Non convex problems of the calculus of variations and differential inclusions

Bernard DACOROGNA (bernard.dacorogna@epfl.ch)
Section of mathematics, EPFL, 1015 Lausanne, Switzerland.

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Abstract

We study existence of minimizers for problems of the type

$$(P)\ \inf\left\{\int_\Omega f\left(x,u\left(x\right),Du\left(x\right)\right)dx : u\in u_0+W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)\right\}$$

where u_0 is a given function.

After recalling some basic facts about existence of minimizers when the function f is convex (quasiconvex), we turn our attention to the case where f is not convex (quasiconvex).

We start by presenting the general tool of relaxation, which gives generalized solutions of (P)

We next discuss some differential inclusions, where we look for solutions $u\in u_0+W_0^{1,\infty}\left(\Omega;\mathbb{R}^N\right)$ of

$$Du\left(x\right)\in E,\text{ a.e. in }\Omega$$

where $E\subset\mathbb{R}^{N\times n}$ is a given compact set.

Finally combining the relaxation theorem and the study of differential inclusions, we give necessary and sufficient conditions for existence of classical minimizers of (P) as well as several examples.

Contents

1 Introduction

We discuss the existence of minimizers for the problem

$$(P) \qquad \inf \left\{ I(u) = \int_{\Omega} f(x, u(x), Du(x))\, dx : u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right) \right\}.$$

where

- $\Omega \subset \mathbb{R}^n$ is a bounded open set, with Lipschitz boundary $\partial\Omega$;
- $u : \Omega \to \mathbb{R}^N$

$$u = u(x) = u(x_1, \cdots, x_n) = \left(u^1(x), \cdots, u^N(x)\right)$$

(if $N = 1$ or, by abuse of language, if $n = 1$, we will say that it is *scalar* valued while if $N, n \geq 2$, we will speak of the *vector* valued case);

- Du denotes its Jacobian matrix, i.e.

$$Du = \left(\frac{\partial u^i}{\partial x_j} \right)_{1 \leq j \leq n}^{1 \leq i \leq N} = \begin{pmatrix} \partial u^1 / \partial x_1 & \cdots & \partial u^1 / \partial x_n \\ \vdots & \vdots & \vdots \\ \partial u^N / \partial x_1 & \cdots & \partial u^N / \partial x_n \end{pmatrix} \in \mathbb{R}^{N \times n};$$

- $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ is continuous, $f = f(x, u, \xi)$;

- $1 \leq p \leq \infty$ and $W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ denotes the usual space of Sobolev maps where

$$u^i, \frac{\partial u^i}{\partial x_j} \in L^p(\Omega), \ i = 1, \cdots, N, \ j = 1, \cdots, n;$$

- $u_0 \in W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ is a given map;

- $u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$, meaning that $u \in W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ and $u = u_0$ on $\partial\Omega$ in the Sobolev sense.

This problem is the fundamental problem of the calculus of variations and it has received a considerable attention since the time of Fermat, Newton, Bernoulli, Euler and all along the 19th and 20th centuries.

The most general way of proving existence of minimizers of (P), meaning to find $\overline{u} \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$ so that

$$I(\overline{u}) \leq I(u)$$

among all admissible $u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$, is the so called *direct methods of the calculus of variations*. These methods rely on some kind of convexity condition of the function $\xi \to f(x, u, \xi)$. There are numerous examples showing that in absence of convexity the problem (P) has no minimizers. At the moment let us quote three elementary examples where non existence occurs.

Example 1 *Let $N = n = 1$,*

$$f(x, u, \xi) = f(\xi) = e^{-\xi^2}$$

and

$$(P) \qquad \inf \left\{ I(u) = \int_0^1 f(u'(x))\, dx : u \in W_0^{1,1}(0,1) \right\}.$$

Example 2 *Let $N = n = 1$,*

$$f(x, u, \xi) = f(u, \xi) = u^4 + \left(\xi^2 - 1\right)^2$$

and

$$(P) \qquad \inf \left\{ I(u) = \int_0^1 f(u(x), u'(x))\, dx : u \in W_0^{1,4}(0,1) \right\}.$$

This example is due to Bolza.

Example 3 *Let* $n = 2,\ N = 1,\ \Omega = (0,1)^2$,

$$f(x,u,\xi) = f(\xi) = f(\xi_1,\xi_2) = \left(\xi_1^2 - 1\right)^2 + \xi_2^4$$

and

$$(P) \qquad \inf\left\{ I(u) = \int_\Omega f(Du(x))\,dx : u \in W_0^{1,4}(\Omega) \right\}.$$

We now continue this introduction by discussing only the scalar case (i.e. when $N = 1$ or $n = 1$), the general vectorial case will be discussed in the next sections. We moreover, in order to simplify the presentation, consider the case where there is no dependence on lower order terms, i.e. $f(x,u,\xi) = f(\xi)$.

When dealing with non convex problems, the *first step* is the *relaxation theorem*, established by LC. Young, Mac Shane, Ekeland and others. This consists in replacing the problem (P) by the so called relaxed problem

$$(QP) \qquad \inf\left\{ \overline{I}(u) = \int_\Omega Cf(Du(x))\,dx : u \in u_0 + W_0^{1,p}(\Omega) \right\}$$

where Cf is the *convex envelope* of f, namely

$$Cf = \sup\{g \le f : g \text{ convex}\}.$$

Therefore the direct methods, which do not apply to (P), apply to (QP). It can be shown (cf. Theorem 15) that

$$\inf(P) = \inf(QP)$$

and that minimizers of (P) are necessarily minimizers of (QP), the converse being false. In the three above examples we have

(i) $Cf(\xi) \equiv 0$, $\inf(P) = \inf(QP) = 0$ and any $u \in W_0^{1,1}(0,1)$ is a solution of (QP);

(ii) $Cf(u,\xi) = u^4 + \left[\xi^2 - 1\right]_+^2$, $\inf(P) = \inf(QP) = 0$ and $u \equiv 0$ is a solution of (QP);

(iii) $Cf(\xi) = \left[\xi_1^2 - 1\right]_+^2 + \xi_2^4$, $\inf(P) = \inf(QP) = 0$ and $u \equiv 0$ is a solution of (QP);

where, for $x \in \mathbb{R}$,

$$[x]_+ = \begin{cases} x & \text{if } x \ge 0 \\ 0 & \text{if } x < 0. \end{cases}$$

The *second step* in proving the existence of minimizers for (P) is to see if among all solutions of (QP), if any, at least one of them is also a solution of (P). This amounts in finding $\overline{u} \in u_0 + W_0^{1,p}(\Omega)$ so that

$$\int_\Omega Cf(D\overline{u}(x))\,dx = \inf(P) = \inf(QP)$$

and at the same time in solving the first order differential equation (called, following Dacorogna-Marcellini [31], *implicit partial differential equation*)

$$Cf\left(D\overline{u}\left(x\right)\right)=f\left(D\overline{u}\left(x\right)\right),\text{ a.e. }x\in\Omega.$$

After this brief and informal introduction, we discuss the organization of the article.

In Section 2 we discuss all the notions of convexity that are involved in the vector valued case, in particular the so called *quasiconvexity*.

In Section 3 we present the relaxation theorem in the vector valued case, introducing all the needed generalization of the notion of convex envelope.

In Section 4, we give some existence theorems for implicit differential equations of the above type.

In Section 5, using the results of the two preceding sections, we discuss necessary and sufficient conditions for the existence of minimizers for non convex problems.

In Section 6, we show how to apply the abstract results to scalar problems; obtaining sharper theorems in the case of single integrals (i.e. $n=1$).

In Section 7, we present several examples involving vector valued functions (i.e. $n,N\geq 2$) which are relevant for applications.

The subject is very large and we do not intend to be complete and we refer to the bibliography for more details. Let us quote some of the significant contributions to the subject.

The scalar case ($n=1$ or $N=1$) has been intensively studied notably by: Aubert-Tahraoui [4], [5], [6], Bauman-Phillips [10], Buttazzo-Ferone-Kawohl [13], Celada-Perrotta [14], [15], Cellina [16], [17], Cellina-Colombo [18], Cesari [20], [21], Cutri [22], Dacorogna [26], Ekeland [39], Friesecke [40], Fusco-Marcellini-Ornelas [41], Giachetti-Schianchi [43], Klötzler [47], Marcellini [50], [51], [52], Mascolo [54], Mascolo-Schianchi [56], [57], Monteiro Marques-Ornelas [58], Ornelas [64], Raymond [67], [68], [69], Sychev [75], Tahraoui [76], [77], Treu [78] and Zagatti [80].

The vectorial case has been investigated for some special examples notably by Allaire-Francfort [3], Cellina-Zagatti [19], Dacorogna-Ribeiro [35], Dacorogna-Tanteri [37], Mascolo-Schianchi [55], Müller-Sverak [61] and Raymond [70]. A more systematic study was achieved by Dacorogna-Marcellini in [27], [31], [32], as well as in Dacorogna-Pisante-Ribeiro [34].

We have always considered in the present article the two important restrictions:

- f does not depend on lower order terms, i.e. $f\left(x,u,\xi\right)=f\left(\xi\right)$;
- the boundary datum u_0 is affine, i.e. there exists $\xi_0\in\mathbb{R}^{N\times n}$ so that

$$Du_0=\xi_0\,.$$

In the above literature, some authors have considered either of these two more general cases. The results are then much less general and essentially apply only to the scalar case.

2 Preliminaries and notations

2.1 The different notions of convexity

We start with the different definitions of convexity that we will use throughout this article and we refer to Dacorogna [26] for more details.

Definition 4 *(i) A function* $f : \mathbb{R}^{N \times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ *is said to be* rank one convex *if*

$$f(\lambda\xi + (1-\lambda)\eta) \le \lambda f(\xi) + (1-\lambda) f(\eta)$$

for every $\lambda \in [0,1]$, $\xi, \eta \in \mathbb{R}^{N \times n}$ *with* $\operatorname{rank}\{\xi - \eta\} \le 1$.

(ii) A Borel measurable and locally integrable function $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ *is said to be* quasiconvex *if*

$$f(\xi) \le \frac{1}{\operatorname{meas} D} \int_D f(\xi + D\varphi(x))\, dx$$

for every bounded domain $D \subset \mathbb{R}^n$, *for every* $\xi \in \mathbb{R}^{N \times n}$ *and for every* $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^N)$.

(iii) A Borel measurable and locally integrable function $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ *is said to be* quasiaffine *if* f *and* $-f$ *are quasiconvex.*

(iv) A function $f : \mathbb{R}^{N \times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ *is said to be* polyconvex *if there exists* $g : \mathbb{R}^{r(n,N)} \to \overline{\mathbb{R}}$ *convex, such that*

$$f(\xi) = g(T(\xi))$$

where $T : \mathbb{R}^{N \times n} \to \mathbb{R}^{r(n,N)}$ *is such that*

$$T(\xi) = (\xi, \operatorname{adj}_2 \xi, \cdots, \operatorname{adj}_{n \wedge N} \xi).$$

In the preceding definition, $\operatorname{adj}_s \xi$ *stands for the matrix of all* $s \times s$ *minors of the matrix* $\xi \in \mathbb{R}^{N \times n}$, $2 \le s \le n \wedge N = \min\{n, N\}$, *and*

$$\tau(n,N) = \sum_{s=1}^{n \wedge N} \sigma(s)$$

where

$$\sigma(s) = \binom{N}{s}\binom{n}{s} = \frac{N!n!}{(s!)^2 (N-s)!(n-s)!}.$$

Remark 5 *(i) The concepts were introduced by Morrey [59] and [60], but the terminology is the one of Ball [7]; note however that Ball calls quasiaffine functions, null Lagrangians.*

(ii) These notions are related through the following diagram

$$f \text{ convex} \Longrightarrow f \text{ polyconvex} \Longrightarrow f \text{ quasiconvex} \Longrightarrow f \text{ rank one convex.}$$

In the scalar case, $n = 1$ or $N = 1$, these notions are all equivalent and reduce therefore to the usual notion of convexity. However in the vectorial case, $n, N \geq 2$, these concepts are all different, meaning that there are counterexamples to all the above implications. The last counter implication being known, thanks to the celebrated example from Sverak [72], only when $n \geq 2$ and $N \geq 3$; the case $N = 2$, $n \geq 2$ being still open.

(iii) Note that in the case $N = n = 2$, the notion of polyconvexity can be read as follows:

$$\begin{cases} \tau(n, N) = \tau(2,2) = 5 \ \ (\textit{since } \sigma(1) = 4, \sigma(2) = 1) \\ \qquad\qquad T(\xi) = (\xi, \det \xi). \end{cases}$$

(iv) Observe that, if we adopt the tensorial notation, the definition of rank one convexity can be read as follows

$$\varphi(t) = f(\xi + t a \otimes b)$$

is convex in t, for every $\xi \in \mathbb{R}^{N \times n}$ and for every $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$ where we have denoted by

$$a \otimes b = \left(a^i b_\alpha\right)_{1 \leq \alpha \leq n}^{1 \leq i \leq N}.$$

(v) One should also note that in the definition of quasiconvexity if the inequality holds for a given domain $D \subset \mathbb{R}^n$, then it holds for every such domain D.

(vi) If the function $f : \mathbb{R}^{N \times n} \to \mathbb{R}$, i.e. f takes only finite values, is convex or polyconvex or quasiconvex or rank one convex, then it is continuous and even locally Lipschitz.

(vii) It can be shown that a quasiaffine function is necessary of the form

$$f(\xi) = \langle \alpha; T(\xi) \rangle + \beta$$

for some constants $\alpha \in \mathbb{R}^\tau$ and $\beta \in \mathbb{R}$ and where $\langle .; . \rangle$ stands for the scalar product in $\mathbb{R}^\tau$; which in the case $N = n = 2$ reads as ($\alpha = (\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \alpha_5) \in \mathbb{R}^5$)

$$f(\xi) = \sum_{i,j=1}^{2} \alpha_{ij} \xi_{ij} + \alpha_5 \det \xi + \beta.$$

(viii) An equivalent characterization of polyconvexity can be given in terms of the separation theorem (cf. Theorem 1.3 page 107 in Dacorogna [26]). A function $f : \mathbb{R}^{N \times n} \longrightarrow \mathbb{R}$ is polyconvex if and only if for every $\xi \in \mathbb{R}^{N \times n}$ there exists $\lambda = \lambda(\xi) \in \mathbb{R}^{\tau(N,n)}$ so that

$$f(\xi + \eta) - f(\xi) - \langle \lambda; T(\xi + \eta) - T(\xi) \rangle \geq 0, \ \textit{for every } \eta \in \mathbb{R}^{N \times n}. \qquad (1)$$

(ix) When the function f depends on lower order terms as in the introduction, i.e. $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ with $f = f(x, u, \xi)$, all the above notions

are understood only with respect to the variable ξ, all the other variables being kept fixed. For example in the case of quasiconvex functions, one should read

$$f\left(x_0, u_0, \xi\right) \leq \frac{1}{\operatorname{meas} D} \int_D f\left(x_0, u_0, \xi + D\varphi\left(x\right)\right) dx$$

for every bounded domain $D \subset \mathbb{R}^n$, for every $\left(x_0, u_0, \xi\right) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n}$ and for every $\varphi \in W_0^{1,\infty}\left(D; \mathbb{R}^N\right)$.

The important concept from the point of view of minimization in the calculus of variations is the notion of quasiconvexity. This condition is equivalent to the fact that the functional I, defined in the introduction, is *(sequentially) weakly lower semicontinuous* in $W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ meaning that

$$I\left(u\right) \leq \liminf_{\nu \to \infty} I\left(u_\nu\right)$$

for every sequence $u_\nu \rightharpoonup u$ in $W^{1,p}$.

Important examples of quasiconvex functions are the following.

(i) The *quadratic case*. Let M be a symmetric matrix in $\mathbb{R}^{(N \times n) \times (N \times n)}$ and

$$f\left(\xi\right) = \left\langle M\xi; \xi\right\rangle$$

where $\xi \in \mathbb{R}^{N \times n}$ and $\langle .; . \rangle$ denotes the scalar product in $\mathbb{R}^{N \times n}$. Then

$$f \text{ quasiconvex} \Longleftrightarrow f \text{ rank one convex.}$$

(ii) The *Alibert-Dacorogna-Marcellini example* (cf. [2]). Here we have $N = n = 2$ and

$$f\left(\xi\right) = \left|\xi\right|^2 \left(\left|\xi\right|^2 - 2\gamma \det \xi\right),$$

where $|\xi|$ stands for the Euclidean norm of the matrix and $\gamma \geq 0$. Then

$$\begin{aligned} f \text{ is convex } &\Longleftrightarrow \gamma \leq \gamma_c = \frac{2}{3}\sqrt{2} \\ f \text{ is polyconvex } &\Longleftrightarrow \gamma \leq \gamma_p = 1 \\ f \text{ is quasiconvex } &\Longleftrightarrow \gamma \leq \gamma_q\,, \text{ where } \gamma_q > 1 \\ f \text{ is rank one convex } &\Longleftrightarrow \gamma \leq \gamma_r = \frac{2}{\sqrt{3}}\,. \end{aligned}$$

(iii) Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$, $\Phi : \mathbb{R}^{N \times n} \to \mathbb{R}$ be quasiaffine and $g : \mathbb{R} \to \mathbb{R}$ be such that

$$f\left(\xi\right) = g\left(\Phi\left(\xi\right)\right)$$

(in particular if $N = n$, one can take $\Phi\left(\xi\right) = \det \xi$), then

$$f \text{ polyconvex} \Longleftrightarrow f \text{ quasiconvex} \Longleftrightarrow f \text{ rank one convex} \Longleftrightarrow g \text{ convex.}$$

(iv) Let $N = n + 1$ and for $\xi \in \mathbb{R}^{(n+1) \times n}$, denote

$$\operatorname{adj}_n \xi = \left(\det \widehat{\xi}^1, -\det \widehat{\xi}^2, \cdots, (-1)^{k+1} \det \widehat{\xi}^k, \cdots, (-1)^{n+2} \det \widehat{\xi}^{n+1}\right)$$

where $\widehat{\xi}^k$ is the $n \times n$ matrix obtained from ξ by suppressing the $k\,th$ line (when $\xi = Du$, $\mathrm{adj}_n Du$ represents, geometrically, the normal to the hypersurface). Let $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that

$$f(\xi) = g(\mathrm{adj}_n \xi)$$

then

$$f \text{ polyconvex} \Longleftrightarrow f \text{ quasiconvex} \Longleftrightarrow f \text{ rank one convex} \Longleftrightarrow g \text{ convex.}$$

(v) Let $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ denote the *singular values* of a matrix $\xi \in \mathbb{R}^{n \times n}$, which are defined as the eigenvalues of the matrix $(\xi\xi^t)^{1/2}$. The functions

$$\xi \to \sum_{i=\nu}^{n} \lambda_i(\xi) \quad \text{and} \quad \xi \to \prod_{i=\nu}^{n} \lambda_i(\xi),\ \nu = 1, \cdots, n,$$

are respectively convex and polyconvex (note that $\prod_{i=1}^{n} \lambda_i(\xi) = |\det \xi|$). In particular the function $\xi \to \lambda_n(\xi)$ is convex and in fact is the operator norm.

2.2 Some function spaces

The following notations will be used throughout.

- For $1 \leq p \leq \infty$, we will let $W^{1,p}(\Omega; \mathbb{R}^N)$ be the space of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ such that

$$u \in L^p(\Omega, \mathbb{R}^N) \text{ and } Du = \left(\frac{\partial u^i}{\partial x_j}\right)_{1 \leq j \leq n}^{1 \leq i \leq N} \in L^p(\Omega, \mathbb{R}^{N \times n}).$$

- For $1 \leq p < \infty$, $W_0^{1,p}(\Omega; \mathbb{R}^N)$ will denote the closure of $C_0^\infty(\Omega; \mathbb{R}^N)$ with respect to the $\|.\|_{W^{1,p}}$ norm.

- $W_0^{1,\infty}(\Omega; \mathbb{R}^N) = W^{1,\infty}(\Omega; \mathbb{R}^N) \cap W_0^{1,1}(\Omega; \mathbb{R}^N)$.

- $Aff_{piec}(\overline{\Omega}; \mathbb{R}^N)$ will stand for the subset of $W^{1,\infty}(\Omega; \mathbb{R}^N)$ consisting of piecewise affine maps.

- $C^1_{piec}(\overline{\Omega}; \mathbb{R}^N)$ will denote the subset of $W^{1,\infty}(\Omega; \mathbb{R}^N)$ consisting of piecewise C^1 maps.

2.3 Statement of the problem

We will be concerned with existence of minimizers for the problem

$$(P) \qquad \inf\left\{\int_\Omega f(Du(x))\,dx : u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)\right\}$$

where:

- $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary,

- $u : \Omega \to \mathbb{R}^N$ and thus $Du \in \mathbb{R}^{N\times n}$,
- $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is lower semicontinuous, locally bounded and non negative,
- $\xi_0 \in \mathbb{R}^{N\times n}$ and u_{ξ_0} is an affine map such that $Du_{\xi_0} = \xi_0$.

The hypothesis $f \geq 0$ can be replaced, with no changes, by

$$f(\xi) \geq \langle \alpha; T(\xi) \rangle + \beta, \text{ for every } \xi \in \mathbb{R}^{N\times n}$$

for some constants $\alpha \in \mathbb{R}^\tau$ and $\beta \in \mathbb{R}$ and where $\langle .; . \rangle$ stands for the scalar product in $\mathbb{R}^\tau$. This hypothesis is made to avoid to have to deal with quasiconvex envelopes $Qf \equiv -\infty$.

3 Relaxation Theorems

We now present the relaxation theorem, which corresponds to the first step described in the introduction. But before that we need to introduce the notions of envelopes corresponding to the different concepts of convexity that we introduced in the previous section. The reference book for this part is still Dacorogna [26].

