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On convexity properties of homogeneous functions of degree one*

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We provide an explicit example of a function that is homogeneous of degree one, rank-one convex, but not convex.

1. Introduction

Let $\mathbf{R}^{2\times 2}$ denote the set of 2×2 real matrices and let $f:\mathbf{R}^{2\times 2}\to\mathbf{R}$ be a continuous function that is homogeneous of degree one, i.e. it satisfies the following condition

$$f(t\xi)=tf(\xi),\quad \text{for every } t\geqq 0 \text{ and } \xi\in\mathbf{R}^{2\times 2}. \tag{1.1}$$

We would like to discuss the convexity properties of such functions. In addition to the usual notion of convexity, we need the following definition:

DEFINITION 1.1. $f:\mathbf{R}^{2\times 2}\to\mathbf{R}$ is *rank-one convex* if

$$f(t\xi+(1-t)\eta)\leqq tf(\xi)+(1-t)f(\eta)$$

for every $t\in[0,1]$, ξ, $\eta\in\mathbf{R}^{2\times 2}$ with $\det(\xi-\eta)=0$ (where det stands for the determinant of the matrix).

Obviously any convex function is rank-one convex, while there are rank-one convex functions (such as $f(\xi)=\det\xi$) which are not convex. Surprisingly, if one imposes condition (1.1), then it is not clear that the two notions are not equivalent.

The first person to produce a counterexample was Müller [**4**], but in a very indirect way. In fact his result gives more than this (see below). Dacorogna [**2**] then showed that if, in addition to (1.1), f is assumed to be rotationally invariant (in particular if $f(\xi)=g(|\xi|,\det\xi)$, where $|\xi|$ denotes the Euclidean norm of the matrix, i.e. $|\xi|^2=\Sigma_{i,j=1}^2\xi_{ij}^2$), then any rank-one convex function is necessarily convex. Thus it remained an open question to find an explicit example of a function that is homogeneous of degree one and rank-one convex, but not convex. We produce here a family of such examples. Before describing our results, we should emphasise that

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these functions and notions are important in the Calculus of Variations (see [**1**]). There, the notion of *quasiconvexity* plays the central role. It is well known that convexity implies quasiconvexity and quasiconvexity implies rank-one convexity. Müller's example gives in fact an example of a quasiconvex function that is not convex. It is not presently known whether our examples are quasiconvex.

We now introduce some notation. It will be more convenient to identify $\mathbf{R}^{2\times 2}$ with $\mathbf{R}^4$ and, therefore, a matrix ξ will be written as a vector $(\xi_1, \xi_2, \xi_3, \xi_4)$. We then let

$$\begin{cases} \langle \xi; \eta \rangle = \sum_{i=1}^{4} \xi_i \eta_i, \quad |\xi|^2 = \langle \xi; \xi \rangle, \\ \tilde{\xi} = (\xi_4, -\xi_3, -\xi_2, \xi_1), \\ \det \xi = \xi_1 \xi_4 - \xi_2 \xi_3 = \frac{1}{2} \langle \xi; \tilde{\xi} \rangle. \end{cases}$$

Note that $\tilde{\xi}$ is just the gradient of det ξ. Consider the matrix $E \in \mathbf{R}^{4\times 4}$ representing the quadratic form det. It is defined as

$$E = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$E\xi = \tilde{\xi} \quad \text{and} \quad \det \xi = \tfrac{1}{2} \langle E\xi; \xi \rangle.$$

Finally, our counterexample will be of the form

$$f(\xi) = \begin{cases} |\xi| + \gamma \dfrac{\langle M\xi; \xi \rangle}{|\xi|} & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases}$$

where $\gamma \geqq 0$ and $M \in \mathbf{R}^{4\times 4}$ is a symmetric matrix whose eigenvalues are $\mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \mu_4$.

We will see in the following theorems that choosing M and γ appropriately will produce rank-one convex functions, f, which are not convex.

