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ON THE SYMMETRY BREAKING PHENOMENON

PETRE BIRTEA* and MIRCEA PUTA†

*Departamentul de Matematică, Universitatea de Vest
RO-1900 Timișoara, Romania*

**birtea@math.uvt.ro*

†puta@math.uvt.ro

TUDOR S. RATIU

*Centre Bernoulli, École Polytechnique Fédérale de Lausanne
CH-1015 Lausanne, Switzerland*

tudor.ratiu@epfl.ch

RĂZVAN MICU TUDORAN

*Centre Bernoulli, École Polytechnique Fédérale de Lausanne
CH-1015 Lausanne, Switzerland*

*Departamentul de Matematică, Universitatea de Vest
RO-1900 Timișoara, Romania*

razvan.tudoran@epfl.ch

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We investigate the problem of symmetry breaking in the framework of dynamical systems with symmetry on a smooth manifold. Two cases will be analyzed: general and Hamiltonian dynamical systems. We give sufficient conditions for symmetry breaking in both cases.

Keywords: Hamiltonian system; symmetry breaking; bifurcation; relative equilibrium.

1. Introduction

In this paper we investigate the problem of symmetry breaking in the framework of dynamical systems with symmetry on a smooth manifold. Two cases will be analyzed: general and Hamiltonian dynamical systems. For general vector fields we translate the problem locally to a symmetry breaking problem on a vector space where the techniques are more developed and easier to apply because of the existence of a Hilbert basis of invariant polynomials for representations of compact Lie groups. The symmetric steady state and Hopf bifurcations are considered as examples. Sufficient conditions for the existence of symmetry breaking bifurcations are known in this case. In the Hamiltonian context we find sufficient conditions

for the existence of symmetry breaking bifurcations for Hamiltonian vector fields on symplectic vector spaces. The Hamiltonian steady state (passing of eigenvalues of the linearization on the imaginary axis at the origin) and the Hamiltonian Hopf bifurcations are analyzed. It turns out that the same methods as in the general case work, because in the Hamiltonian context one can work only with the invariant polynomials of the action and ignore the equivariant polynomials. In addition, it is shown how to locally translate these Hamiltonian bifurcation problems from the manifold to the tangent space at the bifurcation point. This is done by using the equivariant Darboux theorem.

2. Some Basic Results from the Theory of Group Actions

We shall need a few fundamental results from the theory of group actions which we now briefly recall. For proofs and further information see [5, 7, 11, 20] and for the subsection on compact group representations see [12, 13, 23].

2.1. Twisted products

Let G be a Lie group and $H \subset G$ a Lie subgroup. Suppose that H acts on the left on a manifold A . The *twisted action* of H on the product $G \times A$ is defined by

$$h \cdot (g, a) = (gh, h^{-1} \cdot a), \quad h \in H, \quad g \in G, \quad a \in A.$$

Note that this action is free and proper by the freeness and properness of the action on the G -factor. The *twisted product* $G \times_H A$ is defined as the orbit space $(G \times A)/H$ of the twisted action. The elements of $G \times_H A$ will be denoted by $[g, a]$, $g \in G$, $a \in A$. The twisted product $G \times_H A$ is a G -space relative to the left action defined by $g' \cdot [g, a] = [g'g, a]$. Also, the action of H on A is proper if and only if the G -action on $G \times_H A$ is proper. The isotropy subgroups of the G -action on the twisted product $G \times_H A$ satisfy

$$G_{[g,a]} = gH_a g^{-1}, \quad g \in G, \quad a \in A.$$

2.2. Slices

Throughout this paragraph it will be assumed that $\Psi : G \times Q \rightarrow Q$ is a left proper action of the Lie group G on the manifold Q . This action will not be assumed to be free, in general. For $q \in Q$ we will denote by $H := G_q := \{g \in G \mid g \cdot q = q\}$ the isotropy subgroup of the action Ψ at q . We shall introduce also the following convenient notation: if $K \subset G$ is a Lie subgroup of G (possibly equal to G), \mathfrak{k} its Lie algebra, and $q \in Q$, then $\mathfrak{k} \cdot q := \{\eta_Q(q) \mid \eta \in \mathfrak{k}\}$ is the tangent space to the orbit $K \cdot q$ at q . A *tube* around the orbit $G \cdot q$ is a G -equivariant diffeomorphism $\varphi : G \times_H A \rightarrow U$, where U is a G -invariant neighborhood of $G \cdot q$ and A is some manifold on which H acts. Note that the G -action on the twisted product $G \times_H A$ is proper since the isotropy subgroup H is compact and, consequently, its action

on A is proper. Hence the G -action on $G \times_H A$ is proper. Let S be a submanifold of Q such that $q \in S$ and $H \cdot S = S$. We say that S is a *slice* at q if the map

$$\varphi : G \times_H S \ni [g, s] \mapsto g \cdot s \in U$$

is a tube about $G \cdot q$, for some G -invariant open neighborhood of $G \cdot q$. Notice that if S is a slice at q then $g \cdot S$ is a slice at the point $g \cdot q$. The following statements are equivalent:

- (i) There is a tube $\varphi : G \times_H A \rightarrow U$ about $G \cdot q$ such that $\varphi([e, A]) = S$.
- (ii) S is a slice at q .
- (iii) The submanifold S satisfies the following properties:
 - (a) The set $G \cdot S$ is an open neighborhood of the orbit $G \cdot q$ and S is closed in $G \cdot S$.
 - (b) For any $s \in S$ we have $T_s Q = \mathfrak{g} \cdot s + T_s S$. Moreover, $\mathfrak{g} \cdot s \cap T_s S = \mathfrak{h} \cdot s$, where \mathfrak{h} is the Lie algebra of H . In particular $T_q Q = \mathfrak{g} \cdot q \oplus T_q S$.
 - (c) S is H -invariant. Moreover, if $s \in S$ and $g \in G$ are such that $g \cdot s \in S$, then $g \in H$.
 - (d) Let $\sigma : U \subset G/H \rightarrow G$ be a local section of the submersion $G \rightarrow G/H$. Then the map $F : U \times S \rightarrow Q$ given by $F(u, s) := \sigma(u) \cdot s$ is a diffeomorphism onto an open set of Q .
- (iv) $G \cdot S$ is an open neighborhood of $G \cdot q$ and there is an equivariant smooth retraction $r : G \cdot S \rightarrow G \cdot q$ of the injection $G \cdot q \hookrightarrow G \cdot S$ such that $r^{-1}(q) = S$.

Theorem 2.1 (Slice Theorem). *Let Q be a manifold and G be a Lie group acting properly on Q at the point $q \in Q$. Then there exists a slice for the G -action at q .*

Theorem 2.2 (Tube Theorem). *Let Q be a manifold and G a Lie group acting properly on Q at the point $q \in Q$, $H := G_q$. Then there is a tube $\varphi : G \times_H B \rightarrow U$ about $G \cdot q$ such that $\varphi([e, 0]) = q$, $\varphi([e, B]) =: S$ is a slice at q ; B is an open H -invariant neighborhood of 0 in the vector space $T_q Q / T_q(G \cdot q)$, on which H acts linearly by $h \cdot (v_q + T_q(G \cdot q)) := T_q \Psi_h(v_q) + T_q(G \cdot q)$.*

If Q is a Riemannian manifold, then B can be chosen to be a G_q -invariant neighborhood of 0 in $(\mathfrak{g} \cdot q)^\perp$, the orthogonal complement to $\mathfrak{g} \cdot q$ in $T_q Q$. In this case $U = G \cdot \text{Exp}_q(B)$, where $\text{Exp}_q : T_q Q \rightarrow Q$ is the Riemannian exponential map.

2.3. Type submanifolds and fixed point subspaces

Let G be a Lie group acting on a manifold Q . Let H be a closed subgroup of G . We define the following subsets of Q :

$$\begin{aligned} Q_{(H)} &= \{q \in Q \mid G_q = gHg^{-1}, g \in G\}, \\ Q^H &= \{q \in Q \mid H \subset G_q\}, \\ Q_H &= \{q \in Q \mid H = G_q\}. \end{aligned}$$

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All these sets are submanifolds of Q . The set $Q_{(H)}$ is called the (H) -orbit type submanifold, Q_H is the H -isotropy type submanifold, and Q^H is the H -fixed point submanifold. We will collectively call these subsets the *type submanifolds*. We have:

- Q^H is closed in Q ;
- $Q_{(H)} = G \cdot Q_H$;
- Q_H is open in Q^H .
- the tangent space at $q \in Q_H$ to Q_H equals

$$T_q Q_H = \{v_q \in T_q Q \mid T_q \Psi_h(v_q) = v_q, \text{ for all } h \in H\} = (T_q Q)^H = T_q Q^H;$$

- $T_q(G \cdot q) \cap (T_q Q)^H = T_q(N(H) \cdot q)$, where $N(H)$ is the normalizer of H in G ;
- if H is compact, then $Q_H = Q^H \cap Q_{(H)}$ and Q_H is closed in $Q_{(H)}$.

If Q is a vector space on which H acts linearly, the set Q^H is called in the physics literature the *space of singlets* or the *space of invariant vectors*.

