Banach Lie-Poisson spaces and reduction

Anatol Odzijewicz[1] and Tudor S. Ratiu[2]

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Abstract

The category of Banach Lie-Poisson spaces is introduced and studied. It is shown that the category of W^*-algebras can be considered as one of its subcategories. Examples and applications of Banach Lie-Poisson spaces to quantization and integration of Hamiltonian systems are given. The relationship between classical and quantum reduction is discussed.

Contents

[1]Institute of Physics, University of Bialystok, Lipowa 41, PL-15424 Bialystok, Poland. aodzijew@labfiz.uwb.edu.pl

[2]Centre Bernoulli, École Polytechnique Fédérale de Lausanne. CH–1015 Lausanne. Switzerland. Tudor.Ratiu@epfl.ch

1 Introduction

This paper investigates the foundations of Banach Poisson differential geometry, including such topics as Banach Lie-Poisson spaces, classical and quantum reduction, integration and quantization of Hamiltonian systems with the aid of the momentum map. We were inspired to study this circle of problems due to the appearance of formal Poisson structures in a large number of works devoted to the integration of infinite dimensional systems and the crucial role played by the momentum map in these approaches.

The notion of a Lie-Poisson space is as old as the concept of a Lie algebra and both were introduced simultaneously by Lie [Lie]. A Lie-Poisson space is a Poisson vector space with the property that its dual is invariant under the Poisson bracket, which is equivalent to the statement that the Poisson bracket is linear. In the finite dimensional case the notions of Lie algebras and Lie-Poisson spaces are equivalent in the sense that for any Lie algebra $\mathfrak{g}$ its dual $\mathfrak{g}^*$ is a Lie-Poisson space and, conversely, given a Lie-Poisson space its dual is a Lie algebra. This is so because finite dimensional vector spaces are reflexive, the operation of taking the dual defines an isomorphism between these two categories. To generalize this to infinite dimensions, it is reasonable to assume that a Lie-Poisson space is a Banach space $\mathfrak{b}$ endowed with a Poisson bracket $\{\cdot,\cdot\}$ such that the bracket of any two linear continuous functions is again a linear continuous function. This implies that $(\mathfrak{b}^*, \{\cdot,\cdot\})$ is a Banach Lie algebra. In order to preserve the correspondence between Banach Lie-Poisson spaces and Banach Lie algebras it is necessary to restrict to those Banach Lie algebras $(\mathfrak{g}, [\cdot,\cdot])$ that admit a predual $\mathfrak{g}_*$ and satisfy in addition the condition that $\mathrm{ad}^*_{\mathfrak{g}} : \mathfrak{g}^* \to \mathfrak{g}^*$ preserves the predual $\mathfrak{g}_*$. Thus, in the infinite dimensional case, Banach Lie-Poisson spaces form a subcategory of the category of Banach Lie algebras. A crucial example is the Banach space $L^1(\mathcal{M})$ of linear trace class operators on a separable Hilbert space $\mathcal{M}$ which is predual to the Banach Lie algebra $L^\infty(\mathcal{M})$ of all linear bounded operators on $\mathcal{M}$. As far as we know, the Lie-Poisson structure on $L^1(\mathcal{M})$ was first found by Bona [B].

Momentum maps are an efficient way to encode integrals of motion for a Hamiltonian system. In its modern formulation due to Kostant [Ko1] and Souriau [So1, So1] a momentum map is naturally associated to an infinitesimal Poisson action of a Lie algebra on a Poisson manifold and it maps the phase space to the dual of the Lie algebra of symmetries. It turns out, that in finite dimensions, a momentum map is characterized by the property that it is Poisson, when one endows the dual of the Lie algebra with the Lie-Poisson structure (see, e.g. Marsden and Ratiu [M-R2] for a proof of this fact).

In infinite dimensions, due to existence of non-reflexive Banach spaces, we will define a momentum map to be a Poisson map from a Banach Poisson manifold to a Banach Lie-Poisson space, which can always be considered as the predual of the Banach Lie algebra of symmetries. It is shown that the momentum map so defined has all the usual properties, such as being conserved along the flow of any symmetry invariant Hamiltonian vector field (Noether's theorem). Like in finite dimensions, also in the infinite dimensional case, the notion of momentum map is an important tool in the study of Hamiltonian systems. For example, the knowledge of momentum maps leads to integrals of motion of the considered Hamiltonian system, as will be illustrated here through the example of the infinite Toda lattice.

In the special case when one assumes that the momentum map is an injective immersion and its range is linearly dense in the target Banach Lie-Poisson space one discovers that it is the coherent states map in the sense of Odzijewicz [O2]. So, it can be used to quantize the system under consideration. This method of quantization, called Ehrenfest quantization, is a natural unification of the Kostant-Souriau geometric (Kostant [Ko2], Souriau [So1, So1]) and $*$-product quantization; for details see Odzijewicz [O2] and §8.

The structure of the paper is as follows. In §2 the notion of a Banach Poisson manifold is introduced modeled on the example of a strong symplectic manifold. Its elementary properties are presented as well as some comments on the compatibility of the Poisson structure with almost complex, complex, and holomorphic structures.

Classical reduction for Banach Poisson manifolds is discussed in §3. The Poisson reduction theorem of Marsden and Ratiu [M-R1] and its consequences are generalized to the Banach manifold context.

Banach Lie-Poisson spaces and their properties are analyzed in §4. Linear continuous Poisson maps are studied in detail. The realification and complexification of a Banach Lie-Poisson space is also presented. The upshot of this section is the establishment of an isomorphism between a subcategory of Banach Lie-Poisson spaces and a specific subcategory of Banach Lie algebras.

The entirety of §5 is devoted to one crucial example: the predual of a W^*-algebra and the dual to a C^*-algebra are naturally a Banach Lie-Poisson spaces. As a consequence it is shown that various spaces related to operator algebras (for example the space of Hermitian trace class operators on a separable Hilbert space) are Banach Lie-Poisson spaces.

In §6 we show that the quantum measurement operation in the sense of von Neumann can be considered as a Poisson projection. We shall give examples of other physically important Poisson projections. These examples justify the interpretation of Poisson projections as quantum reduction procedures.

The internal structure of Banach Lie-Poisson spaces is presented in §7. If the Lie algebra of a Banach Lie group admits a predual which is invariant under the coadjoint representation it is shown that a large class of coadjoint orbits in the predual, which is naturally a Banach Lie-Poisson space, are symplectic leaves in a weak sense: they are weak symplectic manifolds and are weakly immersed submanifolds (the inclusion is smooth and has injective derivative, but no splitting condition, or even a closed range condition on the derivative, usually imposed in the definition of an immersion, holds). Among these orbits a subclass is determined for which the symplectic form is strong and the orbit is injectively immersed. The section ends with the standard example of a dual pair based on the cotangent bundle (in this case on the precotangent bundle) that illustrates that our definition of a Banach Poisson manifold is violated in this important case and that once one leaves the category of W^*-algebras a weakening of this notion will be needed.

Section 8 introduces momentum maps as Poisson maps from a Banach Poisson manifold to a Banach Lie-Poisson space. It is shown that the coherent states map is a momentum map with certain special properties. In this way the quantization procedure based on the coherent states map has Banach Poisson geometrical interpretation. The relationship between classical and quantum reduction is explored. Using both procedures of reduction, classical and quantum, one can construct a new momentum map from a given one. The description of the infinite Toda lattice in Banach Poisson geometrical terms is presented. Among others, it is shown that the Flaschka transformation is a momentum map of some Banach weak symplectic space into the Banach Lie-Poisson space of the lower triangular trace class operators.

2 Banach Poisson manifolds

Throughout the paper, given a Banach space $\mathfrak{b}$, the notation $\mathfrak{b}^*$ will always mean the Banach space dual to $\mathfrak{b}$. Given $x \in \mathfrak{b}^*$ and $b \in \mathfrak{b}$, we shall denote by $\langle x, b \rangle$ the value of x on b. Thus $\langle \cdot, \cdot \rangle : \mathfrak{b}^* \times \mathfrak{b} \to \mathbb{R}$ (or $\mathbb{C}$, depending on whether we work with real or complex Banach spaces and functions) will denote the natural bilinear continuous duality pairing between $\mathfrak{b}$ and its dual $\mathfrak{b}^*$.

A real finite dimensional Poisson manifold is a pair $(P, \{\cdot, \cdot\})$ consisting of a manifold P whose space of Fréchet smooth functions is endowed with a Lie algebra structure $\{\cdot, \cdot\}$ satisfying the Leibniz property in each factor; this bilinear operation $\{\cdot, \cdot\}$ is called a Poisson bracket. As we shall discuss below, this definition is not appropriate in infinite dimensions and a more stringent condition needs to be imposed.

To see this, assume that on the space $C^\infty(P)$ of smooth functions on the infinite dimensional smooth Banach manifold P there is a Poisson bracket $\{\cdot, \cdot\}$. Due to the Leibniz property, the value of the Poisson bracket at a given point $p \in P$ depends only on the differentials $df(p)$, $dg(p) \in T_p^*P$ which implies that there is a smooth section ϖ of the vector bundle $\bigwedge^2 T^{**}P$ satisfying

$$\{f, g\} = \varpi(df, dg).$$

This means that for each $p \in P$ the map $\varpi_p : T_p^*P \times T_p^*P \to \mathbb{R}$ is a continuous bilinear antisymmetric map that depends smoothly on the base point p. In addition, denoting by $[\cdot, \cdot]_S$ the Schouten bracket on skew symmetric contravariant tensors, the equality (see e.g. Marsden and Ratiu [M-R2], §10.6)

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = i_{[\varpi,\varpi]_S}(df \wedge dg \wedge dh),$$

shows that the Jacobi identity is equivalent to $[\varpi, \varpi]_S = 0$, which is an additional differential quadratic condition on ϖ.

Let $\sharp : T^*P \to T^{**}P$ be the bundle map covering the identity defined by $\sharp_p(dh(p)) := \varpi(\cdot, dh)(p)$, that is, $\sharp_p(dh(p))(dg(p)) = \{g, h\}(p)$, for any locally defined functions g and h.

Denote by $\mathfrak{b}$ the Banach space modeling the Banach manifold P. Thus $T_pP \cong \mathfrak{b}$, $T_p^*P \cong \mathfrak{b}^*$, and $T_p^{**}P \cong \mathfrak{b}^{**}$. If $\mathfrak{b}$ is not reflexive, that is, $\mathfrak{b} \subset \mathfrak{b}^{**}$ and $\mathfrak{b} \neq \mathfrak{b}^{**}$, then

$$X_f := \varpi(\cdot, df) = \sharp(df), \quad \text{or, as a derivation on functions,} \quad X_f = \{\cdot, f\}$$

is a smooth section of $T^{**}P$ and hence is not, in general, a vector field on P. In analogy with the finite dimensional case, we want X_f to be the Hamiltonian vector field defined by the function f. In order to achieve this, we are forced to make the assumption that the Poisson bracket on P satisfies the condition $\sharp(T^*P) \subset TP \subset T^{**}P$. Thus we give the following definition.

