partial_{\xi_i} b \equiv \partial_{\xi_i} \partial_\lambda b$ for $i = 0,1,..,N$ and that

$$\partial_\lambda b(x,0,\lambda) = 0 \text{ for } x \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}. \tag{2.5}$$

Furthermore, it is easy to deduce from (B) that, for each $\lambda \in \mathbb{R}$, $\partial_\lambda b(\cdot,\lambda) : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ is an equicontinuous $C^1_\xi-$bundle map.

Using this terminology we can formulate the following results giving the requisite smoothness of the differential operator (2.1). See Theorem 3 in [17].

Theorem 2.2 *Fix $p \in (N, \infty)$ and consider the operator F defined by (2.1) under the hypothesis (B). Then*
(i) $F \in C^1(I\!R \times X_p, Y_p)$ and the partial derivatives (in the sense of Fréchet) $D_u D_\lambda F$ and $D_\lambda D_u F$ exist and are continuous on $I\!R \times X_p$.
(ii) $F(\cdot, u) : I\!R \to Y_p$ is equicontinuous with respect to u in bounded subsets of X_p.
(iii) In particular, $F(\lambda, 0) = 0$ with

$$[D_u F(\lambda, 0)v](x) =$$

$$-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, 0)\partial_\alpha\partial_\beta v(x) + \sum_{i=1}^{N} \partial_{\xi_i} b(x, 0, \lambda)\partial_i v(x) + \partial_{\xi_0} b(x, 0, \lambda)v(x)$$

and

$$[D_\lambda D_u F(\lambda, 0)v](x) = [D_u D_\lambda F(\lambda, 0)v](x) =$$

$$\sum_{i=1}^{N} \partial_\lambda \partial_{\xi_i} b(x, 0, \lambda)\partial_i v(x) + \partial_\lambda \partial_{\xi_0} b(x, 0, \lambda)v(x).$$

for all $v \in X_p$.

We now turn to a discussion of the bounded linear operator $D_u F(\lambda, u) : X_p \to Y_p$ with a view to ensuring that it is Fredholm of index zero. For this we shall suppose that the differential operator defined by (2.1) is elliptic in the following sense.

(E) The operator F is strictly elliptic in the sense that there exists a lower semicontinuous function, $\nu : I\!R^N \times I\!R^{N+1} \to (0, \infty)$, such that

$$\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, \xi)\eta_\alpha\eta_\beta \geq \nu(x, \xi) |\eta|^2 \tag{2.6}$$

for all $\eta \in I\!R^N$ and $(x, \xi) \in I\!R^N \times I\!R^{N+1}$.

From Theorem 3.1 of [1] we now obtain the following result.

Theorem 2.3 *Consider the operator (2.1) under the hypotheses (B) and (E). For any $p \in (N, \infty)$, $F : I\!R \times X_p \to Y_p$ is a C^1-Fredholm map of index zero on an interval J if and only if for each $\lambda \in J$, there exists an element $u_\lambda \in X_p$ such that $D_u F(\lambda, u_\lambda) : X_p \to Y_p$ is a Fredholm operator of index zero.*

Remark 2.4 We observe that the choice $u_\lambda = 0$ is particularly attractive since $D_uF(\lambda,0)$ does not involve any derivatives of the functions $a_{\alpha\beta}$.

3 Properness

Let X and Y be real Banach spaces. A function $G : X \to Y$ is said to be proper on closed bounded subsets of X provided that $G^{-1}(K) \cap W$ is a compact subset of X for every compact subset K of Y and every closed bounded subset W of X. Let $L(X,Y)$ denote the Banach space of all bounded linear operators from X into Y and let the kernel and range of a linear operator T be denoted by $\ker T$ and rge T, respectively. In the case of a bounded linear operator $L : X \to Y$, it is known that L is proper on closed bounded subsets of X if and only if $L \in \Phi_+(X,Y) \equiv \{L \in L(X,Y) : \text{rge}L \text{ is closed and } \dim\ker L < \infty\}$. This result is due to Yood and appears as on page 78 of [18].

In this section we discuss the properness of the nonlinear differential operator $F(\lambda,\cdot) : X_p \to Y_p$ defined by (2.1) on closed bounded subsets of X_p. For this we shall suppose that it is asymptotically periodic as x tends to infinity in the following sense.

(A) There exist equicontinuous C^0-bundle maps $a^\infty_{\alpha\beta} = a^\infty_{\beta\alpha} : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ for $\alpha,\beta = 1,..,N$ and an equicontinuous $C^1_\eta-$bundle map $b^\infty : \mathbb{R}^N \times \mathbb{R}^{N+2} \to \mathbb{R}$ such that $b^\infty(x,0,\lambda) \equiv 0$ and

$$\lim_{|x|\to\infty} [a_{\alpha\beta}(x,\xi) - a^\infty_{\alpha\beta}(x,\xi)] = \lim_{|x|\to\infty} [\partial_{\xi_i} b(x,\xi,\lambda) - \partial_{\xi_i} b^\infty(x,\xi,\lambda)] = 0$$

uniformly for (ξ,λ) in bounded subsets of $\mathbb{R}^{N+2}$, where $1 \le \alpha,\beta \le N$ and $i = 0,1,..,N$. Furthermore, $a^\infty_{\alpha\beta}(\cdot,\xi)$ and $b^\infty(\cdot,\xi,\lambda) : \mathbb{R}^N \to \mathbb{R}$ are $N-$periodic on $\mathbb{R}^N$ in the sense that, for some $T = (T_1,..T_N)$ with $T_i > 0$ for all $i = 1,..,N$,

$$a^\infty_{\alpha\beta}(x_1,...,x_i+T_i,...,x_N,\xi) = a^\infty_{\alpha\beta}(x_1,...,x_N,\xi)$$

and

$$b^\infty(x_1,...,x_i+T_i,...,x_N,\xi,\lambda) = b^\infty(x_1,...,x_N,\xi,\lambda)$$

for all $(x,\xi,\lambda) \in \mathbb{R}^N \times \mathbb{R}^{N+2}$ and $i = 1,..,N$.