Theorem 2.3 (The stratification theorem). *Let Q be a smooth manifold and G be a Lie group acting properly on it. The connected components of the orbit type manifolds $Q_{(H)}$ and their projections onto the orbit space $Q_{(H)}/G$ constitute a Whitney stratification of Q and Q/G , respectively. This stratification of Q/G is minimal among all Whitney stratifications of Q/G .*

The proof of this result, that can be found in [7] or [22], is based on the Slice Theorem and on a series of extremely important properties of the orbit type manifolds decomposition that we enumerate in what follows. We start by recalling that the set of conjugacy classes of subgroups of a Lie group G admits a partial order by defining $(K) \preceq (H)$ if and only if H is conjugate to a subgroup of K . Also, a point $q \in Q$ in a proper G -subspace of Q (or its corresponding G -orbit, $G \cdot q$) is called *principal* if its corresponding local orbit type manifold is open in Q . The orbit $G \cdot q$ is called *regular* if the dimension of the orbits nearby coincides with the dimension of $G \cdot q$. The set of principal and regular orbits will be denoted by Q_{princ}/G and Q_{reg}/G , respectively. Using this notation we have:

- For any $q \in Q$ there exists a neighborhood U of q that intersects only finitely many connected components of finitely many orbit type manifolds. If Q is compact or a linear space where G acts linearly, then the G -action on Q has only finitely many distinct connected components of orbit type manifolds.
- For any $q \in Q$ there exists an open neighborhood U of q such that $(G_q) \preceq (G_x)$, for all $x \in U$. In particular, this implies that $\dim G \cdot q \leq \dim G \cdot x$, for all $x \in U$.
- *Principal Orbit Theorem:* For every connected component Q^0 of Q the subset $Q_{\text{princ}} \cap Q^0$ is connected, open, and dense in Q^0 . Each connected component $(Q/G)^0$ of Q/G contains only one principal orbit type, which is connected open and dense in $(Q/G)^0$.

2.4. General facts about compact Lie groups representations

Let G be a compact Lie group acting on a finite dimensional vector space V . The compactness of G implies the existence of a G -invariant inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on V , so one can always assume that the G -representation is orthogonal. Denote by $\mathcal{P}_G(V)$ the ring of G -invariant real valued polynomials on V . The Hilbert–Weyl theorem implies that $\mathcal{P}_G(V)$ is finitely generated over \mathbb{R} and it can be shown that one can choose the generators to be homogeneous polynomials. The elements $\{\theta_1, \dots, \theta_l\}$ of such a basis over \mathbb{R} are called *invariant generators*. The following theorem of Schwarz gives a method of writing a G -equivariant smooth map in terms of invariants and equivariants.

Theorem 2.4. *Let V and W be two G -representation spaces. The set of G -equivariant polynomial maps from V to W is a finitely generated $\mathcal{P}_G(V)$ -module. Let $\{F_1, \dots, F_k\}$ be one of its minimal bases. Then, if F is a smooth G -equivariant map from V to W , there exist smooth functions f_1, \dots, f_k on \mathbb{R}^l such that*

$$F(v) = \sum_{i=1}^k f_i(\theta_1(v), \dots, \theta_l(v)) F_i(v).$$

The next theorem gives an identification between the orbit space of a linear action of a compact Lie group G on a vector space V and a semialgebraic subset of \mathbb{R}^l .

Theorem 2.5. *Let π be the Hilbert map $V \ni v \mapsto (\theta_1(v), \dots, \theta_l(v)) \in \mathbb{R}^l$. Then π induces a homeomorphism between the orbit space V/G and $\text{Im}(\pi)$.*

The invariant generators are intimately related to the orthogonal space to the G -orbits. If $v \in V$, let $N_v := \{u \in V \mid \langle\langle u, \xi \cdot v \rangle\rangle = 0, \text{ for all } \xi \in \mathfrak{g}\}$ be the orthogonal space to the orbit $G \cdot v$. Define the *normal invariant subspace* $N_v^{(0)} := \{u \in N_v \mid g \cdot u = u, \text{ for all } g \in G_v\} \subset N_v$. The following theorem is proved in [3].

Theorem 2.6. *The vector subspace of V generated by $\{\text{grad} \theta_i(v) \mid i = 1, \dots, l\}$ coincides with the normal invariant subspace $N_v^{(0)}$.*

Two useful consequences of this fact are the orthogonal sum decomposition

$$T_v V_{(H)} = T_v(G \cdot v) \oplus N_v^{(0)}$$

and the fact that $F_i(v) \in T_v(V_{(G_v)})$ for any equivariant generator F_i , $i = 1, \dots, k$ of the G -representation. Since the orbit space is the image of a finite dimensional vector space by a polynomial map, the Tarski–Seidenberg theorem guarantees that it is a semialgebraic set. Thus, as a semialgebraic set, V/G has a canonical stratification. A theorem of Bierstone implies that this canonical semialgebraic stratification is the same as the stratification by orbit types obtained by projection of those from V . Since the orbit space V/G inherits a smooth structure from V and G , but in general is not a manifold, we have to define the meaning of a smooth vector field on V/G .

Definition 2.7. Let X be a smooth vector field defined on a neighborhood U of $0 \in \mathbb{R}^l$ and tangent to the strata of $\pi(V) \cap U$. Then X is said to be *smooth vector field on V/G* in a neighborhood of the origin.

If one projects a smooth G -equivariant vector field on the orbit space, one obtains a smooth vector field on V/G in the sense defined above. When we speak of a vector field on V/G we mean a vector field which is tangent to each stratum of V/G . The next step gives a method for projecting a G -equivariant vector field on the orbit space. For doing this, let F be a smooth G -equivariant vector field on the finite dimensional vector space V . Let $\{\theta_1, \dots, \theta_l\}$ be a set of generators for $\mathcal{P}_G(V)$ and let $\{F_1, \dots, F_k\}$ be a set of generators for the $\mathcal{P}_G(V)$ -module of G -equivariant polynomial maps from V to V . By the Schwarz theorem we can write F in terms of invariants and equivariants as

$$F(v) = \sum_{i=1}^k f_i(\theta_1(v), \dots, \theta_l(v)) F_i(v),$$

where the functions f_i are smooth functions on \mathbb{R}^l . To obtain the projection of this vector field on the orbit space V/G , one has to compute the derivatives of the invariant generators θ_i with respect to time:

$$\begin{aligned} \dot{\theta}_j &= \langle \langle \text{grad}_v(\theta_j), \dot{v} \rangle \rangle = \left\langle \left\langle \text{grad}_v(\theta_j), \sum_{i=1}^k f_i(\theta(v)) F_i(v) \right\rangle \right\rangle \\ &= \sum_{i=1}^k f_i(\theta(v)) \langle \langle \text{grad}_v(\theta_j), F_i(v) \rangle \rangle. \end{aligned}$$

Since the expression $\langle \langle \text{grad}_v(\theta_j), F_i(v) \rangle \rangle$ is G -invariant one can rewrite it in terms of the invariant generators. Thus the projection \tilde{F} of F on the orbit space V/G has the expression

$$\tilde{F}(\theta) = \sum_{i=1}^k f_i(\theta) \tilde{F}_i(\theta), \quad (2.1)$$

where if v is such that $\pi(v) = \theta$ and

$$\tilde{F}_i(\theta) := \begin{bmatrix} \langle \langle F_i(v), \text{grad}_v \theta_1(v) \rangle \rangle \\ \vdots \\ \langle \langle F_i(v), \text{grad}_v \theta_l(v) \rangle \rangle \end{bmatrix} \quad \text{for } i = 1, \dots, k.$$

3. Formulation of Symmetric Systems on Vector Spaces

This section presents a method of translating a bifurcation problem for a general vector fields family with G -symmetry from a smooth manifold to a vector space. We begin by quickly recalling the necessary notations and statements for a left smooth proper Lie group action $\Phi : G \times Q \rightarrow Q$, where Q is a smooth manifold and G is a Lie group with Lie algebra \mathfrak{g} .

Definition 3.1. A vector field $X : Q \rightarrow TQ$ is said to be G -equivariant if

$$T_q \Phi_g(X(q)) = X(\Phi_g(q))$$

for all $q \in Q$ and $g \in G$. If X is G -equivariant, then G is said to be a *symmetry group* of the dynamical system $\dot{q} = X(q)$.

Definition 3.2. A *relative equilibrium* of a G -equivariant vector field X is a point $q_e \in Q$ at which the values of X and of the infinitesimal generator ξ_Q of some element $\xi \in \mathfrak{g}$, called *velocity* of the relative equilibrium, coincide, that is,

$$X(q_e) = \xi_Q(q_e) := \left. \frac{d}{dt} \right|_{t=0} \Phi(\exp t\xi, q_e),$$

where $\exp : \mathfrak{g} \rightarrow G$ is the Lie group exponential map. A relative equilibrium q_e is said to be *asymmetric* if $\mathfrak{g}_{q_e} = \{0\}$, and *symmetric* otherwise.

From the definition above and the uniqueness of the integral curves of a vector field with given initial condition, it follows that the dynamical orbit $q(t)$ of X starting at the relative equilibrium q_e , that is, $q(0) = q_e$, coincides with the curve $\exp t\xi \cdot q_e := \Phi(\exp t\xi, q_e)$ given by the group action. Note that if ξ is a velocity of the relative equilibrium q_e , then so is $\xi + \eta$, where $\eta \in \mathfrak{g}_{q_e} := \{\zeta \in \mathfrak{g} \mid \zeta_Q(q_e) = 0_{q_e}\}$. The Lie subalgebra \mathfrak{g}_{q_e} is called the *symmetry algebra* of q_e . Denote for each $g \in G$ by $\text{Ad} : G \times \mathfrak{g} \rightarrow \mathfrak{g}$ the adjoint representation of G on \mathfrak{g} ; for each $g \in G$, Ad_g is the Lie algebra isomorphism obtained by taking the derivative of the conjugation isomorphism on G .