Definition 2.1 *A **Banach Poisson manifold** is a pair* $(P, \{\cdot,\cdot\})$ *consisting of a smooth Banach manifold and a bilinear operation* $\{\cdot,\cdot\}$ *satisfying the following conditions:*

(i) $(C^\infty(P), \{\cdot,\cdot\})$ *is a Lie algebra;*

(ii) $\{\cdot,\cdot\}$ *satisfies the Leibniz identity on each factor;*

(iii) *the vector bundle map* $\sharp : T^*P \to T^{**}P$ *covering the identity satisfies* $\sharp(T^*P) \subset TP$.

Condition (iii) allows one to introduce for any function $h \in C^\infty(P)$ the ***Hamiltonian vector field*** by

$$X_h[f] := \langle df, X_h \rangle = \{f, h\}$$

where f is an arbitrary smooth locally defined function on P.

Given two Banach Poisson manifolds $(P_1, \{\,,\}_1)$ and $(P_2, \{\,,\}_2)$, a smooth map $\varphi : P_1 \to P_2$ is said to be ***canonical*** or a ***Poisson map*** if

$$\varphi^*\{f, g\}_2 = \{\varphi^* f, \varphi^* g\}_1 \tag{2.1}$$

for any two smooth locally defined functions f and g on P_2. Condition (iii) in the previous definition implies, like in the finite dimensional case, that (2.1) is equivalent to

$$X_f^2 \circ \varphi = T\varphi \circ X^1_{f\circ\varphi} \tag{2.2}$$

for any smooth locally defined function f on P_2 (for the proof see e.g. Marsden and Ratiu [M-R2], §10.3). Therefore, the flow of a Hamiltonian vector field is a Poisson map and Hamilton's equations in Poisson bracket formulation are valid.

For later applications we shall need the notion of the product of Banach Poisson manifolds. The definition we shall give is the one used in finite dimensions (see, e.g. Weinstein [W1, W2] or Vaisman [V]). However, the proof of the theorem characterizing the product needs some care due to the infinite dimensionality of the manifolds and the additional condition (iii) imposed in Definition 2.1. For this reason we shall sketch it below.

Theorem 2.2 *Given the Banach Poisson manifolds* $(P_1, \{\,,\}_1)$ *and* $(P_2, \{\,,\}_2)$ *there is a unique Banach Poisson structure* $\{\,,\}_{12}$ *on the product manifold* $P_1 \times P_2$ *such that:*

(i) *the canonical projections* $\pi_1 : P_1 \times P_2 \to P_1$ *and* $\pi_2 : P_1 \times P_2 \to P_2$ *are Poisson maps;*

(ii) $\pi_1^*(C^\infty(P_1))$ *and* $\pi_2^*(C^\infty(P_2))$ *are Poisson commuting subalgebras of* $C^\infty(P_1 \times P_2)$.

This unique Poisson structure on $P_1 \times P_2$ *is called the **product** Poisson structure and its bracket is given by the formula*

$$\{f, g\}_{12}(p_1, p_2) = \{f_{p_2}, g_{p_2}\}_1(p_1) + \{f_{p_1}, g_{p_1}\}_2(p_2), \tag{2.3}$$

where $f_{p_1}, g_{p_1} \in C^\infty(P_2)$ *and* $f_{p_2}, g_{p_2} \in C^\infty(P_1)$ *are the partial functions given by* $f_{p_1}(p_2) = f_{p_2}(p_1) = f(p_1, p_2)$ *and similarly for* g.

Proof. Recall that if $f \in C^\infty(P_1 \times P_2)$ then the partial exterior derivative $d_1 f(p_1, p_2)$ relative to P_1 is defined by $d_1 f(p_1,p_2) := df_{p_2}(p_1) = (\pi_1^* df_{p_2})(p_1,p_2) = d(\pi_1^* f_{p_2})(p_1,p_2)$ and similarly $d_2 f(p_1,p_2) = d(\pi_2^* f_{p_1})(p_1,p_2)$. Therefore, $df(p_1,p_2) = d_1 f(p_1,p_2) + d_2 f(p_1,p_2) = d(\pi_1^* f_{p_2})(p_1,p_2) + d(\pi_2^* f_{p_1})(p_1,p_2)$. Thus the functions f and $\pi_1^* f_{p_2} + \pi_2^* f_{p_1}$ have the same derivatives at the point $(p_1,p_2) \in P_1 \times P_2$. Similarly, g and $\pi_1^* g_{p_2} + \pi_2^* g_{p_1}$ have the same derivatives at the point $(p_1,p_2) \in P_1 \times P_2$.

Assume that there is a Poisson bracket $\{\, ,\}_{12}$ on $P_1 \times P_2$ satisfying the conditions in the theorem. Since any Poisson bracket depends only on the first derivatives of the functions we necessarily have

$$\begin{aligned}
\{f,g\}_{12}(p_1,p_2) &= \{\pi_1^* f_{p_2} + \pi_2^* f_{p_1}, \pi_1^* g_{p_2} + \pi_2^* g_{p_1}\}_{12}(p_1,p_2) \\
&= \{\pi_1^* f_{p_2}, \pi_1^* g_{p_2}\}_{12}(p_1,p_2) + \{\pi_1^* f_{p_2}, \pi_2^* g_{p_1}\}_{12}(p_1,p_2) \\
&\qquad + \{\pi_2^* f_{p_1}, \pi_1^* g_{p_2}\}_{12}(p_1,p_2) + \{\pi_2^* f_{p_1}, \pi_2^* g_{p_1}\}_{12}(p_1,p_2) \\
&= (\pi_1^*\{f_{p_2}, g_{p_2}\}_1)(p_1,p_2) + (\pi_2^*\{f_{p_1}, g_{p_1}\}_2)(p_1,p_2) \\
&= \{f_{p_2}, g_{p_2}\}_1(p_1) + \{f_{p_1}, g_{p_1}\}_2(p_2),
\end{aligned}$$

where condition (ii) and (i) were used in the third equality. This shows that the Poisson bracket, if it exists, is unique and is given by (2.3).

Now define $\{\, ,\}_{12}$ by (2.3). It remains to show that the axioms in Definition 2.1 hold. It is obvious that this operation satisfies the Leibniz identity, is bilinear, and skew symmetric. By Definition 2.1 (iii), one can use Hamiltonian vector fields to express $\{\{f,g\},h\}$. A direct computation gives

$$\begin{aligned}
\{\{f,g\},h\}_{12}(p_1,p_2) &= \{\{f_{p_2}, g_{p_2}\}_1, h_{p_2}\}_1(p_1) + \{\{f_{p_1}, g_{p_1}\}_2, h_{p_1}\}_2(p_2) \\
&+ d_1 d_2 f(p_1,p_2)\left(X^1_{h_{p_2}}(p_1), X^2_{g_{p_1}}(p_2)\right) + d_1 d_2 f(p_1,p_2)\left(X^1_{g_{p_2}}(p_1), X^2_{h_{p_1}}(p_2)\right) \\
&- d_1 d_2 g(p_1,p_2)\left(X^1_{h_{p_2}}(p_1), X^2_{f_{p_1}}(p_2)\right) - d_1 d_2 g(p_1,p_2)\left(X^1_{f_{p_2}}(p_1), X^2_{h_{p_1}}(p_2)\right),
\end{aligned}$$

where $d_1 d_2 f$ denotes the second mixed partial derivative of f and where $X^1_{f_{p_2}}$ is the Hamiltonian vector field on P_1 corresponding to the function $f_{p_2} \in C^\infty(P_1)$ and similarly for the other ones. Adding the other two terms obtained by circular permutation gives zero since the first two terms summed with their analogues vanish by the Jacobi identity on P_1 and P_2 respectively and the other terms cancel.

Since Hamiltonian vector fields on P_1 and P_2 exist by Definition 2.1, formula (2.3) shows that the Hamiltonian vector field on $P_1 \times P_2$ exists and is given by

$$X_h^{12}(p_1,p_2) = \left(X^1_{h_{p_2}}(p_1), X^2_{h_{p_1}}(p_2)\right) \in T_{p_1}P_1 \times T_{p_2}P_2, \tag{2.4}$$

where condition (iii) in Definition 2.1 was used on P_1 and P_2; we have identified here $T_{(p_1,p_2)}(P_1 \times P_2)$ with $T_{p_1}P_1 \times T_{p_2}P_2$. Thus all conditions in Definition 2.1 hold which proves that $P_1 \times P_2$ is a Banach Poisson manifold. ■

We remark that (2.3) implies that the product is functorial, that is, if $\varphi_1 : P_1 \to P_1'$ and $\varphi_2 : P_2 \to P_2'$ are Poisson maps then their product $\varphi_1 \times \varphi_2 : P_1 \times P_2 \to P_1' \times P_2'$ is also a Poisson map.

Returning to Definition 2.1, it should be noted that the condition $\sharp(T^*P) \subset TP$ is automatically satisfied in certain cases:

- if P is a smooth manifold modeled on a reflexive Banach space, that is $\mathfrak{b}^{**} = \mathfrak{b}$, or

- P is a strong symplectic manifold with symplectic form ω.

The first condition holds if P is a Hilbert (and, in particular, a finite dimensional) manifold.

Any strong symplectic manifold (P, ω) is a Poisson manifold in the sense of Definition 2.1. Recall that ***strong*** means that for each $p \in P$ the map

$$v_p \in T_pP \mapsto \omega(p)(v_p, \cdot) \in T_p^*P \tag{2.5}$$

is a bijective continuous linear map. Therefore, given a smooth function $f : P \to \mathbb{R}$ there exists a vector field X_f such that $df = \omega(X_f, \cdot)$. The Poisson bracket is defined by $\{f, g\} = \omega(X_f, X_g) = \langle df, X_g \rangle$, thus $\sharp df = X_f$, so $\sharp(T^*P) \subset TP$.

On the other hand, a weak symplectic manifold is not a Poisson manifold in the sense of Definition 2.1. Recall that ***weak*** means that the map defined by (2.5) is an injective continuous linear map that is, in general, not surjective. Therefore, one cannot construct the map that associates to every differential df of a smooth function $f : P \to \mathbb{R}$ the Hamiltonian vector field X_f. Since the definition of the Poisson bracket should be $\{f, g\} = \omega(X_f, X_g)$, one cannot define this operation on functions and hence weak symplectic manifold structures do not define, in general, Poisson manifold structures in the sense of Definition 2.1. There are various ways to deal with this problem. One of them is to restrict the space of functions on which one is working, as is often done in field theory. Another is to deal with densely defined vector fields and invoke the theory of (nonlinear) semigroups; see Chernoff and Marsden [C-M] for this approach. A simple example illustrating the importance of the underlying topology is given by the canonical symplectic structure on $\mathfrak{b} \times \mathfrak{b}^*$, where $\mathfrak{b}$ is a Banach space. This canonical symplectic structure is in general weak; if $\mathfrak{b}$ is reflexive then it is strong.

