An introduction to implicit partial differential equations of the first order

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Abstract

We will discuss here Dirichlet problems for non linear first order partial differential equations. We will present the recent results of Dacorogna-Marcellini on this subject; mentioning briefly, as a matter of introduction, some results on viscosity solutions.

Contents

1 Introduction

We will discuss here Dirichlet problems for first (and incidentally for second) order partial differential equations. Of course this subject is so large that several books would be needed to cover it. We will focus on the work of Dacorogna-Marcellini that is developed in a recent book [12] (following earlier work [9], [10],

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[11]). We have not touched the very closely related work on *convex integration* of Gromov [17] (cf. also Spring [22]) as developed by Müller-Sverak [20], [21] and others. We also will speak only little on the very important tool known as *viscosity method* introduced by Crandall-Lions following earlier work of Hopf, Lax, Kruzkov and others. Some of the results mentioned in the present article have been obtained concurrently or partially by other authors or in collaboration. In order to make this survey as short as possible we did not enter into this matter referring to [12] for details.

2 The scalar case

2.1 Statement of the problem

We will discuss here the following Dirichlet problem for first order partial differential equations

$$\begin{cases} F\left(Du\left(x\right)\right)=0 & \text{a.e. } x\in\Omega \\ u\left(x\right)=\varphi\left(x\right) & x\in\partial\Omega. \end{cases} \tag{1}$$

where
  - $\Omega\subset\mathbb{R}^n$, $n\geq 1$, is a bounded open set,
  - $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}$, $Du\in\mathbb{R}^n$ is the gradient of u,
  - $F:\mathbb{R}^n\to\mathbb{R}$ is a continuous function
  - the boundary datum φ is a given Lipschitz function.

Remark 1 *The method developed here also applies to functions $F = F(x, u, Du)$. In Section 2 we will also consider vector valued function $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m$.*

Very simple examples (cf. below) show that it is not reasonable to expect C^1 solutions of (1). We will however look for *Lipschitz* solutions, $u\in Lip\left(\overline{\Omega}\right)$, which means that there exists $L > 0$ such that

$$\left|u\left(x\right)-u\left(y\right)\right|\leq L\left|x-y\right|,\ \forall x,y\in\overline{\Omega}.$$

Remark 2 *In fact we will look for solutions in the Sobolev space $W^{1,\infty}\left(\Omega\right)$. As well known if the domain Ω is sufficiently regular it can then be identified with $Lip\left(\overline{\Omega}\right)$. We will throughout the present article make this identification. Rademacher theorem implies that any Lipschitz function is differentiable almost everywhere, the equation (1) is therefore understood in the almost everywhere sense.*

We now explain our terminology.

Definition 3 *We will call an equation $F\left(Du\right)=0$ **implicit** if*

$$\text{intco}\, E\neq\emptyset$$

where

$$E=\left\{\xi\in\mathbb{R}^n:F\left(\xi\right)=0\right\},$$

co E denotes its convex hull and intco E the interior of this hull.

Remark 4 *(1) Linear equations are not implicit equations in the above sense since then $E = \operatorname{co} E$ and $\operatorname{intco} E = \emptyset$.*

(2) A better terminology could have been "fully nonlinear equations"; however this could lead to confusion with the so called "fully nonlinear elliptic equations" (such as the Monge-Ampère equation) which are disjoint from ours. For the scalar case one can also use the name of Hamilton-Jacobi equations.

2.2 A trivial example

We will start with a trivial example that has however the merit of explaining the situation.

Consider the problem

$$\begin{cases} \left| u'(x) \right| = 1, \text{ a.e. } x \in (0,1) \\ u(0) = 0, \ u(1) = \alpha. \end{cases} \tag{2}$$

In our notations $E = \{\pm 1\}$, $\operatorname{intco} E = (-1,1)$ and $\varphi(x) = \alpha x$.

Case 1. Let $|\alpha| < 1$. Observe that there is no C^1 solution of (2) but there are infinitely many Lipschitz solutions; in particular the following ones

$$u_+(x) = \begin{cases} x & \text{if } x \in [0, (1+\alpha)/2] \\ -x + (1+\alpha) & \text{if } x \in [(1+\alpha)/2, 1] \end{cases}$$

$$u_-(x) = \begin{cases} -x & \text{if } x \in [0, (1-\alpha)/2] \\ x - (1-\alpha) & \text{if } x \in [(1-\alpha)/2, 1] \end{cases}$$

$$u_N(x) = \begin{cases} x - \frac{n}{N} + \frac{n\alpha}{N} & \text{if } x \in \left[\frac{n}{N}, \frac{n}{N} + \frac{1+\alpha}{2N}\right] \\ -x + \frac{n+1}{N}(1+\alpha) & \text{if } x \in \left[\frac{n}{N} + \frac{1+\alpha}{2N}, \frac{n+1}{N}\right] \end{cases}$$

where $n, N \in \mathbb{N}$ and $0 \leq n \leq N - 1$ (note for further reference that $u_N \to \varphi$ uniformly as $N \to \infty$).