Proposition 3.3. *If q_e is a relative equilibrium with velocity $\xi \in \mathfrak{g}$, then, for any $g \in G$, $\Phi_g(q_e)$ is also a relative equilibrium with velocity $\text{Ad}_g \xi$.*

Proof. Indeed, since X is G -equivariant, we get

$$X(\Phi_g(q_e)) = T_{q_e} \Phi_g(X(q_e)) = (T_{q_e} \Phi_g \circ \xi_Q \circ \Phi_{g^{-1}})(\Phi_g(q_e)) = (\text{Ad}_g \xi)_Q(\Phi_g(q_e)),$$

which proves the statement. □

By one of the remarks following Theorem 2.3, we can choose a G -invariant tubular open neighborhood W_{q_e} of the orbit $G \cdot q_e$ such that the isotropy subgroup G_x for any $x \in W_{q_e}$ is conjugate to a (possibly non-strict) subgroup of G_{q_e} . The next lemma will show that we can do better.

Lemma 3.4. *Let L be a compact Lie group acting on a smooth manifold Q . Assume that $q \in Q$ is a fixed point of the L -action. Then any open neighborhood of q contains a L -invariant open neighborhood of q .*

Proof. Let $\Phi : L \times Q \rightarrow Q$ be the group action and U an arbitrary neighborhood of q . The set $\Phi^{-1}(U)$ is clearly open and contains $L \times \{q\}$. By continuity of Φ , for any $g \in L$, there are open neighborhoods W_g of g in L and V_g of q in Q such that

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$W_g \times V_g \subset \Phi^{-1}(U)$. Since, by hypothesis, L is compact, the open cover $\{W_g \mid g \in L\}$ of L has a finite subcover $\{W_{g_1}, \dots, W_{g_n}\}$. Let $V := \bigcap_{i=1}^n V_{g_i}$. The L -invariant set $W := \Phi(L, V) \subset U$ is clearly open since $W = \bigcup_{g \in L} \Phi_g(V)$ and, by construction, contains the point q . \square

Properness of the G -action on Q implies that the isotropy subgroup G_{q_e} is compact. The above, lemma we can conclude that there exists an open G_{q_e} -invariant neighborhood $W_{q_e}^1$ of q_e contained in W_{q_e} . Thus the local study of the symmetry breaking from G_{q_e} to a smaller symmetry group can be done by restricting the action to $W_{q_e}^1$. Next, assume that Q carries a G -invariant Riemannian metric. The Riemannian exponential map $\text{Exp} : TQ \rightarrow Q$ is also G -equivariant, relative to the naturally lifted G -action on TQ . Since $\text{Exp}_{q_e} : T_{q_e}Q \rightarrow Q$ is a local diffeomorphism around the origin, we can further shrink the neighborhood $W_{q_e}^1$ (keeping the G_{q_e} -invariance), till it is contained in the image under Exp_{q_e} of the neighborhood of the origin where Exp_{q_e} is a diffeomorphism. Denote this shrunk G_{q_e} -invariant neighborhood by U_{q_e} . The inverse map of the Riemannian exponential on U_{q_e} , $\text{Log}_{q_e} : U_{q_e} \rightarrow \text{Log}_{q_e}(U_{q_e}) \subset T_{q_e}Q$ is therefore also G_{q_e} -equivariant and so its image $\text{Log}_{q_e}(U_{q_e}) =: V_{0_{q_e}}$ will be a G_{q_e} -invariant open neighborhood of $0_{q_e} \in T_{q_e}Q$.

Proposition 3.5. *Let $(Q, \langle \langle \cdot, \cdot \rangle \rangle)$ be a smooth Riemannian manifold, $q_e \in Q$, G a Lie group acting properly and smoothly on Q by isometries, and $X \in \mathfrak{X}(Q)$ a smooth G -equivariant vector field. Then, in the notations above, the vector field $\text{Exp}_{q_e}^*(X|_{U_{q_e}}) \in \mathfrak{X}(V_{0_{q_e}})$ is a smooth G_{q_e} -equivariant vector field on the open neighborhood $V_{0_{q_e}}$ of the origin 0_{q_e} in the vector space $T_{q_e}Q$.*

Proof. The proof is a simple consequence of the fact that $\text{Exp}_{q_e} : V_{q_e} \subset T_{q_e}Q \rightarrow U_{q_e} \subset Q$ and $\text{Log}_{q_e} : U_{q_e} \rightarrow V_{q_e}$ are G_{q_e} -equivariant diffeomorphisms inverse to each other. Indeed, since X is G -equivariant, its restriction $X|_{U_{q_e}}$ is G_{q_e} -equivariant. Therefore, for any $h \in G_{q_e}$ and $v_{q_e} \in V_{q_e}$ we have

$$\begin{aligned} & (\text{Exp}_{q_e}^*(X|_{U_{q_e}}))(T_{q_e}\Phi_h(v_{q_e})) \\ &= T_{\text{Exp}_{q_e}(T_{q_e}\Phi_h(v_{q_e}))}\text{Log}_{q_e}(X(\text{Exp}_{q_e}(T_{q_e}\Phi_h(v_{q_e})))) \\ &= T_{\Phi_h(\text{Exp}_{q_e}(v_{q_e}))}\text{Log}_{q_e}(X(\Phi_h(\text{Exp}_{q_e}(v_{q_e})))) \\ &= T_{\text{Exp}_{q_e}(v_{q_e})}(\text{Log}_{q_e} \circ \Phi_h)(X(\text{Exp}_{q_e}(v_{q_e}))) \\ &= T_{\text{Exp}_{q_e}(v_{q_e})}(T_{q_e}\Phi_h \circ \text{Log}_{q_e})(X(\text{Exp}_{q_e}(v_{q_e}))) \\ &= T_{v_{q_e}}(T_{q_e}\Phi_h)((\text{Exp}_{q_e}^*(X|_{U_{q_e}}))(v_{q_e})), \end{aligned}$$

which proves that $\text{Exp}_{q_e}^*(X|_{U_{q_e}})$ is a G_{q_e} -equivariant vector field on $V_{0_{q_e}}$. \square

Proposition 3.6. *Let $(Q, \langle \langle \cdot, \cdot \rangle \rangle)$ be a smooth Riemannian manifold, G a Lie group acting properly on Q by isometries, and $X_\lambda \in \mathfrak{X}(Q)$, $\lambda \in \mathbb{R}$, a family of smooth G -equivariant vector fields. Suppose that $q_e \in Q$ is a (relative) equilibrium for X_{λ_0}*

with symmetry G_{q_e} . Assume the family of vector fields $\text{Exp}_{q_e}^* X_\lambda \in \mathfrak{X}(V_{0_{q_e}})$ has the property that there exist $\epsilon > 0$ and a continuous curve $\gamma : [\lambda_0, \lambda_0 + \epsilon] \rightarrow V_{0_{q_e}}$, $\gamma(\lambda_0) = 0_{q_e}$, such that all the points on the curve $\gamma(\lambda)$ for $\lambda > \lambda_0$ are relative equilibria of $\text{Exp}_{q_e}^* X_\lambda$ with isotropy subgroups conjugate to $H \subset G_{q_e}$. Then the curve $\text{Exp}_{q_e} \circ \gamma : [\lambda_0, \lambda_0 + \epsilon] \rightarrow U_{q_e}$ is formed by relative equilibria of X_λ with isotropy subgroups conjugate to H for all $\lambda > \lambda_0$ and to G_{q_e} for $\lambda = \lambda_0$.

Proof. Let $X_\lambda \in \mathfrak{X}(U_{q_e})$ be one of the vector fields restricted to the G_{q_e} -invariant open neighborhood U_{q_e} . Suppose that $\gamma(\lambda)$ is a relative equilibrium of the vector field $\text{Exp}_{q_e}^* X_\lambda$. This means that there exists an element $\xi(\lambda) \in \mathfrak{g}_{q_e} \subset \mathfrak{g}$ such that

$$(\text{Exp}_{q_e}^* X_\lambda)(\gamma(\lambda)) = (\xi(\lambda))_{T_{q_e} Q}(\gamma(\lambda)),$$

that is,

$$T_{\text{Exp}_{q_e}(\gamma(\lambda))} \text{Log}_{q_e}(X_\lambda(\text{Exp}_{q_e}(\gamma(\lambda)))) = (\xi(\lambda))_{T_{q_e} Q}(\gamma(\lambda)).$$

If we apply to the above equality $T_{\gamma(\lambda)} \text{Exp}_{q_e}$ and use the G_{q_e} -equivariance of the exponential map we get

$$\begin{aligned} X_\lambda(\text{Exp}_{q_e}(\gamma(\lambda))) &= T_{\gamma(\lambda)} \text{Exp}_{q_e}((\xi(\lambda))_{T_{q_e} Q}(\gamma(\lambda))) \\ &= (\xi(\lambda))_Q(\text{Exp}_{q_e}(\gamma(\lambda))). \end{aligned}$$

This means that $\text{Exp}_{q_e}(\gamma(\lambda))$ is a relative equilibrium of X_λ . So we have proved that the curve $\text{Exp}_{q_e} \circ \gamma$ is formed by relative equilibria of X_λ for any $\lambda \in [\lambda_0, \lambda_0 + \epsilon]$. Because the exponential map is a G_{q_e} -equivariant diffeomorphism on the neighborhoods considered, we have $G_{\text{Exp}_{q_e}(\gamma(\lambda))} = G_{\gamma(\lambda)}$, and so all the points on the curve $\gamma(\lambda)$ with the possible exception of $\gamma(\lambda_0)$ are conjugate to H . \square

Since the G_{q_e} -equivariant family of vector fields $\text{Exp}_{q_e}^* X_\lambda \in \mathfrak{X}(V_{0_{q_e}})$ are defined on an open G_{q_e} -invariant neighborhood of the origin in the vector space $T_{q_e} Q$, we can arbitrarily extend their restrictions to a smaller G_{q_e} -invariant neighborhood of the origin to smooth G_{q_e} -equivariant vector fields $\overline{X}_\lambda \in \mathfrak{X}(T_{q_e} Q)$ on the whole space $T_{q_e} Q$. However, on $T_{q_e} Q$ we have an orthogonal representation of the compact group G_{q_e} and thus, as explained earlier, the vector fields $\overline{X}_\lambda \in \mathfrak{X}(T_{q_e} Q)$ can be expressed in terms of the invariant and equivariant polynomials on $T_{q_e} Q$. Thus to study the local symmetry breaking phenomenon around the point $q_e \in Q$, one can translate the problem to the vector space $T_{q_e} Q$ and study this problem for the G_{q_e} -equivariant family $\overline{X}_\lambda \in \mathfrak{X}(T_{q_e} Q)$.