In this paper we shall not address these important questions regarding weak symplectic manifolds and their relation to Poisson structures and we shall exclusively consider Banach Poisson manifolds as given by Definition 2.1. Thus, in some sense, the Poisson manifolds considered in this paper are generalizations of strong symplectic manifolds. However, in §7 and §8 we shall give examples illustrating the need for a weakening of Definition 2.1.

We shall need in the sequel various notions of Poisson structures defined on almost complex and complex manifolds. We briefly summarize the various possibilities below.

Assume that the real Banach manifold P underlying the Poisson structure given by the tensor field ϖ has also the structure of an almost complex manifold, that is, there is a smooth vector bundle map $I : TP \to TP$ covering the identity which satisfies $I^2 = -id$. The question then arises what does it mean for the Poisson and almost complex structures to be compatible. The Poisson structure ϖ is said to be ***compatible with the almost complex structure*** I if the following diagram commutes:

$$
\begin{tikzcd}
T^*P \arrow[r, "\sharp"] & TP \arrow[d, "I"] \\
T^*P \arrow[u, "I^*"] \arrow[r, "\sharp"] & TP
\end{tikzcd}
$$

that is,

$$I \circ \sharp + \sharp \circ I^* = 0. \tag{2.6}$$

The decomposition

$$\varpi = \varpi_{(2,0)} + \varpi_{(1,1)} + \varpi_{(0,2)} \tag{2.7}$$

induced by the almost complex structure I and the reality of ϖ, implies that the compatibility condition (2.6) is equivalent to

$$\varpi_{(1,1)} = 0 \qquad \text{and} \qquad \overline{\varpi}_{(2,0)} = \varpi_{(0,2)}. \tag{2.8}$$

In view of (2.8), $[\varpi, \varpi]_S = 0$ is equivalent to

$$[\varpi_{(2,0)}, \varpi_{(2,0)}]_S = 0 \qquad \text{and} \qquad [\varpi_{(2,0)}, \overline{\varpi}_{(2,0)}]_S = 0. \tag{2.9}$$

If (2.6) holds, the triple $(P, \{\cdot,\cdot\}, I)$ is called an ***almost complex Banach Poisson manifold***. If I is given by a complex analytic structure $P_{\mathbb{C}}$ on P it will be called a ***complex Banach Poisson manifold***. For finite dimensional complex manifolds these structures were introduced and studied by Lichnerowicz [Li].

Denote by $\mathcal{O}\Omega^{(k,0)}(P_{\mathbb{C}})$ and $\mathcal{O}\Omega_{(k,0)}(P_{\mathbb{C}})$ the space of holomorphic k-forms and k-vector fields respectively. If

$$\sharp\left(\mathcal{O}\Omega^{(1,0)}(P_{\mathbb{C}})\right) \subset \mathcal{O}\Omega_{(1,0)}(P_{\mathbb{C}}), \tag{2.10}$$

that is, the Hamiltonian vector field X_f is holomorphic if f is a holomorphic function, then, in addition to (2.8) and (2.9), one has $\varpi_{(2,0)} \in \mathcal{O}\Omega_{(2,0)}(P_{\mathbb{C}})$. As expected, the compatibility condition (2.10) is stronger than (2.6). Note that (2.10) implies the second condition in (2.9). Thus the compatibility condition (2.10) induces on the underlying complex Banach manifold $P_{\mathbb{C}}$ a holomorphic Poisson tensor $\varpi_{\mathbb{C}} := \varpi_{(2,0)}$. A pair $(P_{\mathbb{C}}, \varpi_{\mathbb{C}})$ consisting of an analytic complex manifold $P_{\mathbb{C}}$ and a holomorphic skew symmetric contravariant two-tensor field $\varpi_{\mathbb{C}}$ such that $[\varpi_{\mathbb{C}}, \varpi_{\mathbb{C}}]_S = 0$ and (2.10) holds will be called a ***holomorphic Banach Poisson manifold***.

Consider now a holomorphic Poisson manifold (P, ϖ). Denote by $P_{\mathbb{R}}$ the underlying real Banach manifold and define the real two-vector field $\varpi_{\mathbb{R}} := \operatorname{Re}\varpi$. It is easy to see that $(P_{\mathbb{R}}, \varpi_{\mathbb{R}})$ is a real Poisson manifold compatible with the complex Banach manifold structure of P and $(\varpi_{\mathbb{R}})_{\mathbb{C}} = \varpi$. Summarizing, we have shown that there are two procedures that are inverses of each other: a holomorphic Poisson manifold corresponds in a bijective manner to a real Poisson manifold whose Poisson tensor is compatible with the underlying complex manifold structure. One can call these constructions the ***complexification*** and ***realification*** of Poisson structures on complex manifolds.

3 Classical reduction

We shall review in this section the theory of classical Poisson reduction for Banach Poisson manifolds. Let $(P, \{\cdot,\cdot\}_P)$ be a real Banach Poisson manifold (in the sense of Definition 2.1), $i : N \hookrightarrow P$ be a (locally closed) submanifold, and $E \subset (TP)|_N$ be a subbundle of the tangent bundle of P restricted to N. For simplicity we make the following topological regularity assumption throughout this section: $E \cap TN$ is the tangent bundle to a foliation $\mathcal{F}$ whose leaves are the fibers of a submersion $\pi : N \to M := N/\mathcal{F}$, that is, one assumes that the quotient topological space $N/\mathcal{F}$ admits the quotient manifold structure. The subbundle E is said to be ***compatible with the Poisson structure*** provided the following condition holds: if $U \subset P$ is any open subset and $f, g \in C^\infty(U)$ are two arbitrary functions whose differentials df and dg vanish on E, then $d\{f, g\}_P$ also vanishes on E. The triple (P, N, E) is said to be ***reducible***, if E is compatible with the Poisson structure on P and the manifold $M := N/\mathcal{F}$ carries a Poisson bracket $\{\cdot,\cdot\}_M$ (in the sense of Definition 2.1) such that for any smooth local functions $\bar{f}, \bar{g}$ on M and any smooth local extensions f, g of $\bar{f} \circ \pi$, $\bar{g} \circ \pi$ respectively, satisfying $df|_E = 0$, $dg|_E = 0$, the following relation on the common domain of definition of f and g holds:

$$\{f, g\}_P \circ i = \{\bar{f}, \bar{g}\}_M \circ \pi. \tag{3.1}$$

If (P, N, E) is a reducible triple then $(M = N/\mathcal{F}, \{\cdot,\cdot\}_M)$ is called the ***reduced manifold*** of P via (N, E). Note that (3.1) guarantees that if the reduced Poisson bracket $\{\cdot,\cdot\}_M$ on M exists, it is necessarily unique.

Given a subbundle $E \subset TP$, its ***annihilator*** is defined as the subbundle of T^*P given by $E^\circ := \{\alpha \in T^*P \mid \langle \alpha, v\rangle = 0 \text{ for all } v \in E\}$.

The following statement generalizing the finite dimensional Poisson reduction theorem of Marsden and Ratiu [M-R1] is central for our purposes. The proof in infinite dimensions is a modification of the original one (see the above mentioned paper or Vaisman [V], §7.2, for the finite dimensional proof).

Theorem 3.1 *Let P, N, E be as above and assume that E is compatible with the Poisson structure on P. The triple (P, N, E) is reducible if and only if $\sharp(E_n^\circ) \subset \overline{T_nN + E_n}$ for every $n \in N$.*

Proof. Assume that (P, N, E) is reducible. Thus $M := N/\mathcal{F}$ is a Banach Poisson manifold and (3.1) holds. In addition, recall that N is a (locally closed) submanifold of P and that $E \cap TN$ is the tangent bundle of a foliation on N. For $n \in N$, choose a chart domain U of n in P with the submanifold property relative to N and such that $U \cap N$ is foliated.

Given $\alpha_n \in E_n^\circ$, find a smooth function f on U (shrunk if necessary), such that $df(n) = \alpha_n$ and $\langle df, E\rangle = 0$. This is possible since E is a subbundle of $TP|_N$ and $E \cap TN$ is the tangent bundle to a foliation on N. Let $\bar{f}$ be the smooth function on $\pi(U \cap N) \subset M$ induced by f, that is, $f|_N = \bar{f} \circ \pi$. Therefore, $f : U \to \mathbb{R}$ is a local extension of $\bar{f} \circ \pi$.

Next, take an arbitrary $\beta_n \in (E_n + T_nN)^\circ = E_n^\circ \cap (T_nN)^\circ$ and find a smooth function g on U such that $N \cap U = g^{-1}(0)$, $\langle dg, E\rangle = 0$, and $dg(n) = \beta_n$. Again, the existence of

g is insured by the hypothesis that E is a subbundle of $TP|_N$ and that $E \cap TN$ is the tangent bundle to a foliation on N. Thus $g : U \to \mathbb{R}$ is a local extension of $0 \circ \pi$, where 0 is the identically zero function on M. Then we have by (3.1)

$$\langle \beta_n, \sharp(\alpha_n) \rangle = \langle dg(n), X_f(n) \rangle = \{g, f\}_P(n) = \{0, f\}_M(\pi(n)) = 0.$$

This shows that $\sharp(\alpha_n) \in (E_n + T_nN)^{\circ\circ} = \overline{E_n + T_nN}$, that is, $\sharp(E_n^\circ) \subset \overline{T_nN + E_n}$ for every $n \in N$.

Conversely, assume that $\sharp(E_n^\circ) \subset \overline{T_nN + E_n}$ for every $n \in N$. For $\bar{f}, \bar{g}$ locally defined smooth functions on M we need to define their Poisson bracket, show that (3.1) holds, and that all conditions in Definition 2.1 are satisfied. Let f, g be local extensions of $\bar{f} \circ \pi$ and $\bar{g} \circ \pi$ respectively, such that df and dg vanish on E. Since E is compatible with the Poisson bracket on P, $d\{f, g\}_P$ also vanishes on E and thus $\{f, g\}_P$ is constant on the leaves of $\mathcal{F}$ thereby inducing a smooth locally defined function on M. We take this function to be the definition of $\{\bar{f}, \bar{g}\}_M$. If we show that this function is well defined, that is, is independent on the extensions chosen, then the axioms of a Poisson bracket (that is, conditions (i) and (ii) in Definition 2.1) are trivially verified and, by construction, (3.1) holds.