THEOREM 1.2. *Let f, M and γ be as above. Then*

$$f \text{ is convex} \Leftrightarrow \gamma \leqq \gamma_c,$$

where

$$\gamma_c = \begin{cases} \dfrac{1}{\mu_4 - 2\mu_1} & \text{if } \mu_4 - 2\mu_1 > 0, \\ +\infty & \text{if } \mu_4 - 2\mu_1 \leqq 0. \end{cases}$$

REMARK 1.3. It will be obvious from the proof that if $\gamma \leqq 0$, then f is convex if and only if

$$-\gamma \leqq \frac{1}{2\mu_4 - \mu_1}.$$

THEOREM 1.4. *Let f, M and γ be as above, and let φ_i, $1 \leqq i \leqq 4$, denote an orthonormal set of eigenvectors corresponding to the μ_i's. Assume further that M commutes with E. Then two cases can happen:*

CASE 1. *If* $\det \varphi_4 = -\det \varphi_1$, *then*

$$f \text{ is rank-one convex} \Leftrightarrow f \text{ is convex.}$$

CASE 2. *If* $\det \varphi_4 = \det \varphi_1$, *then*

$$f \text{ is rank-one convex} \Leftrightarrow \gamma \leqq \gamma_r,$$

where

$$\gamma_r = \begin{cases} \min\left\{\dfrac{1}{\gamma_1}, \dfrac{1}{\gamma_2}\right\} & \text{if } \gamma_1 > 0 \text{ and } \gamma_2 > 0, \\ \dfrac{1}{\gamma_2} & \text{if } \gamma_1 \leqq 0 \text{ and } \gamma_2 > 0, \\ \dfrac{1}{\gamma_1} & \text{if } \gamma_1 > 0 \text{ and } \gamma_2 \leqq 0, \\ +\infty & \text{if } \gamma_1 \leqq 0 \text{ and } \gamma_2 \leqq 0 \end{cases}$$

and

$$\gamma_1 = \frac{\mu_4 + \mu_3}{2} - 2\mu_1, \quad \gamma_2 = \mu_4 - (\mu_1 + \mu_2).$$

The following corollary is an immediate consequence of the two theorems.

COROLLARY 1.5. *Consider the function*

$$g(\xi) = \frac{\langle M\xi; \xi\rangle}{|\xi|}.$$

(1) *Let M be as in Theorem 1.2. Then*

$$g \text{ is convex} \Leftrightarrow 2\mu_1 - \mu_4 \geqq 0.$$

(2) *Let M be as in Theorem 1.4 with* $\det \varphi_1 = \det \varphi_4$. (*If* $\det \varphi_1 = -\det \varphi_4$, *see the convex case.*) *Then*

$$g \text{ is rank-one convex} \Leftrightarrow \begin{cases} \mu_1 + \mu_2 - \mu_4 \geqq 0 \\ \quad \text{and} \\ 2\mu_1 - \dfrac{\mu_3 + \mu_4}{2} \geqq 0. \end{cases}$$

REMARK 1.6. It is interesting to compare the corollary with the case of quadratic forms. It is well known that the function

$$q(\xi) = \langle M\xi; \xi\rangle \text{ is convex} \Leftrightarrow \mu_1 \geqq 0.$$

Under the hypotheses of Theorem 1.4, one can show by a similar but simpler argument that

$$q \text{ is rank-one convex} \Leftrightarrow \begin{cases} \dfrac{\mu_1+\mu_2}{2} \geqq 0 & \text{if } \det \varphi_1 = -\det \varphi_2, \\ \dfrac{\mu_1+\mu_3}{2} \geqq 0 & \text{if } \det \varphi_1 = \det \varphi_2. \end{cases}$$

REMARK 1.7. Similar results as in Theorem 1.4 can be derived for $\gamma < 0$. More precisely,

$$f \text{ is rank-one convex} \Leftrightarrow \begin{cases} 1-|\gamma|\left(2\mu_4 - \dfrac{\mu_1+\mu_2}{2}\right) \geqq 0 \\ \quad \text{and} \\ 1-|\gamma|(\mu_4+\mu_3-\mu_1) \geqq 0. \end{cases}$$

REMARK 1.8. The fact that in Theorem 1.4 we have to add the extra hypothesis on M just means that γ_r depends not only on the eigenvalues of M, but also on the eigenvectors, while γ_c depends only on the eigenvalues.