4. Stationary and Hopf Bifurcation with Symmetry

We shall analyze now the local symmetry breaking phenomenon around $0_{q_e} \in T_{q_e} Q$ for the family of G_{q_e} -equivariant vector fields $\overline{X}_\lambda \in \mathfrak{X}(T_{q_e} Q)$ under the following

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two nondegeneracy conditions:

- *Stationary bifurcation with symmetry:*

$$D\overline{X}_\lambda(0_{q_e}) = \sigma_X(\lambda)\text{Id}_{T_{q_e}Q}$$

where $\sigma_X : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ is a smooth map satisfying $\sigma_X(0) = 0$ and $\sigma'_X(0) \neq 0$. In addition, it is assumed that the compact group G_{q_e} acts *absolutely irreducibly* on $T_{q_e}Q$, i.e., that only scalar multiples of the identity commute with the G_{q_e} -action.

- *Hopf bifurcation with symmetry:*

$$D\overline{X}_\lambda(0_{q_e}) = \sigma_X(\lambda)\text{Id}_{T_{q_e}Q} + \rho_X(\lambda)\mathbb{J}_{T_{q_e}Q}$$

where $\sigma_X, \rho_X : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ are smooth maps satisfying $\sigma_X(0) = 0, \sigma'_X(0) \neq 0$, and $\rho_X(0) \neq 0$. In addition, it is assumed that the vector space $T_{q_e}Q$ carries a natural complex structure $\mathbb{J}_{T_{q_e}Q}$ and that the compact group G_{q_e} acts on this complex vector space $T_{q_e}Q$ irreducibly (as a complex representation). In this case the manifold Q is necessarily of even dimension.

In the following will denote by $\mathcal{C}^\infty(T_{q_e}Q \times \mathbb{R}, T_{q_e}Q)^{G_{q_e}}$ the class of G_{q_e} -equivariant vector fields that verifies one of the above nondegeneracy conditions.

Let $\{\theta_1(v_{q_e}), \dots, \theta_l(v_{q_e})\}$ be a set of generators for $\mathcal{P}_{G_{q_e}}(T_{q_e}Q)$ and let $\{F_1, \dots, F_k\}$ be a set of generators for the $\mathcal{P}_{G_{q_e}}(T_{q_e}Q)$ -module of G_{q_e} -equivariant maps from $T_{q_e}Q$ to itself. Then by the Schwarz theorem we have

$$\overline{X}_\lambda(v_{q_e}) = \sum_{j=1}^k f_j(\theta_1(v_{q_e}), \dots, \theta_l(v_{q_e}), \lambda)F_j(v_{q_e}), \tag{4.1}$$

where the functions f_i are smooth functions on \mathbb{R}^l . The projection of this family of vector fields on the orbit space $(T_{q_e}Q)/G_{q_e}$ has the expression:

$$\dot{\theta} = \sum_{i=1}^k f_i(\theta, \lambda)\tilde{F}_i(\theta), \tag{4.2}$$

where, if $v_{q_e} \in T_{q_e}Q$ is such that $\pi(v_{q_e}) = \theta \in (T_{q_e}Q)/G_{q_e}$, we have

$$\tilde{F}_i(\theta) = \begin{bmatrix} \langle \langle F_i(v_{q_e}), \text{grad}_{v_{q_e}} \theta_1(v_{q_e}) \rangle \rangle \\ \vdots \\ \langle \langle F_i(v_{q_e}), \text{grad}_{v_{q_e}} \theta_l(v_{q_e}) \rangle \rangle \end{bmatrix}, \quad \text{for } i = 1, \dots, k.$$

According to the orbit space projection of the vector field \overline{X}_λ we introduce the maps

$$\begin{aligned} \tilde{F} : (T_{q_e}Q)/G_{q_e} \times \mathbb{R}^k \ni (\theta, t) &\mapsto \sum_{i=1}^k t_i \tilde{F}_i(\theta) \in (T_{q_e}Q)/G_{q_e} \\ \gamma : \mathbb{R} \ni \lambda &\mapsto (f_1(0, \lambda), \dots, f_k(0, \lambda)) \in \mathbb{R}^k \end{aligned} \tag{4.3}$$

and define the sets:

$$\begin{aligned}\tilde{\mathcal{E}} &= \{(\theta, t) \in (T_{q_e}Q)/G_{q_e} \times \mathbb{R}^k \mid \tilde{F}(\theta, t) = 0\} \\ \tilde{\mathcal{E}}_{(H)} &= \{(\theta, t) \in \tilde{\mathcal{E}} \mid \theta \in \Pi((T_{q_e}Q)_{(H)})\}, \\ \tilde{\mathcal{A}}_{(H)} &= \overline{\tilde{\mathcal{E}}_{(H)}} \cap (\{0\} \times \mathbb{R}^k),\end{aligned}$$

where $\Pi : T_{q_e}Q \rightarrow (T_{q_e}Q)/G_{q_e}$ is the orbit space projection and $H \subset G_{q_e}$ is an isotropy subgroup of the G_{q_e} -action on $T_{q_e}Q$. Recall that $\tilde{\mathcal{E}}$ is a semialgebraic subset of $\mathbb{R}^l \times \mathbb{R}^k$. Denote by Σ its canonical Whitney stratification. This stratification induces a Whitney semialgebraic stratification $\Sigma_{(H)}$ on each $\tilde{\mathcal{E}}_{(H)}$. Let \mathcal{B} denote the Whitney stratification $\Sigma_{(G_{q_e})}$ of $\tilde{\mathcal{E}}_{(G_{q_e})} = \{0\} \times \mathbb{R}^k$. Next we will define a notion of G_{q_e} -transversality introduced by Koenig and Chossat¹³ that was inspired by Bierstone⁴ and Field [8].

Definition 4.1. The projection to the orbit space $(T_{q_e}Q)/G_{q_e}$ of a G_{q_e} -equivariant smooth family of vector fields $\overline{X_\lambda} \in \mathfrak{X}(T_{q_e}Q)$ is *transverse* to $[0_{q_e}] \in (T_{q_e}Q)/G_{q_e}$ at $\lambda = 0 \in \mathbb{R}$ if $\gamma : \mathbb{R} \rightarrow \mathbb{R}^k$ is transverse at $\lambda = 0$ to the stratum of \mathcal{B} containing $\gamma(0)$, that is, $\text{im}T_0\gamma + T_{\gamma(0)}S = \mathbb{R}^k$, where S is the stratum of \mathcal{B} containing $\gamma(0)$.

Analogous to map (4.3), we define

$$F : T_{q_e}Q \times \mathbb{R}^k \ni (v_{q_e}, t) \mapsto \sum_{i=1}^k t_i F_i(v_{q_e}) \in T_{q_e}Q.$$

This map induces the following sets parameterized by the isotropy type subgroups of the G_{q_e} -action on $T_{q_e}Q$:

$$\begin{aligned}\mathcal{E}_{(H)} &= \{(v_{q_e}, t) \in (T_{q_e}Q)_{(H)} \times \mathbb{R}^k \mid F(v_{q_e}, t) \in \mathfrak{g}_{q_e} \cdot v_{q_e}\} \\ \mathcal{A}_{(H)} &= \overline{\mathcal{E}_{(H)}} \cap (\{0\} \times \mathbb{R}^k).\end{aligned}$$

The sets $\mathcal{E}_{(H)}$ and $\mathcal{A}_{(H)}$ are semialgebraic sets in $T_{q_e}Q \times \mathbb{R}^k$ and \mathbb{R}^k respectively. The next proposition gives an relation between the pair of subsets $\tilde{\mathcal{E}}_{(H)} \subset (T_{q_e}Q)/G_{q_e} \times \mathbb{R}^k$, $\tilde{\mathcal{A}}_{(H)} \subset \mathbb{R}^k$ and $\mathcal{E}_{(H)} \subset (T_{q_e}Q)_{(H)} \times \mathbb{R}^k$, $\mathcal{A}_{(H)} \subset \mathbb{R}^k$.