Since the Poisson bracket is skew symmetric it suffices to show the independence of the extension only for the function f. So let f' be another local extension of $\bar{f} \circ \pi$ such that $\langle df', E \rangle = 0$. On the common domain of definition of f and f', we have hence $(f - f')|_N = 0$; in particular, $d(f - f')$ vanishes on TN. However, since both df and df' vanish on E, it follows that $d(f - f')$ vanishes on $E + TN$. Let $n \in N$ be an arbitrary point in the common domain of definition of f and f'. By continuity, $d(f - f')(n)$ vanishes on $\overline{E_n + T_nN}$. Since $X_g(n) \in \sharp(E_n^\circ)$, using the working hypothesis $\sharp(E_n^\circ) \subset \overline{T_nN + E_n}$, we conclude

$$\{f - f', g\}_P(n) = \langle d(f - f')(n), X_g(n) \rangle = 0,$$

that is, $\{f, g\}_P(n) = \{f', g\}_P(n)$.

It remains to verify condition (iii) of Definition 2.1, that is, $\bar{\sharp}(T^*M) \subset TM$, where $\bar{\sharp} : T^*M \to T^{**}M$ is the vector bundle map covering the identity defined by $\bar{\sharp}_m(d\bar{f}(m)) := \{\cdot, \bar{f}\}_M(m)$ for any smooth locally defined function $\bar{f}$ on M. The idea of the proof below is to use (3.1) to show that $\bar{\sharp}_m(d\bar{f}(m)) = T_n\pi\,(X_f(n)) \in T_mM$ for every $m \in M$ and every locally defined function $\bar{f}$ around m.

To do this, let $\bar{f}, \bar{g} : W \to \mathbb{R}$ be two arbitrary smooth functions, where W is a chart domain on M containing the point m. We shall construct now local extensions of $\bar{f} \circ \pi$ and $\bar{g} \circ \pi$ adapted to our needs. Since we have already shown that that the definition of $\{\cdot, \cdot\}_M$ is independent on the extensions, we can work only with these extensions and conclude the desired result. Since $E \cap TN$ is the tangent bundle to a foliation on N, if $n \in N$ is such that $\pi(n) = m$, there is a foliated chart on N around n whose domain is of the form $W \times W'$ (after an eventual shrinking of W), that is, the leaves of the foliation are given by $\{w\} \times W'$ for all $w \in W$. Since N is a submanifold of P and since E is defined only along N, there is a chart on P whose domain is of the form $W \times W' \times V'$ (after shrinking, if necessary, both W and W'). Define the local extension $f : W \times W' \times V' \to \mathbb{R}$ of $\bar{f} \circ \pi$ by $f(w, w', v') := \bar{f}(w)$. By condition (iii) of Definition

2.1, $\sharp_n(df)(n)$ is a vector of the form $X_f(n) \in T_nP$. Let us show that $X_f(n)$ is tangent to N. This is equivalent to proving that for any linear continuous functional β_n on the ambient Banach space containing V', we have $\langle \beta_n, X_f(n)\rangle = 0$. However, $\beta_n = dk(n)$, for some smooth function $k : W \times W' \times V' \to \mathbb{R}$ that does not depend on the variables from W and W'. But then k is a local extension of $0 \circ \pi$ and, using (3.1), we get

$$\langle \beta_n, X_f(n)\rangle = \langle dk(n), X_f(n)\rangle = \{k, f\}_P(n) = \{0, f\}_M(\pi(n)) = 0,$$

which proves the claim.

Construct in the same fashion a local extension of $\bar{g}\circ\pi$ to the same open neighborhood of n in P. Since $dg(n) \circ T_n i = d\bar{g}(m) \circ T_n\pi$, $m = \pi(n)$, we have by (3.1),

$$\begin{aligned}\bar{\sharp}_m(d\bar{f}(m))(d\bar{g}(m)) &= \{\bar{g}, \bar{f}\}_M(m) = \{g, f\}_P(n) = \sharp_n(df(n))(dg(n)) \\ &= \langle dg(n), X_f(n)\rangle = \langle d\bar{g}(m), T_n\pi\left(X_f(n)\right)\rangle .\end{aligned}$$

Since $\bar{g}$ is an arbitrary smooth function defined on a neighborhood of m, the Hahn-Banach Theorem and the inclusion of the Banach space into its bidual imply that $\bar{\sharp}_m(d\bar{f}(m)) = T_n\pi\left(X_f(n)\right) \in T_mM$ for every $m \in M$, that is $\bar{\sharp}(T^*M) \subset TM$. ■

The behavior of Poisson maps and Hamiltonian dynamics under reduction is given by the following two theorems whose proofs are identical to the ones in finite dimensions (Marsden and Ratiu [M-R1] or Vaisman [V], §7.4).

Theorem 3.2 *Let (P_1, N_1, E_1) and (P_2, N_2, E_2) be Poisson reducible triples and assume that $\varphi : P_1 \to P_2$ is a Poisson map satisfying $\varphi(N_1) \subset N_2$ and $T\varphi(E_1) \subset E_2$. Let $\mathcal{F}_i$ be the regular foliation on N_i defined by the subbundle E_i and denote by $\pi_i : N_i \to M_i := N_i/\mathcal{F}_i$, $i = 1, 2$, the reduced Poisson manifolds. Then there is a unique induced Poisson map $\overline{\varphi} : M_1 \to M_2$, called the **reduction** of φ, such that $\pi_2 \circ \varphi = \overline{\varphi} \circ \pi_1$.*

Proof. The hypotheses imply that $T\varphi(E_1 \cap TN_1) \subset E_2 \cap TN_2$ and hence φ maps the leaves of the foliation $\mathcal{F}_1$ to those of $\mathcal{F}_2$. Therefore φ is a projectable map, that is, there exists a smooth map $\overline{\varphi} : M_1 \to M_2$ such that $\pi_2 \circ \varphi = \overline{\varphi} \circ \pi_1$. It remains to be shown that $\overline{\varphi}$ is a Poisson map.

Let $\bar{f}$ and $\bar{g}$ be two smooth local functions on M_2 and let f and g be the local extensions of $\bar{f} \circ \pi_2$ and $\bar{g} \circ \pi_2$ respectively, such that $df|_{E_2} = dg|_{E_2} = 0$. Since $T\varphi(E_1) \subset E_2$ it follows that $d(f \circ\varphi)|_{E_1} = d(g \circ\varphi)|_{E_1} = 0$. Hence $f \circ\varphi$ is a smooth local extension of $\bar{f} \circ\pi_2 \circ\varphi = \bar{f} \circ\bar{\varphi}\circ\pi_1$. Similarly, $g \circ\varphi$ is a smooth local extension of $\bar{g} \circ\pi_2 \circ\varphi = \bar{g} \circ\bar{\varphi}\circ\pi_1$. Definition (3.1) gives then

$$\begin{aligned}\{\bar{f} \circ \bar{\varphi}, \bar{g} \circ \bar{\varphi}\}_{M_1} \circ \pi_1 &= \{f \circ \varphi, g \circ \varphi\}_{P_1} \circ i_1 = \{f, g\}_{P_2} \circ \varphi \circ i_1 \\ &= \{f, g\}_{P_2} \circ i_2 \circ \varphi = \{\bar{f}, \bar{g}\}_{M_2} \circ \pi_2 \circ \varphi = \{\bar{f}, \bar{g}\}_{M_2} \circ \bar{\varphi} \circ \pi_1,\end{aligned}$$

which implies that $\bar{\varphi}$ is a Poisson map by surjectivity of π_1. ■

Theorem 3.3 *Let (P, N, E) be a Poisson reducible triple and $\pi : N \to M$ be the corresponding Poisson reduced manifold. Assume that $h \in C^\infty(P)$ and the associated flow φ_t of the Hamiltonian vector field X_h satisfies the conditions*

(i) $dh|_E = 0$,

(ii) $\varphi_t(N) \subset N$,

(iii) $T\varphi_t(E) \subset E$

for all t for which the flow φ_t is defined. Then the reduction $\overline{\varphi}_t$ is the flow of the Hamiltonian vector field on M given by the function $\bar{h}$ uniquely determined by the condition $\bar{h} \circ \pi = h|_N$. The Hamiltonian vector fields X_h on $N \subset P$ and $X_{\bar{h}}$ on M are π-related.

Proof. The hypotheses guarantee by Theorem 3.2 that the flow φ_t of X_h reduces to a smooth flow $\overline{\varphi}_t$ on the reduced manifold M. The hypothesis on h insures the existence of the smooth function $\bar{h}$ on M. Let us prove that X_h and $X_{\bar{h}}$ are π-related. If this is done, their flows are necessarily π-related and hence, by surjectivity of π, it follows that the flow of $X_{\bar{h}}$ is $\overline{\varphi}_t$.

To prove that X_h and $X_{\bar{h}}$ are π-related, let $\bar{f}$ be a smooth locally defined function in a neighborhood of $m \in M$ and let f be a smooth local extension of $\bar{f} \circ \pi$ satisfying $df|_E = 0$. Then, since X_h is tangent to N, using the defining identity of the reduced bracket (3.1), for any $n \in N$ satisfying $\pi(n) = m$, we get

$$\begin{aligned} \langle d\bar{f}(m), X_{\bar{h}}(m)\rangle &= \{\bar{f}, \bar{h}\}_M(m) = \{f, h\}_P(n) = \langle df(n), X_h(n)\rangle \\ &= \langle d(\bar{f} \circ \pi)(n), X_h(n)\rangle = \langle d\bar{f}(m), T_n\pi\,(X_h(n))\rangle, \end{aligned}$$

which proves that $X_{\bar{h}} \circ \pi = T\pi \circ X_h$. ∎

In this paper we shall not investigate the consistency of Poisson reduction with other structures such as almost complex, complex, and holomorphic structures. For finite dimensional Poisson manifolds the consistency of Poisson reduction with the complex structure was presented in Nunes da Costa [N].

4 Banach Lie-Poisson spaces

It is well known that the dual of any Lie algebra admits a linear Poisson structure, called the Lie-Poisson structure. In this section we shall extend the definition of this structure to the infinite dimensional case in agreement with Definition 2.1. We shall call such spaces Banach Lie-Poisson spaces and shall investigate their properties.

Recall that a ***Banach Lie algebra*** $(\mathfrak{g}, [\cdot,\cdot])$ is a Banach space that is also a Lie algebra such that the Lie bracket is a bilinear continuous map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Thus the adjoint and coadjoint maps $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_x y := [x, y]$, and $\mathrm{ad}_x^* : \mathfrak{g}^* \to \mathfrak{g}^*$ are also continuous for each $x \in \mathfrak{g}$.