It is easy to see that any solution u of (2) satisfies

$$u_- \leq u \leq u_+.$$

One can also prove that if $\varepsilon > 0$ and u_ε is a solution of

$$\begin{cases} -\varepsilon u_\varepsilon'' + \left| u_\varepsilon' \right| = 1, \text{ a.e. } x \in (0,1) \\ u_\varepsilon(0) = 0, \ u_\varepsilon(1) = \alpha. \end{cases}$$

then $u_\varepsilon \to u_+$ uniformly (similarly if we replace in the equation $-\varepsilon$ by $+\varepsilon$ we get $u_\varepsilon \to u_-$).

Case 2. If $\alpha = \pm 1$ then there is a unique C^1 solution of (2), namely $u(x) = x$ if $\alpha = 1$ and $u(x) = -x$ if $\alpha = -1$.

Case 3. If $|\alpha| > 1$ then no Lipschitz solution exists.

2.3 The eikonal equation

Consider the *eikonal equation* which is one of the most important nonlinear first order equation

$$\begin{cases} |Du(x)| = 1 & \text{a.e. } x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases} \tag{3}$$

In our notations we have $E = \{\xi \in \mathbb{R}^n : |\xi| = 1\} = S^{n-1}$, $\text{int co } E = \{\xi \in \mathbb{R}^n : |\xi| < 1\}$ where we have denoted by $|\xi| = \left(\sum \xi_i^2\right)^{1/2}$ the Euclidean norm. The following theorem is then standard

Theorem 5 *Let Ω be a bounded, open and convex set of $\mathbb{R}^n$ and*

$$|\varphi(x) - \varphi(y)| \leq |x - y|, \ \forall x, y \in \partial\Omega.$$

Then the functions

$$\begin{aligned} u_+(x) &= \inf_{y \in \partial\Omega} \{\varphi(y) + |x - y|\} \\ u_-(x) &= \sup_{y \in \partial\Omega} \{\varphi(y) - |x - y|\} \end{aligned}$$

are Lipschitz solutions of (3) so that any other Lipschitz solution u satisfies $u_- \leq u \leq u_+$. In particular if $\varphi = 0$ then

$$u_\pm(x) = \pm \text{dist}(x, \partial\Omega).$$

The hypothesis of convexity of the domain is made only to have simpler expressions for u_+ and u_-. It is interesting to note that the compatibility condition on φ is also necessary and it corresponds if φ is appropriately defined all over Ω to $D\varphi \in \text{co } E$.

2.4 Viscosity solutions

The above example can be generalized a great deal and it leads us to the so called *viscosity method*. This has now a long history that finds its origins in work of Hopf, Lax, Kruzkov and several others but that was developed and brought to the present framework by Crandall and Lions. In [12] we have a large bibliography (as well as an introduction on this method) and here we just mention recent books that deal with this subject: Bardi-Capuzzo Dolcetta [1], Barles [2], Benton [3], Fleming-Soner [16], Subbotin [23] or the reference article of Crandall-Ishii-Lions [7]. However the most appropriate reference for the theorems either in the preceding section or in the present one is Lions [18].

We start with the definition of viscosity solution for the equation

$$F(Du(x)) = 0, \ \text{a.e. } x \in \Omega. \tag{4}$$

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Definition 6 *A function $u \in C(\Omega)$ is said to be a **viscosity** solution of (4) if the following two conditions are satisfied.*

(1) u is a viscosity subsolution of (4) which means that, for any $w \in C^1(\Omega)$ and for any $y \in \Omega$ local maximum of $u - w$, the following inequality holds

$$F(Dw(y)) \leq 0;$$

(2) u is a viscosity supersolution of (4) which means that, for any $w \in C^1(\Omega)$ and for any $z \in \Omega$ local minimum of $u - w$, the following inequality holds

$$F(Dw(z)) \geq 0.$$

Remark 7 *The terminology "viscosity solution" refers to the fact that in earlier work on the subject these solutions were obtained as limit (when $\varepsilon \to 0^+$) of the approximate problem*

$$\begin{cases} F(Du^\varepsilon(x)) = \varepsilon \Delta u^\varepsilon(x) & a.e. \ x \in \Omega \\ u^\varepsilon(x) = \varphi(x) & x \in \partial\Omega. \end{cases} \tag{5}$$

This is what we also discussed in the above trivial example.