Proposition 4.2. *Let $\Pi : T_{q_e}Q \rightarrow (T_{q_e}Q)/G_{q_e}$ be the projection onto the orbit space and H and isotropy type of the G_{q_e} -action on $T_{q_e}Q$. Then:*

$$\begin{aligned}\mathcal{E}_{(H)} &= (\Pi \times \text{id})^{-1}(\tilde{\mathcal{E}}_{(H)}), \\ \mathcal{A}_{(H)} &= \tilde{\mathcal{A}}_{(H)}.\end{aligned}$$

Proof. The proof is as in [12]. To show the first equality let $v_{q_e} \in \Pi^{-1}(\theta)$. Then

$$\tilde{F}(\theta, t) = 0 \Leftrightarrow \langle F(v_{q_e}, t), \text{grad}_{v_{q_e}} \theta_j(v_{q_e}) \rangle = 0,$$

for any $j \in \{1, \dots, l\}$. As $F(v_{q_e}, t) \in T_{v_{q_e}}(T_{q_e}Q)_{(G_{v_{q_e}})} = \mathfrak{g}_{q_e} \cdot v_{q_e} \oplus N_{v_{q_e}}^{(0)}$ and $N_{v_{q_e}}^{(0)}$ is generated by $\{\text{grad}_{v_{q_e}} \theta_j(v_{q_e}), \text{ for } j \in \{1, \dots, l\}\}$, we must have $F(v_{q_e}, t) \in \mathfrak{g}_{q_e} \cdot v_{q_e}$.

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Thus $(\theta, t) \in \tilde{\mathcal{E}}_{(H)} \Leftrightarrow (v_{q_e}, t) \in \mathcal{E}_{(H)}$. To prove the second equality let $t \in \mathcal{A}_{(H)}$. Then there exists $(v_n, t_n) \in \mathcal{E}_{(H)}$ such that $v_n \rightarrow 0$ and $t_n \rightarrow t$. By the first equality, this implies that $(\Pi(v_n), t_n) \in \tilde{\mathcal{E}}_{(H)}$ and the continuity of Π implies that $t \in \tilde{\mathcal{A}}_{(H)}$. Conversely, let $t \in \tilde{\mathcal{A}}_{(H)}$. Then there exists a sequence $(\theta_n, t_n) \in \tilde{\mathcal{E}}_{(H)}$ whose limit is $(0, t)$. As $\mathcal{E}_{(H)} = (\Pi \times \text{id})^{-1}(\tilde{\mathcal{E}}_{(H)})$, there is a sequence $(v_n, t_n) \in \mathcal{E}_{(H)}$ convergent to a point in $(\Pi^{-1}(0), t)$. Because the action of G_{q_e} on $T_{q_e}Q$ is linear we have that $\Pi^{-1}(0) = 0$ and hence $t \in \mathcal{A}_{(H)}$. \square

Remark 4.3. If H is conjugated to a proper subgroup of G_{q_e} we have $\text{codim}_{\mathbb{R}^k} \mathcal{A}_{(H)} \geq 1$.

The next theorem, due to Koenig and Chossat [13], gives a sufficient criterion for the existence of symmetry breaking branches of relative equilibria for the family of vector fields \overline{X}_λ in a neighborhood of 0_{q_e} .

Theorem 4.4. *Let H be an isotropy group conjugate to a proper subgroup of G_{q_e} and $\overline{X}_\lambda \in \mathcal{C}^\infty(T_{q_e}Q \times \mathbb{R}, T_{q_e}Q)_o^{G_{q_e}}$ verifying the transversality condition given in Definition 4.1. For $v_{q_e} \in (T_{q_e}Q)_{(H)}$ define $n_H := \text{rank } N(G_{v_{q_e}})/G_{v_{q_e}}$.*

- (a) *If $\text{codim}_{\mathbb{R}^k} \mathcal{A}_{(H)} \geq 2$, then the system $\dot{v}_{q_e} = \overline{X}_\lambda(v_{q_e})$ does not have relative equilibria with isotropy conjugate to H in a neighborhood of the origin 0_{q_e} .*
- (b) *If $\text{codim}_{\mathbb{R}^k} \mathcal{A}_{(H)} = 1$ and $\gamma(0) \in \mathcal{A}_{(H)}$, then there exists a branch of tori of dimension n_H bifurcating from 0_{q_e} . The torus corresponding to λ is invariant under the flow of \overline{X}_λ . The points on all of these tori have isotropy type H .*

For the proof of this theorem we will need the following result of Field and Richardson [9]:

Theorem 4.5. *Let H be an isotropy group conjugate to a proper subgroup of G_{q_e} and \widetilde{X}_λ the projection on the orbit space $(T_{q_e}Q)/G_{q_e}$ of the family of vector fields $\overline{X}_\lambda \in \mathcal{C}^\infty(T_{q_e}Q \times \mathbb{R}, T_{q_e}Q)_o^{G_{q_e}}$ verifying the transversality condition given in Definition 4.1.*

- (a) *If $\text{codim}_{\mathbb{R}^k} \tilde{\mathcal{A}}_{(H)} \geq 2$, then $\widetilde{X}_\lambda^{-1}([0])$ does not contain points with isotropy conjugate to H .*
- (b) *If $\text{codim}_{\mathbb{R}^k} \tilde{\mathcal{A}}_{(H)} = 1$ and $\gamma(0) \in \tilde{\mathcal{A}}_{(H)}$, then there exist a \mathcal{C}^1 curve $c : [0, 1] \mapsto \widetilde{X}_\lambda^{-1}([0])$ such that $c(0) = 0$, $c'(0) \neq 0$, and all the points on the curve have isotropy conjugate to H .*

Remark 4.6. In the above theorem the condition $\overline{X}_\lambda \in \mathcal{C}^\infty(T_{q_e}Q \times \mathbb{R}, T_{q_e}Q)_o^{G_{q_e}}$ is used to insure that $f_1(0, 0) = 0$ and $\frac{\partial}{\partial \lambda} f_1(0, \lambda)|_{\lambda=0} \neq 0$, where f_1 is the first coefficient in expansions (4.1) or (4.2). In both cases considered here, this is equivalent to $\sigma_X(0) = 0$ and $\sigma'_X(0) \neq 0$.

Now we give the proof of Theorem 4.4.

Proof. (a) Suppose that the system $\dot{v}_{q_e} = \overline{X_\lambda}(v_{q_e})$ admits relative equilibria with isotropy subgroup conjugated to H in a neighborhood of 0_{q_e} . This means that the projection of this vector field has zeroes whose isotropy subgroups are conjugate to H . By Theorem 4.5 this implies that $\text{codim}_{\mathbb{R}^k} \widetilde{\mathcal{A}}_{(H)} \leq 1$ and $\gamma(0) \in \widetilde{\mathcal{A}}_{(H)}$. This is in contradiction with $\text{codim}_{\mathbb{R}^k} \widetilde{\mathcal{A}}_{(H)} = \text{codim}_{\mathbb{R}^k} \mathcal{A}_{(H)} \geq 2$. (b) Suppose that $\text{codim}_{\mathbb{R}^k} \mathcal{A}_{(H)} = 1$ and $\gamma(0) \in \mathcal{A}_{(H)}$. By Proposition 4.2, this implies that $\text{codim}_{\mathbb{R}^k} \widetilde{\mathcal{A}}_{(H)} = 1$ and $\gamma(0) \in \widetilde{\mathcal{A}}_{(H)}$. Theorem 4.5 guarantees the existence of a \mathcal{C}^1 curve $c : \lambda \in [0, 1] \mapsto \widetilde{X_\lambda}^{-1}([0])$, $c(0) = 0$, $c'(0) \neq 0$, all of whose points have isotropy subgroup conjugate to H . The inverse image of this curve under the orbit space projection Π is the G_{q_e} -orbit of a branch of relative equilibria v_λ . For a fixed λ , using Krupa's theorem,¹⁴ we decompose locally in a tubular neighborhood of the orbit $G_{q_e} \cdot v_\lambda$, the G_{q_e} -equivariant vector field $\overline{X_\lambda}$ into its tangential and normal components. Since the projection of this vector field to the orbit space has $c(\lambda)$ as equilibrium we have that $\overline{X_\lambda}(v_\lambda) \in \mathfrak{g}_{q_e} \cdot v_\lambda$. This means that the local normal component of the vector field vanishes. So, the solution of the Cauchy problem $\dot{w}_{q_e} = \overline{X_\lambda}(w_{q_e})$, $w_{q_e}(0) = v_\lambda$, is given by $w_{q_e}(t) = g(t) \cdot v_\lambda$, where $g(t) = \exp(\xi_\lambda t)$, for small enough $t \geq 0$ and $\xi_\lambda \in \mathfrak{g}_{q_e}$. The existence for each λ of a flow invariant maximal torus of rank n_H on which the solution $\exp(\xi_\lambda t) \cdot v_\lambda$ is generically dense is guaranteed by Theorem 4.1 in [14]. Hence we have obtained a branch $w_{q_e}^\lambda(t) = \exp(\xi_\lambda t) \cdot v_\lambda$ of periodic or quasi-periodic solutions all of whose points have isotropy type conjugate to H . \square

Using Proposition 3.6, we can deduce the same result as the one in Theorem 4.4 for a one-parameter family of vector fields X_λ defined in a small neighborhood around the point $q_e \in Q$. The resulting bifurcation result will give a family of tori bifurcating from q_e whose symmetry is broken to that of the a priori given subgroup $H \subsetneq G_{q_e}$. The bifurcating solutions are periodic or quasi-periodic curves lying on these tori. The method given by Proposition 3.6 allows one to always translate bifurcation results obtained in the linear setting to analogous theorems on the manifolds itself.