3.1 The different envelopes

We now define

$$\begin{aligned} Cf &= \sup \{ g \leq f : g \text{ convex} \}, \\ Pf &= \sup \{ g \leq f : g \text{ polyconvex} \}, \\ Qf &= \sup \{ g \leq f : g \text{ quasiconvex} \}, \\ Rf &= \sup \{ g \leq f : g \text{ rank one convex} \}, \end{aligned}$$

they are respectively the *convex, polyconvex, quasiconvex, rank one convex envelope* of f. In view of the results of the previous section, we have

$$Cf \leq Pf \leq Qf \leq Rf \leq f.$$

As already said, we will always assume, in the sequel, that $f \geq 0$. We then have the following characterizations of the different envelopes.

Theorem 6 *Let* $f : \mathbb{R}^{N\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

Part 1. *Let for any integer* s

$$\Lambda_s = \left\{ \lambda = (\lambda_1, \cdots, \lambda_s) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^{s} \lambda_i = 1 \right\},$$

then

$$Cf(\xi) = \inf \left\{ \sum_{i=1}^{Nn+1} t_i f(\xi_i) : \xi = \sum_{i=1}^{Nn+1} t_i \xi_i, \ t \in \Lambda_{Nn+1} \right\}$$

$$Pf(\xi) = \inf \left\{ \sum_{i=1}^{\tau+1} t_i f(\xi_i) : T(\xi) = \sum_{i=1}^{\tau+1} t_i T(\xi_i), \ t \in \Lambda_{\tau+1} \right\}$$

Part 2. *Let $R_0 f = f$ and define inductively for i an integer*

$$R_{i+1}f(\xi) = \inf \left\{ \begin{array}{c} tR_i f(\xi_1) + (1-t)R_i f(\xi_2) : \ t \in [0,1], \\ \\ \xi = t\xi_1 + (1-t)\xi_2, \ \operatorname{rank}\{\xi_1 - \xi_2\} = 1 \end{array} \right\}$$

then

$$Rf(\xi) = \inf_{i \in \mathbb{N}} R_i f(\xi).$$

Theorem 7 (Dacorogna formula) *If $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ is locally bounded and Borel measurable then*

$$Qf(\xi) = \inf \left\{ \frac{1}{\operatorname{meas}\Omega} \int_\Omega f(\xi + D\varphi(x))\, dx : \varphi \in W_0^{1,\infty}\left(\Omega; \mathbb{R}^N\right) \right\}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. In particular the infimum is independent of the choice of the domain.

Remark 8 *(i) The representation formula for Cf is standard and follows from Carathéodory theorem. The inductive way of representing Rf was found by Kohn-Strang [48]. The formulas for Pf and Qf (and a similar to that of Kohn-Strang for Rf) were established by Dacorogna (cf. [26]).*

(ii) Using the separation theorems one can establish other formulas for Cf and Pf, cf. [26].

3.2 Some examples

We now discuss some examples that will be used in Section 7. We start with the following theorem established by Dacorogna [26] (cf. also [23] and [24]).

Theorem 9 Part 1. *Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$, $\Phi : \mathbb{R}^{N \times n} \to \mathbb{R}$ be quasiaffine and $g : \mathbb{R} \to \mathbb{R}$ be such that*

$$f(\xi) = g(\Phi(\xi))$$

(in particular if $N = n$, one can take $\Phi(\xi) = \det \xi$), then

$$Pf(\xi) = Qf(\xi) = Rf(\xi) = Cg(\Phi(\xi)).$$

Part 2. *Let $N = n+1$, $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that*

$$f(\xi) = g(\operatorname{adj}_n \xi)$$

then

$$Pf(\xi) = Qf(\xi) = Rf(\xi) = Cg(\operatorname{adj}_n \xi).$$

The next result, established by Dacorogna-Pisante-Ribeiro [34], concerns functions depending on singular values. We let $N = n$ and we denote by $\lambda_1(\xi), \cdots, \lambda_n(\xi)$ the singular values of $\xi \in \mathbb{R}^{n \times n}$ with $0 \le \lambda_1(\xi) \le \cdots \le \lambda_n(\xi)$ (which are the eigenvalues of the matrix $\left(\xi\xi^T\right)^{1/2}$) and by Q the set

$$Q = \{x = (x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-2} : \ 0 \le x_2 \le \cdots \le x_{n-1}\}$$

which is the natural set where to consider $(\lambda_2(\xi), \cdots, \lambda_{n-1}(\xi))$ for $\xi \in \mathbb{R}^{n \times n}$.

Theorem 10 *Let $g : Q \times \mathbb{R} \longrightarrow \mathbb{R}$, $g = g(x, s)$, be a function such that $x \to g(x,s)$ is continuous and bounded from below for all $s \in \mathbb{R}$. Let $f : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ be defined by*

$$f(\xi) = g(\lambda_2(\xi), \cdots, \lambda_{n-1}(\xi), \det \xi)$$

then

$$Pf(\xi) = Qf(\xi) = Rf(\xi) = Ch(\det \xi),$$

where $h : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $h(s) = \inf_{x\in Q} g(x, s)$.

Remark 11 *We remark that if some dependence on λ_1 or λ_n is allowed, then no simple and general expression for the envelopes is known; see Dacorogna-Pisante-Ribeiro [34], when there is dependence on λ_1, and Theorem 3.5 in Buttazzo-Dacorogna-Gangbo [12], when there is dependence on λ_n.*

The next result concerns the Saint Venant Kirchhoff energy function, which is particularly important in non linear elasticity. The function, up to rescaling, is given by, $\nu \in (0, 1/2)$ being a parameter,

$$f(\xi) = \left|\xi\xi^t - I\right|^2 + \frac{\nu}{1-2\nu}\left(|\xi|^2 - n\right)^2$$

or in terms of the singular values $0 \le \lambda_1(\xi) \le \cdots \le \lambda_n(\xi)$ of $\xi \in \mathbb{R}^{n\times n}$

$$f(\xi) = \sum_{i=1}^{n}\left(\lambda_i^2 - 1\right)^2 + \frac{\nu}{1-2\nu}\left(\sum_{i=1}^{n}\lambda_i^2 - n\right)^2.$$

Le Dret-Raoult [49] have computed the quasiconvex envelope when $n = 2$ or $n = 3$ and they have shown the following.

Theorem 12 *If $n = 2$ or $n = 3$, then*

$$Qf(\xi) = Cf(\xi).$$

When $n = 2$ it is given by

$$Cf(\xi) = Pf(\xi) = Qf(\xi) = Rf(\xi) = \begin{cases} f(\xi) & \text{if } \xi \notin D_1 \cup D_2 \\ \frac{1}{1-\nu}\left(\lambda_2^2 - 1\right)^2 & \text{if } \xi \in D_2 \\ 0 & \text{if } \xi \in D_1 \end{cases}$$

where

$$\begin{aligned} D_1 &= \left\{\xi \in \mathbb{R}^{2\times 2} : (1-\nu)\left[\lambda_1(\xi)\right]^2 + \nu\left[\lambda_2(\xi)\right]^2 < 1 \text{ and } \lambda_2(\xi) < 1\right\} \\ &= \left\{\xi \in \mathbb{R}^{2\times 2} : \lambda_1(\xi) \le \lambda_2(\xi) < 1\right\} \\ D_2 &= \left\{\xi \in \mathbb{R}^{2\times 2} : (1-\nu)\left[\lambda_1(\xi)\right]^2 + \nu\left[\lambda_2(\xi)\right]^2 < 1 \text{ and } \lambda_2(\xi) \ge 1\right\}. \end{aligned}$$

The last example is related to a problem of optimal design and has been studied by Kohn-Strang [48].

Theorem 13 *Let $n = N = 2$ and*

$$f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases}$$

Then $Pf = Qf = Rf$ and

$$Qf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\xi|^2 + 2|\det \xi| \geq 1 \\ 2\left(|\xi|^2 + 2|\det \xi|\right)^{1/2} - 2|\det \xi| & \text{if } |\xi|^2 + 2|\det \xi| < 1. \end{cases}$$

Remark 14 *The above result is still valid when $N \geq 3$, it suffices to replace* $\det \xi$ *by* $\operatorname{adj}_2 \xi \in \mathbb{R}^{\binom{N}{2}}$.

3.3 The main theorem

We now turn our attention to the relaxation theorem. We recall our minimization problem

$$(P) \qquad \inf\left\{ I(u) = \int_\Omega f(Du(x))\, dx : u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right) \right\}$$

where $1 \leq p \leq \infty$.

We define the *relaxed problem* associated to (P) to be

$$(QP) \qquad \inf\left\{ \overline{I}(u) = \int_\Omega Qf(Du(x))\, dx : u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right) \right\}.$$

Theorem 15 (Relaxation theorem) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be Borel measurable and non negative satisfying, for $1 \leq p < \infty$,*

$$0 \leq f(\xi) \leq \alpha_1 (1 + |\xi|^p), \text{ for every } \xi \in \mathbb{R}^{N\times n} \tag{2}$$

where $\alpha_1 > 0$ is a constant and for $p = \infty$ it is assumed that f is locally bounded.

Let

$$Qf = \sup\{g \leq f : g \text{ quasiconvex}\}$$

be the quasiconvex envelope of f. Then

$$\inf(P) = \inf(QP).$$

More precisely for every $u \in W^{1,p}\left(\Omega; \mathbb{R}^N\right)$, there exists a sequence $\{u^\nu\}_{\nu=1}^\infty \subset u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$ such that

$$\int_\Omega f(Du^\nu(x))\, dx \to \int_\Omega Qf(Du(x))\, dx, \text{ as } \nu \to \infty.$$

Remark 16 *(i) If we add in the theorem a coercivity condition*

$$\alpha_2 \left(-1+|\xi|^p\right) \leq f\left(\xi\right) \leq \alpha_1 \left(1+|\xi|^p\right)$$

where $\alpha_2 > 0$ and $p > 1$, we can infer that (QP) has a minimizer and that the sequence $\{u^\nu\}_{\nu=1}^{\infty}$ further satisfies

$$u^\nu \rightharpoonup u \text{ in } W^{1,p}\left(\Omega; \mathbb{R}^N\right) \text{ as } \nu \to \infty.$$

(ii) The theorem remains also valid if the function f depends on lower order terms, i.e. $f = f\left(x, u, \xi\right)$. The quasiconvex envelope is then to be understood as the quasiconvex envelope only with respect to the variable ξ, the other variables (x, u) being kept fixed.

Proof. We divide the proof into two steps.

Step 1. We start with an approximation of the given function u. Let $\epsilon > 0$ be arbitrary, we can then find disjoint open sets $\Omega_1, \cdots, \Omega_k \subset \Omega$, $\xi_1, \cdots, \xi_k \in \mathbb{R}^{N\times n}$, γ independent of ϵ and $v \in u + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)$ such that

$$\begin{cases} \text{meas}\left[\Omega - \cup_{i=1}^{k}\Omega_i\right] \leq \epsilon \\ \|u\|_{W^{1,p}}\ , \|v\|_{W^{1,p}} \leq \gamma,\ \|u-v\|_{W^{1,1}} \leq \epsilon \\ Dv\left(x\right) = \xi_i\,, \quad \text{if } x \in \Omega_i\,. \end{cases} \tag{3}$$

By taking ϵ smaller if necessary we can also assume, using the continuity of Qf and the growth condition on f, that

$$\int_\Omega \left|Qf\left(Du\left(x\right)\right) - Qf\left(Dv\left(x\right)\right)\right| dx \leq \epsilon \tag{4}$$

$$0 \leq \int_{\Omega - \cup_{i=1}^{k}\Omega_i} \left[f\left(Dv\left(x\right)\right) - Qf\left(Dv\left(x\right)\right)\right] dx \leq \epsilon. \tag{5}$$

Indeed let us discuss the case $1 \leq p < \infty$, the case $p = \infty$ being easy. As well known (cf. Lemma 2.2 page 156 in [26]) any quasiconvex function is locally Lipschitz continuous and if it satisfies (2), then there exists $\beta > 0$ such that

$$\left|Qf\left(Du\right) - Qf\left(Dv\right)\right| \leq \beta\left(1 + |Du|^{p-1} + |Dv|^{p-1}\right)\left|Du - Dv\right|.$$

Using Hölder inequality we obtain

$$\begin{aligned} &\int_\Omega \left|Qf\left(Du\right) - Qf\left(Dv\right)\right| dx \\ &\leq \beta\left[\int_\Omega \left[\left(1 + |Du|^{p-1} + |Dv|^{p-1}\right)\right]^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}} \left[\int_\Omega |Du - Dv|^p\right]^{\frac{1}{p}} \end{aligned}$$

and (4) follows therefore from (3). The inequality (5) follows from (3) and a classical property of the integrals (cf. Lemma 1.4 page 19 in [26]).

Step 2. Now use Theorem 7 on every Ω_i to find $\varphi_i \in W_0^{1,\infty}\left(\Omega_i;\mathbb{R}^N\right)$

$$\frac{1}{\operatorname{meas}\Omega_i}\int_{\Omega_i} f\left(\xi_i + D\varphi_i\left(x\right)\right)dx \geq Qf\left(\xi_i\right) \geq -\epsilon + \frac{1}{\operatorname{meas}\Omega_i}\int_{\Omega_i} f\left(\xi_i + D\varphi_i\left(x\right)\right)dx.$$

Setting

$$w\left(x\right) = \begin{cases} v\left(x\right) + \varphi_i\left(x\right) & \text{if } x \in \Omega_i\,,\ i = 1, \cdots, k \\ v\left(x\right) & \text{if } x \in \Omega - \cup_{i=1}^k \Omega_i \end{cases}$$

we get that $w \in u + W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)$ and (using (5))

$$0 \leq \int_{\cup_{i=1}^k \Omega_i} \left[f\left(Dw\left(x\right)\right) - Qf\left(Dv\left(x\right)\right)\right]dx \leq \epsilon \operatorname{meas}\left[\cup_{i=1}^k \Omega_i\right]$$

$$\begin{aligned} 0 &\leq \int_{\Omega - \cup_{i=1}^k \Omega_i} \left[f\left(Dw\left(x\right)\right) - Qf\left(Dv\left(x\right)\right)\right]dx \\ &= \int_{\Omega - \cup_{i=1}^k \Omega_i} \left[f\left(Dv\left(x\right)\right) - Qf\left(Dv\left(x\right)\right)\right]dx \leq \epsilon. \end{aligned}$$

In other words, combining these inequalities, we have proved that

$$0 \leq \int_\Omega \left[f\left(Dw\left(x\right)\right) - Qf\left(Dv\left(x\right)\right)\right]dx \leq \epsilon\left(1 + \operatorname{meas}\Omega\right).$$

Invoking (4), we find

$$\left|\int_\Omega \left[f\left(Dw\left(x\right)\right) - Qf\left(Du\left(x\right)\right)\right]dx\right| \leq \epsilon\left(2 + \operatorname{meas}\Omega\right).$$

Setting $\epsilon = 1/\nu$ with $\nu \in \mathbb{N}$ and $u^\nu = w$, we have indeed obtained the theorem.

■

We now discuss the history of this theorem (for precise references see [26]).

In the case $N = n = 1$, this result has been proved by L.C. Young and then generalized by others to the scalar case, $N = 1$ or $n = 1$, notably by Berliochi-Lasry, Ekeland, Ioffe-Tihomirov and Marcellini-Sbordone. Note that in this context

$$Qf = Cf = f^{**}$$

where Cf is the usual convex envelope of f. The problem (QP) can then be rewritten as

$$(P^{**}) \qquad \inf\left\{I^{**}\left(u\right) = \int_\Omega f^{**}\left(Du\left(x\right)\right)dx : u \in u_0 + W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)\right\}.$$

The result for the vectorial case (i.e. $N, n > 1$, recall also that, in general, we now have $Qf > Cf$) was established by Dacorogna in [25]. Following a different approach it was later also proved by Acerbi-Fusco [1].

In the present context the equivalence between (QP) and (P^{**}) is not any more valid, one has in general

$$\inf(P) = \inf(QP) > \inf(P^{**}).$$

The inequality is, in general, strict as in the simple example where $N = n \geq 2$ and $f(\xi) = (\det \xi)^2$. We indeed have

$$f(\xi) = Qf(\xi) = (\det \xi)^2 \quad \text{and} \quad f^{**}(\xi) \equiv 0.$$

Therefore if $\det Du_0 > 0$, then, using Jensen inequality, we have

$$\begin{aligned}\inf(P) &= \inf(QP) \\ &\geq \operatorname{meas}\Omega \left(\frac{1}{\operatorname{meas}\Omega} \int_\Omega \det Du_0(x)\, dx \right)^2 \\ &> 0 = \inf(P^{**}).\end{aligned}$$

Closely related to this approach is the notion of parametrized or Young measure, that we do not discuss here.

4 Implicit partial differential equations

4.1 Introduction

We now discuss the existence of solutions, $u \in W^{1,\infty}(\Omega; \mathbb{R}^N)$, for the Dirichlet problem involving differential inclusions of the form

$$\begin{cases} Du(x) \in E & \text{a.e. in } \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega \end{cases}$$

where φ is a given function and $E \subset \mathbb{R}^{N \times n}$ is a given compact set.

To relate this study with what we said in Section 1, one should imagine that

$$E = \left\{ \xi \in \mathbb{R}^{N \times n} : Qf(\xi) = f(\xi) \right\}$$

and therefore the differential inclusion is equivalent to the implicit partial differential equation

$$Qf(Du(x)) = f(Du(x)), \text{ a.e. } x \in \Omega.$$

In the scalar case ($n = 1$ or $N = 1$) a sufficient condition for solving the problem is

$$D\varphi(x) \in E \cup \operatorname{int}\operatorname{co} E, \text{ a.e. in } \Omega$$

where $\operatorname{int}\operatorname{co} E$ stands for the interior of the convex hull of E. This fact was observed by several authors, with different proofs and different levels of generality;

notably in [11], [17], [28], [29], [31], [38] or [40]. It should be noted that this sufficient condition is also necessary, when properly reformulated.

When turning to the vectorial case ($n, N \geq 2$) the problem becomes considerably harder and no result with such a degree of elegancy and generality is available. The first general results were obtained by Dacorogna and Marcellini (see the bibliography, in particular [31]). At the same time Müller and Sverak [61] introduced the method of convex integration of Gromov in this framework, obtaining also similar existence results.

4.2 The different convex hulls

We recall the main notations that we will use throughout the present section and we refer, if necessary, for more details to Dacorogna-Marcellini [31].

Classically the *convex hull* of a given set E is the smallest convex set that contains E and it is denoted by $\operatorname{co} E$. We will now do the same with the other notions of convexity that we have seen earlier. This is not as straightforward as it may seem and there is not a general agreement on the exact definitions. We will not enter in abstract considerations and we will use as definition of the different hulls a consequence of these abstract definitions.

Notation 17 *We let, for $E \subset \mathbb{R}^{N \times n}$,*

$$\overline{\mathcal{F}}_E = \left\{ f : \mathbb{R}^{N \times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} : \, f|_E \leq 0 \right\}$$
$$\mathcal{F}_E = \left\{ f : \mathbb{R}^{N \times n} \to \mathbb{R} : \, f|_E \leq 0 \right\}.$$

We then have respectively, the convex, polyconvex, rank one convex *and (closure of the)* quasiconvex hull *defined by*

$$\begin{aligned}
\operatorname{co} E &= \left\{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \text{ for every convex } f \in \overline{\mathcal{F}}_E \right\} \\
\operatorname{Pco} E &= \left\{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \text{ for every polyconvex } f \in \overline{\mathcal{F}}_E \right\} \\
\operatorname{Rco} E &= \left\{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \text{ for every rank one convex } f \in \overline{\mathcal{F}}_E \right\} \\
\overline{\operatorname{Qco}} E &= \left\{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \text{ for every quasiconvex } f \in \mathcal{F}_E \right\}.
\end{aligned}$$

We should point out that by replacing $\overline{\mathcal{F}}_E$ by $\mathcal{F}_E$ in the definitions of $\operatorname{co} E$ and $\operatorname{Pco} E$ we get their closures denoted by $\overline{\operatorname{co}} E$ and $\overline{\operatorname{Pco}} E$. However if we do so in the definition of $\operatorname{Rco} E$ we get a larger set than the closure of $\operatorname{Rco} E$. We should also draw the attention that some authors call the set

$$\left\{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \text{ for every rank one convex } f \in \overline{\mathcal{F}}_E \right\}$$

the lamination convex hull, while they reserve the name of rank one convex hull to the set

$$\left\{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \text{ for every rank one convex } f \in \mathcal{F}_E \right\}.$$

We think however that our terminology is more consistent with the classical definition of convex hull.

In general we have, for any set $E \subset \mathbb{R}^{N\times n}$,

$$E \subset \operatorname{Rco} E \subset \operatorname{Pco} E \subset \operatorname{co} E$$

$$\overline{E} \subset \overline{\operatorname{Rco}} E \subset \overline{\operatorname{Qco}} E \subset \overline{\operatorname{Pco}} E \subset \overline{\operatorname{co}} E.$$

4.3 Some examples of convex hulls

We now give several examples that will be used in the applications of Sections 6 and 7. Let us start with the scalar case.

Example 18 (Convex Hamiltonian) *Let $F : \mathbb{R}^n \to \mathbb{R}$ be convex and let*

$$E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}$$

then

$$\operatorname{co} E = \{\xi \in \mathbb{R}^n : F(\xi) \leq 0\}.$$

Example 19 (Non convex Hamiltonian) *Consider, for $\xi \in \mathbb{R}^n$, the non convex Hamiltonian*

$$F(\xi) = \sum_{i=1}^{n} \left[(\xi_i)^2 - 1\right]^2$$

and

$$E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}$$

then

$$\operatorname{co} E = [-1, 1]^n.$$

We now turn to some examples in the vectorial case. The following result is due to Dacorogna-Tanteri (cf. [36] and also [31]), it concerns singular values. We recall that we denote by $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ the singular values of a matrix $\xi \in \mathbb{R}^{n\times n}$, which are defined as the eigenvalues of the matrix $(\xi\xi^t)^{1/2}$.