One can then prove the following theorem (this is only one of the simplest results that can be obtained by the viscosity method).

Theorem 8 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set. Let $F : \mathbb{R}^n \to \mathbb{R}$ be convex with $F(0) < 0$ and such that $\lim F(\xi) = +\infty$ if $|\xi| \to +\infty$.*

Let ρ be the gauge associated to $\{\xi : F(\xi) \leq 0\}$ which means that

$$\rho(\xi) = \inf\{\lambda \geq 0 : F(\xi/\lambda) \leq 0\}.$$

and ρ^0 be its polar, which is defined as

$$\rho^0(\xi^*) = \sup_{\xi \neq 0} \left\{ \frac{\langle \xi; \xi^* \rangle}{\rho(\xi)} \right\}.$$

Let $\varphi : \partial\Omega \to \mathbb{R}$ satisfy

$$\varphi(x) - \varphi(y) \leq \rho^0(x - y), \quad \forall x, y \in \partial\Omega; \tag{6}$$

then the function

$$u_+(x) = \inf_{y \in \partial\Omega} \left\{ \varphi(y) + \rho^0(x - y) \right\}$$

is the unique Lipschitz viscosity solution of

$$\begin{cases} F(Du(x)) = 0 & a.e. \ x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases} \tag{7}$$

In addition

$$u_-(x) = \sup_{y \in \partial\Omega} \left\{ \varphi(y) - \rho^0(y - x) \right\}$$

is also a Lipschitz solution of (7) and any other Lipschitz solution u is such that

$$u_- \leq u \leq u_+.$$

Remark 9 *The above theorem is an extension of Theorem 5. In this last case and with the above notations we have $F(\xi) = |\xi| - 1$, $\rho(\xi) = \rho^0(\xi) = |\xi|$.*

2.5   A counterexample

The viscosity method is a very powerful tool, however it is very closely linked with the convexity of the Hamiltonian F. In absence of convexity it may well happen that Lipschitz solutions do exist but no viscosity ones. The following example has been given by Cardaliaguet-Dacorogna-Gangbo-Georgy [5]. In this article a general theorem relates the existence of viscosity solutions, the zeroes set of F and the geometry of the domain Ω.

Theorem 10 *Let $\Omega \subset \mathbb{R}^2$ be convex. Then (writing for $u = u(x,y)$, $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$)*

$$\begin{cases} F(Du) = \left(u_x^2 - 1\right)^2 + \left(u_y^2 - 1\right)^2 = 0, \ a.e. \ in \ \Omega \\ u = 0, \ on \ \partial\Omega \end{cases} \qquad (8)$$

has infinitely many Lipschitz solutions but no viscosity one unless Ω is a rectangle whose faces are orthogonal to the vectors $(1,1)$ and $(1,-1)$.

The existence of Lipschitz solutions follow from the theory developed in the next section. The theorem shows in particular that if Ω is any smooth domain such as the disk then there is no viscosity solution.

In terms of the set E we have

$$\begin{aligned} E &= \left\{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| = |\xi_2| = 1 \right\} \\ \operatorname{co} E &= \left\{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1|, |\xi_2| \le 1 \right\}. \end{aligned}$$

2.6   The Baire category method

The main theorem is the following.

Theorem 11 *Let $\Omega \subset \mathbb{R}^n$ be open and $E \subset \mathbb{R}^n$. Let $\varphi \in Lip\left(\overline{\Omega}\right)$ satisfies*

$$D\varphi(x) \in E \cup \operatorname{int} \operatorname{co} E, \ \ a.e. \ x \in \Omega \qquad (9)$$

(where $\operatorname{int} \operatorname{co} E$ stands for the interior of the convex hull of E); then there exists (a dense set of) $u \in Lip\left(\overline{\Omega}\right)$ such that

$$\begin{cases} Du(x) \in E & a.e. \ x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases} \qquad (10)$$

Remark 12 *(1) If the domain is not sufficiently regular so that $W^{1,\infty}(\Omega) \neq Lip\left(\overline{\Omega}\right)$ then the existence result has to be understood in $W^{1,\infty}(\Omega)$.*

(2) The theorem obviously applies to

$$\begin{cases} F(Du(x)) = 0 & a.e. \ x \in \Omega \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases}$$

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It suffices to write $E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}$. If for example F is convex with $F(0) < 0$ and $\lim F(\xi) = +\infty$ if $|\xi| \to +\infty$, as in Theorem 8, then the compatibility condition (9) reads as follows

$$F(D\varphi(x)) \leq 0, \quad a.e. \ x \in \Omega.$$

(3) The compatibility condition (9), when appropriately interpreted, is also necessary, cf. [12].