In the case of the Hopf bifurcation, several additional remarks are in order. First, the symmetry group of the problem can be considered to be of the form $G \times S^1$. This follows from a normal form result (see e.g. [10, Chapter XVI]) which ensures that there is a natural S^1 -action on $T_{q_e}Q$ such that the family of vector fields $\overline{X_\lambda}$ are $G \times S^1$ -equivariant. Second, given $k \in \mathbb{N}$ and a G -equivariant family of vector fields $\overline{X_\lambda}$ on $T_{q_e}Q$, there is a \mathcal{C}^k λ -dependent change of variables on $T_{q_e}Q$ that maps $\overline{X_\lambda}$ into new family of vector fields $\widetilde{X_\lambda}$ which is $G \times S^1$ -equivariant and that coincides with $\overline{X_\lambda}$ up to order k . Third, one can interpret this result as saying that close enough to 0_{q_e} the S^1 -relative equilibria of the $G \times S^1$ -equivariant family $\widetilde{X_\lambda}$ are mapped by this change of variables to the set of periodic solutions of $\overline{X_\lambda}$. Similarly, the $G \times S^1$ -relative equilibria of $\widetilde{X_\lambda}$ are mapped by this change of variables to G -relative periodic orbits of $\overline{X_\lambda}$.

5. Hamiltonian Steady State and Hamiltonian Hopf Bifurcation with Symmetry

In this section, we study a bifurcation problem similar to the one in Theorem 4.4 in the context of a Hamiltonian system on a symplectic vector space. To obtain the symmetry breaking result the Hamiltonian nature of the vector field plays a crucial role. One cannot deduce Theorem 5.3 below from the generic results in [13] because the non-degeneracy conditions are not satisfied.

Let (V, ω) be a symplectic vector space on which the compact Lie group G acts linearly and symplectically. In this case it is known that the map $G \times V \rightarrow V$ is smooth. Let $F_\lambda : V \rightarrow \mathbb{R}$, $\lambda \in \mathbb{R}$, be a family of smooth G -invariant functions. If $0 \in V$ is an equilibrium (with symmetry G) of the Hamiltonian vector field X_{F_0} , we are interested in finding conditions that guarantee the existence of smooth branches of tori bifurcating from zero such that all the points on these tori have symmetry conjugate to an a priori fixed proper symmetry subgroup H . Analogous to the cases considered in the previous section we shall analyze here the following situations:

- *Hamiltonian steady state bifurcation with symmetry:*

$$DX_{F_\lambda}(0) = \sigma(\lambda)\mathbb{J}$$

where $\sigma : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ is a smooth map satisfying $\sigma(0) = 0$ and $\sigma'(0) \neq 0$. In addition, it is assumed that the vector space V carries a natural complex structure \mathbb{J} and that the compact Lie group G acts on this complex vector space V irreducibly (as a complex representation). In this case a basis for $\mathfrak{gl}(V)^G$, the endomorphisms of V which commute with the G -action, is formed by $\{\mathbb{I}, \mathbb{J}\}$. This implies that a symplectic matrix in $\mathfrak{gl}(V)^G$ should be a real multiple of \mathbb{J} . In this case, we have passing of eigenvalues of $DX_{F_\lambda}(0)$ on the imaginary axis at the origin as λ crosses zero.

- *Hamiltonian Hopf bifurcation with symmetry:*

$$DX_{F_\lambda}(0) = \sigma(\lambda)A_1 + \rho(\lambda)A_2 + \tau(\lambda)A_3 + \psi(\lambda)A_4$$

where

$$A_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}, \quad A_4 = \begin{bmatrix} \mathbb{J} & \mathbf{0} \\ \mathbf{0} & \mathbb{J} \end{bmatrix},$$

and $\sigma, \rho, \tau, \psi : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ are smooth maps satisfying $\sigma(0) = 0$, $\sigma'(0) \neq 0$, $\rho(0) = -1$, $\tau(0) = 0$ and $\psi(0) = -\nu_0$, with $\nu_0 > 0$. In addition, it is assumed that the vector space $V = V_0 \oplus V_1$, where V_0 and V_1 are isomorphic complex dual irreducible representations of the compact Lie group G . The compact Lie group G is considered to be of the form $G = \Gamma \times S^1$. See for example [6].

In the following will denote by $\mathcal{C}^\infty(V \times \mathbb{R}, V)_{oo}^G$ the class of G -equivariant Hamiltonian vector fields that verifies one of the above non-degeneracy conditions.

As opposed to the general case, if the vector field is Hamiltonian, it turns out that one needs to use only invariant polynomials, as the proposition below shows.

Proposition 5.1. *Let (V, ω) be a symplectic representation space of the compact Lie group G . Denote by $\{\theta_1, \theta_2, \dots, \theta_l\}$ a set of generators of $\mathcal{P}_G(V)$. If $F : V \rightarrow \mathbb{R}$ is a smooth G -invariant function then the associated Hamiltonian vector field X_F has the following expression in canonical coordinates, in terms of invariant polynomials and their gradients:*

$$X_F(x) = \sum_{i=1}^l \frac{\partial \tilde{F}}{\partial \theta_i}(\theta_1(x), \dots, \theta_l(x)) \mathbb{J} \operatorname{grad}_x(\theta_i)(x), \quad \text{for any } x \in V, \quad (5.1)$$

where $\tilde{F} : \mathbb{R}^l \rightarrow \mathbb{R}$ is given by:

$$F(x) = \tilde{F}(\theta_1(x), \dots, \theta_l(x)), \quad \text{for any } x \in V.$$

Proof. The proof follows directly from the definition of the Hamiltonian vector field. \square

The next step gives a method for projecting a G -equivariant Hamiltonian vector field on the orbit space. To do this, let F be a G -invariant smooth function and X_F its associated Hamiltonian vector field. Therefore, X_F is G -equivariant on (V, ω) . Note that $\mathbb{J} \operatorname{grad} \theta_i$ are equivariant polynomials. Thus, by the proposition above we can write X_F in terms of invariants and equivariants as

$$X_F(x) = \sum_{i=1}^l \frac{\partial \tilde{F}}{\partial \theta_i}(\theta_1(x), \dots, \theta_l(x)) \mathbb{J} \operatorname{grad}_x \theta_i(x), \quad \text{for any } x \in V,$$

where the function \tilde{F} is smooth on \mathbb{R}^l . To obtain the projection of this vector field on the orbit space V/G , one has to compute the derivatives of the invariant generators θ_j with respect to time:

$$\begin{aligned} \dot{\theta}_j &= \mathbf{d}\theta_j(x)(\dot{x}) = (\operatorname{grad}_x(\theta_j))^t \cdot \left(\sum_{i=1}^l \frac{\partial \tilde{F}}{\partial \theta_i} \mathbb{J} \operatorname{grad}_x(\theta_i) \right) \\ &= \sum_{i=1}^l \frac{\partial \tilde{F}}{\partial \theta_i} \{\theta_j, \theta_i\}. \end{aligned}$$

The projection \tilde{X}_F of X_F on the orbit space V/G has thus the expression

$$\tilde{X}_F(\theta) = \mathcal{P}(\theta) \operatorname{grad}_\theta(\tilde{F}), \quad (5.2)$$

where

$$\mathcal{P}(\theta) := \begin{bmatrix} 0 & \{\theta_1, \theta_2\} & \cdots & \{\theta_1, \theta_l\} \\ \cdots & \cdots & \cdots & \cdots \\ \{\theta_l, \theta_1\} & \cdots & \cdots & 0 \end{bmatrix}.$$

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As in the Riemannian case, one search for symmetry breaking zeroes of the projected vector field. To do this, we need a few preliminary results. Define the following maps associated to the family of smooth G -invariant functions $F : V \times \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{aligned} \tilde{\Phi} : V/G \times \mathbb{R}^l \ni (\theta, t) &\mapsto \sum_{i=1}^l t_i \begin{bmatrix} \{\theta_1, \theta_i\} \\ \{\theta_2, \theta_i\} \\ \dots \\ \{\theta_l, \theta_i\} \end{bmatrix} \in V/G \subset \mathbb{R}^l \\ \gamma : \mathbb{R} \ni \lambda &\mapsto \left(\frac{\partial \tilde{F}}{\partial \theta_1}(0, \lambda), \dots, \frac{\partial \tilde{F}}{\partial \theta_l}(0, \lambda) \right) \in \mathbb{R}^l \end{aligned} \tag{5.3}$$

and the sets

$$\begin{aligned} \tilde{\mathcal{E}} &= \{(\theta, t) \in V/G \times \mathbb{R}^l \mid \tilde{\Phi}(\theta, t) = 0\} \\ \tilde{\mathcal{E}}_{(H)} &= \{(\theta, t) \in \tilde{\mathcal{E}} \mid \theta \in \Pi(V_{(H)})\} \\ \tilde{\mathcal{A}}_{(H)} &= \overline{\tilde{\mathcal{E}}_{(H)}} \cap (\{[0]\} \times \mathbb{R}^l), \end{aligned}$$

where $\Pi : V \rightarrow V/G$ is the orbit space projection and $H \subset G$ is an isotropy subgroup of the G -action on V . Recall that $\tilde{\mathcal{E}}$ is a semialgebraic subset of \mathbb{R}^{2l} . Denote by Σ the canonical Whitney stratification of $\tilde{\mathcal{E}}$. This stratification induces a Whitney semialgebraic stratification $\Sigma_{(H)}$ on each $\tilde{\mathcal{E}}_{(H)}$. In particular, Σ_G stratifies $\tilde{\mathcal{E}}_{(G)} = \{[0]\} \times \mathbb{R}^l$. Next we recall the notion of G -transversality introduced by Koenig and Chossat [13] that was in turn inspired by Bierstone [4] and Field [8]; see Definition 4.1. In the present case, this says that the projection to the orbit space V/G of a G -equivariant smooth family of vector fields $\tilde{X}_{F_\lambda} \in \mathfrak{X}(V)$ is *transverse* to $[0] \in V/G$ at $\lambda = 0 \in \mathbb{R}$ if $\gamma : \mathbb{R} \rightarrow \mathbb{R}^l$ is transverse at $\lambda = 0$ to the stratum of Σ_G containing $\gamma(0)$, that is, $\text{im } T_0\gamma + T_{\gamma(0)}S = \mathbb{R}^l$, where S is the stratum of Σ_G containing $\gamma(0)$. Analogous to map (5.3), we define

$$\Phi : V \times \mathbb{R}^l \ni (x, t) \mapsto \sum_{i=1}^l t_i \mathbb{J} \text{grad}_x \theta_i(x) \in V.$$