Theorem 20 *Let*

$$E = \left\{\xi \in \mathbb{R}^{n\times n} : \lambda_i(\xi) = a_i,\ i = 1, ..., n\right\}$$

where $0 < a_1 \leq ... \leq a_n$. The following then hold

$$\operatorname{co} E = \left\{\xi \in \mathbb{R}^{n\times n} : \sum_{i=\nu}^{n} \lambda_i(\xi) \leq \sum_{i=\nu}^{n} a_i,\ \nu = 1, ..., n\right\}$$

$$\operatorname{Pco} E = \overline{\operatorname{Qco}} E = \operatorname{Rco} E = \left\{\xi \in \mathbb{R}^{n\times n} : \prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} a_i,\ \nu = 1, ..., n\right\}$$

$$\operatorname{int} \operatorname{Rco} E = \left\{\xi \in \mathbb{R}^{n\times n} : \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} a_i,\ \nu = 1, ..., n\right\}.$$

where $\operatorname{int} \operatorname{Rco} E$ *stands for the interior of the rank one convex hull of E.*

This result admits some extensions (it corresponds in the theorem below to $\alpha = -\beta$), cf. Dacorogna-Tanteri [37] and Dacorogna-Ribeiro [35].

Theorem 21 *Let $\alpha \leq \beta$, $0 < a_2 \leq ... \leq a_n$ be constants so that*

$$a_2 \prod_{i=2}^{n} a_i \geq \max \{|\alpha|, |\beta|\}.$$

Let

$$E = \left\{\xi \in \mathbb{R}^{n\times n} : \ \det \xi \in \{\alpha, \beta\}, \ \lambda_i(\xi) = a_i, \ i = 2, ..., n\right\}$$

then

$$\begin{aligned} \operatorname{Pco} E &= \overline{\operatorname{Qco}} E = \operatorname{Rco} E \\ &= \left\{\xi \in \mathbb{R}^{n\times n} : \ \det \xi \in [\alpha, \beta], \ \prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} a_i, \ \nu = 2, ..., n\right\}. \end{aligned}$$

In particular if $\alpha = \beta$

$$\begin{aligned} \operatorname{Pco} E &= \overline{\operatorname{Qco}} E = \operatorname{Rco} E \\ &= \left\{\xi \in \mathbb{R}^{n\times n} : \ \det \xi = \alpha, \ \prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} a_i, \ \nu = 2, ..., n\right\}. \end{aligned}$$

Remark 22 *It is interesting to note some formal analogy between the above result (with $\alpha = \beta$) and some classical theorems of H. Weyl, A. Horn and C.J. Thompson (see [44], [45] page 171 or [53]). Their result states that if we denote, as above, the singular values of a given matrix $\xi \in \mathbb{R}^{n\times n}$ by $0 \leq \lambda_1(\xi) \leq ... \leq \lambda_n(\xi)$ and its eigenvalues, which are complex in general, by $\mu_1(\xi), ..., \mu_n(\xi)$ and if we order them by their modulus ($0 \leq |\mu_1(\xi)| \leq ... \leq |\mu_n(\xi)|$) then the following result holds*

$$\begin{aligned} &\prod_{i=\nu}^{n} |\mu_i(\xi)| \leq \prod_{i=\nu}^{n} \lambda_i(\xi), \ \nu = 2, ..., n \\ &\prod_{i=1}^{n} |\mu_i(\xi)| = \prod_{i=1}^{n} \lambda_i(\xi) \end{aligned}$$

for any matrix $\xi \in \mathbb{R}^{n\times n}$.

We will see several other examples in the next sections.

4.4 An existence theorem

We start with the following definition introduced by Dacorogna-Marcellini in [30] (cf. also [31]), which is the key condition to get existence of solutions.

Definition 23 (Relaxation property) *Let $E, K \subset \mathbb{R}^{N\times n}$. We say that K has the* relaxation property *with respect to E if for every bounded open set $\Omega \subset \mathbb{R}^n$, for every affine function u_ξ satisfying*

$$Du_\xi (x) = \xi \in K,$$

there exists a sequence $u_\nu \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^N\right)$

$$u_\nu \in u_\xi + W_0^{1,\infty}\left(\Omega; \mathbb{R}^N\right),\ Du_\nu (x) \in E \cup K,\ a.e.\ in\ \Omega$$

$$u_\nu \overset{*}{\rightharpoonup} u_\xi\ in\ W^{1,\infty},\ \int_\Omega \operatorname{dist}(Du_\nu (x); E)\, dx \to 0\ as\ \nu \to \infty.$$

Remark 24 *(i) It is interesting to note that in the scalar case ($n = 1$ or $N = 1$) then $K = \operatorname{int}\operatorname{co} E$ has the relaxation property with respect to E.*

(ii) In the vectorial case we have that, if K has the relaxation property with respect to E, then necessarily

$$K \subset \overline{\mathrm{Qco}} E.$$

Indeed first recall that the definition of quasiconvexity implies that, for every quasiconvex $f \in \mathcal{F}_E$,

$$f(\xi) \operatorname{meas} \Omega \le \int_\Omega f(Du_\nu (x))\, dx.$$

Combining this last result with the fact that $\{Du_\nu\}$ is uniformly bounded, the fact that any quasiconvex function is continuous and the last property in the definition of the relaxation property, we get the inclusion $K \subset \overline{\mathrm{Qco}} E$.

The main theorem is then.

Theorem 25 *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E, K \subset \mathbb{R}^{N\times n}$ be such that E is compact and K is bounded. Assume that K has the relaxation property with respect to E. Let $\varphi \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^N\right)$ be such that*

$$D\varphi (x) \in E \cup K,\ a.e.\ in\ \Omega.$$

Then there exists (a dense set of) $u \in \varphi + W_0^{1,\infty}\left(\Omega; \mathbb{R}^N\right)$ such that

$$Du(x) \in E,\ a.e.\ in\ \Omega.$$

Remark 26 *(i) According to Chapter 10 in [31], the boundary datum φ can be more general if we make the following extra hypotheses:*

- in the scalar case, if K is open, φ can be even taken in $W^{1,\infty}\left(\Omega; \mathbb{R}^N\right)$, with $D\varphi (x) \in E \cup K$ (cf. Corollary 10.11 in [31]);

- in the vectorial case, if the set K is open, φ can be taken in $C^1_{piec}\left(\overline{\Omega}; \mathbb{R}^N\right)$ (cf. Corollary 10.15 or Theorem 10.16 in [31]), with $D\varphi (x) \in E \cup K$. While if K is open and convex, φ can be taken in $W^{1,\infty}\left(\Omega; \mathbb{R}^N\right)$ provided

$$D\varphi (x) \in C,\ a.e.\ in\ \Omega$$

where $C \subset K$ is compact (cf. Corollary 10.21 in [31]).

(ii) In the scalar case (cf. Theorem 29) the hypothesis on the compactness of E can be dropped.

(iii) This theorem was first proved by Dacorogna-Marcellini in [30] (cf. also Theorem 6.3 in [31]) under the further hypothesis that

$$E = \left\{ \xi \in \mathbb{R}^{N\times n} : F_i\left(\xi\right) = 0,\ i = 1, 2, ..., I \right\}$$

where $F_i : \mathbb{R}^{N\times n} \to \mathbb{R}, i = 1, 2, ..., I$, are quasiconvex. This hypothesis was later removed by Sychev in [74] using the theory of convex integration (see also Müller and Sychev [63]). Kirchheim in [46] pointed out that using a classical result (Theorem 38) then the proof of Dacorogna-Marcellini was still valid without the extra hypothesis on E. Kirchheim's idea, combined with the proof of [31], was then used by Dacorogna-Pisante [33] and we will follow this last approach.

Proof. We let $\overline{V}$ be the closure in $L^\infty\left(\Omega; \mathbb{R}^N\right)$ of

$$V = \left\{ u \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^N\right) : u = \varphi \text{ on } \partial\Omega \text{ and } Du\left(x\right) \in E \cup K \right\}.$$

V is non empty since $\varphi \in V$. Let, for $k \in \mathbb{N}$,

$$V^k = \operatorname{int}\left\{ u \in \overline{V} : \int_\Omega \operatorname{dist}\left(Du\left(x\right); E\right) dx \leq \frac{1}{k} \right\}$$

where int stands for the interior of the set. We claim that V^k, in addition to be open, is dense in the complete metric space $\overline{V}$. Postponing the proof of the last fact for the end of the proof, we conclude by Baire category theorem that

$$\bigcap_{k=1}^{\infty} V^k \subset \left\{ u \in \overline{V} : \operatorname{dist}\left(Du\left(x\right), E\right) = 0, \text{ a.e. in } \Omega \right\} \subset \overline{V}$$

is dense, and hence non empty, in $\overline{V}$. The result then follows, since E is compact.

We now show that V^k is dense in $\overline{V}$. So let $u \in \overline{V}$ and $\epsilon > 0$ be arbitrary. We wish to find $v \in V^k$ so that

$$\|u - v\|_{L^\infty} \leq \epsilon.$$

We recall (cf. Appendix 4.6) that

$$\omega_D(\alpha) = \lim_{\delta \to 0} \sup_{v,w \in B_\infty(\alpha,\delta)} \|Dv - Dw\|_{L^1(\Omega)}$$

where

$$B_\infty(\alpha, \delta) = \left\{ u \in \overline{V} : \|u - \alpha\|_{L^\infty} < \delta \right\}.$$

- We start by finding $\alpha \in \overline{V}$ a point of continuity of the operator D so that

$$\|u - \alpha\|_{L^\infty} \leq \frac{\epsilon}{3}.$$

This is always possible by virtue of Corollary 40. In particular we have that the oscillation $\omega_D(\alpha)$ of the gradient operator at α is zero.

- We next approximate $\alpha \in \overline{V}$ by $\beta \in V$ so that

$$\|\beta - \alpha\|_{L^\infty} \leq \frac{\epsilon}{3} \text{ and } \omega_D(\beta) < \frac{1}{2k}.$$

This is possible since by Proposition 37 we know that for every $\varepsilon > 0$ the set

$$\Omega_D^\varepsilon := \{u \in \overline{V} \ : \ \omega_D(u) < \varepsilon\}$$

is open in $\overline{V}$.

- Finally we use the relaxation property on every piece where $D\beta$ is constant and we then construct $v \in V$, by patching all the pieces together, such that

$$\|\beta - v\|_{L^\infty} \leq \frac{\epsilon}{3}\ ,\ \omega_D(v) < \frac{1}{2k} \text{ and } \int_\Omega \operatorname{dist}\left(Dv\left(x\right); E\right) dx < \frac{1}{k}.$$

Moreover since $\omega_D(v) < \frac{1}{2k}$ we can find $\delta = \delta(k, v) > 0$ so that

$$\|v - \psi\|_{L^\infty} \leq \delta \Rightarrow \|Dv - D\psi\|_{L^1} \leq \frac{1}{2k}$$

and hence

$$\int_\Omega \operatorname{dist}(D\psi(x); E)\, dx \leq \int_\Omega \operatorname{dist}(Dv(x); E)\, dx + \|Dv - D\psi\|_{L^1} < \frac{1}{k}$$

for every $\psi \in B_\infty(v, \delta)$; which implies that $v \in V^k$.

Combining these three facts we have indeed obtained the desired density result. ■

To conclude this section we give a sufficient condition that ensures the relaxation property. In concrete examples this condition is usually much easier to check than the relaxation property. We start with a definition.

Definition 27 (Approximation property) *Let $E \subset K(E) \subset \mathbb{R}^{N \times n}$. The sets E and $K(E)$ are said to have the* approximation property *if there exists a family of closed sets E_δ and $K(E_\delta)$, $\delta > 0$, such that*

(1) $E_\delta \subset K(E_\delta) \subset \operatorname{int} K(E)$ for every $\delta > 0$;

(2) for every $\epsilon > 0$ there exists $\delta_0 = \delta_0(\epsilon) > 0$ such that $\operatorname{dist}(\eta; E) \leq \epsilon$ for every $\eta \in E_\delta$ and $\delta \in [0, \delta_0]$;

(3) if $\eta \in \operatorname{int} K(E)$ then $\eta \in K(E_\delta)$ for every $\delta > 0$ sufficiently small.

We therefore have the following theorem (cf. Theorem 6.14 in [31] and for a slightly more flexible one see Theorem 6.15).

Theorem 28 *Let $E \subset \mathbb{R}^{N \times n}$ be compact and $\operatorname{Rco} E$ has the approximation property with $K(E_\delta) = \operatorname{Rco} E_\delta$, then $\operatorname{int} \operatorname{Rco} E$ has the relaxation property with respect to E.*

4.5 Some examples of existence of solutions

We now give several examples of existence theorems that follow from the abstract ones.

The first one concerns the scalar case, where we can even get sharper results (cf. [11], [17], [28], [29], [31], [38] or [40]).

Theorem 29 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $E \subset \mathbb{R}^n$. Let $\varphi \in W^{1,\infty}(\Omega)$ satisfy*

$$D\varphi(x) \in E \cup \operatorname{int}\operatorname{co} E, \ \text{a.e. } x \in \Omega \tag{6}$$

(where $\operatorname{int}\operatorname{co} E$ stands for the interior of the convex hull of E); then there exists $u \in \varphi + W_0^{1,\infty}(\Omega)$ such that

$$Du(x) \in E, \ \text{a.e. } x \in \Omega. \tag{7}$$

Remark 30 *The theorem is in fact much less restrictive than the abstract one, here we do not need, for example, E to be compact. For a proof we refer to Dacorogna-Marcellini [31].*

We now show that (6) is in fact also a necessary condition, at least when φ is affine, for the general case see Section 2.4 in Dacorogna-Marcellini [31]. For the affine case the result is implicit in the above mentioned articles, but we follow here Bandyopadhyay-Barroso-Dacorogna-Matias [9].

Theorem 31 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $E \subset \mathbb{R}^n$, $\xi_0 \in \mathbb{R}^n$ and $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ (u_{ξ_0} being such that $Du_{\xi_0} = \xi_0$) so that*

$$Du(x) \in E, \ \text{a.e. } x \in \Omega$$

then

$$\xi_0 \in E \cup \operatorname{int}\operatorname{co} E.$$

Proof. Assume that $\xi_0 \notin E$, otherwise nothing is to be proved. It is easy to see that, by Jensen inequality and since $Du(x) \in E$,

$$\xi_0 = \frac{1}{\operatorname{meas}\Omega} \int_\Omega Du(x)\, dx \in \overline{\operatorname{co}} E.$$

Let us show that we cannot have $\xi_0 \in \partial(\overline{\operatorname{co}} E)$. If we can prove this, we will deduce that $\xi_0 \in \operatorname{int} \overline{\operatorname{co}} E$. Since $\operatorname{int} \overline{\operatorname{co}} E = \operatorname{int}\operatorname{co} E$ (cf. Theorem 6.3 in Rockafellar [71]) we will have the result.

If $\xi_0 \in \partial(\overline{\operatorname{co}} E)$, we find from the separation theorem that there exists $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$, such that

$$\langle \alpha; z - \xi_0 \rangle \geq 0, \ \forall z \in \overline{\operatorname{co}} E.$$

We therefore have that

$$\langle \alpha; Du(x) - \xi_0 \rangle \geq 0, \ \text{a.e. } x \in \Omega.$$

Recalling that $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$, we find that

$$\int_\Omega \langle \alpha; Du(x) - \xi_0 \rangle \, dx = 0,$$

which coupled with the above inequality leads to

$$\langle \alpha; Du(x) - \xi_0 \rangle = 0, \text{ a.e. } x \in \Omega.$$

Applying Lemma 57, we get that $u \equiv u_{\xi_0}$ and hence $\xi_0 \in E$, a contradiction with the hypothesis made at the beginning of the proof. Therefore $\xi_0 \notin \partial(\overline{\text{co}} E)$ as claimed and hence the theorem is proved. ■

Theorem 29 applies to the following case.

Corollary 32 (Convex Hamiltonian) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $F : \mathbb{R}^n \to \mathbb{R}$ be convex and such that $\lim_{|\xi|\to\infty} F(\xi) = +\infty$. Let $\varphi \in W^{1,\infty}(\Omega)$ be such that*

$$F(D\varphi(x)) \leq 0, \text{ a.e. } x \in \Omega.$$

Then there exists $u \in \varphi + W_0^{1,\infty}(\Omega)$ such that

$$F(Du(x)) = 0, \text{ a.e. } x \in \Omega.$$

The next one deals with the singular values case that we have encountered in Subsection 4.3. The next theorem is due to Dacorogna-Ribeiro [35].

Theorem 33 (Singular values) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\alpha < \beta$ and $0 < a_2 \leq ... \leq a_n$ be such that*

$$\max\{|\alpha|, |\beta|\} \leq a_2 \prod_{i=2}^{n} a_i.$$

Let $\varphi \in C^1_{piec}(\overline{\Omega}; \mathbb{R}^n)$ be such that, for almost every $x \in \Omega$,

$$\alpha < \det D\varphi(x) < \beta, \ \prod_{i=\nu}^{n} \lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n} a_i, \ \nu = 2, ..., n,$$

then there exists $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ so that, for almost every $x \in \Omega$,

$$\det Du(x) \in \{\alpha, \beta\}, \ \lambda_\nu(Du(x)) = a_\nu, \ \nu = 2, ..., n.$$

Remark 34 *(i) If $\alpha = -\beta < 0$ and if we set*

$$a_1 = \beta \left[\prod_{i=2}^{n} a_i \right]^{-1},$$

we recover the result of Dacorogna-Marcellini [31], namely that if

$$\prod_{i=\nu}^{n} \lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n} a_i, \ \nu = 1, ..., n,$$

then there exists $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ *so that*

$$\lambda_\nu(Du) = a_\nu, \ \nu = 1, ..., n, \ a.e. \ in \ \Omega.$$

(ii) If $\alpha = \beta \neq 0$ *we can also prove, as in Dacorogna-Tanteri [37], that if*

$$\det D\varphi(x) = \alpha, \ \prod_{i=\nu}^{n} \lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n} a_i, \ \nu = 2, ..., n,$$

then there exists $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ *so that*

$$\lambda_\nu(Du) = a_\nu, \ \nu = 2, ..., n \ and \ \det Du = \alpha, \ a.e. \ in \ \Omega.$$

4.6 Appendix

In this appendix we recall some well known facts about the so called functions of first class in the sense of Baire, with particular interest in their application to the gradient operator.

We start recalling some definitions.

Definition 35 *Let* X*,* Y *be metric spaces and* $f : X \to Y$*. We define the* oscillation *of* f *at* $x_0 \in X$ *as*

$$\omega_f(x_0) = \lim_{\delta \to 0} \sup_{x,y \in B_X(x_0,\delta)} d_Y(f(y), f(x))$$

where $B_X(x_0, \delta) := \{x \in X : d_X(x, x_0) < \delta\}$ *is the open ball centered at* x_0 *and* d_X*,* d_Y *are the metric on the spaces* X *and* Y *respectively.*

Definition 36 *A function* f *is said to be of* first class *(in the sense of Baire) if it can be represented as the pointwise limit of an everywhere convergent sequence of continuous functions.*

In the next proposition we recall some elementary properties of the oscillation function ω_f.

Proposition 37 *Let* X*,* Y *be metric spaces, and* $f : X \to Y$*.*

(i) f *is continuous at* $x_0 \in X$ *if and only if* $\omega_f(x_0) = 0$*.*

(ii) The set $\Omega_f^\epsilon := \{x \in X : \omega_f(x) < \epsilon\}$ *is an open set in* X*.*

Using the notion of oscillation and Proposition 37 we can write the set $\mathcal{D}_f$ of all points at which a given function f is discontinuous as an F_σ set as follows

$$\mathcal{D}_f = \bigcup_{n=1}^{\infty} \left\{ x \in X : \omega_f(x) \geq \frac{1}{n} \right\}. \tag{8}$$

We therefore have the following Baire theorem for functions of first class (for a proof see Theorem 7.3 in Oxtoby [65], Yosida [79] page 12, or Dacorogna-Pisante [33]).

Theorem 38 *Let X, Y be metric spaces let X be complete and $f : X \to Y$. If f is a function of first class, then $\mathcal{D}_f$ is a set of first category.*

Remark 39 *From Theorem 38 and the Baire category theorem follows in particular that the set of points of continuity of a function of first class from a complete metric space X to any metric space Y, i.e. the set $\mathcal{D}_f^c$ complement of $\mathcal{D}_f$, is a dense G_δ set. Indeed for any $\epsilon > 0$, the set*

$$\Omega_f^\epsilon := \{x \in X : \omega_f(x) < \epsilon\}$$

is open and dense in X.

In the proof of our main theorem we have used Theorem 38 applied to the following, quite surprising, special case of function of first class. This result was observed by Kirchheim in [46] for complete sets of Lipschitz functions and the same argument gives in fact the result for general complete subsets $W^{1,\infty}(\Omega)$ functions.

Corollary 40 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $V \subset W^{1,\infty}(\Omega)$ be a non empty complete space with respect to the L^∞ metric. Then the gradient operator $D : V \to L^p(\Omega; \mathbb{R}^n)$ is a function of first class for any $1 \le p < \infty$.*

Proof. For $h \neq 0$, we let

$$D^h = \left(D_1^h, ..., D_n^h\right) : V \to L^p(\Omega; \mathbb{R}^n)$$

be defined, for every $u \in V$ and $x \in \Omega$, by

$$D_i^h u\,(x) = \begin{cases} \frac{u(x+he_i)-u(x)}{h} & \text{if } \operatorname{dist}(x, \Omega^c) > |h| \\ 0 & \text{elsewhere} \end{cases}$$

for $i = 1, \ldots, n$, where $e_1, ..., e_n$ stand for the vectors from the Euclidean basis.

The claim will follow once we will have proved that for any fixed h the operator D^h is continuous and that, for any sequence $h \to 0$,

$$\lim_{h \to 0} \left\|D_i^h u - D_i u\right\|_{L^p(\Omega)} = 0$$

for any $i = 1, \ldots, n$, $u \in V$.