REMARK 1.9. With the help of the theorems we may now give an explicit example. Let

$$M = \begin{pmatrix} 9 & 0 & 0 & 1 \\ 0 & 6 & 2 & 0 \\ 0 & 2 & 6 & 0 \\ 1 & 0 & 0 & 9 \end{pmatrix};$$

its eigenvalues are $\mu_1 = 4 \leqq \mu_2 = \mu_3 = 8 \leqq \mu_4 = 10$, with eigenvectors

$$\varphi_1 = \frac{1}{\sqrt{2}}(0, 1, -1, 0), \quad \varphi_2 = \frac{1}{\sqrt{2}}(0, 1, 1, 0), \quad \varphi_3 = \frac{1}{\sqrt{2}}(1, 0, 0, -1).$$

and

$$\varphi_4 = \frac{1}{\sqrt{2}}(1, 0, 0, 1).$$

It commutes with E and

$$\gamma_c = \tfrac{1}{2}, \quad \gamma_r = 1.$$

Therefore, choosing $\gamma \in (\frac{1}{2}, 1]$ gives the explicit counterexample.

REMARK 1.10. Theorems 1.2 and 1.4 should be compared with [2] when f is rotationally invariant. In particular, if $M = E$, i.e.

$$f(\xi) = \begin{cases} |\xi| + 2\gamma \dfrac{\det \xi}{|\xi|} & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases}$$

we find (as in [2]) that

$$\gamma_c = \gamma_r = \tfrac{1}{3},$$

since the eigenvalues are $\mu_1 = \mu_2 = -1$ and $\mu_3 = \mu_4 = 1$.

REMARK 1.11. Of course our results do not settle the quasiconvexity of f. It is easy to see that f is quasiconvex if and only if $\gamma \leqq \gamma_q$ for a certain γ_q. By the general fact, we have $\gamma_c \leqq \gamma_q \leqq \gamma_r$. The question is to decide whether $\gamma_q = \gamma_r$ or $\gamma_q < \gamma_r$. If equality holds, then we would have an explicit example of a quasiconvex function satisfying (1.1) which is not convex (as in [4]). Obviously, the second possibility would be much more interesting and would settle the long-standing question of the equivalence of quasiconvexity and rank-one convexity (see [5] for a counterexample in higher dimensions). Our numerical results, presented in a forthcoming paper [3], tend to show that $\gamma_q = \gamma_r$.

2. Proof of Theorem 1.2

We start by computing the Hessian of f at $\xi \neq 0$. Observe first that

$$\begin{aligned}\frac{\partial f(\xi)}{\partial \xi_\alpha} &= \frac{\xi_\alpha}{|\xi|} + \gamma \frac{2|\xi|(M\xi)_\alpha - \langle M\xi; \xi\rangle \dfrac{\xi_\alpha}{|\xi|}}{|\xi|^2} \\ &= \frac{\xi_\alpha}{|\xi|} + \gamma \frac{2|\xi|^2 (M\xi)_\alpha - \langle M\xi; \xi\rangle \xi_\alpha}{|\xi|^3}.\end{aligned}$$

We then have

$$\begin{aligned}\frac{\partial^2 f(\xi)}{\partial \xi_\alpha \, \partial \xi_\beta} &= \frac{\delta_{\alpha\beta}}{|\xi|} - \frac{\xi_\alpha \xi_\beta}{|\xi|^3} \\ &\quad + \frac{\gamma}{|\xi|^6} \{[4(M\xi)_\alpha \xi_\beta + 2|\xi|^2 M_{\alpha\beta} - \langle M\xi;\xi\rangle \delta_{\alpha\beta} - 2(M\xi)_\beta \xi_\alpha]|\xi|^3 \\ &\quad - 3|\xi|\xi_\beta[2|\xi|^2(M\xi)_\alpha - \langle M\xi; \xi\rangle \xi_\alpha]\},\end{aligned}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol. Thus, we get that, for any $\xi \neq 0$,