Definition 4.1 *A* ***Banach Lie-Poisson space*** $(\mathfrak{b}, \{\cdot,\cdot\})$ *is a real or holomorphic Poisson manifold such that $\mathfrak{b}$ is a Banach space and the dual $\mathfrak{b}^* \subset C^\infty(\mathfrak{b})$ is a Banach Lie algebra under the Poisson bracket operation.*

Throughout this section we shall treat the real and the holomorphic cases simultaneously. Denote by $[\cdot,\cdot]$ the restriction of the Poisson bracket $\{\cdot,\cdot\}$ from $C^\infty(\mathfrak{b})$ to the Lie subalgebra $\mathfrak{b}^*$. For any $x, y \in \mathfrak{b}^*$ and $b \in \mathfrak{b}$ we have

$$\begin{aligned}\langle y, \operatorname{ad}_x^* b\rangle &= \langle [x,y], b\rangle = \{x,y\}(b) = -\{y,x\}(b)\\ &= -X_x[y](b) = -\langle Dy(b), X_x(b)\rangle = -\langle y, X_x(b)\rangle,\end{aligned}$$

where we have used the linearity of $y \in C^\infty(\mathfrak{b})$ to conclude that the Fréchet derivative $Dy(b) = y$. Thus we obtain the following identity in the bidual $\mathfrak{b}^{**}$:

$$X_x(b) = -\operatorname{ad}_x^* b \qquad \text{for} \qquad x \in \mathfrak{b}^*, \quad b \in \mathfrak{b}. \tag{4.1}$$

Theorem 4.2 *The Banach space $\mathfrak{b}$ is a Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot,\cdot\})$ if and only if its dual $\mathfrak{b}^*$ is a Banach Lie algebra $(\mathfrak{b}^*, [\cdot,\cdot])$ satisfying $\operatorname{ad}_x^* \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{b}^{**}$ for all $x \in \mathfrak{b}^*$. Moreover, the Poisson bracket of $f, g \in C^\infty(\mathfrak{b})$ is given by*

$$\{f,g\}(b) = \langle [Df(b), Dg(b)], b\rangle, \tag{4.2}$$

where $b \in \mathfrak{b}$ and D denotes the Fréchet derivative. If h is a smooth function on $\mathfrak{b}$, the associated Hamiltonian vector field is given by

$$X_h(b) = -\operatorname{ad}_{Dh(b)}^* b. \tag{4.3}$$

Proof. Assume that $\mathfrak{b}$ is a Banach Lie-Poisson space relative to the bracket $\{\cdot,\cdot\}$. By Definition 4.1, its dual $\mathfrak{b}^*$ is a Banach Lie algebra relative to the bracket $[\cdot,\cdot] := \{\cdot,\cdot\}|_{\mathfrak{b}^*}$. However, $\mathfrak{b}$ is also a Poisson manifold and thus, by definition, $X_x(b) \in \mathfrak{b}$ for all $x \in \mathfrak{b}^*$ and all $b \in \mathfrak{b}$. Formula (4.1) implies then that $\operatorname{ad}_x^*(b) \in \mathfrak{b}$ for all $x \in \mathfrak{b}^*$ and all $b \in \mathfrak{b}$ which is the required condition.

Conversely, assume that $(\mathfrak{b}^*, [\cdot,\cdot])$ is a Banach Lie algebra satisfying $\operatorname{ad}_x^* \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{b}^{**}$ for all $x \in \mathfrak{b}^*$. Define the bracket $\{f,g\}$ of $f, g \in C^\infty(b)$ by (4.2). All properties of the Poisson bracket are trivially satisfied by (4.2) except for the Jacobi identity. For this, we note that from $\operatorname{ad}_x^* \mathfrak{b} \subset \mathfrak{b}$, $x \in \mathfrak{b}^*$, one has

$$D\{f,g\}(b) = \langle [Df(b), Dg(b)], \cdot\rangle - D^2f(b)\left(\operatorname{ad}_{Dg(b)}^* b, \cdot\right) + D^2g(b)\left(\operatorname{ad}_{Df(b)}^* b, \cdot\right). \tag{4.4}$$

for $f, g \in C^\infty(\mathfrak{b})$. Using (4.4) we obtain

$$\begin{aligned}\{\{f,g\},h\}(b) &= \langle [D\{f,g\}(b), Dh(b)], b\rangle\\ &= \langle [[Df(b), Dg(b)], Dh(b)], b\rangle + D^2f(b)\left(\operatorname{ad}_{Dg(b)}^* b, \operatorname{ad}_{Dh(b)}^* b\right)\\ &\qquad - D^2g(b)\left(\operatorname{ad}_{Df(b)}^* b, \operatorname{ad}_{Dh(b)}^* b\right).\end{aligned}$$

Taking the two other terms obtained by circular permutation of f, g, and h, using the Jacobi identity for the Lie bracket in the sum of the first three terms and the symmetry of the second derivative in the sum of the remaining terms, proves that (4.2) satisfies the Jacobi identity.

Since

$$\langle Df(b), X_h(b)\rangle = \{f,h\}(b) = \langle [Df(b), Dh(b)], b\rangle = -\left\langle Df(b), \operatorname{ad}_{Dh(b)}^* b\right\rangle$$

for every $f \in C^\infty(\mathfrak{b})$ and $\operatorname{ad}_x^* \mathfrak{b} \subset \mathfrak{b}$ for every $x \in \mathfrak{b}^*$, it follows that the Hamiltonian vector field X_h is given by (4.3). ■

Example 4.3 Let $\mathfrak{b}$ be a reflexive Banach Lie algebra, that is, $\mathfrak{b}^{**} = \mathfrak{b}$. Then its dual $\mathfrak{b}^*$ is a Banach Lie-Poisson space. To see this, note that $\mathfrak{b}^{**} = \mathfrak{b}$ is a Banach Lie algebra and that $\mathrm{ad}_x^*(\mathfrak{b}^*) \subset \mathfrak{b}^*$ for all $x \in \mathfrak{b}$, so Theorem 4.2 applies. ♦

Example 4.4 Since every finite dimensional Lie algebra is reflexive Example 4.3 yields the following classical result: the dual of any finite dimensional Lie algebra is a Lie-Poisson space. ♦

Definition 4.5 *A* ***morphism*** *between two Banach Lie-Poisson spaces $\mathfrak{b}_1$ and $\mathfrak{b}_2$ is a continuous linear map $\phi : \mathfrak{b}_1 \to \mathfrak{b}_2$ that preserves the Poisson bracket structure, that is,*

$$\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi$$

for any $f, g \in C^\infty(\mathfrak{b}_2)$. Such a map ϕ is also called a ***linear Poisson map****.*

We consider now the category $\mathfrak{B}$ whose objects are the Banach Lie-Poisson spaces and whose morphisms are the linear Poisson maps.

Let $\mathfrak{L}$ denote the category of Banach Lie algebras and continuous Lie algebra homomorphisms. Denote by $\mathfrak{L}_0$ the following subcategory of $\mathfrak{L}$. An object of $\mathfrak{L}_0$ is a Banach Lie algebra $\mathfrak{g}$ admitting a predual $\mathfrak{g}_*$, that is, $(\mathfrak{g}_*)^* = \mathfrak{g}$, and satisfying $\mathrm{ad}_{\mathfrak{g}}^* \mathfrak{g}_* \subset \mathfrak{g}_*$ where ad^* is the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$; note that $\mathfrak{g}_* \subset \mathfrak{g}^*$. A morphism in the category $\mathfrak{L}_0$ is a Banach Lie algebra homomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that the dual map $\psi^* : \mathfrak{g}_2^* \to \mathfrak{g}_1^*$ preserves at least on choice of the corresponding preduals, that is, $\psi^* : (\mathfrak{g}_2)_* \to (\mathfrak{g}_1)_*$. Let $\mathfrak{L}_{0u}$ be the subcategory of $\mathfrak{L}_0$ whose objects have a *unique* predual.

Theorem 4.6 *There is a contravariant functor $\mathfrak{F} : \mathfrak{B} \to \mathfrak{L}_0$ defined by $\mathfrak{F}(\mathfrak{b}) = \mathfrak{b}^*$ and $\mathfrak{F}(\phi) = \phi^*$. On the subcategory $\mathfrak{F}^{-1}(\mathfrak{L}_{0u})$ this functor is invertible. The inverse of $\mathfrak{F}$ is given by $\mathfrak{F}^{-1}(\mathfrak{g}) = \mathfrak{g}_*$ and $\mathfrak{F}^{-1}(\psi) = \psi^*|_{(\mathfrak{g}_2)_*}$, where $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$.*

Proof. If $\mathfrak{b}$ is a Banach Lie-Poisson space, then $\mathfrak{F}(\mathfrak{b}) = \mathfrak{b}^*$ is a Banach Lie algebra that admits $\mathfrak{b}$ as a predual and, according to Theorem 4.2, $\mathrm{ad}_{\mathfrak{b}^*}^* \mathfrak{b} \subset \mathfrak{b}$. Thus $\mathfrak{F}(\mathfrak{b})$ is indeed an object in the category $\mathfrak{L}_0$. If $\phi : \mathfrak{b}_1 \to \mathfrak{b}_2$ is a linear Poisson map let us show that $\mathfrak{F}(\phi) = \phi^* : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ is a Banach Lie algebra homomorphism. First, ϕ^* is a linear continuous map between Banach spaces. Second, since the Lie bracket on $\mathfrak{b}_2^*$ is defined by $[x, y]_2 = \{x, y\}_2$ and similarly for $\mathfrak{b}_1$, we get

$$\phi^*[x, y]_2 = \phi^*\{x, y\}_2 = \{\phi^* x, \phi^* y\}_1 = [\phi^* x, \phi^* y]_1,$$

which shows that ϕ^* a homomorphism of Banach Lie algebras. Finally, the dual of $\mathfrak{F}(\phi)$, that is, $\phi^{**} : \mathfrak{b}_1^{**} \to \mathfrak{b}_2^{**}$ satisfies $\phi^{**}|_{\mathfrak{b}_1} = \phi$. Thus $\mathfrak{F}(\phi)$ is indeed a morphism in the category $\mathfrak{L}_0$. Since duality reverses the direction of the arrows and the order of the composition, $\mathfrak{F}$ is a contravariant functor.