(4) The fact that there is a dense set of solutions has to be understood in the sense of Baire category theorem (cf. below for more details). It implies, in particular, that for every $\varepsilon > 0$ there is a Lipschitz solution of (10) such that $\|u - \varphi\|_{L^\infty} < \varepsilon$.

(5) This theorem has a long history and we refer to [12] for more historical background. Note only that when the boundary datum is affine, this result was established by Cellina and Friesecke with a very explicit construction that we call "pyramidal". The method of proof that we will outline below follows works of Cellina [6], Bressan-Flores [4], De Blasi-Pianigiani [14], [15] and Dacorogna-Marcellini [12].

Proof. We very roughly outline the idea of the proof only in a very special case that has the advantage to make the procedure transparent and not burdened by too many technical details. It should be pointed out that, in this particular case, the result can be obtained by several other methods, notably the viscosity one.

We will assume, as in Theorem 8, that F is convex with $F(0) < 0$ and $\lim F(\xi) = +\infty$ if $|\xi| \to +\infty$ and that

$$E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}.$$

We then let

$$V = \left\{ u \in \varphi + W_0^{1,\infty}(\Omega) : \ F(Du(x)) \leq 0, \quad a.e. \ x \in \Omega \right\}$$

which in this particular case is the set of subsolutions. Note also that V is non empty since $\varphi \in V$. We next endow V with the C^0 metric. V is then a *complete metric space*. This follows from the coercivity ($\lim F(\xi) = +\infty$ if $|\xi| \to +\infty$) and the convexity of F. Indeed the coercivity condition ensures that any Cauchy sequence in V has uniformly bounded gradient and therefore has a subsequence that converges weak * in $W^{1,\infty}$ to a limit. The convexity of F implying lower semicontinuity, we get that the limit is indeed in V.

We next introduce, for every integer k, the subset V^k of V

$$V^k = \left\{ u \in V : \ \int_\Omega F(Du(x))\, dx > -\frac{1}{k} \right\}.$$

By the very same above argument we find that V^k is *open* in V. It can then be proved (and this is the difficult step) that V^k is *dense* in V, by means of the

relaxation theorems of the calculus of variations (involving constructions similar to that of the functions u_N in the trivial example of Section 2.2).

The Baire category theorem implies then that

$$\bigcap_k V^k = \left\{ u \in V : \int_\Omega F\left(Du\left(x\right)\right) dx \geq 0 \right\}$$

$$= \left\{ u \in \varphi + W_0^{1,\infty}\left(\Omega\right) : F\left(Du\left(x\right)\right) = 0, \quad \text{a.e. } x \in \Omega \right\}$$

is dense, and hence non empty, in V, which is just to say that the set of solutions is non empty. ∎

3 The vectorial case

3.1 Statement of the problem

We now want to discuss the analogue of Theorem 11 in the vectorial case. The problem is then

$$\begin{cases} F_1\left(Du\right) = ... = F_N\left(Du\right) = 0, \text{ a.e. in } \Omega \\ u = \varphi, \text{ on } \partial\Omega \end{cases} \tag{11}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $n, m > 1$, so that now $Du \in \mathbb{R}^{m \times n}$ (the set of $m \times n$ matrices) and $F_i : \mathbb{R}^{m \times n} \to \mathbb{R}$, $i = 1, ..., N$.

We then let

$$E = \left\{ \xi \in \mathbb{R}^{m \times n} : F_i\left(\xi\right) = 0, i = 1, ..., N \right\}.$$

No simple analogue to Theorem 11 exists and in order explain some results that can be applied to vectorial problems we need to introduce some terminology (cf. [8] or [12]).