$$\begin{aligned}\sum_{\alpha,\beta=1}^{4} \frac{\partial^2 f(\xi)}{\partial \xi_\alpha \, \partial \xi_\beta} \lambda_\alpha \lambda_\beta &= \frac{|\lambda|^2}{|\xi|} - \frac{(\langle \xi; \lambda\rangle)^2}{|\xi|^3} \\ &\quad + \frac{\gamma}{|\xi|^5} \{|\xi|^2[2\langle M\xi; \lambda\rangle\langle \xi; \lambda\rangle + 2|\xi|^2 \langle M\lambda; \lambda\rangle - \langle M\xi; \xi\rangle |\lambda|^2] \\ &\quad - 3\langle \xi; \lambda\rangle[2|\xi|^2\langle M\xi; \lambda\rangle - \langle M\xi; \xi\rangle\langle \xi; \lambda\rangle]\} \\ &= \frac{|\lambda|^2}{|\xi|} - \frac{(\langle \xi; \lambda\rangle)^2}{|\xi|^3} \\ &\quad + \frac{\gamma}{|\xi|^5} \{2|\xi|^4 \langle M\lambda; \lambda\rangle - 4|\xi|^2 \langle M\xi; \lambda\rangle\langle \xi; \lambda\rangle \\ &\quad - |\xi|^2 |\lambda|^2 \langle M\xi; \xi\rangle + 3(\langle \xi; \lambda\rangle)^2 \langle M\xi; \xi\rangle\}. \end{aligned} \tag{2.1}$$

It is clear that f will be convex if and only if the quadratic form in (2.1) is positive for every $\xi \neq 0$ and λ. The fact that f is not differentiable at 0 does not cause any trouble in this case. Therefore

$$f \text{ is convex} \Leftrightarrow \inf_{\xi \neq 0} \inf_{\lambda} \left\{ \sum_{\alpha,\beta=1}^{4} \frac{\partial^2 f(\xi)}{\partial \xi_\alpha \partial \xi_\beta} \lambda_\alpha \lambda_\beta = \langle \nabla^2 f(\xi)\lambda; \lambda \rangle \right\} \geqq 0.$$

Since the quadratic form is homogeneous of degree -1 in ξ, we may assume that $|\xi| = 1$. We may also write

$$\lambda = t\xi + s\eta, \quad \text{with } t, s \in \mathbf{R},\ |\eta| = 1, \quad \text{and} \quad \langle \xi; \eta \rangle = 0.$$

We then obtain that

$$|\lambda|^2 = t^2 + s^2, \quad \langle \xi; \lambda \rangle = t, \quad \langle M\xi, \lambda \rangle = t\langle M\xi; \xi \rangle + s\langle M\xi; \eta \rangle$$
$$\langle M\lambda; \lambda \rangle = t^2 \langle M\xi; \xi \rangle + 2st\langle M\xi; \eta \rangle + s^2 \langle M\eta; \eta \rangle.$$

So, coming back to the quadratic form, we have for $|\xi| = |\eta| = 1$,

$$\langle \nabla^2 f(\xi)\lambda; \lambda \rangle = s^2 \{1 + \gamma[2\langle M\eta; \eta \rangle - \langle M\xi; \xi \rangle]\}.$$

We finally get that

$$f \text{ is convex} \Leftrightarrow \inf_{\substack{|\xi| = |\eta| = 1 \\ \langle \xi; \eta \rangle = 0}} \{1 + \gamma[2\langle M\eta; \eta \rangle - \langle M\xi; \xi \rangle]\} \geqq 0. \tag{2.2}$$

Since $\gamma \geqq 0$, it is clear that the minimum is attained when $\langle M\eta; \eta \rangle$ is minimum and $\langle M\xi; \xi \rangle$ is maximum, i.e. when $\eta = \varphi_1$ (the eigenvector corresponding to the smallest eigenvalue μ_1) and $\xi = \varphi_4$ (the eigenvector corresponding to the largest eigenvalue μ_4). Thus

$$f \text{ is convex} \Leftrightarrow 1 + \gamma(2\mu_1 - \mu_4) \geqq 0.$$

The conclusion of the theorem follows at once. One also notices that if $\gamma \leqq 0$, then the same argument leads to

$$f \text{ is convex} \Leftrightarrow 1 + \gamma(2\mu_4 - \mu_1) \geqq 0. \quad \square$$

3. Proof of Theorem 1.4

We divide the proof into three steps.