Under the assumptions (B) and (A) we define a differential operator, F^∞, by

$$F^\infty(\lambda, u) = \tag{3.1}$$
$$- \sum_{\alpha,\beta=1}^{N} a^\infty_{\alpha\beta}(x, u(x), \nabla u(x))\partial_\alpha\partial_\beta u(x) + b^\infty(x, u(x), \nabla u(x), \lambda)$$

Remark 3.1 The assumptions (B) and (A) imply that $F^\infty(\lambda, u) \in Y_p$ for all $(\lambda, u) \in I\!R \times X_p$ and that $F^\infty(\lambda, 0) = 0$ for all $\lambda \in I\!R$.

In this context Theorem 6.1 of [1] yields the following result.

Theorem 3.1 *Consider the differential operator defined by (2.1) under the hypotheses (B), (E) and (A) and let $p \in (N, \infty)$. Then $F(\lambda, \cdot) : X_p \to Y_p$ is proper on closed bounded subsets of X_p if and only if*
$$\begin{cases} (i) \text{ there is an element } u_\lambda \in X_p \text{ such that } D_uF(\lambda, u_\lambda) \in \Phi_+(X_p, Y_p), \\ (ii) \text{ the equation } F^\infty(\lambda, u) = 0 \text{ has no non-trivial solution } u \text{ in } X_p. \end{cases}$$

4 Abstract bifurcation results

Let X and Y be real Banach spaces. A topological degree for C^1-Fredholm maps of index zero from X to Y has been developed in [9, 10]. That degree is based on the notion of the parity, denoted by $\pi\left(A(\lambda) : \lambda \in [a, b]\right)$, of a continuous path, $\lambda \mapsto A(\lambda)$, of bounded linear Fredholm operators of index zero from X into Y, which was introduced in [8]. For such a path, a parametrix is any continuous function $B : [a, b] \to GL(Y, X)$ such that $B(\lambda)A(\lambda) : X \to X$ is a compact perturbation of the identity for each $\lambda \in [a, b]$. If $A(a)$ and $A(b) \in GL(X, Y)$, the parity of the path A on $[a, b]$ is defined by

$$\pi\left(A(\lambda) : \lambda \in [a, b]\right) = d_{LS}(B(a)A(a))d_{LS}(B(b)A(b)) \tag{4.1}$$

where d_{LS} denotes the Leray-Schauder degree. This definition is justified by showing that a parametrix always exists and that $d_{LS}(B(a)A(a))d_{LS}(B(b)A(b))$ is independent of the choice of parametrix B. (Note that in deriving the required properties of parity, the Leray - Schauder degree is only used in the very special case of linear homeomorphisms of the form $I - K$ where K is compact. Thus the degree is given by $(-1)^m, m$ being the sum of the multiplicities of the eigenvalues of K which are real and greater than 1.) In some circumstances the parity can be expressed in a form which is easier to evaluate directly. In formulating our bifurcation theorems for (1.1), (1.2) we shall only use the following criterion.

Proposition 4.1 *Let $A : [a,b] \to L(X,Y)$ be a continuous path of bounded linear operators having the following properties.*

(i) $A \in C^1([a,b], L(X,Y))$.

(ii) $A(\lambda) : X \to Y$ is a Fredholm operator of index zero for all $\lambda \in [a,b]$.

(iii) There exists $\lambda_0 \in (a,b)$ such that

$$A'(\lambda_0)[\ker\ A(\lambda_0)] \oplus rge\ A(\lambda_0) = Y \tag{4.2}$$

in the sense of a topological direct sum.
Then there exists $\varepsilon > 0$ such that $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset [a,b]$,

$$A(\lambda) \in GL(X,Y) \text{ for all } \lambda \in [\lambda_0 - \varepsilon, \lambda_0) \cup (\lambda_0, \lambda_0 + \varepsilon] \tag{4.3}$$

and

$$\pi\left(A(\lambda) : \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]\right) = (-1)^k \tag{4.4}$$

where $k = \dim\ker A(\lambda_0)$.

This result is a combination of Proposition 2.1 of [13] and Theorem 6.18 of [8].

Note that for any continuous path $A : [a,b] \to L(X,Y)$ and any $\lambda_0 \in (a,b)$ such that $A(\lambda) \in GL(X,Y)$ for all $\lambda \in [a,b] \backslash \{\lambda_0\}$, the parity $\pi\left(A(\lambda) : \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]\right)$ is the same for all $\varepsilon > 0$ provided that $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset [a,b]$. This quantity is called the parity of the path A across λ_0. The preceding proposition provides one way of calculating the parity across λ_0.

We can now state the main result about global bifurcation which can be derived using the above notions.