The continuity of D^h follows easily by observing that for every $i = 1, ..., n$, $\epsilon > 0$ and $u, v \in V$ we have that

$$\begin{aligned} \left\|D_i^h u - D_i^h v\right\|_{L^p(\Omega)} &\le \frac{1}{|h|} \left(\int_{\Omega_h} |u(x) - v(x) + u\,(x + he_i) - v\,(x + he_i)|^p \, dx \right)^{\frac{1}{p}} \\ &\le \frac{2(\operatorname{meas}\Omega)^{\frac{1}{p}}}{|h|} \left\|u - v\right\|_{L^\infty(\Omega)}, \end{aligned}$$

where $\Omega_h = \{x \in \Omega \ : \ \operatorname{dist}(x, \Omega^c) > |h|\}$.

For the second claim we start observing that for any $x \in \Omega_h$ and for any $u \in V$ we have

$$|u(x + he_i) - u(x)| \leq \|D_i u\|_{L^\infty(\Omega)} |h| .$$

This implies that

$$\left\| D_i^h u \right\|_{L^\infty(\Omega)} \leq \|D_i u\|_{L^\infty(\Omega)} < +\infty.$$

Moreover by Rademacher theorem, for any sequence $h \to 0$,

$$\lim_{h \to 0} D_i^h u(x) = D_i u(x) \quad \text{a.e. } x \in \Omega.$$

The result follows by Lebesgue dominated convergence theorem. ■

5 Existence of minimizers

5.1 Introduction

We now discuss the existence of minimizers for the problem

$$(P) \qquad \inf \left\{ \int_\Omega f(Du(x))\, dx : u \in u_{\xi_0} + W_0^{1,\infty}\left(\Omega; \mathbb{R}^N\right) \right\}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $u : \Omega \to \mathbb{R}^N$, $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ is lower semicontinuous, locally bounded and non negative and u_{ξ_0} is a given affine map (i.e., $Du_{\xi_0} = \xi_0$, where $\xi_0 \in \mathbb{R}^{N\times n}$ is a fixed matrix).

If the function f is quasiconvex, i.e.

$$\int_U f(\xi + D\varphi(x))\, dx \geq f(\xi) \text{ meas}(U)$$

for every bounded domain $U \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^{N\times n}$, and $\varphi \in W_0^{1,\infty}\left(U; \mathbb{R}^N\right)$, then the problem (P) trivially has u_{ξ_0} as a minimizer. We also recall that in the scalar case ($n = 1$ or $N = 1$), quasiconvexity and ordinary convexity are equivalent.

We now study the case where f fails to be quasiconvex. The first step in dealing with such problems is the relaxation theorem (cf. Theorem 15). It has as a direct consequence (cf. Theorem 41) that (P) has a solution $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ if and only if

$$f(D\overline{u}(x)) = Qf(D\overline{u}(x)), \text{ a.e. } x \in \Omega$$

$$\int_\Omega Qf(D\overline{u}(x))\, dx = Qf(\xi_0) \text{ meas}\, \Omega$$

where Qf is the *quasiconvex envelope* of f, namely

$$Qf = \sup\{g \leq f : g \text{ quasiconvex}\}.$$

The problem is then to discuss the existence or non existence of a $\overline{u}$ satisfying the two equations. The two equations are not really of the same nature. The first one is what we called in Section 4 an implicit partial differential equation. The second one is more geometric in nature and has to do with some *"quasiaffinity"* of the quasiconvex envelope Qf.

In the present section we will discuss some abstract necessary and sufficient conditions for the existence of minimizers for (P) and in Sections 6 and 7 we will see several examples. We will follow in the present section the approach of Dacorogna-Pisante-Ribeiro [34].

5.2 Sufficient conditions

With the help of the relaxation theorem and of Theorem 25, we are now in a position to discuss some existence results for the problem (P). The following Theorem (cf. [27]) is elementary and gives a necessary and sufficient condition for existence of minima. It will be crucial in several of our arguments.

Theorem 41 *Let* Ω, f *and* u_{ξ_0} *be as above, in particular* $Du_{\xi_0} = \xi_0$. *The problem* (P) *has a solution if and only if there exists* $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ *such that*

$$f\left(D\overline{u}\left(x\right)\right) = Qf\left(D\overline{u}\left(x\right)\right),\ a.e.\ x \in \Omega \tag{9}$$

$$\int_\Omega Qf\left(D\overline{u}\left(x\right)\right) dx = Qf\left(\xi_0\right) \operatorname{meas}\Omega. \tag{10}$$

Proof. By the relaxation theorem and since u_{ξ_0} is affine, we have

$$\inf\left(P\right) = \inf\left(QP\right) = Qf\left(\xi_0\right) \operatorname{meas}\Omega.$$

Moreover, since we always have $f \geq Qf$ and we have a solution of (9) satisfying (10), we get that $\overline{u}$ is a solution of (P). The fact that (9) and (10) are necessary for the existence of a minimum for (P) follows in the same way. ■

The previous theorem explains why the set

$$K = \left\{\xi \in \mathbb{R}^{N\times n} : Qf\left(\xi\right) < f\left(\xi\right)\right\}$$

plays a central role in the existence theorems that follow. In order to ensure (9) we will have to consider differential inclusions of the form studied in the previous section, namely: find $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ such that

$$D\overline{u}\left(x\right) \in \partial K,\ \text{a.e. } x \in \Omega.$$

In order to deal with the second condition (10) we will have to impose some hypotheses of the type "Qf is *quasiaffine* on K".

The main abstract theorem is the following.

Theorem 42 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set,* $\xi_0 \in \mathbb{R}^{N\times n}$, $f : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ *a lower semicontinuous function, locally bounded and non negative function and let*

$$K = \left\{\eta \in \mathbb{R}^{N\times n} : Qf\left(\eta\right) < f\left(\eta\right)\right\}.$$

Assume that there exists $K_0 \subset K$ such that

- $\xi_0 \in K_0$,
- K_0 *is bounded and has the relaxation property with respect to* $\overline{K}_0 \cap \partial K$,
- Qf *is quasiaffine on* $\overline{K}_0$.

Let $u_{\xi_0}(x) = \xi_0 x$. *Then the problem*

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(Du(x))\, dx : \; u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N) \right\}$$

has a solution $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$.

Remark 43 *(i) Although this theorem applies only to functions* f *that takes only finite values, it can sometimes be extended to functions* $f : \mathbb{R}^{N\times n} \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

(ii) The last hypothesis in the theorem means that

$$\int_\Omega Qf(\xi + D\varphi(x))\, dx = Qf(\xi)\, \mathrm{meas}\, \Omega$$

for every $\xi \in \overline{K}_0$, *every* $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ *with*

$$\xi + D\varphi(x) \in \overline{K}_0\,, \text{ a.e. in } \Omega.$$

Proof. Since $\xi_0 \in K_0$ and K_0 is bounded and has the relaxation property with respect to $\overline{K}_0 \cap \partial K$, we can find, appealing to Theorem 25, a map $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ satisfying

$$D\bar{u} \in \overline{K}_0 \cap \partial K, \text{ a.e. in } \Omega,$$

which means that (9) of Theorem 41 is satisfied. Moreover, since Qf is quasiaffine on $\overline{K}_0$, we have that (10) of Theorem 41 holds and thus the claim. ■

The second hypothesis in the theorem is clearly the most difficult to verify, nevertheless there are some cases when it is automatically satisfied. For example if K is bounded, we can take $K_0 = K$.

We will see that, in many applications, the set K turns out to be unbounded and in order to apply Theorem 42 we need to find some weaker conditions on K that guarantees the existence of a subset K_0 of K satisfying the requested properties. With this aim in mind we give the following notations and definitions.

Notation 44 *Let* $K \subset \mathbb{R}^{N\times n}$ *be open and* $\lambda \in \mathbb{R}^{N\times n}$.

(i) For $\xi \in K$, *we denote by* $L_K(\xi, \lambda)$ *the largest segment of the form* $[\xi + t\lambda, \xi + s\lambda]$, $t < 0 < s$, *so that* $(\xi + t\lambda, \xi + s\lambda) \subset K$.

(ii) If $L_K(\xi, \lambda)$ *is bounded, we denote by* $t_-(\xi) < 0 < t_+(\xi)$ *the elements so that* $L_K(\xi, \lambda) = [\xi + t_-\lambda, \xi + t_+\lambda]$. *They therefore satisfy*

$$\xi + t_\pm \lambda \in \partial K \;\; \text{and} \;\; \xi + t\lambda \in K, \; \forall t \in (t_-, t_+).$$

(iii) If $H \subset K$, we let

$$L_K(H,\lambda) = \underset{\xi \in H}{\cup} L_K(\xi,\lambda).$$

Definition 45 *Let $K \subset \mathbb{R}^{N\times n}$ be open, $\xi_0 \in K$ and $\lambda \in \mathbb{R}^{N\times n}$.*

(i) We say that K is bounded at ξ_0 in the direction λ *if $L_K(\xi_0,\lambda)$ is bounded.*

(ii) We say that K is stably bounded at ξ_0 in the rank-one direction $\lambda = \alpha\otimes\beta$ *(with $\alpha \in \mathbb{R}^N$ and $\beta \in \mathbb{R}^n$) if there exists $\epsilon > 0$ so that $L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda)$ is bounded, where we have denoted by*

$$\xi_0 + \alpha \otimes B_\epsilon = \left\{\xi \in \mathbb{R}^{N\times n} : \xi = \xi_0 + \alpha \otimes b \text{ with } |b| < \epsilon\right\}.$$

Clearly a bounded open set K is bounded at every point $\xi \in K$ and in any direction λ and consequently it is also stably bounded.

We now give an example of a globally unbounded set which is bounded in certain directions.

Example 46 *Let $N = n = 2$ and*

$$K = \left\{\xi \in \mathbb{R}^{2\times 2} : \alpha < \det \xi < \beta\right\}.$$

The set K is clearly unbounded.

(i) If $\xi_0 = I$ then K is bounded, and even stably bounded, at ξ_0, in a direction of rank one, for example with

$$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \lambda = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) However if $\xi_0 = 0$, then K is unbounded in any rank one direction, but is bounded in any rank two direction.

In the following result we deal with sets K that are bounded in a rank-one direction only. This corollary says, roughly speaking, that if K is bounded at ξ_0 in a rank-one direction λ and this boundedness (in the same direction) is preserved under small perturbations of ξ_0 along rank one λ-compatible directions, then we can ensure the relaxation property required in the main existence theorem.

Corollary 47 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ a lower semicontinuous function, locally bounded and non negative and let $\xi_0 \in K$ where*

$$K = \left\{\xi \in \mathbb{R}^{N\times n} : Qf(\xi) < f(\xi)\right\}.$$

If there exist a rank-one direction $\lambda \in \mathbb{R}^{N\times n}$ such that

(i) K is stably bounded at ξ_0 in the direction $\lambda = \alpha \otimes \beta$,

(ii) Qf is quasiaffine on the set (cf. Definition 45) $L_K(\xi_0 + \alpha \otimes \overline{B}_\epsilon, \lambda)$,

then the problem

$$(P) \qquad \inf\left\{I(u) = \int_\Omega f(Du(x))\, dx : \; u \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^N)\right\}$$

has a solution $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^N)$.

Proof. We divide the proof into two steps.

Step 1. Assume that $|\beta| = 1$, otherwise replace it by $\beta / |\beta|$, and let $\beta_k \in \mathbb{R}^n$, $k \geq n$, with $|\beta_k| = 1$, be such that

$$0 \in H := \operatorname{int} \operatorname{co}\{\beta, -\beta, \beta_3, \ldots, \beta_k\} \subset B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}.$$

Let then, for $\epsilon > 0$ as in the hypothesis,

$$K_0 := (\xi_0 + \alpha \otimes \epsilon H) \cup \left[\partial K \cap L_K(\xi_0 + \alpha \otimes \epsilon \overline{H}, \lambda)\right].$$

We therefore have that $\xi_0 \in K_0$ and, by hypothesis, that K_0 is bounded, since

$$K_0 \subset \overline{K}_0 \subset L_K(\xi_0 + \alpha \otimes \overline{B}_\epsilon, \lambda).$$

Furthermore we have

$$\overline{K}_0 \cap \partial K = \partial K \cap L_K(\xi_0 + \alpha \otimes \epsilon \overline{H}, \lambda).$$

In order to deduce the corollary from Theorem 42, we only need to show that K_0 has the relaxation property with respect to $\overline{K}_0 \cap \partial K$. This will be achieved in the next step.

Step 2. We now prove that K_0 has the relaxation property with respect to $\overline{K}_0 \cap \partial K$. Let $\xi \in K_0$ and let us find a sequence $u_\nu \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^N\right)$ so that

$$\begin{aligned} &u_\nu \in u_\xi + W_0^{1,\infty}\left(\Omega; \mathbb{R}^N\right),\ Du_\nu(x) \in \left(\overline{K}_0 \cap \partial K\right) \cup K_0, \text{ a.e. in } \Omega \\ &u_\nu \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{1,\infty},\ \int_\Omega \operatorname{dist}\left(Du_\nu(x); \overline{K}_0 \cap \partial K\right) dx \to 0 \text{ as } \nu \to \infty. \end{aligned} \tag{11}$$

If $\xi \in \partial K \cap L_K(\xi_0 + \alpha \otimes \epsilon \overline{H}, \lambda)$, nothing is to be proved; so we assume that $\xi \in \xi_0 + \alpha \otimes \epsilon H$. By hypothesis (i), we can find $t_-(\xi) < 0 < t_+(\xi)$ so that

$$\xi_\pm := \xi + t_\pm \lambda \in \partial K \quad \text{and} \quad \xi + t\lambda \in K \quad \forall t \in (t_-, t_+)$$

and hence $\xi_\pm \in \overline{K}_0 \cap \partial K$. We moreover have that

$$\xi = \frac{-t_-}{t_+ - t_-}\xi_+ + \frac{t_+}{t_+ - t_-}\xi_- \quad \text{with} \quad \xi_\pm \in \overline{K}_0 \cap \partial K. \tag{12}$$

Furthermore, since $\xi \in \xi_0 + \alpha \otimes \epsilon H$, we can find $\gamma \in \epsilon H$ such that

$$\xi = \xi_0 + \alpha \otimes \gamma.$$

The set H being open we have that $\overline{B}_\delta(\gamma) \subset \epsilon H$, for every sufficiently small $\delta > 0$. Moreover since for every $\delta > 0$, we have

$$0 \in \delta H = \operatorname{int} \operatorname{co}\{\pm \delta\beta, \delta\beta_3, \ldots, \delta\beta_k\}$$

and since for every sufficiently small $\delta > 0$, we have

$$\pm\delta\beta \in \operatorname{co}\{\pm(t_+ - t_-)\beta\} \subset \operatorname{co}\{\pm(t_+ - t_-)\beta, \delta\beta_3, \ldots, \delta\beta_k\},$$

we get that

$$0 \in \delta H = \operatorname{int} \operatorname{co}\{\pm\delta\beta, \delta\beta_3, \ldots, \delta\beta_k\} \subset \operatorname{int} \operatorname{co}\{\pm (t_+ - t_-)\, \beta, \delta\beta_3, \ldots, \delta\beta_k\}.$$

We are therefore in a position to apply Lemma 48 to

$$\begin{aligned}
&a = \alpha,\ b = (t_+ - t_-)\, \beta,\ b_j = \delta\beta_j \text{ for } j = 3, \cdots, k,\ t = \frac{-t_-}{t_+ - t_-},\\
&A = \xi_+ = \xi + \frac{t_+}{t_+ - t_-} \alpha \otimes (t_+ - t_-)\, \beta = \xi + (1-t)\, a \otimes b,\\
&B = \xi_- = \xi + \frac{t_-}{t_+ - t_-} \alpha \otimes (t_+ - t_-)\, \beta = \xi - t a \otimes b
\end{aligned}$$

and find $u_\delta \in Aff_{piec}\left(\overline{\Omega}; \mathbb{R}^N\right)$, open sets $\Omega_+, \Omega_- \subset \Omega$, such that

$$\begin{cases}
|\text{meas } (\Omega_+ \cup \Omega_-) - \text{meas } \Omega| \leq \delta \\
u_\delta(x) = u_\xi(x),\ x \in \partial\Omega \ \text{ and } \ |u_\delta(x) - u_\xi(x)| \leq \delta,\ x \in \Omega \\
Du_\delta(x) = \xi_\pm \ \text{ a.e. in } \ \Omega_\pm \\
Du_\delta(x) \in \xi + \{t_+ \alpha \otimes \beta, t_- \alpha \otimes \beta, \alpha \otimes \delta\beta_3, \ldots, \alpha \otimes \delta\beta_k\},\ \text{a.e. in } \Omega.
\end{cases} \tag{13}$$

Since $\xi_\pm \in \overline{K}_0 \cap \partial K$ and

$$\xi + \alpha \otimes \delta\beta_j \in \xi + \alpha \otimes \delta\overline{H} = \xi_0 + \alpha \otimes \left(\gamma + \delta\overline{H}\right) \subset \xi_0 + \alpha \otimes \epsilon H \subset K_0 \ \text{ for } j = 3, \cdots, k,$$

we deduce, by choosing $\delta = 1/\nu$ as $\nu \to \infty$, from (13), the relaxation property (12). This achieves the proof of Step 2 and thus of the corollary. ■

We finally want to point out that, as a particular case of Corollary 47, we find the existence theorem (Theorem 3.1) proved by Dacorogna-Marcellini in [27].

We have used the following result due to Müller-Sychev [63] and which is a refinement of a classical result.

Lemma 48 (Approximation lemma) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $t \in [0,1]$ and $A, B \in \mathbb{R}^{N \times n}$ such that*

$$A - B = a \otimes b$$

with $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$. Let $b_3, \ldots, b_k \in \mathbb{R}^n$, $k \geq n$, such that $0 \in \operatorname{int} \operatorname{co}\{b, -b, b_3, \ldots, b_k\}$. Let φ be an affine map such that

$$D\varphi(x) = \xi_0 = tA + (1-t)B, \quad x \in \overline{\Omega}$$

(i.e. $A = \xi_0 + (1-t)\, a \otimes b$ and $B = \xi_0 - ta \otimes b$). Then, for every $\varepsilon > 0$, there exists a piecewise affine map u and there exist disjoint open sets Ω_A, $\Omega_B \subset \Omega$,

such that

$$\begin{cases} \left|\text{meas }\Omega_A - t\text{ meas }\Omega\right|, \ \left|\text{meas }\Omega_B - (1-t)\text{ meas }\Omega\right| \leq \varepsilon \\ u(x) = \varphi(x),\ x \in \partial\Omega \quad \textit{and} \quad |u(x) - \varphi(x)| \leq \varepsilon,\ x \in \Omega \\ Du(x) = \begin{cases} A & \textit{in } \Omega_A \\ B & \textit{in } \Omega_B \end{cases} \\ Du(x) \in \xi_0 + \{(1-t)\, a \otimes b, -ta \otimes b, a \otimes b_3, \ldots, a \otimes b_k\},\ \textit{a.e. in } \Omega. \end{cases}$$

5.3 Necessary conditions

Recall that we are considering the minimization problem

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(Du(x))\, dx : \ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N) \right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, u_{ξ_0} is affine, i.e. $Du_{\xi_0} = \xi_0$ and $f : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ is a lower semicontinuous, locally bounded and non negative function. In order to avoid the trivial case we will always assume that

$$Qf(\xi_0) < f(\xi_0).$$

Most non existence results for problem (P) follow by showing that the relaxed problem (QP) has a *unique* solution, namely u_{ξ_0}, which is by hypothesis not a solution of (P). This approach was strongly used in Marcellini [51], Dacorogna-Marcellini [27] and Dacorogna-Pisante-Ribeiro [34]; we will follow here this last article. We should point out that we will give an example (see Proposition 78 in Section 7.5) related to minimal surfaces, where non existence occurs, while the relaxed problem has infinitely many solutions, none of them being a solution of (P).

The right notion in order to have uniqueness of the relaxed problem is

Definition 49 *A quasiconvex function $f : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ is said to be* strictly quasiconvex *at $\xi_0 \in \mathbb{R}^{N\times n}$, if for some bounded domain $U \subset \mathbb{R}^n$ the following equality holds*

$$\int_U f(\xi_0 + D\varphi(x))\, dx = f(\xi_0)\ \text{meas}(U)$$

for some $\varphi \in W_0^{1,\infty}\left(U; \mathbb{R}^N\right)$, then necessarily $\varphi \equiv 0$.

We should observe that as in Remark 5 (v) the notion of strict quasiconvexity is independent of the choice of the domain U, more precisely we have.

Proposition 50 *If a function $f : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ is strictly quasiconvex at $\xi_0 \in \mathbb{R}^{N\times n}$ for one bounded domain $U \subset \mathbb{R}^n$ it is so for any such domain.*

Proof. Let $V \subset \mathbb{R}^n$ be a bounded domain and $\psi \in W_0^{1,\infty}\left(V;\mathbb{R}^N\right)$ be such that

$$\int_V f\left(\xi_0 + D\psi\left(x\right)\right) dx = f\left(\xi_0\right) \operatorname{meas}(V) \tag{14}$$

and let us conclude that we necessarily have $\psi \equiv 0$.

Choose first $a > 0$ sufficiently large so that

$$V \subset Q_a = \left(-a, a\right)^n$$

and then define

$$v\left(x\right) = \begin{cases} \psi\left(x\right) & \text{if } x \in V \\ 0 & \text{if } x \in Q_a - V \end{cases}$$

so that $v \in W_0^{1,\infty}\left(Q_a;\mathbb{R}^N\right)$.