Step 1. It is clear that, even though f is not differentiable at 0, we have

$$f \text{ is rank-one convex} \Leftrightarrow \inf_{\xi \neq 0} \inf_{\det \lambda = 0} \{\langle \nabla^2 f(\xi)\lambda; \lambda \rangle\} \geqq 0.$$

Writing $\lambda = t\xi + s\eta$ with $t, s \in \mathbf{R}$, $\langle \xi; \eta \rangle = 0$, $|\xi| = |\eta| = 1$, we find as in the proof of Theorem 1.2 that (see (2.2)) f is a rank-one convex if and only if

$$\inf \{1 + \gamma[2\langle M\eta; \eta \rangle - \langle M\xi; \xi \rangle] : |\xi| = |\eta| = 1, \langle \xi; \eta \rangle = \det(t\xi + s\eta) = 0,$$
$$t, s \in \mathbf{R} \text{ with } t^2 + s^2 \neq 0\} \geqq 0.$$

Since $\gamma \geqq 0$, we finally deduce that if

$$m = \inf \{2\langle M\eta; \eta \rangle - \langle M\xi; \xi \rangle : |\xi| = |\eta| = 1, \langle \xi; \eta \rangle = \det(t\xi + s\eta) = 0,$$
$$t, s \in \mathbf{R} \text{ with } t^2 + s^2 \neq 0\}, \tag{3.1}$$

then

$$f \text{ is rank-one convex} \Leftrightarrow 1 + \gamma m \geqq 0. \tag{3.2}$$

(If $\gamma \leqq 0$, then one has to compute, analogously, the sup in (3.1).) We will show in the next steps that, in case 1,

$$m = 2\mu_1 - \mu_4, \tag{3.3}$$

while in case 2,

$$m = \min\left\{\mu_1 + \mu_2 - \mu_4,\, 2\mu_1 - \frac{\mu_3 + \mu_4}{2}\right\}. \tag{3.4}$$

Combining (3.2), (3.3), and (3.4) will then give the claimed result.

Step 2. Since M commutes with E, we necessarily have

$$\tilde{\varphi}_i = E\varphi_i = \pm\varphi_i,$$
$$\det \varphi_i = \tfrac{1}{2}\langle\tilde{\varphi}_i; \varphi_i\rangle = \pm\tfrac{1}{2},$$

for $1 \leqq i \leqq 4$. Therefore, cases 1 and 2 do cover all possibilities, since two of the $\det \varphi_i$ are $+\frac{1}{2}$ and two are $-\frac{1}{2}$.

We then immediately get the theorem in case 1. Indeed, choose $\eta = \varphi_1$, $\xi = \varphi_4$, $s = t = 1$, and observe that they are admissible for the minimisation in (3.1). This choice leads to

$$m = 2\mu_1 - \mu_4.$$

Hence, we get (cf. Theorem 1.2)

$$f \text{ is rank-one convex} \Rightarrow 1 + \gamma(2\mu_1 - \mu_4) \geqq 0 \Rightarrow \gamma \leqq \gamma_c.$$

Since, by Theorem 1.2, we have

$$\gamma \leqq \gamma_c \Rightarrow f \text{ is convex}$$

and, trivially, f convex implies that f is rank-one convex, we have indeed established the theorem in case 1.

Step 3. From now on, we will assume that $\det \varphi_1 = \det \varphi_4$, so that the choice $\eta = \varphi_1$ and $\xi = \varphi_4$ is no longer admissible in (3.1) for any choice of $s, t \in \mathbf{R}$ with $s^2 + t^2 \neq 0$. We will prove that the right choice is either

$$\eta = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2), \quad \xi = \varphi_4, \quad \text{and} \quad t = 0,\, s = 1,$$

or

$$\eta = \varphi_1, \quad \xi = \frac{1}{\sqrt{2}}(\varphi_3 + \varphi_4), \quad \text{and} \quad t = 1,\, s = 0.$$

Let us write any $\xi, \eta \in \mathbf{R}^4$ as

$$\xi = \sum_{i=1}^{4} \xi_i\varphi_i, \quad \eta = \sum_{i=1}^{4} \eta_i\varphi_i,$$

and observe that

$$2\langle M\eta;\eta\rangle-\langle M\xi;\xi\rangle=\sum_{i=1}^{4}\mu_i(2\eta_i^2-\xi_i^2).$$