Theorem 4.2 *Let X and Y be real Banach spaces and consider a function $F \in C^1(J \times X, Y)$ where J is an open interval such that $F(\lambda, 0) = 0$ for all $\lambda \in J$ and F is a Fredholm map of index zero on J. Suppose that $\lambda_0 \in J$ and that there exists $\varepsilon > 0$ such that $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset J$,*

$$D_u F(\lambda, 0) \in GL(X,Y) \text{ for } \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \backslash \{\lambda_0\}$$

and

$$\pi\left(D_uF(\lambda,0),[\lambda_0-\varepsilon,\lambda_0+\varepsilon]\right)=-1.$$

Let $Z=\{(\lambda,u)\in J\times X: u\neq 0$ *and* $F(\lambda,u)=0\}$ *and let* C *denote the connected component of* $Z\cup\{(\lambda_0,0)\}$ *containing* $(\lambda_0,0)$. *Then* C *has at least one of the following properties.*
(i) C *is not a compact subset of* $J\times X$.
(ii) The closure of C *contains a point* $(\lambda_1,0)$ *where* $\lambda_1\in J\backslash[\lambda_0-\varepsilon,\lambda_0+\varepsilon]$ *and* $D_uF(\lambda_1,0)\notin GL(X,Y)$.

The above statements refer to Z and C with the metric inherited from $I\!R\times X$. The basic procedure for proving a result like this is to suppose that C has neither of the properties stated in the conclusion and then to derive a contradiction using the properties of the degree for C^1-Fredholm maps. For the special case $J=I\!R$, this result appears as Theorem 6.1 of [10], but the same arguments yield the version we have stated.

The ways in which C can fail to be compact can be made more precise provided that F is proper on closed bounded subsets of X.

Theorem 4.3 *Let* X *and* Y *be real Banach spaces and let* J *be an open interval. Consider a function* $F\in C^1(J\times X,Y)$ *such that the maps* $F(\cdot,u): J\to Y$ *are equicontinuous for* u *in bounded subsets of* X. *Suppose that* $F(\lambda,0)=0$ *for all* $\lambda\in J$ *and that* F *is a Fredholm map of index zero on* J, *with the property that* $F(\lambda,\cdot):X\to Y$ *is proper on closed bounded subsets of* X *for all* $\lambda\in J$. *Suppose also that* $\lambda_0\in J$ *and that there exists* $\varepsilon>0$ *such that* $[\lambda_0-\varepsilon,\lambda_0+\varepsilon]\subset J$,

$$D_uF(\lambda,0)\in GL(X,Y) \text{ for } \lambda\in[\lambda_0-\varepsilon,\lambda_0+\varepsilon]\backslash\{\lambda_0\}$$

and

$$\pi\left(D_uF(\lambda,0),[\lambda_0-\varepsilon,\lambda_0+\varepsilon]\right)=-1.$$

Let $Z=\{(\lambda,u)\in J\times X: u\neq 0$ *and* $F(\lambda,u)=0\}$ *and let* C *denote the connected component of* $Z\cup\{(\lambda_0,0)\}$ *containing* $(\lambda_0,0)$. *Then* C *has at least one of the following properties.*
(i) C *is an unbounded subset of* $J\times X$.

(ii) The closure of $PC = \{\lambda \in I\!R : (\lambda, u) \in C\}$ *intersects the boundary of* J.
(iii) The closure of C *contains a point* $(\lambda_1, 0)$ *where* $\lambda_1 \in J \backslash [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ *and* $D_u F(\lambda_1, 0) \notin GL(X, Y)$.

Proof Suppose that C has none of the properties (i), (ii), (iii). We shall show that this implies that C is a compact subset of $J \times X$, contradicting the previous theorem.
Let $\{(\lambda_n, u_n)\} \subset C$. For the compactness of C, it is enough to show that this sequence contains a subsequence converging to a point (λ, u) in $J \times X$. Since C is a bounded subset of $I\!R \times X$ there is a closed bounded subset W of X such that $\{u_n\} \subset W$. Passing to a subsequence we can suppose immediately that $\lambda_n \to \lambda$ and $\lambda \in J$ since $\lambda_n \in J$ and C does not have the property (ii). Furthermore since the functions $F(\cdot, u_n)$ are equicontinuous at λ, $F(\lambda_n, u_n) - F(\lambda, u_n) \to 0$ as $n \to \infty$. Since $F(\lambda_n, u_n) = 0$, it follows that $F(\lambda, u_n) \to 0$ and so $K = \{F(\lambda, u_n)\} \cup \{0\}$ is a compact subset of Y. Hence $[F(\lambda, \cdot)^{-1} K] \cap W$ is a compact subset of X. This implies that $\{u_n\}$ contains a convergent subsequence in X which in turn establishes the compactness of C.

5 Bifurcation for quasilinear elliptic equations on $I\!R^N$

Throughout this section we fix $p \in (N, \infty)$ and consider the differential operator $F : I\!R \times X_p \to Y_p$ defined by (2.1) under the assumption (B). By Theorem 2.2 this already ensures that

$$F \in C^1(I\!R \times X_p, Y_p) \text{ with } F(\lambda, 0) = 0$$

and

$$D_u F(\lambda, 0) v = - \sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(\cdot, 0) \partial_\alpha \partial_\beta v + \sum_{i=1}^{N} \partial_{\xi_i} b(\cdot, 0, \lambda) \partial_i v + \partial_{\xi_0} b(\cdot, 0, \lambda) v$$

for all $\lambda \in I\!R$.

Our next results deal with a situation where the parity of the path $\lambda \mapsto D_u F(\lambda, 0)$ across a value λ_0 can be determined in a relatively explicit way. This approach uses the following assumption (L) which ensures that $D_uF(\lambda, 0)$ has a particularly simple form. More general behaviour at $u = 0$ can be handled provided that the asymptotic behaviour required for the discussion of properness is assumed and we shall return to this in due course.