Definition 13 *(1) A function $f : \mathbb{R}^{m \times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is said to be **rank one convex** if*

$$f\left(tA + \left(1 - t\right)B\right) \leq tf\left(A\right) + \left(1 - t\right)f\left(B\right)$$

for every $t \in [0, 1]$ and every $A, B \in \mathbb{R}^{m \times n}$ with $\text{rank}\left\{A - B\right\} = 1$.
*(2) Given a set $E \subset \mathbb{R}^{m \times n}$ we define the **rank one convex hull** of E to be*

$$\text{Rco}\, E = \left\{ \begin{array}{l} A \in \mathbb{R}^{m \times n} : f(A) \leq 0, \quad \forall f : \mathbb{R}^{m \times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}, \\ \left. f \right|_E = 0, \quad f \text{ rank one convex} \end{array} \right\}.$$

In the scalar case $m = 1$ the usual notion of convexity coincide with that of rank one convexity (since, by definition, vectors are rank one matrices). However in the vectorial context convex functions are obviously rank one convex but not conversely as the classical example $f\left(A\right) = \det A$ (when $m = n$) shows. This implies in particular that given a set $E \subset \mathbb{R}^{m \times n}$ we have in general that

$$\text{Rco}\, E \underset{\neq}{\subseteq} \text{co}\, E$$

while in the scalar case they are both equal. Note also that rank one convex sets are much wilder than convex sets; for example, just to quote one peculiarity, such sets may even be disconnected (choose $E = \{A, B\}$ with rank $\{A - B\} \geq 2$).

All the theorems that can generalize Theorem 11 to the vectorial case are considerably harder, even in their statements. They involve $\operatorname{Rco} E$ (or some other sets, such as the *quasiconvex hull* that we do not define here) but also need other properties, difficult to verify, such as the so called *relaxation property* and that we do not discuss here; we refer for details to [12].

We will therefore not give, in the following sections, any general theorem but we will only present some relevant examples where we can obtain existence theorems.

3.2   The singular values case

We now present a problem that can be applied to some geometrical problems or to non linear elasticity. Let $\xi \in \mathbb{R}^{n \times n}$ and denote by $0 \leq \lambda_1(\xi) \leq ... \leq \lambda_n(\xi)$ the singular values of the matrix ξ (i.e. the eigenvalues of $(\xi^t \xi)^{1/2}$). This implies in particular that

$$|\xi|^2 = \sum_{i,j=1}^{n} \xi_{ij}^2 = \sum_{i=1}^{n} (\lambda_i(\xi))^2, \ |\det \xi| = \prod_{i=1}^{n} \lambda_i(\xi).$$

Theorem 14 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $0 < a_1 \leq ... \leq a_n$. Let $\varphi \in C^1\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfy*

$$\prod_{i=\nu}^{n} \lambda_i(D\varphi(x)) < \prod_{i=\nu}^{n} a_i, \ x \in \Omega, \ \nu = 1, ..., n \qquad (12)$$

(in particular $\varphi \equiv 0$), then there exists $u \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ such that

$$\begin{cases} \lambda_i(Du(x)) = a_i & a.e. \ x \in \Omega, \ i = 1, ..., n \\ u(x) = \varphi(x) & x \in \partial\Omega. \end{cases} \qquad (13)$$

Remark 15 *(1) Using the notations of the previous section we have*

$$E = \left\{\xi \in \mathbb{R}^{n \times n} : \lambda_i(\xi) = a_i, i = 1, ..., n\right\}$$

$$\operatorname{co} E = \left\{\xi \in \mathbb{R}^{n \times n} : \sum_{i=\nu}^{n} \lambda_i(\xi) \leq \sum_{i=\nu}^{n} a_i, \ \nu = 1, ..., n\right\}$$

$$\operatorname{Rco} E = \left\{\xi \in \mathbb{R}^{n \times n} : \prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} a_i, \ \nu = 1, ..., n\right\}.$$

(2) If $a_i \equiv 1$, for every $i = 1, ..., n$, then

$$\operatorname{Rco} E = \operatorname{co} E = \left\{\xi \in \mathbb{R}^{n \times n} : \lambda_n(\xi) \leq 1\right\}$$

and (12) becomes

$$\lambda_n(D\varphi(x)) < 1, \ x \in \Omega.$$

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3.3 The complex eikonal equation