This map induces the following sets parameterized by the isotropy type subgroups of the G -action on V :

$$\begin{aligned} \mathcal{E}_{(H)} &= \{(x, t) \in V_{(H)} \times \mathbb{R}^l \mid \Phi(x, t) \in \mathfrak{g} \cdot x\}, \\ \mathcal{A}_{(H)} &= \overline{\mathcal{E}_{(H)}} \cap (\{0\} \times \mathbb{R}^l). \end{aligned}$$

The sets $\mathcal{E}_{(H)}$ and $\mathcal{A}_{(H)}$ are semialgebraic sets in $V \times \mathbb{R}^l$ and \mathbb{R}^l , respectively. A proof analogous to that of Proposition 4.2 shows that $\mathcal{E}_{(H)} = (\Pi \times \text{id})^{-1}(\tilde{\mathcal{E}}_{(H)})$ and $\tilde{\mathcal{A}}_{(H)} = \mathcal{A}_{(H)}$.

We shall prove below that if $H \neq G$, then $\text{codim}_{\mathbb{R}^l} \mathcal{A}_{(H)} \geq 1$.

Theorem 5.2. *Let V be a symplectic representation space of a compact Lie group G and $\{\theta_1, \dots, \theta_l\}$ a Hilbert base. Suppose there exist $i_0, i_1 \in \{1, \dots, l\}$, $i_0 \neq i_1$, $D_{i_0, i_1} := \{x \in V \mid \{\theta_{i_0}, \theta_{i_1}\}(x) \neq 0\} \neq \emptyset$, and $x_0 \in V$ such that $\lim_{x \rightarrow x_0} \frac{\{\theta_{i_0}, \theta_{i_1}\}(x)}{\{\theta_{i_0}, \theta_{i_1}\}(x)} = 0$, for any $i \in \{1, \dots, l\} \setminus \{i_1\}$. Let H be an isotropy subgroup of the G -representation on V . If $(H) \neq (G)$, then we have $\tilde{\mathcal{A}}_{(H)} \subset (\mathbb{R}^l)^\perp = \{t \in \mathbb{R}^l \mid t_{i_1} = 0\}$ and $\tilde{\mathcal{A}}_{(G)} = \mathbb{R}^l$. In particular, $\text{codim}_{\mathbb{R}^l} \tilde{\mathcal{A}}_{(H)} \geq 1$.*

Proof. From (5.3), it follows that the i_0 -component of the map $\tilde{\Phi}$ is given by

$$t_1\{\theta_{i_0}, \theta_1\} + t_2\{\theta_{i_0}, \theta_2\} + \dots + t_l\{\theta_{i_0}, \theta_l\}.$$

Let $\theta := (\theta_1, \dots, \theta_l) \in V/G \subset \mathbb{R}^l$. If $(\theta, t) \in \tilde{\mathcal{E}}_{(H)}$, the above relation becomes

$$t_1\{\theta_{i_0}, \theta_1\} + t_2\{\theta_{i_0}, \theta_2\} + \dots + t_l\{\theta_{i_0}, \theta_l\} = 0.$$

This implies that

$$t_1\{\theta_{i_0}, \theta_1\}(x) + t_2\{\theta_{i_0}, \theta_2\}(x) + \dots + t_l\{\theta_{i_0}, \theta_l\}(x) = 0, \quad (\forall)x \in D_{i_0, i_1}.$$

Dividing this relation by $\{\theta_{i_0}, \theta_{i_1}\}(x)$ and letting $x \rightarrow x_0$ one concludes that $t_{i_1} = 0$. This proves that $\tilde{\mathcal{A}}_{(H)} \subset \{t \in \mathbb{R}^l \mid t_{i_1} = 0\}$. The identity $\tilde{\mathcal{A}}_{(G)} = \mathbb{R}^l$ follows directly from the definitions. \square

Adapting the Proposition 5.16 of Field and Richardson⁹ to the family of vector fields X_{F_λ} , yields the following result. Note that $X_{F_\lambda} \in \mathcal{C}^\infty(V \times \mathbb{R}, V)_{oo}^G$ implies the condition from Remark 4.6, this condition being equivalent to $\sigma(0) = 0$ and $\sigma'(0) \neq 0$.

Theorem 5.3. *Let V be a symplectic representation space of a compact Lie group G and $\{\theta_1, \dots, \theta_l\}$ a Hilbert base. Suppose there exist $i_0, i_1 \in \{1, \dots, l\}$, $i_0 \neq i_1$, $D_{i_0, i_1} := \{x \in V \mid \{\theta_{i_0}, \theta_{i_1}\}(x) \neq 0\} \neq \emptyset$, and $x_0 \in V$ such that $\lim_{x \rightarrow x_0} \frac{\{\theta_{i_0}, \theta_{i_1}\}(x)}{\{\theta_{i_0}, \theta_{i_1}\}(x)} = 0$, for any $i \in \{1, \dots, l\} \setminus \{i_1\}$. Let H be a given proper isotropy subgroup of G and assume that the family of Hamiltonian vector fields $X_{F_\lambda} \in \mathcal{C}^\infty(V \times \mathbb{R}, V)_{oo}^G$ satisfies the transversality condition given in Definition 4.1. For $x \in V_{(H)}$ define $n_H := \text{rank } N(G_x)/G_x$.*

- (a) *If $\text{codim}_{\mathbb{R}^l} \mathcal{A}_{(H)} \geq 2$, then the system $\dot{x} = X_{F_\lambda}(x)$ does not have relative equilibria with isotropy conjugate to H in a neighborhood of the origin 0.*
- (b) *If $\text{codim}_{\mathbb{R}^l} \mathcal{A}_{(H)} = 1$ and $\gamma(0) \in \mathcal{A}_{(H)}$, then there exists a branch of tori of dimension n_H , formed by points of isotropy type H , bifurcating from 0. The torus corresponding to λ is invariant under the flow of X_{F_λ} .*

We shall make some remarks regarding the Hamiltonian Hopf bifurcation. We have made the hypothesis that the symmetry group of the problem is of the form $G \times S^1$. This always holds if we assume only that the system has a compact Lie group

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G of symmetries represented on the symplectic vector space V . The reason for this is that in the case of the Hamiltonian Hopf bifurcation there is a natural S^1 -action on V given by the exponential of the semisimple part of the linearization on the resonance space corresponding to ν_0 . More precisely, let (V, ω, F_λ) be a λ -parameter family ($\lambda \in \mathbb{R}$ in a neighborhood of the origin) of G -Hamiltonian systems such that $F_0(0) = 0$, $\mathbf{d}F_0(0) = 0$, and the G -equivariant infinitesimally symplectic linear map $DX_{F_0}(0)$ is nonsingular and has $\pm i\nu_0$ as eigenvalues. Let $(U_{\nu_0}, \omega|_{U_{\nu_0}})$ be the resonance space of $DX_{F_0}(0)$ with primitive period $T_{\nu_0} := 2\pi/\nu_0$. Recall that the resonance space is the direct sum of the real eigenspaces of $DX_{F_0}(0)$ corresponding to eigenvalues of the form $\pm k i \nu_0$ with $k = 1, 2, \dots$. A theorem of Vanderbauwhede and van der Meer [26] states that for each $k \geq 0$ one can find a C^k -mapping $\psi : U_{\nu_0} \times \mathbb{R} \rightarrow V$ and a C^{k+1} -function $\widehat{F}_\lambda : U_{\nu_0} \times \mathbb{R} \rightarrow \mathbb{R}$ such that $\psi(0, \lambda) = 0$, for all λ , $D_{U_{\nu_0}} \psi(0, 0) = \mathbb{I}_{U_{\nu_0}}$, and \widehat{F}_λ is a $G \times S^1$ -invariant function that coincides with F_λ up to order $k + 1$. In the same paper it is shown (Theorem 3.2) that if λ is close to 0 then the S^1 -relative equilibria of the $G \times S^1$ -invariant Hamiltonian \widehat{F}_λ that lie in a sufficiently small neighborhood of the origin in U_{ν_0} are mapped by $\psi(\cdot, \lambda)$ to the set of periodic solutions of (V, ω, F_λ) in a neighborhood of $0 \in V$ with periods close to T_{ν_0} . Thus, one can replace the search of periodic orbits of (V, ω, F_λ) by the search of S^1 -relative equilibria of the $G \times S^1$ -invariant family of Hamiltonian systems $(U_{\nu_0}, \omega|_{U_{\nu_0}}, \widehat{F}_\lambda)$. Similarly, the $G \times S^1$ -relative equilibria of $X_{\widehat{F}_\lambda}$ are mapped by $\psi(\cdot, \lambda)$ to G -relative periodic orbits of X_{F_λ} .