(L) There is a (constant) positive definite matrix $[A_{\alpha\beta}]$ such that

$$a_{\alpha\beta}(x, 0) = A_{\alpha\beta} = A_{\beta\alpha} \text{ for all } x \in I\!\!R^N$$

and

$$\partial_{\xi_\alpha} b(x, 0, \lambda) = 0 \text{ for all } x \in I\!\!R^N \text{ and } \lambda \in I\!\!R$$

for all $\alpha, \beta = 1, .., N$,

Under the hypotheses (B) and (L), it follows from Theorem 2.2 that

$$[D_uF(\lambda, 0)v](x) = -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}\partial_\alpha\partial_\beta v(x) + \partial_{\xi_0} b(x, 0, \lambda)v(x)$$

which can be reduced to the form

$$[D_uF(\lambda, 0)v](x) = -\Delta v(x) + \partial_{\xi_0} b(x, 0, \lambda)v(x) \tag{5.1}$$

by a linear change of the variable x.

By Theorem 2.3, $F : I\!\!R \times X_p \to Y_p$ is a C^1-Fredholm map of index zero on an interval J if and only if $D_uF(\lambda, 0) : X_p \to Y_p$ linear Fredholm operator of index zero for all $\lambda \in J$. Using the hypothesis (L) the latter property can be expressed in terms of the linear Schrödinger operator $-\Delta + \partial_{\xi_0} b(x, 0, \lambda)$ on $L^2(I\!\!R^N)$.

We refer to [12] for the notions of spectrum, discrete spectrum and essential spectrum of an unbounded self-adjoint operator acting on a Hilbert space. The discrete spectrum consists of the isolated points in the spectrum which are eigenvalues of finite multiplicity. Those points in the spectrum which do not belong to the discrete spectrum form the essential spectrum.

In [15] we discussed the Fredholm properties of the operator $-\Delta + V$ in $L^p(I\!\!R^N)$ for a class of potentials admitting singularities. To deal with (5.1) it is sufficient to recall the following special case which appears as Theorem 1 in [15].

Theorem 5.1 *Let $V \in L^\infty(\mathbb{R}^N)$. Then $-\Delta + V : W^{2,2}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a self-adjoint operator whose spectrum and discrete spectrum are denoted by σ and σ_d respectively. For $p \in (1,\infty)$, consider also the operator $S_p : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ defined by*

$$S_p u = (-\Delta + V)u \text{ for } u \in W^{2,p}(\mathbb{R}^N).$$

For every $p \in (1,\infty)$, the following conclusions are valid.
(i) $S_p - \lambda I : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is an isomorphism if $\lambda \notin \sigma$,
whereas, if $\lambda \in \sigma_d$, then
(ii) $S_p - \lambda I : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is a Fredholm operator of index zero,
(iii) ker $(S_p - \lambda I) =$ ker $(S_2 - \lambda I)$, and
(iv) $L^p(\mathbb{R}^N) =$ ker $(S_p - \lambda I) \oplus$ rge $(S_p - \lambda I)$ where $\oplus$ denotes a topological direct sum.

We now use this result in conjunction with Proposition 4 to discuss the parity of the path $\lambda \longmapsto D_u F(\lambda, 0)$.

Lemma 5.2 *Suppose that the conditions (B) and (L) are satisfied and consider a point $\lambda_0 \in \mathbb{R}$ such that 0 does not belong to the essential spectrum of the self-adjoint operator $-\Delta + \partial_{\xi_0} b(x, 0, \lambda_0) : X_2 \subset Y_2 \to Y_2$. Let $p \in (N, \infty)$.*
(i) Then $D_u F(\lambda_0, 0) : X_p \to Y_p$ is a Fredholm operator of index zero.
(ii) Furthermore, if either

$$\partial_\lambda \partial_{\xi_0} b(\cdot, 0, \lambda_0) \geq 0 \text{ but } \not\equiv 0 \text{ on } \mathbb{R}^N,$$

or

$$\partial_\lambda \partial_{\xi_0} b(\cdot, 0, \lambda_0) \leq 0 \text{ but } \not\equiv 0 \text{ on } \mathbb{R}^N,$$

then there exists $\varepsilon > 0$ such that $D_u F(\lambda, 0) \in GL(X_p, Y_p)$ for all $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \setminus \{\lambda_0\}$ and

$$\pi(D_u F(\lambda, 0), [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) = (-1)^k$$

where $k = \dim \ker[-\Delta + \partial_{\xi_0} b(x, 0, \lambda_0)]$.

Remark 5.1 By Theorem 5.1(iii), the kernel of the linear operator $-\Delta + \partial_{\xi_0} b(x,0,\lambda) : X_p \to Y_p$ does not depend on the choice of $p \in (1,\infty)$.

We can now formulate our first bifurcation theorem for the problem (1.1), (1.2).

Theorem 5.3 *Let the conditions (B), (E) and (L) be satisfied and let $p \in (N,\infty)$. Consider the operator $F : \mathbb{R} \times X_p \to Y_p$ defined by (2.1) and suppose that J is an open interval having the following properties.*
(a) For $\lambda \in J$, the essential spectrum of the self-adjoint operator $-\Delta + \partial_{\xi_0} b(x,0,\lambda) : X_2 \subset Y_2 \to Y_2$ does not contain 0.
(b) There is a point $\lambda_0 \in J$ such that dim ker$[-\Delta + \partial_{\xi_0} b(x,0,\lambda_0)]$ is odd and either

$$\partial_\lambda \partial_{\xi_0} b(\cdot,0,\lambda_0) \geq 0 \text{ but } \not\equiv 0 \text{ on } \mathbb{R}^N,$$

or

$$\partial_\lambda \partial_{\xi_0} b(\cdot,0,\lambda_0) \leq 0 \text{ but } \not\equiv 0 \text{ on } \mathbb{R}^N.$$

Let $Z = \{(\lambda,u) \in J \times X_p : u \neq 0 \text{ and } F(\lambda,u) = 0\}$ and let C denote the connected component of $Z \cup \{(\lambda_0,0)\}$ containing $(\lambda_0,0)$. Then C has at least one of the following properties.
(i) C is a non-compact subset of $J \times X_p$.
(ii) The closure of C in $\overline{J} \times X_p$ contains a point $(\lambda_1,0)$ where $\lambda_1 \neq \lambda_0$.