Conversely, consider the functor $\mathfrak{F}^{-1} : \mathfrak{L}_{0u} \to \mathfrak{B}$ and let $\mathfrak{g}$ be an object of $\mathfrak{L}_{0u}$. By Theorem 4.2, $\mathfrak{F}^{-1}(\mathfrak{g}) = \mathfrak{g}_*$ is a Banach Lie-Poisson space, that is, an object of $\mathfrak{B}$. If $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphism in the category $\mathfrak{L}_{0u}$, then let us show that $\mathfrak{F}^{-1}(\psi) = \psi^*|_{(\mathfrak{g}_2)_*}$ is a linear Poisson map. Let f, g be smooth functions on $(\mathfrak{g}_1)_*$. From (4.2) and using the fact that ψ is morphism of Banach Lie algebras and that $\psi^*|_{(\mathfrak{g}_2)_*}$ is a linear map, we get for every $b \in (\mathfrak{g}_2)_*$

$$
\begin{aligned}
\{f \circ \psi^*|_{(\mathfrak{g}_2)_*}, g \circ \psi^*|_{(\mathfrak{g}_2)_*}\}_2(b) &= \left\langle [D(f \circ \psi^*|_{(\mathfrak{g}_2)_*})(b), D(g \circ \psi^*|_{(\mathfrak{g}_2)_*})(b)]_2, b \right\rangle \\
&= \left\langle [Df(\psi^*(b)) \circ \psi^*|_{(\mathfrak{g}_2)_*}, Dg(\psi^*(b)) \circ \psi^*|_{(\mathfrak{g}_2)_*}]_2, b \right\rangle \\
&= \langle [\psi\left(Df(\psi^*(b))\right), \psi\left(Dg(\psi^*(b))\right)]_2, b\rangle \\
&= \langle \psi\left([Df(\psi^*(b)), Dg(\psi^*(b))]_1\right), b\rangle \\
&= \langle [Df(\psi^*(b)), Dg(\psi^*(b))]_1, \psi^*(b)\rangle \\
&= \{f, g\}_1(\psi^*|_{(\mathfrak{g}_2)_*}(b))
\end{aligned}
$$

which shows that $\psi^*|_{(\mathfrak{g}_2)_*} : (\mathfrak{g}_2)_* \to (\mathfrak{g}_1)_*$ is a morphism of Banach Lie-Poisson spaces. The functor $\mathfrak{F}^{-1}$ is contravariant since its action on morphisms is given by duality.

Finally, it is clear the functors $\mathfrak{F} : \mathfrak{F}^{-1}(\mathfrak{L}_{0u}) \to \mathfrak{L}_{0u}$ and $\mathfrak{F}^{-1} : \mathfrak{L}_{0u} \to \mathfrak{F}^{-1}(\mathfrak{L}_{0u})$ are inverses of each other. $\blacksquare$

We turn now to the study of the internal structure of morphisms of Banach Lie-Poisson spaces.

Proposition 4.7 *Let $\phi : \mathfrak{b}_1 \to \mathfrak{b}_2$ be a linear Poisson map between Banach Lie-Poisson spaces and assume that $\operatorname{im}\phi$ is closed in $\mathfrak{b}_2$. Then the Banach space $\mathfrak{b}_1/\ker\phi$ is predual to $\mathfrak{b}_2^*/\ker\phi^*$, that is, $(\mathfrak{b}_1/\ker\phi)^* \cong \mathfrak{b}_2^*/\ker\phi^*$. In addition, $\mathfrak{b}_2^*/\ker\phi^*$ is a Banach Lie algebra satisfying the condition $\operatorname{ad}^*_{[x]}(\mathfrak{b}_1/\ker\phi) \subset \mathfrak{b}_1/\ker\phi$ for all $[x] \in \mathfrak{b}_2^*/\ker\phi^*$ and $\mathfrak{b}_1/\ker\phi$ is a Banach Lie-Poisson space. Moreover, the following properties hold:*

(i) *the quotient map $\pi : \mathfrak{b}_1 \to \mathfrak{b}_1/\ker\phi$ is a surjective linear Poisson map;*

(ii) *the map $\iota : \mathfrak{b}_1/\ker\phi \to \mathfrak{b}_2$ defined by $\iota([b]) := \phi(b)$, where $b \in \mathfrak{b}_1$ and $[b] \in \mathfrak{b}_1/\ker\phi$ is an injective linear Poisson map;*

(iii) *the decomposition $\phi = \iota \circ \pi$ into a surjective and an injective linear Poisson map is valid.*

Proof. We define the pairing $\langle\cdot,\cdot\rangle : \mathfrak{b}_2^*/\ker\phi^* \times \mathfrak{b}_1/\ker\phi \to \mathbb{C}$ (or $\mathbb{R}$) by

$$
\langle [x], [b]\rangle := \langle x, \phi(b)\rangle_2 = \langle \phi^*(x), b\rangle_1 \tag{4.5}
$$

where $[x] \in \mathfrak{b}_2^*/\ker\phi^*$, $[b] \in \mathfrak{b}_1/\ker\phi$, and $\langle\cdot,\cdot\rangle_i : \mathfrak{b}_i^* \times \mathfrak{b}_i \to \mathbb{C}$ (or $\mathbb{R}$), $i = 1, 2$, are the pairings between the given Banach Lie-Poisson spaces and their duals. This pairing is correctly defined since it does not depend on the choice of the representatives $x \in \mathfrak{b}_2$ and $b \in \mathfrak{b}_1$. One has

$$
|\langle [x], [b]\rangle| \leq \|\phi\| \, \|[x]\| \, \|[b]\|
$$

and if $\langle [x], [b]\rangle = 0$ for each $x \in \mathfrak{b}_2$ ($b \in \mathfrak{b}_1$) then $[b] = [0]$ ($[x] = [0]$). Thus (4.5) defines a continuous weakly non degenerate pairing and therefore the map

$$
[x] \in \mathfrak{b}_2^*/\ker\phi^* \mapsto \langle [x], [\cdot]\rangle = \langle \phi^*(x), \cdot\rangle_1 = \langle x, \phi(\cdot)\rangle_2 \in (\mathfrak{b}_1/\ker\phi)^*
$$

is a continuous linear injective map of Banach spaces. To show that this map is surjective, we need to find for a given $\alpha \in (\mathfrak{b}_1/\ker\phi)^*$ an $[x] \in \mathfrak{b}_2^*/\ker\phi^*$ such that $\langle [x], [b]\rangle =$

$\langle \phi^*(x), b\rangle_1 = \langle x, \phi(b)\rangle_1 = \alpha([b])$ for all $b \in \mathfrak{b}_1$. Since the range $\operatorname{im}\phi$ is closed in $\mathfrak{b}_2$, it is a Banach subspace and hence the map $\Phi : [b] \in \mathfrak{b}_1/\ker\phi \mapsto \phi(b) \in \operatorname{im}\phi$ is a Banach space isomorphism. Thus $\alpha \circ \Phi^{-1} \in (\operatorname{im}\phi)^*$. Let $x \in \mathfrak{b}_2^*$ be an extension of $\alpha\circ\Phi^{-1}$ to $\mathfrak{b}_2$. Then we have for any $b \in \mathfrak{b}_1$

$$\langle [x], [b]\rangle = \langle x, \phi(b)\rangle_2 = \langle \alpha\circ\Phi^{-1}, \phi(b)\rangle_2 = \alpha([b]).$$

Thus the Banach space $\mathfrak{b}_2^*/\ker\phi^*$ is isomorphic to the dual of $\mathfrak{b}_1/\ker\phi$.

The space $\mathfrak{b}_2^*/\ker\phi^*$ is a Banach Lie algebra because $\ker\phi^*$ is an ideal in the Banach Lie algebra $\mathfrak{b}_2^*$ (since $\phi^* : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ is a morphism of Banach Lie algebras). Finally, since $\phi : \mathfrak{b}_1 \to \mathfrak{b}_2$ is a linear Poisson map, we have

$$\operatorname{ad}^{2*}_x \phi(b) = \phi\left(\operatorname{ad}^{1*}_{x\circ\phi} b\right), \tag{4.6}$$

for any $x \in \mathfrak{b}_2^*$, $b \in \mathfrak{b}_1$, and where ad^i denotes the adjoint operator in the Banach Lie algebra $\mathfrak{b}_i^*$, $i = 1, 2$. Here we have used the fact that $\operatorname{ad}^{1*}_{x\circ\phi}\mathfrak{b}_1 \subset \mathfrak{b}_1$ for any $x \in \mathfrak{b}_2^*$. From (4.6) and $\operatorname{ad}^{2*}_x\mathfrak{b}_2 \subset \mathfrak{b}_2$ for all $x \in \mathfrak{b}_2^*$ we conclude that for all $b \in \mathfrak{b}_1$ we have

$$\begin{aligned}\left\langle [y], \operatorname{ad}^*_{[x]}[b]\right\rangle &= \langle [[x],[y]], [b]\rangle = \langle [[x,y]], [b]\rangle = \langle [x,y], \phi(b)\rangle_2 \\ &= \left\langle y, \operatorname{ad}^{2*}_x\phi(b)\right\rangle_2 = \left\langle y, \phi\left(\operatorname{ad}^{1*}_{x\circ\phi}b\right)\right\rangle_2 = \left\langle [y], \left[\operatorname{ad}^{1*}_{x\circ\phi}b\right]\right\rangle\end{aligned}$$

for each $y \in \mathfrak{b}_2^*$. This implies that

$$\operatorname{ad}^*_{[x]}[b] = \left[\operatorname{ad}^{1*}_{x\circ\phi}b\right] \in \mathfrak{b}_1/\ker\phi$$

for all $[x] \in \mathfrak{b}_2^*/\ker\phi^*$, $[b] \in \mathfrak{b}_1/\ker\phi$, and thus

$$\operatorname{ad}^*_{[x]}\left(\mathfrak{b}_1/\ker\phi\right) \subset \mathfrak{b}_1/\ker\phi$$

for all $[x] \in \mathfrak{b}_2^*/\ker\phi^*$. Thus $\mathfrak{b}_2^*/\ker\phi^*$ is an object in the category $\mathfrak{L}_0$. Theorem 4.2 (or Theorem 4.6) guarantees then that the quotient Banach space $\mathfrak{b}_1/\ker\phi$ is a Banach Lie-Poisson space.

Endow the Banach subspace $\operatorname{im}\phi$ with the Banach Lie-Poisson structure making the Banach space isomorphism $\Phi : \mathfrak{b}_1/\ker\phi \to \operatorname{im}\phi$ into a linear Poisson isomorphism. Thus $\Phi^* : (\operatorname{im}\phi)^* \to (\mathfrak{b}_1/\ker\phi)^*$ is an isomorphism in the category $\mathfrak{L}_0$. Since $\phi = \Phi\circ\pi : \mathfrak{b}_1 \to \operatorname{im}\phi$ is a linear Poisson map by hypothesis, it follows that $\phi^* = \pi^*\circ\Phi^*$ is a morphism in the category $\mathfrak{L}_0$ which then implies that π^* is also a morphism in the category $\mathfrak{L}_0$. By Theorem 4.6 this is equivalent to the fact that π is a linear Poisson map thereby proving property (i) in the statement of the proposition.

Define $\iota : \mathfrak{b}_1/\ker\phi \to \mathfrak{b}_2$ to be the composition of the inclusion $\operatorname{im}\phi \hookrightarrow \mathfrak{b}_2$ with the isomorphism $\Phi : \mathfrak{b}_1/\ker\phi \to \operatorname{im}\phi$. The definition of Φ is equivalent to the equality $\phi = \Phi\circ\pi$ thought of as a map from $\mathfrak{b}_1$ to $\operatorname{im}\phi$. Composing this identity on the left with the inclusion $\operatorname{im}\phi \hookrightarrow \mathfrak{b}_2$ yields $\phi = \iota\circ\pi$ which proves property (iii).