The problem (coming from geometrical optics, cf. Magnanini-Talenti [19]) is to find a complex function

$$w(x) = u(x) + iv(x)$$

(where $x = (x_1, ..., x_n)$) satisfying

$$\begin{cases} \sum_{i=1}^n w_{x_i}^2 = f^2 & \text{a.e. in } \Omega \\ w = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $f \in \mathbb{R}$ is a constant (one can also handle continuous functions $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, $f = f(x, u, v)$). The stated problem is equivalent to the real system (setting $\langle .;. \rangle$ for the scalar product in $\mathbb{R}^n$)

$$\begin{cases} |Du|^2 = |Dv|^2 + f^2 & \text{a.e in } \Omega \\ \langle Du; Dv \rangle = 0 & \text{a.e in } \Omega \\ (u, v) = (\varphi_1, \varphi_2) & \text{on } \partial\Omega. \end{cases} \tag{14}$$

Note that when $\text{Im}\, w = v \equiv 0$ then we recover the classical eikonal equation.

In the terminology introduced before we have

$$\begin{aligned} E &= \left\{ A = (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi|^2 = |\eta|^2 + f^2, \ \langle \xi; \eta \rangle = 0 \right\} \\ \text{Rco}\, E &= \text{co}\, E = \mathbb{R}^{2 \times n}. \end{aligned}$$

We can then prove (cf. [12], [13])

Theorem 16 *Let* $\varphi \in W^{1,\infty}\left(\Omega; \mathbb{R}^2\right)$*; then there exists* $w = (u, v) \in W^{1,\infty}\left(\Omega; \mathbb{R}^2\right)$ *satisfying (14).*

3.4 The second order case

The method can also be applied to problems with constraints [13] or to *second order* equations. We give here just one typical theorem that can be established.

Theorem 17 *Let* $\Omega \subset \mathbb{R}^n$ *be bounded and open. Let* $F : \mathbb{R}_s^{n \times n} \to \mathbb{R}$ *(where* $\mathbb{R}_s^{n \times n}$ *stands for the set of* $n \times n$ *symmetric matrices) be convex and coercive (which means that* $\lim F(\xi) = +\infty$ *if* $|\xi| \to +\infty$*). Let* $\varphi \in C^2(\overline{\Omega})$ *be such that*

$$F\left(D^2\varphi(x)\right) \leq 0, \quad a.e. \ x \in \Omega.$$

Then there exists $u \in W^{2,\infty}(\Omega)$ *such that*

$$\begin{cases} F(D^2 u(x)) = 0 & a.e. \ in \ \Omega \\ u = \varphi, \ Du = D\varphi & on \ \partial\Omega. \end{cases} \tag{15}$$

By $W^{2,\infty}(\Omega)$ we denote the Sobolev space that can be identified, if Ω is regular enough, to the set of functions $u \in C^1(\overline{\Omega})$ whose Hessian D^2u (in the distributional sense) has entries that are bounded; or if Ω is sufficiently regular, this means that the partial derivatives u_{x_i}, $i = 1, ..., n$, are Lipschitz continuous.

Note the simultaneous *Dirichlet-Neumann* condition which excludes from our analysis "fully non linear elliptic equations", as mentioned previously.

Some examples can be handled by this theorem or by some of its generalizations, in particular the following ones.

(1) If $\varphi \in C^2(\overline{\Omega})$ is such that

$$|\Delta\varphi(x)| \leq 1, \quad \forall x \in \Omega$$

then there exists $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} |\Delta u| = 1 & \text{a.e. in } \Omega \\ u = \varphi, \ Du = D\varphi & \text{on } \partial\Omega. \end{cases}$$

Observe that because of the boundary data one cannot solve this problem just by solving the linear Poisson equation $\Delta u = 1$ with the Dirichlet boundary datum $u = \varphi$.

(2) If $\varphi \in C^2(\overline{\Omega})$ satisfies $\left|\det D^2\varphi(x)\right| < f$, for every $x \in \Omega$; then there exists $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} \left|\det D^2u(x)\right| = f & \text{a.e. in } \Omega \\ u = \varphi, \ Du = D\varphi & \text{on } \partial\Omega. \end{cases}$$

As in the previous example the above problem cannot be solved by the Monge-Ampère equation $\det D^2u(x) = f$ because of the simultaneous Dirichlet-Neumann conditions $u = \varphi$ and $Du = D\varphi$.

(3) Consider the generalization of the eikonal equation to second derivatives,

$$\begin{cases} \left|D^2u(x)\right| = a & \text{a.e. in } \Omega \\ u = \varphi, \ Du = D\varphi & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Then the compatibility condition on $\varphi \in C^2(\overline{\Omega})$ for existence is that

$$\left|D^2\varphi(x)\right| \leq a, \quad \forall x \in \Omega.$$

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