Let then $x_0 \in U$ and choose ν sufficiently large so that

$$x_0 + \frac{1}{\nu} Q_a = x_0 + \left(-\frac{a}{\nu}, \frac{a}{\nu}\right)^n \subset U.$$

Define next

$$\varphi\left(x\right) = \begin{cases} \frac{1}{\nu} v\left(\nu\left(x - x_0\right)\right) & \text{if } x \in x_0 + \frac{1}{\nu} Q_a \\ 0 & \text{if } x \in U - \left[x_0 + \frac{1}{\nu} Q_a\right]. \end{cases}$$

Observe that $\varphi \in W_0^{1,\infty}\left(U;\mathbb{R}^N\right)$ and

$$\begin{aligned} \int_U f\left(\xi_0 + D\varphi\left(x\right)\right) dx &= f\left(\xi_0\right) \operatorname{meas}(U - \left[x_0 + \frac{1}{\nu} Q_a\right]) \\ &\quad + \int_{\left[x_0 + \frac{1}{\nu} Q_a\right]} f\left(\xi_0 + Dv\left(\nu\left(x - x_0\right)\right)\right) dx \\ &= f\left(\xi_0\right) \left[\operatorname{meas}(U) - \frac{\operatorname{meas}(Q_a)}{\nu^n}\right] + \frac{1}{\nu^n} \int_{Q_a} f\left(\xi_0 + Dv\left(y\right)\right) dy \\ &= f\left(\xi_0\right) \left[\operatorname{meas}(U) - \frac{\operatorname{meas}(Q_a)}{\nu^n} + \frac{\operatorname{meas}(Q_a - V)}{\nu^n}\right] \\ &\quad + \frac{1}{\nu^n} \int_V f\left(\xi_0 + D\psi\left(y\right)\right) dy. \end{aligned}$$

Appealing to (14), we deduce that

$$\int_U f\left(\xi_0 + D\varphi\left(x\right)\right) dx = f\left(\xi_0\right) \operatorname{meas}(U).$$

Since f is strictly quasiconvex at $\xi_0 \in \mathbb{R}^{N \times n}$ for the domain U, we deduce that $\varphi \equiv 0$, which in turn implies that

$$v\left(y\right) \equiv 0, \text{ for every } y \in Q_a.$$

This finally implies that $\psi \equiv 0$ as claimed. ■

We will see below some sufficient conditions that can ensure strict quasiconvexity, but let us start with the elementary following non existence theorem.

Theorem 51 *Let $f : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ be lower semicontinuous, locally bounded and non negative, $\xi_0 \in \mathbb{R}^{N\times n}$ with $Qf(\xi_0) < f(\xi_0)$ and Qf be strictly quasiconvex at ξ_0. Then the relaxed problem (QP) has a unique solution, namely u_{ξ_0}, while (P) has no solution.*

Proof. The fact that (QP) has only one solution follows by definition of the strict quasiconvexity of Qf and Proposition 50. Assume for the sake of contradiction that (P) has a solution $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^N)$. We should have from Theorem 41 that (writing $\overline{u}(x) = \xi_0 x + \varphi(x)$)

$$f(\xi_0 + D\varphi(x)) = Qf(\xi_0 + D\varphi(x)), \text{ a.e. } x \in \Omega$$

$$\int_\Omega Qf(\xi_0 + D\varphi(x))\, dx = Qf(\xi_0) \operatorname{meas} \Omega.$$

Since Qf is strictly quasiconvex at ξ_0, we deduce from the last identity that $\varphi \equiv 0$. Hence we have, from the first identity, that $Qf(\xi_0) = f(\xi_0)$, which is in contradiction with the hypothesis. ■

We now want to give some criteria that can ensure the strict quasiconvexity of a given function. The first one has been introduced by Dacorogna-Marcellini in [27].

Definition 52 *A convex function $f : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ is said to be* strictly convex at $\xi_0 \in \mathbb{R}^{N\times n}$ in at least N directions *if there exists $\alpha = \left(\alpha^i\right)^{1\le i\le N} \in \mathbb{R}^{N\times n}$, $\alpha^i \neq 0$ for every $i = 1, \cdots, N$, such that: if for some $\eta \in \mathbb{R}^{N\times n}$ the identity*

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) = f\left(\xi_0 + \frac{1}{2}\eta\right)$$

holds, then necessarily

$$\left\langle \alpha^i; \eta^i \right\rangle = 0,\ i = 1, \cdots, N.$$

In order to understand better the generalization of this notion to polyconvex functions (cf. Proposition 58), it might be enlightening to state the definition in the following way.

Proposition 53 *Let $f : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ be a convex function and, for $\xi \in \mathbb{R}^{N\times n}$, denote by $\partial f(\xi)$ the subdifferential of f at ξ. The two following conditions are then equivalent:*

(i) f is strictly convex at $\xi_0 \in \mathbb{R}^{N\times n}$ in at least N directions

(ii) there exists $\alpha = \left(\alpha^i\right)^{1\le i\le N} \in \mathbb{R}^{N\times n}$ with $\alpha^i \neq 0$ for every $i = 1, ..., N$, so that whenever

$$f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda; \eta \rangle = 0$$

for some $\eta \in \mathbb{R}^{N\times n}$ *and for some* $\lambda \in \partial f(\xi_0)$ *, then*

$$\left\langle \alpha^i ; \eta^i \right\rangle = 0,\ i = 1, ..., N.$$

Proof. *Step 1.* We start with a preliminary observation that if

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) = f\left(\xi_0 + \frac{1}{2}\eta\right) \tag{15}$$

then, for every $t \in [0,1]$, we have

$$t f(\xi_0 + \eta) + (1-t) f(\xi_0) = f(\xi_0 + t\eta). \tag{16}$$

Let us show this under the assumption that $t > 1/2$ (the case $t < 1/2$ is handled similarly). We can therefore find $\alpha \in (0,1)$ such that

$$\frac{1}{2} = \alpha t + (1-\alpha) 0 = \alpha t.$$

From the convexity of f and by hypothesis, we obtain

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) = f\left(\xi_0 + \frac{1}{2}\eta\right) \leq \alpha f(\xi_0 + t\eta) + (1-\alpha) f(\xi_0).$$

Assume, for the sake of contradiction, that

$$f(\xi_0 + t\eta) < t f(\xi_0 + \eta) + (1-t) f(\xi_0).$$

Combine then this inequality with the previous one to get

$$\begin{gathered} \tfrac{1}{2} f(\xi_0 + \eta) + \tfrac{1}{2} f(\xi_0) < \\ \alpha \left[t f(\xi_0 + \eta) + (1-t) f(\xi_0) \right] + (1-\alpha) f(\xi_0) \\ = \tfrac{1}{2} f(\xi_0 + \eta) + \tfrac{1}{2} f(\xi_0) \end{gathered}$$

which is clearly a contradiction. Therefore the convexity of f and the above contradiction implies (16). This also implies that

$$f'(\xi_0, \eta) := \lim_{t \to 0^+} \frac{f(\xi_0 + t\eta) - f(\xi_0)}{t} = f(\xi_0 + \eta) - f(\xi_0).$$

Applying Theorem 23.4 in Rockafellar [71], combined with the fact that $\partial f(\xi_0)$ is non empty and compact, we get that there exists $\lambda \in \partial f(\xi_0)$ so that $f(\xi_0 + \eta) - f(\xi_0) = \langle \lambda ; \eta \rangle$ and hence

$$f(\xi_0 + t\eta) - f(\xi_0) - t \langle \lambda ; \eta \rangle = 0,\ \forall t \in [0,1]. \tag{17}$$

We have therefore proved that (15) implies (17). Since the converse is obviously true, we conclude that they are equivalent.

Step 2. Let us show the equivalence of the two conditions.

(i) $\Longrightarrow$ (ii). We first observe that for any $\mu \in \mathbb{R}^{N\times n}$ we have

$$\begin{gathered}\tfrac{1}{2}f\left(\xi_0+\eta\right)+\tfrac{1}{2}f\left(\xi_0\right)-f\left(\xi_0+\tfrac{1}{2}\eta\right)= \\ \tfrac{1}{2}\left[f\left(\xi_0+\eta\right)-f\left(\xi_0\right)-\langle\mu;\eta\rangle\right]-\left[f\left(\xi_0+\tfrac{1}{2}\eta\right)-f\left(\xi_0\right)-\tfrac{1}{2}\langle\mu;\eta\rangle\right].\end{gathered} \tag{18}$$

Assume that, for $\lambda \in \partial f\left(\xi_0\right)$, we have

$$f\left(\xi_0+\eta\right)-f\left(\xi_0\right)-\langle\lambda;\eta\rangle=0.$$

From (18) applied to $\mu=\lambda$, from the definition of $\partial f\left(\xi_0\right)$ and from the convexity of f, we have

$$\begin{aligned}0 &\leq \tfrac{1}{2}f\left(\xi_0+\eta\right)+\tfrac{1}{2}f\left(\xi_0\right)-f\left(\xi_0+\tfrac{1}{2}\eta\right) \\ &= -\left[f\left(\xi_0+\tfrac{1}{2}\eta\right)-f\left(\xi_0\right)-\tfrac{1}{2}\langle\lambda;\eta\rangle\right]\leq 0.\end{aligned}$$

Using the above identity, we then are in the framework of (i) and we deduce that $\left\langle\alpha^i;\eta^i\right\rangle=0$, $i=1,\cdots,N$, and thus (ii).

(ii) $\Longrightarrow$ (i). Assume now that we have (15), namely

$$\frac{1}{2}f\left(\xi_0+\eta\right)+\frac{1}{2}f\left(\xi_0\right)-f\left(\xi_0+\frac{1}{2}\eta\right)=0$$

which, by Step 1, implies that there exists $\lambda \in \partial f\left(\xi_0\right)$ so that

$$f\left(\xi_0+t\eta\right)-f\left(\xi_0\right)-t\langle\lambda;\eta\rangle=0,\ \forall t\in[0,1].$$

We are therefore, choosing $t=1$, in the framework of (ii) and we get $\left\langle\alpha^i;\eta^i\right\rangle=0$, $i=1,\cdots,N$, as wished. ■

Of course any strictly convex function is strictly convex in at least N directions, but the above condition is much weaker. For example in the scalar case, $N=1$, it is enough that the function is not affine in a neighborhood of ξ_0, to guarantee the condition (see below).

We now have the following result established by Dacorogna-Marcellini in [27].

Proposition 54 *If a convex function $f:\mathbb{R}^{N\times n}\longrightarrow\mathbb{R}$ is strictly convex at $\xi_0\in\mathbb{R}^{N\times n}$ in at least N directions, then it is strictly quasiconvex at ξ_0.*

Theorem 51, combined with the above proposition, gives immediately a sharp result for the scalar case, namely

Corollary 55 *Let $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ be lower semicontinuous, locally bounded and non negative, $\xi_0\in\mathbb{R}^n$ with $Cf\left(\xi_0\right)<f\left(\xi_0\right)$ and Cf not affine in the neighborhood of ξ_0. Then (P) has no solution.*

Remark 56 *In the scalar case this result has been obtained by several authors, in particular Cellina [16], Friesecke [40] and Dacorogna-Marcellini [27]. It also gives (cf. Theorem 66), combined with the result of the preceding section, that, provided some appropriate boundedness is assumed, a necessary and sufficient condition for existence of minima for* (P) *is that* f *be affine on the connected component of* $\{\xi : Cf(\xi) < f(\xi)\}$ *that contains* ξ_0*.*

Before proceeding with the proof of Proposition 54 we need the following elementary lemma.

Lemma 57 *Let* Ω *be a bounded open set of* $\mathbb{R}^n$ *and* $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ *be such that*

$$\left\langle \alpha^i ; D\varphi^i(x) \right\rangle = 0, \text{ a.e. } x \in \Omega,\ i = 1, \cdots, N$$

for some $\alpha^i \neq 0$, $i = 1, \cdots, N$*, then* $\varphi \equiv 0$*.*

Proof. (Lemma 57). Working component by component we can assume that $N = 1$ and therefore we will drop the indices. So let $\varphi \in W_0^{1,\infty}(\Omega)$ satisfy for some $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$,

$$\langle \alpha; D\varphi(x) \rangle = 0, \text{ a.e. } x \in \Omega.$$

We then choose $\alpha_2, \cdots, \alpha_n \in \mathbb{R}^n$ so that $\{\alpha, \alpha_2, \cdots, \alpha_n\}$ generate a basis of $\mathbb{R}^n$. Let $a > 0$ and for m an integer

$$Q_a^m = (-a, a)^m .$$

Let $x \in \Omega$ and let a and t be sufficiently small so that

$$x + \tau\alpha + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n \in \Omega, \text{ for every } \tau \in (0,t) \text{ and } (\tau_2, \cdots, \tau_n) \in Q_a^{n-1}.$$

Observe then that if $\varphi \in C_0^1(\Omega)$, then

$$\begin{aligned}
&\int_{Q_a^{n-1}} \left[\varphi(x + t\alpha + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n) - \varphi(x + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n)\right] d\tau_2 \cdots d\tau_n \\
&= \int_{Q_a^{n-1}} \int_0^t \frac{d}{d\tau}\left[\varphi(x + \tau\alpha + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n)\right] d\tau d\tau_2 \cdots d\tau_n \\
&= \int_{Q_a^{n-1}} \int_0^t \langle D\varphi(x + \tau\alpha + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n); \alpha\rangle\, d\tau d\tau_2 \cdots d\tau_n .
\end{aligned}$$

By a standard regularization procedure the above identity also holds for any $\varphi \in W_0^{1,\infty}(\Omega)$. Since $\langle \alpha; D\varphi \rangle = 0$, we deduce that

$$\int_{Q_a^{n-1}} \left[\varphi(x + t\alpha + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n) - \varphi(x + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n)\right] d\tau_2 \cdots d\tau_n = 0.$$

Since φ is continuous, we deduce, by dividing by the measure of Q_a^{n-1} and letting $a \to 0$, that, for every t sufficiently small so that $x + t\alpha \in \Omega$,

$$\varphi(x + t\alpha) = \varphi(x).$$

Choosing t so that

$$x + \tau\alpha \in \Omega, \ \forall \tau \in [0, t) \ \text{ and } x + t\alpha \in \partial\Omega$$

we obtain the claim, namely

$$\varphi(x) = 0, \ \forall x \in \Omega.$$

■

Proof. (Proposition 54). Assume that for a certain bounded domain $U \subset \mathbb{R}^n$ and for some $\varphi \in W_0^{1,\infty}(U; \mathbb{R}^N)$ we have

$$\int_U f(\xi_0 + D\varphi(x))\, dx = f(\xi_0) \text{ meas}(U)$$

and let us show that $\varphi \equiv 0$.

Since f is convex and the above identity holds, we find

$$\begin{aligned} f(\xi_0) \text{ meas}(U) &= \int_U \left[\frac{1}{2} f(\xi_0) + \frac{1}{2} f(\xi_0 + D\varphi(x))\right] dx \\ &\geq \int_U f\left(\xi_0 + \frac{1}{2} D\varphi(x)\right) dx \geq f(\xi_0) \text{ meas}(U), \end{aligned}$$

which implies that

$$\int_U \left[\frac{1}{2} f(\xi_0) + \frac{1}{2} f(\xi_0 + D\varphi(x)) - f\left(\xi_0 + \frac{1}{2} D\varphi(x)\right)\right] dx = 0.$$

The convexity of f implies then that, for almost every x in U, we have

$$\frac{1}{2} f(\xi_0) + \frac{1}{2} f(\xi_0 + D\varphi(x)) - f\left(\xi_0 + \frac{1}{2} D\varphi(x)\right) = 0.$$

The strict convexity in at least N directions leads to

$$\left\langle \alpha^i; D\varphi^i(x) \right\rangle = 0, \ \text{ a.e. } x \in \Omega, \, i = 1, \cdots, N.$$

Lemma 57 gives the claim. ■

We will now generalize Proposition 54. Since the notations in the next result are involved, we will first write the proposition when $N = n = 2$.

Proposition 58 *Let $f : \mathbb{R}^{N \times n} \longrightarrow \mathbb{R}$ be polyconvex, $\xi_0 \in \mathbb{R}^{N \times n}$ and $\lambda = \lambda(\xi_0) \in \mathbb{R}^{\tau(N,n)}$ so that*

$$f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda; T(\xi_0 + \eta) - T(\xi_0) \rangle \geq 0, \textit{ for every } \eta \in \mathbb{R}^{N \times n}.$$

(i) Let $N = n = 2$ and assume that there exist $\alpha^{1,1}, \alpha^{1,2}, \alpha^{2,2} \in \mathbb{R}^2$, $\alpha^{1,1} \neq 0$, $\alpha^{2,2} \neq 0$, $\beta \in \mathbb{R}$, so that if for some $\eta \in \mathbb{R}^{2 \times 2}$ the following equality holds

$$f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda; T(\xi_0 + \eta) - T(\xi_0) \rangle = 0$$

then necessarily

$$\left\langle \alpha^{2,2};\eta^{2}\right\rangle =0 \text{ and } \left\langle \alpha^{1,1};\eta^{1}\right\rangle +\left\langle \alpha^{1,2};\eta^{2}\right\rangle +\beta\det\eta=0.$$

Then f is strictly quasiconvex at ξ_0.

(ii) Let $N,n\geq 2$ and assume that there exist, for every $\nu=1,\cdots,N$,

$$\alpha^{\nu,\nu},\alpha^{\nu,\nu+1},\cdots,\alpha^{\nu,N}\in\mathbb{R}^{n},\ \alpha^{\nu,\nu}\neq 0,\ \beta^{\nu,s}\in\mathbb{R}^{\binom{n}{s}},\ 2\leq s\leq n\wedge(N-\nu+1)$$

so that if for some $\eta\in\mathbb{R}^{N\times n}$ the following equality holds

$$f\left(\xi_0+\eta\right)-f\left(\xi_0\right)-\left\langle \lambda;T\left(\xi_0+\eta\right)-T\left(\xi_0\right)\right\rangle =0$$

then necessarily

$$\sum_{s=\nu}^{N}\left\langle \alpha^{\nu,s};\eta^{s}\right\rangle +\sum_{s=2}^{n\wedge(N-\nu+1)}\left\langle \beta^{\nu,s};\operatorname{adj}_{s}\left(\eta^{\nu},\cdots,\eta^{N}\right)\right\rangle =0,\ \nu=1,\cdots,N.$$

Then f is strictly quasiconvex at ξ_0.

Remark 59 *(i) The existence of a λ as in the hypotheses of the proposition is automatically guaranteed by the polyconvexity of f (see (1) in Section 2, it corresponds in the case of a convex function to an element of $\partial f\left(\xi_0\right)$).*

(ii) We have adopted the convention that if $l>k>0$ are integers, then

$$\sum_{l}^{k}=0.$$

Example 60 *Let $N=n=2$ and consider the function*

$$f\left(\eta\right)=\left(\eta_{2}^{2}\right)^{2}+\left(\eta_{1}^{1}+\det\eta\right)^{2}.$$

This function is trivially polyconvex and according to the proposition it is also strictly quasiconvex at $\xi_0=0$ (choose $\lambda=0\in\mathbb{R}^{5}$, $\alpha^{2,2}=(0,1)$, $\alpha^{1,2}=(0,0)$, $\alpha^{1,1}=(1,0)$, $\beta=1$).

Proof. We will prove the proposition only in the case $N=n=2$, the general case being handled similarly.

Assume that for a certain bounded domain $U\subset\mathbb{R}^{2}$ and for some $\varphi\in W_{0}^{1,\infty}\left(U;\mathbb{R}^{2}\right)$ we have

$$\int_{U}f\left(\xi_0+D\varphi\left(x\right)\right)dx=f\left(\xi_0\right)\operatorname{meas}(U)$$

and let us prove that $\varphi\equiv 0$. This is equivalent, for every $\mu\in\mathbb{R}^{\tau(2,2)}$, to

$$\left[\int_{U}f\left(\xi_0+D\varphi\left(x\right)\right)-f\left(\xi_0\right)-\left\langle \mu;T\left(\xi_0+D\varphi\left(x\right)\right)-T\left(\xi_0\right)\right\rangle \right]dx=0.$$

Choosing $\mu = \lambda$ (λ as in the statement of the proposition) in the previous equation and using the polyconvexity of the function f, we get

$$f(\xi_0 + D\varphi(x)) - f(\xi_0) - \langle \lambda; T(\xi_0 + D\varphi(x)) - T(\xi_0) \rangle = 0, \text{ a.e. } x \in \Omega.$$

We hence infer that, for almost every $x \in \Omega$, we have

$$\left\langle \alpha^{2,2}; D\varphi^2 \right\rangle = 0 \text{ and } \left\langle \alpha^{1,1}; D\varphi^1 \right\rangle + \left\langle \alpha^{1,2}; D\varphi^2 \right\rangle + \beta \det D\varphi = 0.$$

Lemma 57, applied to the first equation, implies that $\varphi^2 \equiv 0$. Using this result in the second equation we get

$$\left\langle \alpha^{1,1}; D\varphi^1 \right\rangle = 0$$

and hence, appealing once more to the lemma, we have the claim, namely $\varphi^1 \equiv 0$. ■

Summarizing the results of Theorem 51, Proposition 54 and Proposition 58, we get

Corollary 61 *Let $f : \mathbb{R}^{N \times n} \longrightarrow \mathbb{R}$ be lower semicontinuous, locally bounded and non negative, $\xi_0 \in \mathbb{R}^{N \times n}$ with*

$$Qf(\xi_0) < f(\xi_0).$$

If either one of the two following conditions hold

(i) $Qf(\xi_0) = Cf(\xi_0)$ and Cf is strictly convex at ξ_0 in at least N directions;

(ii) $Qf(\xi_0) = Pf(\xi_0)$ and Pf is strictly polyconvex at ξ_0 (in the sense of Proposition 58);

then (QP) has a unique solution, namely u_{ξ_0}, while (P) has no solution.

Proof. The proof is almost identical under both hypotheses and so we will establish the corollary only in the first case. The result will follow from Theorem 51 if we can show that Qf is strictly convex at ξ_0. So assume that

$$\int_\Omega Qf(\xi_0 + D\varphi(x))\, dx = Qf(\xi_0) \operatorname{meas} \Omega$$

for some $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ and let us prove that $\varphi \equiv 0$. Using Jensen inequality combined with the hypothesis $Qf(\xi_0) = Cf(\xi_0)$ and the fact that $Qf \geq Cf$, we find that the above identity implies

$$\int_\Omega Cf(\xi_0 + D\varphi(x))\, dx = Cf(\xi_0) \operatorname{meas} \Omega.$$

The hypotheses on Cf and Proposition 54 imply that $\varphi \equiv 0$, as wished. ■

We now conclude this section with a different necessary condition that is based on Carathéodory theorem.