To prove part (ii), let $f, g \in C^\infty(\mathfrak{b}_2)$. Then $f\circ\iota, g\circ\iota \in C^\infty(\mathfrak{b}_1/\ker\phi)$. Since $\pi : \mathfrak{b}_1 \to \mathfrak{b}_1/\ker\phi$ is a surjective linear Poisson map and $\phi = \iota\circ\pi$, the relation

$$\{f\circ\iota, g\circ\iota\}\circ\pi = \{f\circ\iota\circ\pi, g\circ\iota\circ\pi\}_1 = \{f\circ\phi, g\circ\phi\}_1 = \{f,g\}_2\circ\phi = \{f,g\}_2\circ\iota\circ\pi$$

implies that $\{f \circ \iota, g \circ \iota\} = \{f, g\}_2 \circ \iota$, that is, $\iota : \mathfrak{b}_1/\ker\phi \to \mathfrak{b}_2$ is an injective linear Poisson map. $\blacksquare$

Proposition 4.7 reduces the investigation of linear Poisson maps with closed range between Banach Lie-Poisson spaces to the study of surjective and injective linear Poisson maps.

Consider therefore the surjective linear continuous map $\pi : \mathfrak{b}_1 \to \mathfrak{b}_2$, where $\mathfrak{b}_1$ is a Banach Lie-Poisson space and $\mathfrak{b}_2$ is just a Banach space with no additional structure. The dual map $\pi^* : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ is therefore an injective continuous linear map of Banach spaces. The space $\operatorname{im}\pi^*$ coincides with the Banach subspace of linear continuous functionals on $\mathfrak{b}_1$ that vanish on $\ker\pi$, which is closed in $\mathfrak{b}_1$. Thus $\operatorname{im}\pi^*$ is a closed subspace of $\mathfrak{b}_1^*$.

Assume next that $\operatorname{im}\pi^*$ is also closed under the Lie bracket operation $[\cdot,\cdot]_1$ of $\mathfrak{b}_1^*$. Then $\pi^* : \mathfrak{b}_2^* \to \operatorname{im}\pi^*$ is a Banach space isomorphism and, declaring it to be also a Lie algebra morphism, it follows that there is a Banach Lie algebra structure $[\cdot,\cdot]_2$ on $\mathfrak{b}_2^*$ and that $\pi^* : (\mathfrak{b}_2^*, [\cdot,\cdot]_2) \to (\mathfrak{b}_1^*, [\cdot,\cdot]_1)$ is a Banach Lie algebra morphism.

Let $\tilde{x}, \tilde{y} \in \mathfrak{b}_2^*$ and let $\pi(b) = \tilde{b} \in \mathfrak{b}_2$. Then

$$\begin{aligned}\left\langle \tilde{y}, \operatorname{ad}^{2*}_{\tilde{x}}\tilde{b}\right\rangle_2 &= \left\langle [\tilde{x}, \tilde{y}]_2, \tilde{b}\right\rangle_2 = \langle [\tilde{x}, \tilde{y}]_2, \pi(b)\rangle_2 = \langle \pi^*([\tilde{x}, \tilde{y}]_2), b\rangle_1 \\ &= \langle [\pi^*(\tilde{x}), \pi^*(\tilde{y})]_1, b\rangle_1 = \left\langle \pi^*(\tilde{y}), \operatorname{ad}^{1*}_{\pi^*(\tilde{x})} b\right\rangle_1 = \left\langle \tilde{y}, \pi\left(\operatorname{ad}^{1*}_{\pi^*(\tilde{x})} b\right)\right\rangle_2;\end{aligned}$$

the last equality is a consequence of the inclusion $\operatorname{ad}^{1*}_{\pi^*(\tilde{x})}\mathfrak{b}_1 \subset \mathfrak{b}_1$ for all $\tilde{x} \in \mathfrak{b}_2^*$ which is insured by the fact that $\mathfrak{b}_1$ is a Banach Lie-Poisson space. Since this relation holds for any $\tilde{y} \in \mathfrak{b}_2$, we conclude that

$$\operatorname{ad}^{2*}_{\tilde{x}}\tilde{b} = \pi\left(\operatorname{ad}^{1*}_{\pi^*(\tilde{x})} b\right)$$

for any $\tilde{x} \in \mathfrak{b}_2^*$ and any $\tilde{b} \in \mathfrak{b}_2$. This shows that $\operatorname{ad}^{2*}_{\tilde{x}}\mathfrak{b}_2 \subset \mathfrak{b}_2$ for any $\tilde{x} \in \mathfrak{b}_2^*$, and hence, by Theorem 4.2, $\mathfrak{b}_2$ is a Banach Lie-Poisson space or, equivalently, $\mathfrak{b}_2^*$ is an object in the category $\mathfrak{L}_0$.

The map π^* is a morphism of Banach Lie algebras. In addition, its dual $\pi^{**} : \mathfrak{b}_1^{**} \to \mathfrak{b}_2^{**}$ has the property $\pi^{**}(\mathfrak{b}_1) \subset \mathfrak{b}_2$. Indeed, for any $b_1 \in \mathfrak{b}_1$ and $\beta_2 \in \mathfrak{b}_2^*$, the definition of the dual of a linear map gives

$$\langle \pi^{**}(b_1), \beta_2\rangle_2' = \langle b, \pi^*(\beta_2)\rangle_1 = \langle \pi(b_1), \beta_2\rangle_2$$

where $\langle\cdot,\cdot\rangle_2' : \mathfrak{b}_2^{**} \times \mathfrak{b}_2^* \to \mathbb{R}$ (or $\mathbb{C}$) is the canonical pairing between a Banach space and its dual and similarly for $\langle\cdot,\cdot\rangle_1 : \mathfrak{b}_1 \times \mathfrak{b}_1^* \to \mathbb{R}$ (or $\mathbb{C}$) and $\langle\cdot,\cdot\rangle_2 : \mathfrak{b}_2 \times \mathfrak{b}_2^* \to \mathbb{R}$ (or $\mathbb{C}$). This shows that $\pi^{**}(b_1) = \pi(b_1) \in \mathfrak{b}_2$. Therefore π^* is a morphism in the category $\mathfrak{L}_0$ and, by Theorem 4.6, $\pi : \mathfrak{b}_1 \to \mathfrak{b}_2$ is a linear Poisson map. In addition, since π is also surjective, the Banach Lie-Poisson structure on $\mathfrak{b}_2$ is unique. Therefore, following e.g. Vaisman [V], we shall call this Banach Lie-Poisson structure on $\mathfrak{b}_2$ ***coinduced*** by the surjective mapping π.

The above proves the "only if" part of the following proposition; the converse is an easy verification.

Proposition 4.8 *Let $(\mathfrak{b}_1, \{\cdot,\cdot\})$ be a Banach Lie-Poisson space and let $\pi : \mathfrak{b}_1 \to \mathfrak{b}_2$ be a continuous linear surjective map onto the Banach space $\mathfrak{b}_2$. Then $\mathfrak{b}_2$ carries the Banach*

Lie-Poisson structure coinduced by π *if and only if* $\operatorname{im}\pi^* \subset \mathfrak{b}_1^*$ *is closed under the Lie bracket* $[\cdot,\cdot]_1$ *of* $\mathfrak{b}_1^*$*. The map* $\pi^* : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ *is a Banach Lie algebra morphism whose dual* $\pi^{**} : \mathfrak{b}_1^{**} \to \mathfrak{b}_2^{**}$ *maps* $\mathfrak{b}_1$ *into* $\mathfrak{b}_2$*.*

Example 4.9 Let $(\mathfrak{g}, [\cdot,\cdot])$ be a complex Banach Lie algebra admitting a predual $\mathfrak{g}_*$ satisfying $\operatorname{ad}_x^* \mathfrak{g}_* \subset \mathfrak{g}_*$ for every $x \in \mathfrak{g}$. Then, by Theorem 4.2, the predual $\mathfrak{g}_*$ admits a holomorphic Banach Lie-Poisson structure, whose holomorphic Poisson tensor ϖ is given by (4.2). We shall work with the realification $(\mathfrak{g}_{*\mathbb{R}}, \varpi_{\mathbb{R}})$ of $(\mathfrak{g}_*, \varpi)$ in the sense of §2. We want to construct a real Banach space $\mathfrak{g}_*^\sigma$ with a real Banach Lie-Poisson structure ϖ_σ such that $\mathfrak{g}_*^\sigma \otimes \mathbb{C} = \mathfrak{g}_*$ and ϖ_σ is coinduced from $\varpi_{\mathbb{R}}$ in the sense of Proposition 4.8. To achieve this, introduce a continuous $\mathbb{R}$-linear map $\sigma : \mathfrak{g}_{*\mathbb{R}} \to \mathfrak{g}_{*\mathbb{R}}$ satisfying the properties:

(i) $\sigma^2 = id$;

(ii) the dual map $\sigma^* : \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ defined by

$$\langle \sigma^* z, b\rangle = \overline{\langle z, \sigma b\rangle} \tag{4.7}$$

for $z \in \mathfrak{g}_{\mathbb{R}}$, $b \in \mathfrak{g}_{*\mathbb{R}}$ and where $\langle\cdot,\cdot\rangle$ is the pairing between the complex Banach spaces $\mathfrak{g}$ and $\mathfrak{g}_*$, is a homomorphism of the Lie algebra $(\mathfrak{g}_{\mathbb{R}}, [\cdot,\cdot])$;

(iii) $\sigma \circ I + I \circ \sigma = 0$, where $I : \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ is defined by

$$\langle z, Ib\rangle := \langle I^* z, b\rangle := i\langle z, b\rangle \tag{4.8}$$

for $z \in \mathfrak{g}_{\mathbb{R}}$, $b \in \mathfrak{g}_{*\mathbb{R}}$.

Consider the projectors

$$R := \frac{1}{2}(id + \sigma) \qquad R^* := \frac{1}{2}(id + \sigma^*) \tag{4.9}$$

and define $\mathfrak{g}_*^\sigma := \operatorname{im} R$, $\mathfrak{g}^\sigma := \operatorname{im} R^*$. Then one has the splittings

$$\mathfrak{g}_{*\mathbb{R}} = \mathfrak{g}_*^\sigma \oplus I\mathfrak{g}_*^\sigma \quad \text{and} \quad \mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^\sigma \oplus I\mathfrak{g}^\sigma \tag{4.10}$$

into real Banach subspaces. One can identify canonically the splittings (4.10) with the splittings

$$\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{C} = (\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{R}) \oplus (\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{R}i)\,. \tag{4.11}$$

Thus one obtains isomorphisms $\mathfrak{g}_*^\sigma \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}_*$ and $\mathfrak{g}^\sigma \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}$ of complex Banach spaces.