Proof Using Lemma 5.2(i) and Theorem 2.3, we see that $F : \mathbb{R} \times X_p \to Y_p$ is a Fredholm map of index zero on the interval J. By Lemma 8(ii) its parity across λ_0 is equal to -1. The conclusion follows from Theorem 4.2.

To resolve the non-compactness property into a global one we suppose that F is asymptotically periodic in the sense of condition (A) and then we discuss the equation $F^\infty(\lambda,u) = 0$ with a view to showing that $u = 0$ is the only solution in X_p. We can also ensure that the operator (2.1) is a C^1-Fredholm map on an interval by imposing conditions on the linearization of F^∞.

First of all we recall from Section 6 of [1] that, although the assumption (A) does not guarantee the differentiability of the operator $F^\infty : \mathbb{R} \times X_p \to Y_p$, it does imply that $F^\infty(\lambda,\cdot) : X_p \to Y_p$ is differentiable (in the sense of Fréchet) at 0 with

$$D_u F^\infty(\lambda,0)v =$$

$$-\sum_{\alpha,\beta=1}^{N} a^{\infty}_{\alpha\beta}(\cdot,0)\partial_\alpha\partial_\beta v + \sum_{\alpha=1}^{N} \partial_{\xi_\alpha} b^{\infty}(\cdot,0,\lambda)\partial_\alpha v + \partial_{\xi_0} b^{\infty}(\cdot,0,\lambda)v$$

for all $v \in X_p$ and $\lambda \in I\!R$.
We note that $D_u F^{\infty}(\lambda,0)$ is a linear second order differential operator with continuous $N-$periodic coefficients. In [1], Lemma 6.6 and Remark 6.2 describe some situations where it is a Fredholm operator of index zero. The following assumption isolates a particularly agreeable situation.

($\mathbf{L}^{\infty}$) There is a (constant) positive definite matrix $\left[A^{\infty}_{\alpha\beta}\right]$ such that

$$a^{\infty}_{\alpha\beta}(x,0) = A^{\infty}_{\alpha\beta} = A^{\infty}_{\beta\alpha} \text{ for all } x \in I\!R^N$$

and

$$\partial_{\xi_\alpha} b^{\infty}(x,0,\lambda) = 0 \text{ for all } x \in I\!R^N \text{ and } \lambda \in I\!R$$

for $1 \leq \alpha, \beta \leq N$.

Remark 5.2 When this condition is satisfied we can assume that $A^{\infty}_{\alpha\beta} = \delta_{\alpha\beta}$ for $1 \leq \alpha,\beta \leq N$ (by making a linear change of variable) and hence that

$$D_u F^{\infty}(\lambda,0)v = -\Delta v + \partial_{\xi_0} b^{\infty}(\cdot,0,\lambda)v$$

for $v \in X_p$ and $\lambda \in I\!R$.

In this context we have the following result which appears as Lemma 11 of [17].

Lemma 5.4 *Suppose that the conditions (B), (A) and (L^{∞}) are satisfied and consider $\lambda_0 \in I\!R$ such that the self-adjoint operator, $S(\lambda_0) : X_2 \subset Y_2 \to Y_2$, defined by*

$$S(\lambda_0)v = -\Delta v + \partial_{\xi_0} b^{\infty}(\cdot,0,\lambda_0)v \text{ for } v \in X_2$$

is an isomorphism. Let $p \in (N,\infty)$.
(i) Then $D_u F(\lambda_0,0) : X_p \to Y_p$ is a Fredholm operator of index zero.
(ii) Let $\{\varphi_i \in X_p : i = 1,..,k\}$ and $\{\psi_i \in Y_q : i = 1,..,k\}$ be bases for $\ker D_u F(\lambda_0,0)$ and $\ker\,[D_u F(\lambda_0,0)]^*$ respectively, with $k = \dim\ker D_u F(\lambda_0,0)$ and $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\det\left[\int_{I\!R^N} \psi_i \left\{D_\lambda D_u F(\lambda_0,0)\varphi_j\right\} dx\right] \neq 0 \qquad (5.2)$$

if and only if

$$(5.3) \qquad D_\lambda D_u F(\lambda_0, 0)\,[\ker D_u F(\lambda_0, 0)] \oplus rge\ D_u F(\lambda_0, 0) = Y_p.$$

When (5.2) is satisfied there exists $\varepsilon > 0$ *such that* $D_u F(\lambda, 0) \in GL(X_p, Y_p)$ *for all* $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \setminus \{\lambda_0\}$ *and*

$$\pi\,(D_u F(\lambda, 0), [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) = (-1)^k.$$

Remark 5.3 This result gives the same conclusions as Lemma 5.2 without requiring $D_u F(\lambda, 0)$ to be a formally symmetric differential operator. Note that when the conditions (B), (E), (A) and (L) are satisfied then so is (L^∞) with $A^\infty_{\alpha\beta} = A_{\alpha\beta}$. We also observe that, since $\partial_{\xi_0} b^\infty(\cdot, 0, \lambda_0)$ is an $N-$ periodic function, $S(\lambda_0)$ is an isomorphism if and only if 0 does not belong to its essential spectrum. (See Theorem 5.4 of Chapter 3 in [3].) Moreover when there exists a continuous $N-$ periodic function P such that $\partial_{\xi_0} b^\infty(x, 0, \lambda_0) \equiv P(x) - \lambda_0, S(\lambda_0) = -\Delta + P - \lambda_0$ and it is an isomorphism if and only if λ_0 does not belong to the spectrum of the $N-$ periodic Schrödinger operator $-\Delta + P$.