Recall first that for any integer s, we let

$$\Lambda_s = \left\{ \lambda = (\lambda_1, \cdots, \lambda_s) : \lambda_i \geq 0 \text{ and } \sum\nolimits_{i=1}^{s} \lambda_i = 1 \right\}.$$

Theorem 62 *If (P) has a solution* $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}\left(\Omega;\mathbb{R}^N\right)$, *then there exist* $\mu \in \Lambda_{Nn+1}$ *and* $\xi_\nu \in \mathbb{R}^{N\times n}$, $|\xi_\nu| \leq \|\overline{u}\|_{W^{1,\infty}}$, $1 \leq \nu \leq Nn+1$ *such that*

$$Qf(\xi_0) \geq \sum_{\nu=1}^{Nn+1} \mu_\nu f(\xi_\nu) \ \text{ and } \xi_0 = \sum_{\nu=1}^{Nn+1} \mu_\nu \xi_\nu \,.$$

Moreover if either $n = 1$ *or* $N = 1$, *the inequality becomes an equality, namely*

$$Cf(\xi_0) = \sum_{\nu=1}^{Nn+1} \mu_\nu f(\xi_\nu) \ \text{ and } \xi_0 = \sum_{\nu=1}^{Nn+1} \mu_\nu \xi_\nu \,.$$

Remark 63 *The theorem is just a curiosity in the vectorial case* $n, N > 1$. *However in the scalar case* $n > N = 1$ *under some extra hypotheses (cf. Theorem 66), one of them being*

$$\xi_0 \in \operatorname{int}\operatorname{co}\{\xi_1, \cdots, \xi_{n+1}\}\,,$$

it turns out that the necessary condition is also sufficient. But it is in the case $N \geq n = 1$ *that it is particularly interesting since then this condition is also sufficient, cf. Theorem 64.*

Proof. We decompose the proof into three steps.

Step 1. Let $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}\left(\Omega;\mathbb{R}^N\right)$ be a solution of (P). It should therefore satisfy

$$\frac{1}{\operatorname{meas}\Omega}\int_\Omega f(D\overline{u}(x))\,dx = \inf(P) = \inf(QP) = Qf(\xi_0)\,. \tag{19}$$

Let $r = \|\overline{u}\|_{W^{1,\infty}}$ and use the fact that f is locally bounded to find $R = R(r)$ so that

$$0 \leq f(D\overline{u}(x)) \leq R, \text{ a.e. } x \in \Omega.$$

Denote by

$$\begin{aligned} K_r &= \left\{(\xi, y) \in \mathbb{R}^{N\times n} \times \mathbb{R} : |\xi| \leq r \text{ and } |y| \leq R\right\} \\ \operatorname{epi} f &= \left\{(\xi, y) \in \mathbb{R}^{N\times n} \times \mathbb{R} : f(\xi) \leq y\right\} \\ E &= \operatorname{epi} f \cap K_r\,. \end{aligned}$$

Note that since f is lower semicontinuous then $\operatorname{epi} f$ is closed and hence E is compact. Therefore its convex hull $\operatorname{co} E$ is also compact.

Observe that, for almost every $x \in \Omega$, we have

$$(D\overline{u}(x), f(D\overline{u}(x))) \in E$$

and thus by Jensen inequality and (19) we deduce that

$$(\xi_0, Qf(\xi_0)) = \frac{1}{\operatorname{meas}\Omega}\int_\Omega (D\overline{u}(x), f(D\overline{u}(x)))\,dx \in \operatorname{co} E.$$

Appealing to Carathéodory theorem we can find $\lambda \in \Lambda_{Nn+2}$, $(\xi_i, y_i) \in E$, $1 \leq i \leq Nn+2$ (in particular $f(\xi_i) \leq y_i$) such that

$$Qf(\xi_0) = \sum_{i=1}^{Nn+2} \lambda_i y_i \geq \sum_{i=1}^{Nn+2} \lambda_i f(\xi_i) \text{ and } \xi_0 = \sum_{i=1}^{Nn+2} \lambda_i \xi_i \,.$$

(Note, in passing, that if f is continuous, we can replace in the above argument $\operatorname{epi} f$ by

$$\operatorname{graph} f = \left\{ (x,y) \in \mathbb{R}^{N\times n} \times \mathbb{R} : f(x) = y \right\}$$

obtaining therefore equality instead of inequality in the above statement.)

Step 2. To obtain the theorem it therefore remains to show that one can take only $(Nn+1)$ elements. This is a classical procedure in convex analysis. The result is equivalent to showing that there exist μ_i, $1 \leq i \leq Nn+2$, such that

$$\begin{cases} \mu_i \geq 0, \ \sum\limits_{i=1}^{Nn+2} \mu_i = 1, \text{ at least one of the } \mu_i = 0 \\ \sum\limits_{i=1}^{Nn+2} \mu_i f(\xi_i) \leq \sum\limits_{i=1}^{Nn+2} \lambda_i f(\xi_i) \,, \ \xi_0 = \sum\limits_{i=1}^{Nn+2} \mu_i \xi_i \,. \end{cases} \tag{20}$$

meaning in fact that $\mu \in \Lambda_{Nn+1}$ as wished

Assume that $\lambda_i > 0$, $1 \leq i \leq Nn+2$, otherwise nothing is to be proved. Observe first that $\xi_0 \in \operatorname{co}\{\xi_1, \cdots, \xi_{Nn+2}\} \subset \mathbb{R}^{N\times n}$. Thus it follows from Carathéodory theorem that there exist $\nu \in \Lambda_{Nn+2}$, with at least one of the $\nu_i = 0$, (i.e. $\nu \in \Lambda_{Nn+1}$) such that

$$\xi_0 = \sum_{i=1}^{Nn+2} \nu_i \xi_i \,.$$

Assume, without loss of generality, that

$$\sum_{i=1}^{Nn+2} \nu_i f(\xi_i) > \sum_{i=1}^{Nn+2} \lambda_i f(\xi_i) \,; \tag{21}$$

otherwise choosing $\mu_i = \nu_i$ we would have immediately (20). Let

$$J = \{ i \in \{1, \cdots, Nn+2\} : \lambda_i - \nu_i < 0 \} \,.$$

Observe that $J \neq \emptyset$, since otherwise $\lambda_i \geq \nu_i \geq 0$ for every i and since at least one of the $\nu_i = 0$, we would have a contradiction with $\sum \nu_i = \sum \lambda_i = 1$ and $\lambda_i > 0$ for every i. We then define

$$\gamma = \min_{i \in J} \left\{ \frac{\lambda_i}{\nu_i - \lambda_i} \right\}.$$

We clearly have that $\gamma > 0$. Finally let

$$\mu_i = \lambda_i + \gamma(\lambda_i - \nu_i) \,, \ 1 \leq i \leq Nn+2.$$

We immediately get that

$$\mu_i \geq 0, \ \sum_{i=1}^{Nn+2} \mu_i = 1, \text{ at least one of the } \mu_i = 0. \tag{22}$$

From (21) we obtain

$$\begin{aligned}\sum_{i=1}^{Nn+2} \mu_i f(\xi_i) &= \sum_{i=1}^{Nn+2} \lambda_i f(\xi_i) + \gamma \left(\sum_{i=1}^{Nn+2} \lambda_i f(\xi_i) - \sum_{i=1}^{Nn+2} \nu_i f(\xi_i) \right) \\ &\leq \sum_{i=1}^{Nn+2} \lambda_i f(\xi_i) .\end{aligned}$$

The combination of the above with (22) (assuming for the sake of notations that $\mu_{Nn+2} = 0$) gives immediately

$$Qf(\xi_0) \geq \sum_{i=1}^{Nn+1} \mu_i f(\xi_i) \text{ and } \xi_0 = \sum_{i=1}^{Nn+1} \mu_i \xi_i .$$

Step 3. The result for the scalar case follows from the fact that $Qf(\xi_0) = Cf(\xi_0)$ and from Theorem 6. ■

6 The scalar case

We now see how to apply the above abstract considerations to the case where either $n = 1$ or $N = 1$. We recall that

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(Du(x))\, dx : u \in u_{\xi_0} + W_0^{1,\infty}\left(\Omega; \mathbb{R}^N\right) \right\}.$$

We will first treat the more elementary case where $n = 1$ and then the case $N = 1$.

6.1 The case of single integrals

In this very elementary case we can get much simpler and sharper results.

Theorem 64 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be non negative, locally bounded and lower semicontinuous. Let $a < b$, $\alpha, \beta \in \mathbb{R}^N$, $N \geq 1$, and*

$$(P) \qquad \inf \left\{ I(u) = \int_a^b f(u'(x))\, dx : u \in X \right\}$$

where

$$X = \left\{ u \in W^{1,\infty}\left((a,b); \mathbb{R}^N\right) : u(a) = \alpha,\ u(b) = \beta \right\}.$$

The two following statements are then equivalent:

(i) problem (P) has a minimizer;

(ii) there exist $\lambda_\nu \geq 0$ *with* $\sum_{\nu=1}^{N+1} \lambda_\nu = 1$, $\gamma_\nu \in \mathbb{R}^N$, $1 \leq \nu \leq N+1$ *such that*

$$Cf\left(\frac{\beta-\alpha}{b-a}\right) = \sum_{\nu=1}^{N+1} \lambda_\nu f(\gamma_\nu) \text{ and } \frac{\beta-\alpha}{b-a} = \sum_{\nu=1}^{N+1} \lambda_\nu \gamma_\nu \tag{23}$$

where $Cf = \sup\{g \leq f : g \text{ convex}\}$.

Furthermore if (23) is satisfied and if

$$I_p = \left[a + (b-a)\sum_{\nu=1}^{p-1} \lambda_\nu, a + (b-a)\sum_{\nu=1}^{p} \lambda_\nu\right],\ 1 \leq p \leq N+1$$

then

$$\overline{u}(x) = \gamma_p(x-a) + (b-a)\sum_{\nu=1}^{p} \lambda_\nu(\gamma_\nu - \gamma_p) + \alpha,\ x \in I_p,\ 1 \leq p \leq N+1$$

is a solution of (P).

Remark 65 *(i) The sufficiency of (23) is implicitly or explicitly proved in the papers mentioned in the bibliography. The necessity is less known but is also implicit in the literature. The theorem as stated can be found in Dacorogna [25].*

(ii) Recall that by Carathéodory theorem (cf. Theorem 6) we always have

$$Cf\left(\frac{\beta-\alpha}{b-a}\right) = \inf\left\{\sum_{\nu=1}^{N+1} \lambda_\nu f(\gamma_\nu) : \sum_{\nu=1}^{N+1} \lambda_\nu \gamma_\nu = \frac{\beta-\alpha}{b-a}\right\}. \tag{24}$$

Therefore (23) states that a necessary and sufficient condition for existence of solutions is that the infimum in (24) be attained. Note also that if f *is convex* or f *coercive (in the sense that* $f(\xi) \geq a|\xi|^p + b$ *with* $p > 1$, $a > 0$*) then the infimum in (24) is always attained.*

(iii) Therefore if $f(x,u,\xi) = f(\xi)$, *counterexamples to existence must be non convex and non coercive; cf. Example 1 where*

$$(P) \qquad \inf\left\{I(u) = \int_0^1 e^{-\left(u'(x)\right)^2} dx : u \in W_0^{1,\infty}(0,1)\right\}$$

i.e. $f(\xi) = e^{-\xi^2}$, *then* $Cf(\xi) \equiv 0$ *and therefore by the relaxation theorem*

$$\inf(P) = \inf(QP) = 0.$$

However it is obvious that $I(u) \neq 0$ *for every* $u \in W_0^{1,\infty}(0,1)$ *and hence the infimum of (P) is not attained.*

(iv) A similar proof to that of Theorem 64 (see for example Marcellini [50]) shows that a sufficient condition to ensure existence of minima to

$$(P) \qquad \inf\left\{I(u) = \int_a^b f(x, u'(x))\, dx : u \in X\right\}$$

is (23) where λ_ν *and* γ_ν *are then measurable functions. Of course if* f *depends explicitly on* u, *the example of Bolza (cf. Example 2) shows that the theorem is then false.*

Proof. (Theorem 64). It is easy to see that we can reduce our study to the case where

$$a = 0,\ b = 1 \text{ and } \alpha = 0.$$

Sufficient condition. The sufficiency part is elementary. Let

$$(QP) \qquad \inf \left\{ \overline{I}(u) = \int_0^1 Cf(u'(x))\, dx : u \in X \right\}$$

where now

$$X = \left\{ u \in W^{1,\infty}\left((0,1);\mathbb{R}^N\right) : u(0) = 0,\ u(1) = \beta \right\}.$$

Then $\widetilde{u}(x) = \beta x$ is trivially a solution of (QP) and therefore

$$\inf (QP) = Cf(\beta).$$

Let now $\overline{u}$ be as in the statement of the theorem. Observe first that $\overline{u} \in W^{1,\infty}\left((0,1);\mathbb{R}^N\right)$ and $\overline{u}(0) = 0$, $\overline{u}(1) = \beta$. We now compute

$$\begin{aligned} \overline{I}(\overline{u}) &= \int_0^1 f(\overline{u}'(x))\, dx = \sum_{p=1}^{N+1} \int_{I_p} f(\overline{u}'(x))\, dx = \sum_{p=1}^{N+1} f(\gamma_p) \operatorname{meas} I_p \\ &= \sum_{p=1}^{N+1} \lambda_p f(\gamma_p) = Cf(\beta) = \inf (QP) \le \inf (P). \end{aligned}$$

Necessary condition. This has already been proved in Theorem 62. ■

6.2 The case of multiple integrals

We now discuss the case $n > N = 1$. This is of course a more difficult case than the preceding one and no such simple result as Theorem 64 is available. However we immediately have from Sections 5.2 and 5.3 (Theorem 29 and Corollary 55) the theorem stated below. For some historical comments on this theorem, see the remark following Corollary 55.

But let us first recall the problem and the notations. We have

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(Du(x))\, dx :\ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega) \right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, u_{ξ_0} is affine, i.e. $Du_{\xi_0} = \xi_0$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a lower semicontinuous, locally bounded and non negative function. Let

$$Cf = \sup \{ g \le f : g \text{ convex} \}.$$

In order to avoid the trivial situation we assume that

$$Cf(\xi_0) < f(\xi_0).$$

We next set

$$K = \{ \xi \in \mathbb{R}^n : Cf(\xi) < f(\xi) \}$$

and we assume that it is connected, otherwise we replace it by its connected component that contains ξ_0.

Theorem 66 Necessary condition. *If (P) has a minimizer, then Cf is affine in a neighborhood of ξ_0.*

Sufficient condition. *If there exists $E \subset \partial K$ such that $\xi_0 \in \operatorname{int} \operatorname{co} E$ and $Cf|_{E\cup\{\xi_0\}}$ is affine, then (P) has a solution.*

Remark 67 *(i) By $Cf|_{E\cup\{\xi_0\}}$ affine we mean that there exist $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ such that*

$$Cf(\xi) = \langle \alpha; \xi \rangle + \beta \text{ for every } \xi \in E \cup \{\xi_0\}.$$

Usually one proves that Cf is affine on the whole of $\operatorname{co} E$.

(ii) The theorem applies, of course, to the case where $E = \partial K$ and Cf is affine on the whole of K (since K is open and $\xi_0 \in K \subset \operatorname{int} \operatorname{co} K$). However in many simple examples such as the one given below, it is not realistic to assume that $E = \partial K$.

Proof. The necessary part is just Corollary 55. We therefore discuss only the sufficient part. We use Theorem 29 to find $\overline{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ such that

$$D\overline{u}(x) \in E \subset \partial K, \text{ a.e. } x \in \Omega$$

and hence

$$f(D\overline{u}(x)) = Cf(D\overline{u}(x)), \text{ a.e. } x \in \Omega.$$

Then use the fact that $Cf|_{E\cup\{\xi_0\}}$ is affine to deduce that

$$\int_\Omega Cf(D\overline{u}(x))\, dx = Cf(\xi_0) \operatorname{meas} \Omega.$$

The conclusion then follows from Theorem 41. ■

We now would like to give two simple examples. The first one generalizes Example 3.

Example 68 *Let $N = 1$, $n = 2$, $\Omega = (0,1)^2$, $u_0 \equiv 0$, $a \geq 0$ and*

$$f(\xi) = \left(\xi_1^2 - 1\right)^2 + \left(\xi_2^2 - a^2\right)^2.$$

We find that

$$Cf(\xi) = \left[\xi_1^2 - 1\right]_+^2 + \left[\xi_2^2 - a^2\right]_+^2$$

where

$$[x]_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

We therefore have that

$$K = \left\{\xi \in \mathbb{R}^2 : \xi_1^2 < 1 \text{ or } \xi_2^2 < a^2\right\}$$

and note that it is unbounded and that Cf is not affine on the whole of K.

Let us discuss the two different cases.

Case 1: $a = 0$. *This corresponds to Example 3. Then clearly* Cf *is not affine in the neighborhood of* $\xi_0 = 0$, *since it is strictly convex in the direction* $e_2 = (0,1)$. *Hence (P) has no solution.*

Case 2: $a > 0$. *We let*

$$E = \left\{ \xi \in \mathbb{R}^2 : |\xi_1| = 1 \text{ and } |\xi_2| = a \right\} \subset \partial K.$$

Note that $\xi_0 = 0 \in \operatorname{int} \operatorname{co} E$ *and* $Cf|_{\operatorname{co} E} \equiv 0$ *is affine. Therefore the theorem applies and we obtain that (P) has a solution.*

Example 69 *We conclude with the following example (cf. Marcellini [51] and Dacorogna-Marcellini [27]). Let* $n \geq 2$ *and*

$$f(Du) = g(|Du|)$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ *is lower semicontinuous, locally bounded and non negative with*

$$g(0) = \inf \{ g(t) : t \geq 0 \}.$$

It is easy to see that $Cf = Cg$. *Let*

$$\begin{aligned} S &= \{ t \geq 0 : Cg(t) < g(t) \} \\ K &= \{ \xi \in \mathbb{R}^n : Cf(\xi) < f(\xi) \} = \{ \xi \in \mathbb{R}^n : |\xi| \in S \}. \end{aligned}$$

Assume that $\xi_0 \in K$ *and that* S *is connected, otherwise replace it by its connected component containing* $|\xi_0|$.

We then have to consider two cases.

Case 1: Cg *is strictly increasing at* $|\xi_0|$. *Then clearly* Cf *is not affine in any neighborhood of* ξ_0 *and hence (P) has no solution.*

Case 2: Cg *is constant on* S. *Assume that* S *is bounded, this can be guaranteed if, for example,*

$$\lim_{t \to +\infty} \frac{g(t)}{t} = +\infty.$$

So let $|\xi_0| \in S = (\alpha, \beta)$ *and choose in the sufficient part of the theorem*

$$E = \{ \xi \in \mathbb{R}^n : |\xi| = \beta \}$$

and apply the theorem to find a minimizer for (P).

7 The vectorial case

We now consider several examples of the form studied in the previous sections, namely

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(Du(x))\, dx : \ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N) \right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, u_{ξ_0} is affine, i.e. $Du_{\xi_0} = \xi_0$ and $f : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ is a lower semicontinuous, locally bounded and non negative function.

1) We consider in Subsection 7.1 the case where $N = n$ and

$$f(\xi) = g(\lambda_2(\xi), \cdots, \lambda_{n-1}(\xi), \det \xi)$$

where $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ are the singular values of $\xi \in \mathbb{R}^{n\times n}$.

2) In Subsection 7.2, we deal with the case

$$f(\xi) = g(\Phi(\xi))$$

where $\Phi : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ is quasiaffine (so in particular we can have, when $N = n$, $\Phi(\xi) = \det \xi$, as in the previous case).

3) We next discuss in Subsection 7.3 the Saint Venant-Kirchhoff energy functional. Up to rescaling, the function under consideration is (here $N = n$ and $\nu \in (0, 1/2)$ is a parameter)

$$f(\xi) = \left|\xi\xi^t - I\right|^2 + \frac{\nu}{1-2\nu}\left(|\xi|^2 - n\right)^2$$

or in terms of the singular values, $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$, of $\xi \in \mathbb{R}^{n\times n}$

$$f(\xi) = \sum_{i=1}^{n}\left(\lambda_i^2 - 1\right)^2 + \frac{\nu}{1-2\nu}\left(\sum_{i=1}^{n}\lambda_i^2 - n\right)^2.$$

4) In Subsection 7.4 we consider a problem of optimal design where $N = n = 2$ and

$$f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases}$$

5) In Subsection 7.5 we deal with the minimal surface case, namely when $N = n + 1$ and $f(\xi) = g(\operatorname{adj}_n \xi)$.

6) Finally in Subsection 7.6 we discuss the problem of potential wells.

7.1 The case of singular values

In this section we let $N = n$ and we denote by $\lambda_1(\xi), \cdots, \lambda_n(\xi)$ the singular values of $\xi \in \mathbb{R}^{n\times n}$ with $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ and by Q the set

$$Q = \{x = (x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-2} : \ 0 \leq x_2 \leq \cdots \leq x_{n-1}\}$$

which is the natural set where to consider $(\lambda_2(\xi), \cdots, \lambda_{n-1}(\xi))$ for $\xi \in \mathbb{R}^{n\times n}$.

The functions under consideration are of the form studied in Theorem 10, namely

$$f(\xi) = g(\lambda_2(\xi), \cdots, \lambda_{n-1}(\xi), \det \xi)$$

and we have

$$Pf(\xi) = Qf(\xi) = Rf(\xi) = Ch(\det \xi),$$

where $h : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $h(s) = \inf_{x \in Q} g(x, s)$.

We next apply the theory of Section 5.2 to get the following existence result, established by Dacorogna-Pisante-Ribeiro [34].