If (3.4) holds, this means that for every $\xi,\eta\in\mathbf{R}^4$, $s,t\in\mathbf{R}$ with $s^2+t^2\neq 0$ such that

$$\begin{cases}|\xi|^2=\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2=|\eta|^2=\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2=1,\\ \langle\xi;\eta\rangle=0\Leftrightarrow\xi_1\eta_1+\xi_4\eta_4=-(\xi_2\eta_2+\xi_3\eta_3),\\ \det(t\xi+s\eta)=0\Leftrightarrow t^2(\xi_1^2+\xi_4^2-\xi_2^2-\xi_3^2)+s^2(\eta_1^2+\eta_4^2-\eta_2^2-\eta_3^2)\\ \qquad\qquad+2ts(\xi_1\eta_1+\xi_4\eta_4-\xi_2\eta_2-\xi_3\eta_3)=0,\end{cases}\tag{3.5}$$

we have to prove that

$$\begin{cases}\sum_{i=1}^{4}\mu_i(2\eta_i^2-\xi_i^2)\geqq\mu_1+\mu_2-\mu_4,\\ \quad\text{or}\\ \sum_{i=1}^{4}\mu_i(2\eta_i^2-\xi_i^2)\geqq 2\mu_1-\dfrac{\mu_3+\mu_4}{2}.\end{cases}\tag{3.6}$$

So we now have to show that (3.6) holds whenever (3.5) does. We will transform (3.6) into more amenable inequalities. Using the facts that $|\xi|=|\eta|=1$, we get that (3.6) is equivalent to

$$\begin{cases}\mu_1(\eta_1^2-\eta_2^2-\eta_3^2-\eta_4^2-\xi_1^2)+\mu_2(\eta_2^2-\eta_1^2-\eta_3^2-\eta_4^2-\xi_2^2)+\mu_3(2\eta_3^2-\xi_3^2)\\ \quad+\mu_4(2\eta_4^2+\xi_1^2+\xi_2^2+\xi_3^2)\geqq 0,\\ \quad\text{or}\\ \mu_1(-2\eta_2^2-2\eta_3^2-2\eta_4^2-\xi_1^2)+\mu_2(2\eta_2^2-\xi_2^2)+\mu_3(2\eta_3^2-\tfrac12\xi_3^2+\tfrac12\xi_2^2+\tfrac12\xi_1^2+\tfrac12\xi_4^2)\\ \quad+\mu_4(2\eta_4^2-\tfrac12\xi_4^2+\tfrac12\xi_1^2+\tfrac12\xi_2^2+\tfrac12\xi_3^2)\geqq 0.\end{cases}$$

Rewriting the above inequalities, we find that

$$\begin{cases}(\mu_2-\mu_1)(\eta_2^2-\eta_1^2)+(2\mu_3-\mu_1-\mu_2)\eta_3^2+(2\mu_4-\mu_1-\mu_2)\eta_4^2\\ \quad+(\mu_4-\mu_1)\xi_1^2+(\mu_4-\mu_2)\xi_2^2+(\mu_4-\mu_3)\xi_3^2\geqq 0,\\ \quad\text{or}\\ 2(\mu_2-\mu_1)\eta_2^2+2(\mu_3-\mu_1)\eta_3^2+2(\mu_4-\mu_1)\eta_4^2+\left(\dfrac{\mu_4+\mu_3}{2}-\mu_1\right)\xi_1^2\\ +\left(\dfrac{\mu_4+\mu_3}{2}-\mu_2\right)\xi_2^2+\tfrac12(\mu_4-\mu_3)(\xi_3^2-\xi_4^2)\geqq 0.\end{cases}$$

Since $\mu_4\geqq\mu_3\geqq\mu_2\geqq\mu_1$, we find that if $\mu_2=\mu_1$ or $\mu_4=\mu_3$, then one of the above inequalities is satisfied. So we may assume that $\mu_3\neq\mu_4$ and $\mu_1\neq\mu_2$.