Combining the above results we obtain the following rather general global bifurcation theorem for (1.1), (1.2). See Theorem 12 in [17].

Theorem 5.5 *Let the conditions (B),(E),(A) and (L^∞) be satisfied. Choose* $p \in (N, \infty)$ *and consider the operator* $F : \mathbb{R} \times X_p \to Y_p$ *defined by (2.1). Suppose that* J *is an open interval having the following properties.*
(a) For all $\lambda \in J, \{u \in X_p : F^\infty(\lambda, u) = 0\} = \{0\}$.
(b) For all $\lambda \in J$*, the self-adjoint operator* $-\Delta + \partial_{\xi_0} b^\infty(\cdot, 0, \lambda) : X_2 \subset Y_2 \to Y_2$ *is an isomorphism.*
(c) There is a point $\lambda_0 \in J$ *such that* $\dim\ker D_u F(\lambda_0, 0)$ *is odd and the condition (5.2) is satisfied.*
Let C *denote the connected component of* $Z \cup \{(\lambda_0, 0)\}$ *containing* $(\lambda_0, 0)$ *where* $Z = \{(\lambda, u) \in J \times X_p : u \neq 0$ *and* $F(\lambda, u) = 0\}$ *and* $Z \cup \{(\lambda_0, 0)\}$ *has the metric inherited from* $\mathbb{R} \times X_p$*. Then* C *has at least one of the following properties.*
(i) C *is an unbounded subset of* $J \times X_p$.
(ii) The closure of $\{\lambda : (\lambda, u) \in C$ *for some* $u \in X_p\}$ *intersects the boundary of* J.
(iii) The closure of C *in* $\overline{J} \times X_p$ *contains a point* $(\lambda_1, 0)$ *where* $\lambda_1 \neq \lambda_0$.

There are several approaches which can be used to check the condition (a) in this result and we shall present three of them. In order to give relatively explicit hypotheses on the operator F we shall strengthen the hypothesis (L).

(LL) The condition (L) is satisfied and there is a constant $c \neq 0$ such that

$$\partial_\lambda \partial_{\xi_0} b(x,0,\lambda) = c \text{ for all } x \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}.$$

When (B) and (LL) are satisfied we can and shall suppose (by redefining λ) that there exists a function $V \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$\partial_{\xi_0} b(x,0,\lambda) = V(x) - \lambda \text{ for all } x \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}.$$

In this case

$$D_u F(\lambda, 0)v = (-\Delta + V)v - \lambda v \tag{5.4}$$

where $-\Delta + V : X_2 \subset Y_2 \to Y_2$ is a self-adjoint operator. When the condition (LL) is satisfied with $\partial_{\xi_0} b(x,0,\lambda) \equiv V(x) - \lambda$, the condition ($L^\infty$) is also satisfied and $\partial_{\xi_0} b^\infty(x,0,\lambda_0) \equiv P(x) - \lambda$ where P is a continuous $N-$periodic function. Since $\lim_{|x|\to\infty} \{V(x) - P(x)\} = 0$, it follows that the essential spectrum of the Schrödinger operator $-\Delta + V : X_2 \subset Y_2 \to Y_2$ is equal to the whole spectrum of the $N-$periodic Schrödinger operator $-\Delta + P : X_2 \subset Y_2 \to Y_2$.

Henceforth we use σ and σ_e to denote its spectrum and essential spectrum, respectively.

5.1 Using the maximum principle

The maximum principle can be used to establish the condition (a) of Theorem 5.5 provided that the functions $a^\infty_{\alpha\beta}$ and b^∞ have the following properties.

(M_λ) There exists a continuous function $\nu : \mathbb{R}^{N+1} \to (0,\infty)$ such that

$$\sum_{\alpha,\beta=1}^{N} a^\infty_{\alpha\beta}(x,s,0)\eta_\alpha\eta_\beta \geq \nu(x,s)\,|\eta|^2 \text{ for all } \eta \in \mathbb{R}^N$$

and for all $(x,s) \in \mathbb{R}^{N+1}$ and

$$b^\infty(x,s,0,\lambda)s > 0 \text{ for all } (x,s) \in \mathbb{R}^{N+1} \text{ with } s \neq 0.$$

Remark 5.4 It follows from this that $\partial_{\xi_0} b^\infty(x,0,\lambda) \geq 0$ for all $x \in \mathbb{R}^N$.

Theorem 5.6 *Let the conditions (B),(E) and (A) be satisfied and let* $p \in (N, \infty)$. *Suppose that* J *is an open interval such that* (M_λ) *is satisfied for all* $\lambda \in J$.

(i) Suppose that the condition (L^∞) *is satisfied and that the operator* $S(\lambda) = -\Delta + \partial_{\xi_0} b^\infty(\cdot, 0, \lambda) : X_2 \to Y_2$ *is an isomorphism for all* $\lambda \in J$. *Then the conclusion of Theorem 5.5 is valid for the operator* $F : I\!\!R \times X_p \to Y_p$ *defined by (2.1) at any point* $\lambda_0 \in J$ *such that* $\dim \ker D_u F(\lambda_0, 0)$ *is odd and the condition (5.2) is satisfied with* $\lambda = \lambda_0$.