Theorem 70 *Let*

$$f(\xi) = g(\lambda_2(\xi), \cdots, \lambda_{n-1}(\xi)) + h(\det \xi)$$

where $g : Q \longrightarrow \mathbb{R}$ *is non negative, lower semicontinuous, locally bounded and verifies*

$$\inf g = g(m_2, \cdots, m_{n-1}), \text{ with } 0 < m_2 \leq \cdots \leq m_{n-1}$$

and $h : \mathbb{R} \longrightarrow \mathbb{R}$ *is a non negative, lower semi-continuous and locally bounded function such that*

$$\lim_{|t| \to +\infty} \frac{h(t)}{|t|} = +\infty. \tag{25}$$

Then (P) *has a solution.*

Proof. We note that, by Theorem 10, $Qf(\xi) = \inf g + Ch(\det \xi)$. Letting

$$K = \left\{ \xi \in \mathbb{R}^{n \times n} : Qf(\xi) < f(\xi) \right\}$$

we see that

$$K = L_1 \cup L_2$$

where

$$L_1 = \left\{ \xi \in \mathbb{R}^{n \times n} : \ Ch(\det \xi) < h(\det \xi) \right\}$$

$$L_2 = \left\{ \xi \in \mathbb{R}^{n \times n} : \ Ch(\det \xi) = h(\det \xi), \ \inf g < g(\lambda_2(\xi), ..., \lambda_{n-1}(\xi)) \right\}.$$

We now prove the result. Clearly, if $\xi_0 \notin K$ then u_{ξ_0} is a solution of (P), so from now on we assume that $\xi_0 \in K$. There are three different cases to consider, one of them will be treated with Theorem 42 and the two others with Theorem 41.

Case 1: $\xi_0 \in L_1$. We first observe that hypothesis (25) allows us to write

$$S = \{t \in \mathbb{R} : Ch(t) < h(t)\} = \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j),$$

Ch being affine in each interval (α_j, β_j); thus Qf is quasiaffine on each connected component of L_1 and

$$L_1 = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi \in \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j) \right\}.$$

Let (α_j, β_j) be an interval as above such that $\det \xi_0 \in (\alpha_j, \beta_j)$. We get the result applying Theorem 42 with

$$K_0 = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi \in (\alpha_j, \beta_j) \, , \ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} m_i, \ \nu = 2, ..., n \right\},$$

where m_n is chosen sufficiently large so that

$$m_{n-1} \leq m_n, \tag{26}$$

$$\prod_{i=\nu}^{n} \lambda_i(\xi_0) < \prod_{i=\nu}^{n} m_i, \ \nu = 2, ..., n, \tag{27}$$

$$\max \{|\alpha_j|, |\beta_j|\} < m_2 \prod_{i=2}^{n} m_i. \tag{28}$$

Clearly $K_0 \subset L_1 \subset K$, moreover (27) ensures that $\xi_0 \in K_0$ and (28) ensures the relaxation property of K_0 with respect to

$$E = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi \in \{\alpha_j, \beta_j\} \, , \ \lambda_\nu(\xi) = m_\nu, \ \nu = 2, ..., n \right\} \subset \overline{K}_0 \cap \partial K$$

through Theorems 21, 28 and the family of sets

$$E_\delta = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi \in \{\alpha_j + \delta, \beta_j - \delta\} \, , \ \lambda_i(\xi) = m_i - \delta, \ i = 2, ..., n \right\},$$

(cf. the proof of Theorem 1.1 of Dacorogna-Ribeiro [35] for details). Consequently K_0 has the relaxation property with respect to $\overline{K}_0 \cap \partial K$.

Case 2: $\xi_0 \in L_2$ and $\det \xi_0 \neq 0$. We consider in this case the set

$$K_1 = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi = \det \xi_0, \ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} m_i, \ \nu = 2, ..., n \right\}$$

where m_n satisfies the conditions (26) and (27) of the first case (with strict inequality for the first one: $m_n > m_{n-1}$). It was shown by Dacorogna-Tanteri [37] that K_1 has the relaxation property with respect to

$$E = \left\{ \xi \in \mathbb{R}^{n \times n} : \ \det \xi = \det \xi_0, \ \lambda_\nu(\xi) = m_\nu, \ \nu = 2, ..., n \right\}$$

and moreover there exists $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ such that $Du \in E$, a.e. in Ω. Since $Qf = f$ in E and $Qf(\xi_0) = Qf(Du)$, we can apply Theorem 41 and get the result.

Case 3: $\xi_0 \in L_2$ and $\det \xi_0 = 0$. We here just briefly outline the idea and we refer to Dacorogna-Pisante-Ribeiro [34] for details. Since any matrix $\xi \in \mathbb{R}^{n \times n}$ can be decomposed in the form RDQ, where $R, Q \in O(n)$ and $D = diag(\lambda_1(\xi), ..., \lambda_n(\xi))$ (cf. [45]) we can reduce ourselves to the case of

$\xi_0 = diag(\lambda_1(\xi_0), ..., \lambda_n(\xi_0))$. In particular, as $\det \xi_0 = 0$, we have $\lambda_1(\xi_0) = 0$ and thus the first line of ξ_0 equal to zero. Let $m_n \geq m_{n-1}$ and define

$$K_1 = \left\{ \xi \in \mathbb{R}^{n\times n} :\ \xi^1 = 0,\ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} m_i,\ \nu = 2, ..., n \right\}$$
$$E = \left\{ \xi \in \mathbb{R}^{n\times n} :\ \xi^1 = 0,\ \lambda_i(\xi) = m_i,\ i = 2, ..., n \right\}$$

we get that K_1 has the relaxation property with respect to E. If we choose m_n sufficiently large such that $\xi_0 \in K_1$ we can apply Theorem 25 to get the existence of $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ such that $Du \in E$. Finally, as $Qf = f$ in E and $Qf(\xi_0) = Qf(Du)$, applying Theorem 41, we conclude the proof. ■

7.2 The case of quasiaffine functions

We next study the minimization problem

$$(P) \qquad \inf \left\{ \int_\Omega g(\Phi(Du(x)))\, dx :\ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N) \right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $Du_{\xi_0} = \xi_0$ and

– $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a lower semicontinuous, locally bounded and non negative function,

– $\Phi : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ is quasiaffine and non constant.

We recall that in particular we can have, when $N = n$, $\Phi(\xi) = \det \xi$.

The relaxed problem is then

$$(QP) \qquad \inf \left\{ \int_\Omega Cg(\Phi(Du(x)))\, dx :\ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N) \right\},$$

where Cg is the convex envelope of g (here $f(\xi) = g(\Phi(\xi))$ and we get $Qf = Cg$, cf. Theorem 9).

The existence result is the following.

Theorem 71 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $g : \mathbb{R} \longrightarrow \mathbb{R}$ a non negative, lower semicontinuous and locally bounded function such that*

$$\lim_{|t|\to+\infty} \frac{g(t)}{|t|} = +\infty \tag{29}$$

and $u_{\xi_0}(x) = \xi_0 x$, with $\xi_0 \in \mathbb{R}^{N\times n}$. Then there exists $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ solution of (P).

Remark 72 *This result has first been established by Mascolo-Schianchi [55] and later by Dacorogna-Marcellini [27] for the case of the determinant. The general case is due to Cellina-Zagatti [19] and later to Dacorogna-Ribeiro [35]. Here we see that it can be obtained as a particular case of Theorem 42.*

Proof. We will here only sketch the proof and we refer for details to Dacorogna-Ribeiro [35]. We first let

$$S = \{t \in \mathbb{R} : \; Cg(t) < g(t)\}.$$

From the hypothesis on g we can write

$$S = \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j)$$

with Cg affine in each interval (α_j, β_j) and thus Qf is quasiaffine on each connected component of K, where

$$K = \left\{\xi \in \mathbb{R}^{N \times n} : \; \Phi(\xi) \in S\right\}.$$

If $\Phi(\xi_0) \notin S$ then u_{ξ_0} is a solution of (P). In the other case, $\Phi(\xi_0) \in (\alpha, \beta) \subset S$ for some α and β and we apply Theorem 42 with

$$K_0 = \left\{\xi \in \mathbb{R}^{N \times n} : \; \Phi(\xi) \in (\alpha, \beta), \; \left|\xi_j^i\right| < c_j^i, \; i = 1, \cdots, N, \; j = 1, \cdots, n\right\},$$

where c_j^i are constants sufficiently large so that $\xi_0 \in K_0$ and satisfying

$$\inf\{|\Phi(\xi)| : \; \left|\xi_j^i\right| = c_j^i\} > \max\{|\alpha|, |\beta|\}.$$

This condition allows us to obtain the relaxation property of K_0 with respect to

$$\overline{K}_0 \cap \partial K = \left\{\xi \in \mathbb{R}^{N \times n} : \; \Phi(\xi) \in \{\alpha, \beta\}, \; \left|\xi_j^i\right| \leq c_j^i, \; i = 1, \cdots, N, \; j = 1, \cdots, n\right\}.$$

The relaxation property is obtained using the approximation property (cf. Definition 27 and Theorem 28) considering the sets, here $\delta > 0$ is sufficiently small,

$$H_\delta = \left\{\xi \in \mathbb{R}^{N \times n} : \begin{array}{c} \Phi(\xi) \in \{\alpha + \delta, \beta - \delta\}, \\ \\ \left|\xi_j^i\right| \leq c_j^i - \delta, \; i = 1, \cdots, N, \; j = 1, \cdots, n \end{array}\right\}.$$

This concludes the proof of the theorem.

The problem under consideration is sufficiently flexible that we could also proceed as in Dacorogna-Marcellini [27], using Corollary 47. Indeed if $D\Phi(\xi_0) \neq 0$ (in the case $\Phi(\xi) = \det \xi$ this means that $\operatorname{rank} \xi_0 \geq n - 1$), we can apply the corollary, since the connected component of K containing ξ_0 is bounded, in the neighborhood of ξ_0, in a direction of rank one. We do not discuss the details of this different approach. ■

7.3 The Saint Venant Kirchhoff energy

The problem is now of the form

$$(P) \qquad \inf\left\{\int_\Omega f\left(Du\left(x\right)\right) dx : \; u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^n)\right\},$$

where, upon rescaling, the function under consideration is, $\nu \in (0, 1/2)$ being a parameter,

$$f(\xi) = \left|\xi\xi^t - I\right|^2 + \frac{\nu}{1-2\nu}\left(|\xi|^2 - n\right)^2$$

or in terms of the singular values $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ of $\xi \in \mathbb{R}^{n\times n}$

$$f(\xi) = \sum_{i=1}^{n}\left(\lambda_i^2 - 1\right)^2 + \frac{\nu}{1-2\nu}\left(\sum_{i=1}^{n}\lambda_i^2 - n\right)^2.$$

According to Le Dret-Raoult [49] the quasiconvex envelope and the convex envelope coincide, at least when $n = 2$ or $n = 3$, i.e.

$$Qf(\xi) = Cf(\xi).$$

In the case $n = 2$ it is given by

$$Qf(\xi) = \begin{cases} f(\xi) & \text{if } \xi \notin D_1 \cup D_2 \\ \frac{1}{1-\nu}\left(\lambda_2^2 - 1\right)^2 & \text{if } \xi \in D_2 \\ 0 & \text{if } \xi \in D_1 \end{cases}$$

where

$$\begin{aligned} D_1 &= \left\{\xi \in \mathbb{R}^{2\times 2} : (1-\nu)\left[\lambda_1(\xi)\right]^2 + \nu\left[\lambda_2(\xi)\right]^2 < 1 \text{ and } \lambda_2(\xi) < 1\right\} \\ &= \left\{\xi \in \mathbb{R}^{2\times 2} : \lambda_1(\xi) \leq \lambda_2(\xi) < 1\right\} \\ D_2 &= \left\{\xi \in \mathbb{R}^{2\times 2} : (1-\nu)\left[\lambda_1(\xi)\right]^2 + \nu\left[\lambda_2(\xi)\right]^2 < 1 \text{ and } \lambda_2(\xi) \geq 1\right\}. \end{aligned}$$

The existence theorem is then.

Theorem 73 *Let $\Omega \subset \mathbb{R}^2$, f and ξ_0 be as above.*

(i) If $\xi_0 \notin D_2$ then (P) has a solution.

(ii) If $\xi_0 \in \operatorname{int} D_2$ then (P) has no solution.

Remark 74 *The non existence part has been proved by Dacorogna-Marcellini in [27].*

Proof. (i) The case where $\xi_0 \notin D_1 \cup D_2$ corresponds to the trivial case where $Qf(\xi_0) = f(\xi_0)$. So we now assume that $\xi_0 \in D_1$. Note that Qf is quasiaffine on D_1 (in fact $Qf(\xi) \equiv 0$). Apply then Theorem 33 (and the remark following it) to get $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^2)$ such that

$$\lambda_1(Du) = \lambda_2(Du) = 1, \text{ a.e. in } \Omega.$$

This implies that $Qf(Du) = f(Du) = Qf(\xi_0) = 0$ and hence the claim follows from Theorem 41.

(ii) It was shown in [27], and we do not discuss here the details, that if $\xi_0 \in \operatorname{int} D_2$ then the function Qf is strictly quasiconvex at ξ_0 and therefore (P) has no solution. ■

7.4 An optimal design problem

We now consider the case, studied by many authors following the pioneering work of Kohn-Strang [48], where

$$(P) \qquad \inf\left\{\int_{\Omega} f\left(Du\left(x\right)\right)\,dx:\ u\in u_{\xi_0}+W_0^{1,\infty}(\Omega;\mathbb{R}^2)\right\},$$

$\Omega\subset\mathbb{R}^2$ is a bounded open set with Lipschitz boundary, $Du_{\xi_0}=\xi_0$, and

$$f\left(\xi\right)=\begin{cases} 1+\left|\xi\right|^2 & \text{if } \xi\neq 0 \\ 0 & \text{if } \xi=0. \end{cases}$$

It was shown by Kohn-Strang [48] that the quasiconvex envelope is then

$$Qf\left(\xi\right)=\begin{cases} 1+\left|\xi\right|^2 & \text{if } \left|\xi\right|^2+2\left|\det\xi\right|\geq 1 \\ 2\left(\left|\xi\right|^2+2\left|\det\xi\right|\right)^{1/2}-2\left|\det\xi\right| & \text{if } \left|\xi\right|^2+2\left|\det\xi\right|<1. \end{cases}$$

The existence of minimizers for problem (P) was then established by Dacorogna-Marcellini in [27] and [31]. Later Dacorogna-Tanteri [37] gave a different proof which is more in the spirit of the present report and we follow here this last approach.

Theorem 75 *Let $\Omega\subset\mathbb{R}^2$, f and ξ_0 be as above. Then a necessary and sufficient condition for (P) to have a solution is that one of the following conditions hold:*

(i) $\xi_0=0$ *or* $\left|\xi_0\right|^2+2\left|\det\xi_0\right|\geq 1$, *(i.e.* $f\left(\xi_0\right)=Qf\left(\xi_0\right)$*)*

(ii) $\det\xi_0\neq 0$.

Proof. We do not discuss the details and in particular not the necessary part (see [27] for details). So we assume that we are in the non trivial case

$$\det\xi_0\neq 0 \text{ and } \left|\xi_0\right|^2+2\left|\det\xi_0\right|<1. \tag{30}$$

We just point out how to define the set K_0 of Theorem 42. We have (denoting by $\mathbb{R}_s^{2\times 2}$ the set of 2×2 symmetric matrices)

$$\begin{aligned} K &= \left\{\xi\in\mathbb{R}^{2\times 2}:\left|\xi\right|^2+2\left|\det\xi\right|<1\right\}\setminus\{0\} \\ K_0 &= \left\{\xi\in\mathbb{R}_s^{2\times 2}:\det\xi>0 \text{ and } \operatorname{trace}\xi\in(0,1)\right\} \\ \overline{K}_0\cap\partial K &= \{0\}\cup\left\{\xi\in\mathbb{R}_s^{2\times 2}:\det\xi\geq 0 \text{ and } \operatorname{trace}\xi=1\right\} \\ &= \left\{\xi\in\mathbb{R}_s^{2\times 2}:\det\xi\geq 0 \text{ and } \operatorname{trace}\xi\in\{0,1\}\right\}. \end{aligned}$$

Since f is invariant under rotations and symmetries and (30) holds, we can assume, without loss of generality, that $\xi_0\in K_0$. Furthermore Qf is quasiaffine on K_0 ($Qf\left(\xi\right)=2\operatorname{trace}\xi-2\det\xi$), while it is not so on K. It remains to prove that K_0 has the relaxation property with respect to $\overline{K}_0\cap\partial K$; and this is easily established as in Dacorogna-Tanteri [37]. ■

7.5 The minimal surface case

Following Dacorogna-Pisante-Ribeiro [34], we now deal with the case where $N = n + 1$ and

$$f(\xi) = g(\mathrm{adj}_n \xi).$$

The minimization problem is then

$$(P) \qquad \inf \left\{ \int_\Omega g(\mathrm{adj}_n (Du(x)))\, dx : \; u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^{n+1}) \right\}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $Du_{\xi_0} = \xi_0$ and $g : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is a non negative, lower semicontinuous and locally bounded non convex function.

From Theorem 9 we have

$$Qf(\xi) = Cg(\mathrm{adj}_n \xi).$$

We next set

$$S = \{ y \in \mathbb{R}^{n+1} : \; Cg(y) < g(y) \}$$

and assume, in order to avoid the trivial situation, that $\mathrm{adj}_n \xi_0 \in S$. We also assume that S is connected, otherwise we replace it by its connected component that contains $\mathrm{adj}_n \xi_0$.

Observe that

$$K = \{ \xi \in \mathbb{R}^{(n+1)\times n} : \; Qf(\xi) < f(\xi) \} = \left\{ \xi \in \mathbb{R}^{(n+1)\times n} : \mathrm{adj}_n \xi \in S \right\}.$$

Theorem 76 *If S is bounded, Cg is affine in S and* rank $\xi_0 \geq n - 1$, *then (P) has a solution.*

Remark 77 *The fact that Cg be affine in S is not a necessary condition for existence of minima, as seen in Proposition 78.*

Proof. The result follows if we choose a convenient rank one direction $\lambda = \alpha \otimes \beta \in \mathbb{R}^{(n+1)\times n}$ satisfying the hypothesis of Corollary 47. We remark that, since we suppose Cg affine in S, Qf is quasiaffine in $L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda)$ (cf. Notation 44 and Definition 45) independently of the choice of λ. So we only have to prove that K is stably bounded at ξ_0 in a direction $\lambda = \alpha \otimes \beta$.

Firstly we observe that we can find (cf. Theorem 3.1.1 in [45]) $P \in O\,(n+1)$, $Q \in SO\,(n)$ and $0 \leq \lambda_1 \leq ... \leq \lambda_n$, so that

$$\xi_0 = PLQ, \text{ where } L = (\lambda_j \delta_{ij})_{1 \leq j \leq n}^{1 \leq i \leq n+1} \;;$$

in particular when $n = 2$ we have

$$L = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 0 & 0 \end{pmatrix}.$$

Since rank $\xi_0 \geq n-1$ we have that $\lambda_2 > 0$. We also note that

$$adj_n \xi_0 = adj_n P \, . \, adj_n L \text{ and } adj_n L = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^n \lambda_1 ... \lambda_n \end{pmatrix}.$$

Without loss of generality we assume $\xi_0 = L$. We then choose $\lambda = \alpha \otimes \beta$ where $\alpha = (1, 0, ..., 0) \in \mathbb{R}^{n+1}$ and $\beta = (1, 0, ..., 0) \in \mathbb{R}^n$. We will see that $L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda)$ is bounded for some $\epsilon > 0$. Let $\eta \in L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda)$ then we can write $\eta = \xi_0 + \alpha \otimes \gamma_\epsilon + t\lambda$ for some $\gamma_\epsilon \in B_\epsilon$ and $t \in \mathbb{R}$. By definition of $L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda)$ we have $adj_n \eta \in \overline{S}$. Since S is bounded and

$$|adj_n \eta| = \left| \lambda_1 + \gamma_\epsilon^1 + t \right| \lambda_2 ... \lambda_n$$

it follows, using the fact that rank $\xi_0 \geq n-1$, that $|t|$ is bounded by a constant depending on S, ξ_0 and ϵ. Consequently $|\eta| \leq |\xi_0| + |\alpha \otimes \gamma_\epsilon| + |t| \ |\lambda|$ is bounded for any fixed positive ϵ and we get the result. ■

As already alluded in Section 5.3, we obtain now a result of non existence although the integrand of the relaxed problem is not strictly quasiconvex. We will consider the case where $N = 3$, $n = 2$ and $f : \mathbb{R}^{3 \times 2} \to \mathbb{R}$ is given by

$$f(\xi) = g(\mathrm{adj}_2 \xi)$$

where $g : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$g(\nu) = \left(\nu_1^2 - 4\right)^2 + \nu_2^2 + \nu_3^2 \, .$$

We therefore get $Qf(\xi) = Cg(\mathrm{adj}_2 \xi)$ and

$$Cg(\nu) = \left[\nu_1^2 - 4\right]_+^2 + \nu_2^2 + \nu_3^2$$

where

$$[x]_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

We will choose the boundary datum as follows

$$u_{\xi_0}(x) = \begin{pmatrix} u_{\xi_0}^1(x) = \alpha_1 x_1 + \alpha_2 x_2 \\ \\ u_{\xi_0}^2(x) = 0 \\ \\ u_{\xi_0}^3(x) = 0 \end{pmatrix}$$

and hence

$$Du_{\xi_0}(x) = \xi_0 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathrm{adj}_2 Du_{\xi_0}(x) = \mathrm{adj}_2 \xi_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The problem is then

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(Du(x))\, dx : \; u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^3) \right\}.$$

Note also that $Qf(\xi_0) = 0 < f(\xi_0) = 16$.

In terms of the preceding notations we have

$$\begin{aligned} S &= \{ y \in \mathbb{R}^3 : \; Cg(y) < g(y) \} = \{ y = (y_1, y_2, y_3) \in \mathbb{R}^3 : \; |y_1| < 2 \} \\ K &= \{ \xi \in \mathbb{R}^{3\times 2} : \; Qf(\xi) < f(\xi) \} = \left\{ \xi \in \mathbb{R}^{3\times 2} : \operatorname{adj}_2 \xi \in S \right\} \end{aligned}$$

and we observe that Cg is not affine on S, which in turn implies that Qf is not quasiaffine on K.