We transform the inequalities again and get the following formulation, still equivalent to (3.6):

$$\begin{cases} \eta_2^2+\eta_3^2+\eta_4^2-\eta_1^2+\xi_1^2+\dfrac{\mu_4-\mu_2}{\mu_2-\mu_1}(2\eta_4^2+\xi_1^2+\xi_2^2)+2\dfrac{\mu_3-\mu_2}{\mu_2-\mu_1}\eta_3^2+\dfrac{\mu_4-\mu_3}{\mu_2-\mu_1}\xi_3^2 \geqq 0, \\ \quad \text{or} \\ \tfrac{1}{2}(\xi_1^2+\xi_2^2+\xi_3^2-\xi_4^2)+2\eta_4^2+\dfrac{\mu_3-\mu_1}{\mu_4-\mu_3}(2\eta_3^2+2\eta_4^2+\xi_1^2)+2\dfrac{\mu_2-\mu_1}{\mu_4-\mu_3}\eta_2^2+\dfrac{\mu_3-\mu_2}{\mu_4-\mu_3}\xi_2^2 \geqq 0. \end{cases} \tag{3.7}$$

From now on, we proceed by contradiction and assume that there exist ξ, η, s, t as in (3.5) but that (3.6) does not hold. This means, using (3.7), that, in addition to (3.5), ξ and η satisfy

$$\begin{cases} \eta_1^2-\eta_2^2-\eta_3^2 > \eta_4^2+\xi_1^2+\dfrac{\mu_4-\mu_2}{\mu_2-\mu_1}(2\eta_4^2+\xi_1^2+\xi_2^2)+2\dfrac{\mu_3-\mu_2}{\mu_2-\mu_1}\eta_3^2+\dfrac{\mu_4-\mu_3}{\mu_2-\mu_1}\xi_3^2 \\ \qquad \geqq \dfrac{(\mu_4-\mu_2)\xi_2^2+(\mu_4-\mu_3)\xi_3^2}{\mu_2-\mu_1} \\ \qquad \geqq \dfrac{\mu_4-\mu_3}{\mu_2-\mu_1}(\xi_2^2+\xi_3^2) \\ \quad \text{and} \\ \xi_4^2-\xi_2^2-\xi_3^2 > \xi_1^2+4\eta_4^2+2\dfrac{\mu_3-\mu_1}{\mu_4-\mu_3}(2\eta_3^2+2\eta_4^2+\xi_1^2)+4\dfrac{\mu_2-\mu_1}{\mu_4-\mu_3}\eta_2^2+2\dfrac{\mu_3-\mu_2}{\mu_4-\mu_3}\xi_2^2 \\ \qquad \geqq 4\dfrac{(\mu_3-\mu_1)\eta_3^2+(\mu_2-\mu_1)\eta_2^2}{\mu_4-\mu_3} \\ \qquad \geqq 4\dfrac{\mu_2-\mu_1}{\mu_4-\mu_3}(\eta_2^2+\eta_3^2). \end{cases}$$

We now use the facts that $\det(t\xi+s\eta)=0$, $\langle \xi;\eta\rangle=0$, and the strict inequalities above, to get

$$\begin{aligned} 0 &= t^2(\xi_1^2+\xi_4^2-\xi_2^2-\xi_3^2)+s^2(\eta_1^2+\eta_4^2-\eta_2^2-\eta_3^2)-4ts(\xi_2\eta_2+\xi_3\eta_3) \\ &> t^2\left[\xi_1^2+4\frac{\mu_2-\mu_1}{\mu_4-\mu_3}(\eta_2^2+\eta_3^2)\right]+s^2\left[\eta_4^2+\frac{\mu_4-\mu_3}{\mu_2-\mu_1}(\xi_2^2+\xi_3^2)\right]-4|ts|\,|\xi_2\eta_2+\xi_3\eta_3| \\ &\geqq \frac{\mu_4-\mu_3}{\mu_2-\mu_1}s^2(\xi_2^2+\xi_3^2)-4|ts|\,|\xi_2\eta_2+\xi_3\eta_3|+4\frac{\mu_2-\mu_1}{\mu_4-\mu_3}t^2(\eta_2^2+\eta_3^2). \end{aligned}$$

Hence, using the Cauchy–Schwarz inequality on the middle term, we get

$$0 > [(\mu_4-\mu_3)|s|\sqrt{\xi_2^2+\xi_3^2}-2(\mu_2-\mu_1)|t|\sqrt{\eta_2^2+\eta_3^2}]^2,$$

which is absurd. Therefore, if (3.5) holds, then (3.6) does also. This concludes the proof of the theorem. □

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