(ii) If the condition (LL) is satisfied with $\partial_{\xi_0} b(x, 0, \lambda) \equiv V(x) - \lambda$, *then* $J \subset (-\infty, \omega)$ *where* $\omega \equiv \liminf_{|x| \to \infty} V(x)$. *Furthermore, for every eigenvalue* $\lambda_0 \in J$ *of odd multiplicity of* $-\Delta + V : X_2 \subset Y_2 \to Y_2$, *the conclusion of Theorem 5.5 is valid for the operator* $F : I\!\!R \times X_p \to Y_p$ *defined by (2.1).*

This result appears as Theorem 14 of [17].

Remark 5.5 Under the hypotheses of part (ii), there exists a continuous $N-$periodic function P such that $\lim_{|x| \to \infty} \{V(x) - P(x)\} = 0$ and $\partial_{\xi_0} b^\infty(x, 0, \lambda) \equiv P(x) - \lambda$. Thus $\omega = \inf \left\{ P(x) : x \in I\!\!R^N \right\}$ and so the condition (M$_\lambda$) implies that $\lambda \leq \omega$ for all $\lambda \in J$. But J is open so in fact $J \subset (-\infty, \omega)$. Furthermore, the essential spectrum, σ_e, of the Schrödinger operator $-\Delta + V : X_2 \subset Y_2 \to Y_2$ is equal to the whole spectrum of the $N-$periodic Schrödinger operator $-\Delta + P : X_2 \subset Y_2 \to Y_2$ and so $J \subset I\!\!R \backslash \sigma_e$.

5.2 Using variational identities

When the condition (M$_\lambda$) is not satisfied an alternative is offered under the following conditions which ensure that all solutions of the equation $F^\infty(\lambda, u) = 0$, with $u \in X_p$ for some $p > N$, satisfy an integral identity of the type found by Pohozaev, [2, 14]. Under appropriate conditions this can be used to show that $u = 0$. We refer to Sections 5 and 6 of our paper [16] for these results.

(V) There exist two functions

$$Q = Q(\xi) \in C^3(I\!\!R^{N+1}) \text{ and } g = g(\xi_0, \lambda) \in C^1(I\!\!R^2)$$

such that

$$a^\infty_{\alpha\beta}(x, \xi) = \partial_{\xi_\alpha} \partial_{\xi_\beta} Q(\xi)$$

and

$$b^{\infty}(x,\xi,\lambda)=\partial_{\xi_0}Q(\xi)-\sum_{\alpha=1}^{N}\xi_\alpha\partial_{\xi_\alpha}\partial_{\xi_0}Q(\xi)+g(\xi_0,\lambda)$$

for all $x\in I\!R^N, \xi=(\xi_0,\xi_1,..\xi_N)$ and $\lambda\in I\!R$. Furthermore,

$$Q(\xi_0,0)=\partial_{\xi_0}Q(\xi_0,0)=0 \text{ for all } \xi_0\in I\!R,$$
$$\partial_{\xi_\alpha}Q(0)=0 \text{ for } \alpha=1,...,N, \tag{5.5}$$

and there exists a continuous function $\nu: I\!R^{N+1}\to(0,\infty)$ such that

$$\sum_{\alpha,\beta=1}^{N}\partial_{\xi_\alpha}\partial_{\xi_\beta}Q(0)\eta_\alpha\eta_\beta\geq\nu(\xi)\left|\eta\right|^2$$

for all $\eta\in I\!R^N$ and $\xi\in I\!R^{N+1}$.

Remark 5.6 The condition (V) means that the equation $F^{\infty}(\lambda,u)=0$ has the variational form

$$-\sum_{\alpha=1}^{N}\partial_\alpha\left\{\partial_{\xi_\alpha}Q\left(u(x),\nabla u(x)\right)\right\}+\partial_{\xi_0}Q\left(u(x),\nabla u(x)\right)$$
$$+g(u(x),\lambda)=0 \tag{5.6}$$

associated with the formal Euler-Lagrange equation of the functional

$$\int_{I\!R^N}\left\{Q\left(u(x),\nabla u(x)\right)+\int_0^{u(x)}g(s,\lambda)ds\right\}dx. \tag{5.7}$$

Under the assumption (V) and the condition (5.9) introduced below, we show in Theorem 5.2 of [16] that any solution, $u\in X_p$ for some $p\in(N,\infty)$, of the equation (5.6) satisfies the following energy identity,

$$\int_{I\!R^N}\sum_{\alpha=1}^{N}\partial_{\xi_\alpha}Q\left(u,\nabla u\right)\partial_\alpha u+\partial_{\xi_0}Q\left(u,\nabla u\right)u+g(u,\lambda)dx=0,$$

and Pohozaev identity,

$$\int_{I\!R^N}\sum_{\alpha=1}^{N}\partial_{\xi_\alpha}Q\left(u,\nabla u\right)\partial_\alpha u dx=$$
$$N\int_{I\!R^N}\left\{Q\left(u(x),\nabla u(x)\right)+\int_0^{u(x)}g(s,\lambda)ds\right\}dx.$$

Remark 5.7 As is shown in Section 5 of [16], the properties of Q required in (5.5) involve no real restriction. If they are not satisfied, they can be recovered by replacing Q by

$$\widetilde{Q}(\xi) = Q(\xi) - Q(\xi_0, 0) - \sum_{\alpha=1}^{N} \partial_{\xi_\alpha} Q(0)\xi_\alpha$$

and g by

$$\widetilde{g}(\xi_0, \lambda) = g(\xi_0, \lambda) + \partial_{\xi_0} Q(\xi_0, 0),$$

since $\widetilde{Q}$ and $\widetilde{g}$ generate the same functions $a^{\infty}_{\alpha\beta}$ and b^{∞}as Q and g.