The following result shows that the hypothesis of strict quasiconvexity of Qf is not necessary for *non existence.*

Proposition 78 *(P) has a solution if and only if $u_{\xi_0} \equiv 0$. Moreover Qf is not strictly quasiconvex at any $\xi_0 \in \mathbb{R}^{3\times 2}$ of the form*

$$\xi_0 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. *Step 1.* We first show that if (P) has a solution then $u_{\xi_0} \equiv 0$. If $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^3)$ is a solution of (P) we necessarily have, denoting by $\nu(\xi) = \operatorname{adj}_2 \xi$,

$$|\nu_1(Du)| = 2 \,, \nu_2(Du) = \nu_3(Du) = 0 \,,$$

since

$$Qf(Du_{\xi_0}) = Cg(\operatorname{adj}_2 Du_{\xi_0}) = Cg(0) = 0.$$

The three equations read as

$$\begin{cases} \left| u^2_{x_1} u^3_{x_2} - u^2_{x_2} u^3_{x_1} \right| = 2 \\ u^1_{x_1} u^3_{x_2} - u^1_{x_2} u^3_{x_1} = 0 \\ u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1} = 0. \end{cases} \tag{31}$$

Multiplying the second equation of (31) first by $u^2_{x_1}$, then by $u^2_{x_2}$, using the third equation of (31), we get

$$\begin{aligned} 0 &= u^2_{x_1} u^1_{x_1} u^3_{x_2} - u^2_{x_1} u^1_{x_2} u^3_{x_1} = u^2_{x_1} u^1_{x_1} u^3_{x_2} - u^1_{x_1} u^2_{x_2} u^3_{x_1} = u^1_{x_1} \left(u^2_{x_1} u^3_{x_2} - u^2_{x_2} u^3_{x_1} \right) \\ 0 &= u^2_{x_2} u^1_{x_1} u^3_{x_2} - u^2_{x_2} u^1_{x_2} u^3_{x_1} = u^2_{x_1} u^1_{x_2} u^3_{x_2} - u^2_{x_2} u^1_{x_2} u^3_{x_1} = u^1_{x_2} \left(u^2_{x_1} u^3_{x_2} - u^2_{x_2} u^3_{x_1} \right). \end{aligned}$$

Combining these last equations with the first one of (31), we find

$$u^1_{x_1} = u^1_{x_2} = 0, \text{ a.e.}$$

We therefore find that any solution of (P) should have $Du^1 = 0$ a.e. and hence $u^1 \equiv$constant on each connected component of Ω. Since u^1 agrees with $u^1_{\xi_0}$ on the boundary of Ω, we deduce that $u^1_{\xi_0} \equiv 0$ and thus $u_{\xi_0} \equiv 0$, as claimed.

Step 2. We next show that if $u_{\xi_0} \equiv 0$, then (P) has a solution. It suffices to choose $u^1 \equiv 0$ and to solve

$$\begin{cases} \left| u^2_{x_1} u^3_{x_2} - u^2_{x_2} u^3_{x_1} \right| = 2 & \text{a.e. in } \Omega \\ \\ u^2 = u^3 = 0 & \text{on } \partial\Omega. \end{cases}$$

This is possible by virtue of, for example, Corollary 7.30 in [31].

Step 3. We finally prove that Qf is not strictly quasiconvex at any $\xi_0 \in \mathbb{R}^{3\times 2}$ of the form given in the statement of the proposition. Indeed let $0 < R_1 < R_2 < R$ and denote by B_R the ball centered at 0 and of radius R. Choose $\lambda, \mu \in C^\infty(B_R)$ such that

1) $\lambda = 0$ on ∂B_R and $\lambda \equiv 1$ on B_{R_2}.

2) $\mu \equiv 0$ on $B_R \diagdown \overline{B}_{R_2}$, $\mu \equiv 1$ on B_{R_1} and

$$\left| \mu^2 + \mu \left(x_1 \mu_{x_1} + x_2 \mu_{x_2} \right) \right| < 2 \text{ for every } x \in B_R\,.$$

This last condition (which is a restriction only in $B_{R_2} \diagdown \overline{B}_{R_1}$) is easily ensured by choosing appropriately R_1, R_2 and R.

We then choose $u(x) = u_{\xi_0}(x) + \varphi(x)$ where

$$\varphi^1(x) = -\lambda(x)\, u^1_{\xi_0}(x)\,,\ \varphi^2(x) = \mu(x)\, x_1 \text{ and } \varphi^3(x) = \mu(x)\, x_2\,.$$

We therefore have that $\varphi \in W^{1,\infty}_0(B_R;\mathbb{R}^3)$, $\mathrm{adj}_2 Du \equiv 0$ on $B_R \diagdown \overline{B}_{R_2}$, while on B_{R_2} we have

$$\mathrm{adj}_2 Du = \left(\mu^2 + \mu \left(x_1 \mu_{x_1} + x_2 \mu_{x_2} \right), 0, 0 \right).$$

We have thus obtained that $Cg\left(\mathrm{adj}_2 Du\right) \equiv 0$ and hence

$$Qf\left(\xi_0 + D\varphi\right) \equiv Qf\left(\xi_0\right) = 0.$$

This implies that (QP) has infinitely many solutions. However since φ does not vanish identically, we deduce that Qf is not strictly quasiconvex at any ξ_0 of the given form. ■

7.6 The problem of potential wells

The general problem of potential wells has been intensively studied by many authors in conjunction with crystallographic models involving fine microstructures. The reference paper on the subject is Ball and James [8]. It has then been studied by many authors including Bhattacharya-Firoozye-James-Kohn, Dacorogna-Marcellini, De Simone-Dolzmann, Dolzmann-Müller, Ericksen, Firoozye-Kohn, Fonseca-Tartar, Kinderlehrer-Pedregal, Kohn, Luskin, Müller-Sverak, Pipkin, Sverak and we refer to [31] for exact bibliographic references.

In mathematical terms the problem of *potential wells* can be described as follows. Find a minimizer of the problem

$$(P) \qquad \inf\left\{\int_{\Omega} f\left(Du\left(x\right)\right)dx : u \in u_{\xi_0} + W_0^{1,\infty}\left(\Omega;\mathbb{R}^n\right)\right\},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, u_{ξ_0} is an affine map with $Du_{\xi_0} = \xi_0$ and $f : \mathbb{R}^{n\times n} \to \mathbb{R}_+$ is such that

$$f\left(\xi\right) = 0 \iff \xi \in E = \bigcup_{i=1}^{m} SO\left(n\right)A_i\,.$$

The m wells are $SO\left(n\right)A_i$, $1 \leq i \leq m$ (and $SO\left(n\right)$ denotes the set of matrices U such that $U^tU = UU^t = I$ and $\det U = 1$).

The interesting case is when

$$\xi_0 \in \operatorname{int}\operatorname{Rco}E$$

and we have then that

$$Qf\left(\xi_0\right) = 0.$$

Therefore by the relaxation theorem we have

$$\inf\left(P\right) = \inf\left(QP\right) = 0.$$

The existence of minimizers, since Qf is affine on $\operatorname{Rco}E$ (indeed $Qf \equiv 0$), for (P) is then reduced to finding a function $u \in u_{\xi_0} + W_0^{1,\infty}\left(\Omega;\mathbb{R}^n\right)$ so that

$$Du\left(x\right) \in E = \bigcup_{i=1}^{m} SO\left(n\right)A_i\,.$$

The problem is relatively well understood only in the cases of two wells, i.e. $m = 2$, and in dimension $n = 2$. It is this case that we briefly discuss now. We therefore have now $A, B \in \mathbb{R}^{2\times 2}$ with $0 < \det A < \det B$ and we want to find $u \in u_{\xi_0} + W_0^{1,\infty}\left(\Omega;\mathbb{R}^2\right)$, where $\Omega \subset \mathbb{R}^2$ is a bounded open set, satisfying

$$Du(x) \in SO(2)A \cup SO(2)B, \text{ a.e. in } \Omega$$

The first important result is to identify the set where the gradient of the boundary datum, ξ_0, should lie. This was resolved by Sverak [73] who showed that

$$\operatorname{Rco}E = \left\{\xi \in \mathbb{R}^{2\times 2} : \begin{array}{c} \text{there exist } 0 \leq \alpha \leq \frac{\det B - \det \xi}{\det B - \det A},\ 0 \leq \beta \leq \frac{\det \xi - \det A}{\det B - \det A} \\ \\ R, S \in SO\left(2\right),\ \text{so that } \xi = \alpha RA + \beta SB \end{array}\right\}$$

while the interior is given by the same formulas with strict inequalities in the right hand side.

We therefore have

Theorem 79 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set*

$$\xi_0 \in \operatorname{int} \operatorname{Rco} E.$$

Then there exists $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^2)$ such that

$$Du(x) \in E = SO(2)A \cup SO(2)B \quad a.e. \ in \ \Omega$$

and therefore (P) has a solution.

This result was proved by Müller-Sverak [61] using the so called method of convex integration of Gromov [42] and by Dacorogna-Marcellini in [28] and [31] following the approach presented in Section 4.4 and we refer to [31] for details.

The case where $\det A = \det B > 0$ can also be handled (cf. Müller-Sverak [62], see also Dacorogna-Tanteri [37]), using the representation formula of Sverak [73], namely

$$\operatorname{Rco} E = \left\{ \xi \in \mathbb{R}^{2\times 2} : \begin{array}{c} \text{there exist } R, S \in SO\left(2\right),\ 0 \leq \alpha, \beta \leq \alpha+\beta \leq 1, \\ \\ \det \xi = \det A = \det B \text{ so that } \xi = \alpha RA + \beta SB \end{array} \right\}.$$

References

[1] Acerbi E. and Fusco N., Semicontinuity problems in the calculus of variations, *Arch. Rational Mech. Anal.* **86** (1984), 125-145.

[2] Alibert J.J. and Dacorogna B., An example of a quasiconvex function that is not polyconvex in two dimensions; *Arch. Rational Mech. Anal.* **117** (1992), 155-166.

[3] Allaire G. and Francfort G., Existence of minimizers for nonquasiconvex functionals arising in optimal design; *Annales Institut Henri Poincaré, Analyse Non Linéaire* **15** (1998), 301–339.

[4] Aubert G. and Tahraoui R., Théorèmes d'existence pour des problèmes du calcul des variations; *J. Differential Equations* **33** (1979), 1-15.

[5] Aubert G. and Tahraoui R., Sur la minimisation d'une fonctionelle nonconvexe, non différentiable en dimension 1; *Boll. Un. Mat. Ital.* **17** (1980), 244-258.

[6] Aubert G. and Tahraoui R., Théorèmes d'existence en optimisation non convexe; *Appl. Anal.* **18** (1984), 75-100.

[7] Ball J.M., Convexity conditions and existence theorems in nonlinear elasticity; *Arch. Rational Mech. Anal.* **63** (1977), 337-403.

[8] Ball J.M. and James R.D., Fine phase mixtures as minimizers of energy; *Arch. Rational Mech. Anal.* **100** (1987), 15-52.

[9] Bandyopadhyay S., Barroso A.C., Dacorogna B. and Matias J., Differential inclusions for differential forms and applications to the calculus of variations; to appear.

[10] Bauman P. and Phillips D., A nonconvex variational problem related to change of phase; *J. Appl. Math. Optimization* **21** (1990), 113-138.

[11] Bressan A. and Flores F., On total differential inclusions; *Rend. Sem. Mat. Univ. Padova* **92** (1994), 9-16.

[12] Buttazzo G., Dacorogna B. and Gangbo W., On the envelopes of functions depending on singular values of matrices; *Boll. Un. Mat. Ital.* **8B** (1994), 17-35.

[13] Buttazzo G., Ferone V. and Kawohl B., Minimum problems over sets of concave functions and related questions; *Math. Nachr.* **173** (1995), 71–89.

[14] Celada P. and Perrotta S., Minimizing nonconvex multiple integrals: a density result; *Proc. Roy. Soc. Edinburgh* **130A** (2000), 721–741

[15] Celada P. and Perrotta S., On the minimum problem for nonconvex multiple integrals of product type; *Calc. Var. Partial Differential Equations* **12** (2001), 371–398.

[16] Cellina A., On minima of a functional of the gradient: necessary conditions; *Nonlinear Analysis* **20** (1993), 337-341.

[17] Cellina A., On minima of a functional of the gradient, sufficient conditions; *Nonlinear Analysis* **20** (1993), 343-347.

[18] Cellina A. and Colombo G., On a classical problem of the calculus of variations without convexity conditions; *Annales Institut Henri Poincaré, Analyse Non Linéaire* **7** (1990), 97-106.

[19] Cellina A. and Zagatti S., An existence result in a problem of the vectorial case of the calculus of variations; *SIAM J. Control Optimization* **33** (1995), 960-970.

[20] Cesari L., An existence theorem without convexity conditions; *SIAM J. Control Optimization* **12** (1974), 319-331.

[21] Cesari L., *Optimization - theory and applications*; Springer Verlag, Berlin, 1983.

[22] Cutrì A., Some remarks on Hamilton-Jacobi equations and nonconvex minimization problems; *Rendiconti di Matematica* **13** (1993), 733-749.

[23] Dacorogna B., A relaxation theorem and its applications to the equilibrium of gases; *Arch. Rational Mech. Anal.* **77** (1981), 359-386.

[24] Dacorogna B., Minimal hypersurfaces in parametric form with non convex integrands; *Indiana Univ. Math. J.* **31** (1982), 531-552.

[25] Dacorogna B., Quasiconvexity and relaxation of non convex variational problems; *J. Funct. Anal.* **46** (1982), 102-118.

[26] Dacorogna B., *Direct methods in the calculus of variations*; Applied Math. Sciences, 78, Springer, Berlin, 1989.

[27] Dacorogna B. and Marcellini P., Existence of minimizers for non quasiconvex integrals; *Arch. Rational Mech. Anal.* **131** (1995), 359-399.

[28] Dacorogna B. and Marcellini P., Théorèmes d'existence dans le cas scalaire et vectoriel pour les équations de Hamilton-Jacobi; *C. R. Acad. Sci. Paris* **322** (1996), 237-240.

[29] Dacorogna B. and Marcellini P., General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial case; *Acta Mathematica* **178** (1997), 1-37.

[30] Dacorogna B. and Marcellini P., On the solvability of implicit nonlinear systems in the vectorial case; in the *AMS Series of Contemporary Mathematics*, edited by G.Q. Chen and E. DiBenedetto, 1999, 89-113.

[31] Dacorogna B. and Marcellini P., *Implicit partial differential equations*; Birkhäuser, Boston, 1999.

[32] Dacorogna B. and Marcellini P., Attainment of minima and implicit partial differential equations; *Ricerche di Matematica*, **48** (1999), 311-346.

[33] Dacorogna B. and Pisante G., A general existence theorem for differential inclusions in the vector valued case; to appear in *Portugaliae Mathematica*.

[34] Dacorogna B., Pisante G. and Ribeiro A.M., On non quasiconvex problems of the calculus of variations; to appear.

[35] Dacorogna B. and Ribeiro A.M., Existence of solutions for some implicit pdes and applications to variational integrals involving quasiaffine functions; *Proc. Royal Soc. Edinburgh* **134A** (2004), 1-15.

[36] Dacorogna B. and Tanteri C., On the different convex hulls of sets involving singular values; *Proc. Royal Soc. Edinburgh* **128A** (1998), 1261-1280.

[37] Dacorogna B. and Tanteri C., Implicit partial differential equations and the constraints of nonlinear elasticity; *J. Math. Pures Appl.* **81** (2002), 311-341.

[38] De Blasi F.S. and Pianigiani G., On the Dirichlet problem for Hamilton-Jacobi equations. A Baire category approach; *Nonlinear Differential Equations Appl.* **6** (1999), 13–34.

[39] Ekeland I., Nonconvex minimization problems; *Bull. Amer. Math. Soc.* **1** (1979), 443-475.

[40] Friesecke G., A necessary and sufficient condition for non attainment and formation of microstructure almost everywhere in scalar variational problems; *Proc. Royal Soc. Edinburgh* **124A** (1994), 437-471.

[41] Fusco N., Marcellini P. and Ornelas A., Existence of minimizers for some nonconvex one-dimensional integrals; *Portugaliae Mathematica* **55** (1998),167-185.

[42] Gromov M., *Partial differential relations*; Springer-Verlag, Berlin, 1986.

[43] Giachetti D. and Schianchi R., Minima of some nonconvex noncoercive problems; *Ann. Mat. Pura Appl.* **165** (1993), 109-120.

[44] Horn R.A. and Johnson C.A., *Matrix Analysis*; Cambridge University Press (1985).

[45] Horn R.A. and Johnson C.A., *Topics in Matrix Analysis*; Cambridge University Press (1991).

[46] Kirchheim B., Deformations with finitely many gradients and stability of quasiconvex hulls; *C. R. Acad. Sci. Paris*, **332** (2001), 289–294.

[47] Klötzler R., On the existence of optimal processes; *Banach Center of Warsaw Publications* (1976), 125-130.

[48] Kohn R.V. and Strang G., Optimal design and relaxation of variational problems I, II, III; *Comm. Pure Appl. Math.* **39** (1986), 113-137, 139-182, 353-377.

[49] Le Dret H. and Raoult A., Enveloppe quasi-convexe de la densité d'énergie de Saint Venant-Kirchhoff; *C. R. Acad. Sci. Paris* **318** (1994), 93-98.

[50] Marcellini P., Alcune osservazioni sull'esistenza del minimo di integrali del calcolo delle variazioni senza ipotesi di convessità; *Rend. Mat.* **13** (1980), 271-281.

[51] Marcellini P., A relation between existence of minima for nonconvex integrals and uniqueness for not strictly convex integrals of the calculus of variations; in: *Math. Theories of Optimization*, ed. Cecconi J.P. et al., Lecture Notes in Math. 979, Springer-Verlag, Berlin, 1983, 216-231.

[52] Marcellini P., Non convex integrals of the calculus of variations; in: *Methods of nonconvex analysis*, ed. Cellina A., Lecture Notes in Math. 1446, Springer-Verlag, Berlin, 1990, 16-57.

[53] Marshall A. W. and Olkin I., *Inequalities: theory of majorisation and its applications*; Academic Press, 1979.

[54] Mascolo E., Some remarks on nonconvex problems; in: *Material instabilities in continuum mechanics*, ed. Ball J.M., Oxford Univ. Press, 1988, 269-286.

[55] Mascolo E. and Schianchi R., Existence theorems for nonconvex problems; *J. Math. Pures Appl.* **62** (1983), 349-359.

[56] Mascolo E. and Schianchi R., Nonconvex problems in the calculus of variations; *Nonlinear Analysis, Theory Meth. Appl.* **9** (1985), 371-379.

[57] Mascolo E. and Schianchi R., Existence theorems in the calculus of variations; *J. Differential Equations* **67** (1987), 185-198.

[58] Monteiro Marques M.D.P. and Ornelas A., Genericity and existence of a minimum for scalar integral functionals; *J. Optimization Th. Appl.* **86** (1995), 421-431.

[59] Morrey C.B., Quasiconvexity and the lower semicontinuity of multiple integrals, *Pacific J. Math.* **2** (1952), 25-53.

[60] Morrey C.B., *Multiple integrals in the calculus of variations*; Springer-Verlag, Berlin, 1966.

[61] Müller S. and Sverak V., Attainment results for the two-well problem by convex integration; ed. Jost J., *International Press*, 1996, 239-251.

[62] Müller S. and Sverak V.: Convex integration with constraints and applications to phase transitions and partial differential equations; *J. Eur. Math.Soc.* **1** (1999), 393-422.

[63] Müller S. and Sychev M., Optimal existence theorems for nonhomogeneous differential inclusions; *J. Funct. Anal.* **181** (2001), 447–475.

[64] Ornelas A., Existence of scalar minimizers for nonconvex simple integrals of sum type; *J. Math. Anal. Appl.* **221** (1998), 559–573.

[65] Oxtoby J.C., *Measure and category*; Springer, Berlin, 1971.

[66] Pisante G., *Implicit partial differential equations: different approaching methods*, PhD Thesis, (2004).

[67] Raymond J.P., Champs Hamiltoniens, relaxation et existence de solutions en calcul des variations; *J. Differential Equations* **70** (1987), 226-274.

[68] Raymond J.P., Conditions nécessaires et suffisantes de solutions en calcul des variations; *Annales Institut Henri Poincaré, Analyse Non Linéaire* **4** (1987), 169-202.

[69] Raymond J.P., Théorème d'existence pour des problèmes variationnels non convexes; *Proc. Royal Soc. Edinburgh* **107A** (1987), 43-64.

[70] Raymond J.P., Existence of minimizers for vector problems without quasiconvexity conditions; *Nonlinear Analysis, Theory Meth. Appl.* **18** (1992), 815-828.

[71] Rockafellar R.T., *Convex Analysis*; Princeton University Press, Princeton, 1970.

[72] Sverak V., Rank one convexity does not imply quasiconvexity; *Proc. Royal Soc. Edinburgh* **120A** (1992), 185-189.

[73] Sverak V., On the problem of two wells; in: *Microstructure and phase transitions*, IMA Vol. Appl. Math. 54, ed. Ericksen J. et al., Springer-Verlag, Berlin, 1993, 183-189.

[74] Sychev M., Comparing two methods of resolving homogeneous differential inclusions; *Calc. Var. Partial Differential Equations* **13** (2001), 213–229.

[75] Sychev M.A., Characterization of homogeneous scalar variational problems solvable for all boundary data; *Proc. Roy. Soc. Edinburgh* **130A** (2000), 611–631.

[76] Tahraoui R., Théorèmes d'existence en calcul des variations et applications à l'élasticité non linéaire; *C. R. Acad. Sci. Paris* **302** (1986), 495-498. Also in: *Proc. Royal Soc. Edinburgh* **109A** (1988), 51-78.

[77] Tahraoui R., Sur une classe de fonctionnelles non convexes et applications; *SIAM J. Math. Anal.* **21** (1990), 37-52.

[78] Treu G., An existence result for a class of nonconvex problems of the calculus of variations; *J. Convex Analysis* **5** (1998), 31–44.

[79] Yosida K., *Functional analysis*; sixth edition, Springer, Berlin, 1980.

[80] Zagatti S., Minimization of functionals of the gradient by Baire's theorem; *SIAM J. Control Optim.* **38** (2000), 384–399.