6. Bifurcation of Relative Equilibria in Hamiltonian Systems on Symplectic Vector Spaces

In this section, we will prove a bifurcation result in symplectic vector spaces in the context of Hamiltonian systems. We consider a symplectic representation space (V, ω) of a compact Lie group G and let $\{\theta_1, \dots, \theta_l\}$ a Hilbert base. We will denote by $V_{(H)} \subset V$ the principal stratum of the action of G on V and by $\Pi : V \rightarrow V/G$ the orbit space projection. Recall that each connected component of $V_{(H)}$ is an open dense subset of the vector space V . We consider a family of G -invariant smooth functions $F_\lambda : V \rightarrow \mathbb{R}$, $\lambda \in \mathbb{R}$, such that there exists $\lambda_0 \in \mathbb{R}$ and $v_0 \in V_{(H)}$ a relative equilibrium for the Hamiltonian vector field $X_{F_{\lambda_0}}$. We will prove that under a certain non-degeneracy condition we have the existence of a smooth branch of relative equilibria (starting from v_0) for the Hamiltonian vector fields X_{F_λ} for λ close to λ_0 . To do this we recall some of the techniques necessary to find the expression of the projected Hamiltonian vector field on V/G . More precisely, if $F_\lambda(x) = \tilde{F}_\lambda(\theta_1(x), \dots, \theta_l(x))$, for any $x \in V$, where $\tilde{F}_\lambda : \mathbb{R}^l \rightarrow \mathbb{R}$ is smooth, then the j -component of the projection on the orbit space V/G of the Hamiltonian vector field X_{F_λ} is given by:

$$\sum_{i=1}^l \frac{\partial \tilde{F}_\lambda}{\partial \theta_i} \{\theta_j, \theta_i\}.$$

Searching for relative equilibria of a vector field is the same as searching for equilibria for the projected vector field on the orbit space. Thus, we reduced the problem to find branches in λ of solutions of the system:

$$\sum_{i=1}^l \frac{\partial \tilde{F}_\lambda}{\partial \theta_i}(\theta_1, \dots, \theta_l) \{\theta_j, \theta_i\} = 0, \quad \text{for } j \in \{1, \dots, l\}.$$

To do this, we introduce the function $g : (V/G \subset \mathbb{R}^l) \times \mathbb{R} \rightarrow \mathbb{R}^l$, $g = (g_1, \dots, g_l)$, where the components g_j are defined by:

$$g_j((\theta_1, \dots, \theta_l), \lambda) := \sum_{i=1}^l \frac{\partial \tilde{F}_\lambda}{\partial \theta_i}(\theta_1, \dots, \theta_l) \{\theta_j, \theta_i\}, \quad \text{for } j \in \{1, \dots, l\}.$$

As $v_0 \in V_{(H)}$ is a relative equilibrium of $X_{F_{\lambda_0}}$ this implies that $\theta_0 =: \Pi(v_0)$ is an equilibrium point for the projection on the orbit space $V/G \subset \mathbb{R}^l$ of the Hamiltonian vector field $X_{F_{\lambda_0}}$. This means that $\theta_0 \in V/G \subset \mathbb{R}^l$ is a solution of the equation $g(\theta, \lambda_0) = 0$. The next theorem will give a sufficient condition that guarantees the existence of a smooth curve of solutions of the equation $g(\theta, \lambda) = 0$, in an open neighborhood of the point (θ_0, λ_0) .

Theorem 6.1. *Let (V, ω) be a symplectic representation space of a compact Lie group G and $\Pi : V \rightarrow V/G$ be the orbit space projection. Let $F_\lambda : V \rightarrow \mathbb{R}$, $\lambda \in \mathbb{R}$, be a family of G -invariant smooth functions. Suppose that $v_0 \in V$ is a relative equilibrium of the Hamiltonian vector field $X_{F_{\lambda_0}}$ which belongs to the principal stratum of the G -action on V . Denote $\theta_0 =: \Pi(v_0)$. If $\mathbf{d}_\theta g(\theta_0, \lambda_0)$ is non-degenerate, then there exists an open subset I of \mathbb{R} containing λ_0 and a smooth curve $\gamma : \lambda \in I \mapsto \gamma(\lambda) \in V/G \subset \mathbb{R}^l$, such that $\gamma(\lambda) \in V/G \subset \mathbb{R}^l$ lifts to relative equilibria of X_{F_λ} and $\gamma(\lambda_0) = \theta_0 \in V/G \subset \mathbb{R}^l$.*

Proof. The proof is a consequence of the implicit function theorem and of the fact that $v_0 \in V_{(H)} \subset V$ is an element of the principal stratum; the connected component containing it is an open subset of V . \square

Note that in this case the only thing we can say about the symmetry of the relative equilibria on the branch is that for λ small enough all of them are of principal type. This theorem can be used in the following way to study bifurcating branches of relative equilibria from a given one with isotropy K . Let $v_0 \in V$ be a relative equilibrium with isotropy K . Since G is compact, so is K and hence the subspace V^K of K -fixed vectors is a symplectic subspace of (V, ω) . Let us assume that the normalizer $N(K)$ of K in G is closed, which implies that $N(K)/K$ is a compact Lie group that acts naturally on V^K . Note that the restriction of this action on the K -invariant open subset V_K is free. Then we can apply the previous theorem to V^K , $N(K)/K$, and the restrictions of the vector fields X_{F_λ} to the open subset V_K of V^K . The corresponding function g from before will be restricted to the open set $V_K/(N(K)/K)$ and, under the same non-degeneracy hypothesis, we

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obtain λ -branches of relative equilibria for X_{F_λ} consisting of points with isotropy exactly K .

7. From Symplectic Vector Spaces to Manifolds

In this short section, we will give a method of translating the results obtained in the previous two sections from symplectic vector spaces to symplectic manifolds. As all the symmetry breaking results in Sections 5 and 6 are of local type we will make use of the Equvariant Darboux Theorem to pass locally from a symplectic manifold to a symplectic vector space.

Theorem 7.1. *Suppose that Γ is a compact Lie group acting on a finite-dimensional symplectic manifold Q and let ω be a Γ -invariant symplectic form on Q . Let $q \in Q$ be a Γ -invariant point. Then there are open Γ -invariant neighborhoods $U \subset Q$ of q and $V \subset T_q Q$ of $0_q \in T_q Q$ and a Γ -invariant symplectic diffeomorphism $f : (U, \omega) \rightarrow (V, \omega(q))$, such that $f(q) = 0_q$.*

This theorem follows from the G -relative Darboux Theorem (see, for example, [20]).

Theorem 7.2. *Let Q be a smooth manifold and ω_0 and ω_1 two symplectic forms on it. Let G be a Lie group acting properly on Q and symplectically with respect to both ω_0 and ω_1 . Assume that*

$$\omega_0(g \cdot q)(v_{g \cdot q}, w_{g \cdot q}) = \omega_1(g \cdot q)(v_{g \cdot q}, w_{g \cdot q})$$

for all $g \in G$ and $v_{g \cdot q}, w_{g \cdot q} \in T_{g \cdot q} Q$. Then there exist two open G -invariant neighborhoods U_0 and U_1 of $G \cdot q$ and a G -equivariant diffeomorphism $\Psi : U_0 \rightarrow U_1$ such that $\Psi|_{G \cdot q} = \text{Id}$ and $\Psi^* \omega_1 = \omega_0$.

Theorem 7.1 is proved as follows. On the symplectic manifold (Q, ω) consider around the point q an open neighborhood $W \subset Q$ contained in a Darboux chart. Thus, on this neighborhood the symplectic form ω_1 is the pull back of the symplectic form $\omega(q)$ on $T_q Q$. Recall that $\omega(q)$ is Γ -invariant, where Γ acts by linearization of the original action on $T_q Q$. Since the point q is Γ -invariant we can shrink the neighborhood W to a Γ -invariant open neighborhood $B \subset W$ of q . So, on the Γ -invariant manifold B we have two Γ -invariant symplectic forms: ω and ω_1 . Now using the Γ -relative Darboux theorem we get the existence of two open Γ -invariant neighborhoods $U_0 \subset B$ and $U_1 \subset B$ of q and a Γ -equivariant diffeomorphism $\Psi : U_0 \rightarrow U_1$ such that $\Psi|_{\Gamma \cdot q} = \text{Id}$ and $\Psi^* \omega = \omega_1$.

The nonlinear version of the results from Secs. 5 and 6 can be stated in the following theorem.

Theorem 7.3. *Let G be a Lie group acting properly on a symplectic finite dimensional manifold (Q, ω) . Let $F_\lambda : Q \rightarrow \mathbb{R}$, $\lambda \in \mathbb{R}$, be a family of G -invariant smooth functions and $q_e \in Q$ be a relative equilibrium of the Hamiltonian vector field $X_{F_{\lambda_0}}$.*

Then, in the conditions of Theorems 5.3 and 6.1, the same conclusions hold on the manifold Q around the point q_e .

Proof. The proof is a consequence of Theorem 7.1 by considering the compact Lie group G_{q_e} acting on the symplectic manifold (Q, ω) . \square

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