Remark 5.8 If the conditions (B),(A) and (V) are satisfied then so is (L$^{\infty}$) with $A^{\infty}_{\alpha\beta} = \partial_{\xi_\alpha}\partial_{\xi_\beta} Q(0)$. Thus we can suppose that

$$D_u F(\lambda, 0) = -\Delta + \partial_{\xi_0} b^{\infty}(\cdot, 0, \lambda) \tag{5.8}$$

where $\partial_{\xi_0} b^{\infty}(\cdot, 0, \lambda)$ is equal to the constant $\partial_{\xi_0} g(0, \lambda)$.
If, in addition, the condition (LL) is satisfied with $\partial_{\xi_0} b(x, 0, \lambda) \equiv V(x) - \lambda$, then $V(\infty) = \lim_{|x|\to\infty} V(x)$ exists and $\partial_{\xi_0} b^{\infty}(\cdot, 0, \lambda) \equiv V(\infty) - \lambda$.

Theorem 5.7 *Let the conditions (B), (E) and (A) be satisfied and let* $p \in (N, \infty)$*. Consider an open interval* J *such that,*

(5.9) (*a*) $g(0, \lambda) = 0$ *and* $\partial_{\xi_0} g(0, \lambda) > 0$ *for all* $\lambda \in J$,

(b) there exists $a \in \mathbb{R}$ *such that*

$$NQ(\xi) \geq (a + 1) \sum_{\alpha=1}^{N} \xi_\alpha \partial_{\xi_\alpha} Q(\xi) + a\xi_0 \partial_{\xi_0} Q(\xi) \text{ for all } \xi \in \mathbb{R}^{N+1} \tag{5.10}$$

and

$$N \int_0^s g(t, \lambda) dt \geq a g(s, \lambda) s \text{ for all } (s, \lambda) \in \mathbb{R} \times J. \tag{5.11}$$

(i) Then the operator $S(\lambda) = -\Delta + \partial_{\xi_0} b^{\infty}(\cdot, 0, \lambda) : X_2 \to Y_2$ *is an isomorphism for all* $\lambda \in J$*. Furthermore, the conclusion of Theorem 5.5 is valid for the operator* $F : \mathbb{R} \times X_p \to Y_p$ *defined by (2.1) at any point* $\lambda_0 \in J$ *such that* $\dim \ker D_u F(\lambda_0, 0)$ *is odd and the condition (5.2) is satisfied with* $\lambda = \lambda_0$.
(ii) If the condition (LL) is satisfied with $\partial_{\xi_0} b(x, 0, \lambda) \equiv V(x) - \lambda$*, then the condition (5.9) is satisfied if and only if* $\lambda < V(\infty) = \lim_{|x|\to\infty} V(x)$*. The conclusion of Theorem 5.5 is also valid for the operator* $F : \mathbb{R} \times X_p \to Y_p$ *defined by (2.1) at any eigenvalue* $\lambda_0 \in J$ *of* $-\Delta + V : X_2 \subset Y_2 \to Y_2$ *which has odd multiplicity, provided that (5.10) and (5.11) are satisfied.*

This result appears as Theorem 15 of [17].

Remark 5.9 The condition (V) restricts the applicability of this result to cases where the differential operator F^∞ has no explicit dependence on the variable x. In particular, the condition (5.9) means that $\partial_{\xi_0} b^\infty(x,0,\lambda) = \partial_{\xi_0} g(0,\lambda) > 0$ for all $x \in I\!R$. Since the spectrum of $-\Delta : X_2 \subset Y_2 \to Y_2$ is the interval $[0,\infty)$, it follows that $-\Delta + \partial_{\xi_0} b^\infty(\cdot,0,\lambda) : X_2 \to Y_2$ is an isomorphism whenever (5.9) holds. When (LL) is satisfied the condition (5.9) becomes $V(\infty) - \lambda > 0$. In this case, $\sigma_e = [V(\infty),\infty)$.

5.3 Using asymptotic linearity

In Theorems 5.6(ii) and 5.7(ii) the admissible interval lies below the essential spectrum of the linearization about the trivial solution. We now present a situation where there is global bifurcation in gaps of the essential spectrum of this linearization.

Theorem 5.8 *Let the conditions (B), (E), (A) and (LL) be satisfied and let* $p \in (N,\infty)$. *Suppose that there is an* $N-$*periodic function* $P \in C(I\!R^N)$ *such that*

$$\lim_{|x|\to\infty} \{V(x) - P(x)\} = 0$$

and that

$$a^\infty_{\alpha\beta}(x,\xi) = \delta_{\alpha\beta} \text{ and } b^\infty(x,\xi,\lambda) = \{P(x) - \lambda\}\,\xi_0$$

for all $(x,\xi,\lambda) \in I\!R^N \times I\!R^{N+2}$ *with* $\xi = (\xi_0,\xi_1,..\xi_N)$. *Consider an open interval* $J \subset I\!R\backslash\sigma_e$ *and an eigenvalue* $\lambda_0 \in J$ *of odd multiplicity of* $-\Delta + V : X_2 \subset Y_2 \to Y_2$. *Then the conclusion of Theorem 5.5 is valid for the operator* $F : I\!R \times X_p \to Y_p$ *defined by (2.1).*

This result appears as Theorem 16 of [17].

Remark 5.10 Since $\lim_{|x|\to\infty} \{V(x) - P(x)\} = 0$, the essential spectrum, σ_e, of the operator $-\Delta + V : X_2 \subset Y_2 \to Y_2$ is equal to the entire spectrum, Σ, of the periodic Schrödinger operator, $-\Delta + P : X_2 \subset Y_2 \to Y_2$. Typically, it is a finite union of closed intervals. See [3].

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