For any $x, y \in \mathfrak{g}_{\mathbb{R}}$ one has

$$[R^* x, R^* y] = R^*[x, R^* y] \tag{4.12}$$

and thus $\mathfrak{g}^\sigma$ is a real Banach Lie subalgebra of $\mathfrak{g}_{\mathbb{R}}$. From

$$\begin{aligned} \operatorname{Re}\langle z, b\rangle &= \langle R^* z, Rb\rangle + \langle I^* R^* I^* z, IRIb\rangle \\ &= \langle R^* z, Rb\rangle + \langle (1 - R^*)z, (1 - R)b\rangle \end{aligned} \tag{4.13}$$

for all $z \in \mathfrak{g}_{\mathbb{R}}$ and all $b \in \mathfrak{g}_{*\mathbb{R}}$, where for the last equality we used $R = 1 + IRI$ and $R^* = 1 + I^*R^*I^*$, one concludes that the annihilator $(\mathfrak{g}_*^\sigma)^\circ$ of $\mathfrak{g}_*^\sigma$ in $\mathfrak{g}_{\mathbb{R}}$ equals $I^*\mathfrak{g}^*$. Therefore $\mathfrak{g}_*^\sigma$ is the predual of $\mathfrak{g}^\sigma$.

Taking into account all of the above facts we conclude from Proposition 4.8 that $\mathfrak{g}_*^\sigma$ carries a real Banach Lie-Poisson structure $\{\cdot,\cdot\}_{\mathfrak{g}_*^\sigma}$ coinduced by $R : \mathfrak{g}_{*\mathbb{R}} \to \mathfrak{g}_*^\sigma$. According to (4.13), the bracket $\{\cdot,\cdot\}_{\mathfrak{g}_*^\sigma}$ is given by

$$\{f, g\}_{\mathfrak{g}_*^\sigma}(\rho) = \langle [df(\rho), dg(\rho)], \rho\rangle, \tag{4.14}$$

where $\rho \in \mathfrak{g}_*^\sigma$ and the pairing on the right is between $\mathfrak{g}_*^\sigma$ and $\mathfrak{g}^\sigma$. In addition, for any real valued functions $f, g \in C^\infty(\mathfrak{g}_*^\sigma)$ and any $b \in \mathfrak{g}_{*\mathbb{R}}$ we have

$$\begin{aligned}
&\{f\circ R, g\circ R\}_{\mathfrak{g}_{\mathbb{R}}}(b) = \mathrm{Re}\langle [d(f\circ R)(b), d(g\circ R)(b)], b\rangle \\
&\ = \langle R^*[d(f\circ R)(b), d(g\circ R)(b)], R(b)\rangle + \langle (1-R^*)[d(f\circ R)(b), d(g\circ R)(b)], (1-R)b\rangle \\
&\ = \langle R^*[R^*df(R(b)), R^*dg(R(b))], R(b)\rangle + \langle (1-R^*)[R^*df(R(b)), R^*dg(R(b))], (1-R)b\rangle \\
&\ = \langle [df(R(b)), dg(R(b))], R(b)\rangle = \{f, g\}_{\mathfrak{g}_*^\sigma}(R(b)),
\end{aligned}$$

where we have used (4.12). The above computation proves, independently of Proposition 4.8, that $R : \mathfrak{g}_{*\mathbb{R}} \to \mathfrak{g}_*^\sigma$ is a linear Poisson map. ♦

Next we investigate the case of injective linear Poisson maps.

Proposition 4.10 *Let $\mathfrak{b}_1$ be a Banach space, $(\mathfrak{b}_2, \{\cdot,\cdot\}_2)$ be a Banach Lie-Poisson space, and $\iota : \mathfrak{b}_1 \to \mathfrak{b}_2$ be an injective continuous linear map with closed range. Then $\mathfrak{b}_1$ carries a unique Banach Lie-Poisson structure such that ι is a linear Poisson map if and only if $\ker \iota^*$ is an ideal in the Banach Lie algebra $\mathfrak{b}_2^*$.*

Proof. Assume that $\ker \iota^*$ is an ideal in the Banach Lie algebra $\mathfrak{b}_2^*$. Denote by $[\cdot,\cdot]_2$ the Lie bracket of the Banach Lie algebra $\mathfrak{b}_2^*$. Since $\iota : \mathfrak{b}_1 \to \mathfrak{b}_2$ is an injective linear continuous map, its adjoint $\iota^* : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ is a surjective linear continuous map inducing the Banach space isomorphism $[\iota^*] : \mathfrak{b}_2^*/\ker \iota^* \xrightarrow{\sim} \mathfrak{b}_1^*$. Since $\ker \iota^*$ is an ideal in the Banach Lie algebra $\mathfrak{b}_2^*$, it follows that $\mathfrak{b}_2^*/\ker \iota^*$ is a Banach Lie algebra. The isomorphism $[\iota^*]$ induces a Banach Lie algebra structure $[\cdot,\cdot]_1$ on $\mathfrak{b}_1^*$. The linear map $\iota^* : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ becomes a Banach Lie algebra homomorphism.

For each $x, y \in \mathfrak{b}_2^*$ and each $b \in \mathfrak{b}_1$ we have

$$\begin{aligned}
\left\langle y,\, \iota^{**}\left(\mathrm{ad}^{1*}_{\iota^*(x)} b\right)\right\rangle_2 &= \left\langle \iota^*(y),\, \mathrm{ad}^{1*}_{\iota^*(x)} b\right\rangle_1 = \langle [\iota^*(x), \iota^*(y)]_1, b\rangle_1 \\
&= \langle \iota^*\left([x,y]_2\right), b\rangle_1 = \langle [x,y]_2, \iota(b)\rangle = \left\langle y,\, \mathrm{ad}^{2*}_x \iota(b)\right\rangle
\end{aligned}$$

and hence we get the following identity in $\mathfrak{b}_2^{**}$

$$\iota^{**}\left(\mathrm{ad}^{1*}_{\iota^*(x)} b\right) = \mathrm{ad}^{2*}_x \iota(b) \tag{4.15}$$

for any $x \in \mathfrak{b}_2^*$ and any $b \in \mathfrak{b}_1$.

Let us now prove that

$$\mathrm{ad}^{2*}_{\mathfrak{b}_2^*} \iota(\mathfrak{b}_1) \subset \iota(\mathfrak{b}_1), \tag{4.16}$$

where ad^{2*} denotes the coadjoint action of $\mathfrak{b}_2^*$ on $\mathfrak{b}_2^{**}$. We begin by noticing that $\ker \iota^* = [\iota(\mathfrak{b}_1)]^\circ$, where $[\iota(\mathfrak{b}_1)]^\circ$ is the annihilator of $\iota(\mathfrak{b}_1)$ in $\mathfrak{b}_2^*$. Taking the annihilator of this identity in $\mathfrak{b}_2$ one obtains

$$[\ker \iota^*]^\circ = [\iota(\mathfrak{b}_1)]^{\circ\circ} = \iota(\mathfrak{b}_1), \tag{4.17}$$

where the last equality follows by the closedness of $\iota(\mathfrak{b}_1)$ in $\mathfrak{b}_2$. By the definition of Banach Lie-Poisson spaces, ${\mathrm{ad}^{2*}}_{\mathfrak{b}_2^*}\mathfrak{b}_2 \subset \mathfrak{b}_2$. Since $\ker \iota^*$ is an ideal in $\mathfrak{b}_2^*$ it follows that ${\mathrm{ad}^{2*}}_{\mathfrak{b}_2^*}[\ker \iota^*]^\circ \subset [\ker \iota^*]^\circ$ and thus (4.17) implies (4.16).

By (4.15) and (4.16), we have ${\mathrm{ad}^{2*}}_x\iota(b) \in \iota(\mathfrak{b}_1)$ and thus

$$\iota^{**}\left({\mathrm{ad}^{1*}}_{\iota^*(x)}b\right) \in \iota(\mathfrak{b}_1)$$

for any $x \in \mathfrak{b}_2^*$ and any $b \in \mathfrak{b}_1$. The double adjoint $\iota^{**} : \mathfrak{b}_1^{**} \to \mathfrak{b}_2^{**}$ is a injective continuous linear map (since $\iota^* : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ is a surjective continuous linear map) and ι maps $\mathfrak{b}_1 \subset \mathfrak{b}_1^{**}$ into $\mathfrak{b}_2 \subset \mathfrak{b}_2^{**}$. This shows that ${\mathrm{ad}^{1*}}_{\iota^*(x)}\mathfrak{b}_1 \subset \mathfrak{b}_1$. Applying Theorem 4.2 we conclude that $\mathfrak{b}_1$ is a Banach Lie-Poisson space and that $\iota : \mathfrak{b}_1 \to \mathfrak{b}_2$ is an injective linear Poisson map. Uniqueness of the Poisson structure on $\mathfrak{b}_1$ follows from the injectivity of ι.

Conversely, let us assume that $\mathfrak{b}_1$ is a Banach Lie-Poisson space and that $\iota : \mathfrak{b}_1 \to \mathfrak{b}_2$ is a linear Poisson map. Then $\iota^* : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ is a homomorphism of Banach Lie algebras and therefore its kernel is an ideal in $\mathfrak{b}_2^*$. ■

Proposition 4.10 allows one to introduce a unique Banach Lie-Poisson structure on $\mathfrak{b}_1$ relative to which ι is a linear Poisson map. In analogy to the previous case, this Poisson structure on $\mathfrak{b}_1$ will be said to be the Banach Lie-Poisson structure ***induced*** by the mapping ι. Proposition 5.4 in §5 gives an example of an induced Poisson structure.

Proposition 4.11 *Let $\mathfrak{b}_1$ be a Banach space, $(\mathfrak{b}_2, \{\cdot,\cdot\}_2)$ be a Banach Lie-Poisson space, and $\iota : \mathfrak{b}_1 \to \mathfrak{b}_2$ be an injective continuous linear map with closed range. Then the equality*

$${\mathrm{ad}^{2*}}_{\mathfrak{b}_2^*}\iota(\mathfrak{b}_1) = \iota(\mathfrak{b}_1) \tag{4.18}$$

implies that $\ker \iota^*$ *is an ideal in the Banach Lie algebra $\mathfrak{b}_2^*$ and thus the map $\iota : \mathfrak{b}_1 \to \mathfrak{b}_2$ induces a Banach Lie Poisson structure on $\mathfrak{b}_1$.*

Proof. To show that (4.18) implies that $\ker \iota^*$ is an ideal, we prove first the following equality:

$$\ker \iota^* = \left\{x \in \mathfrak{b}_2^* \mid {\mathrm{ad}^{2*}}_x\iota(\mathfrak{b}_1) = 0\right\}.$$

To see this, let $x, y \in \mathfrak{b}_2^*$ and note that

$$\left\langle y, {\mathrm{ad}^{2*}}_x\iota(\mathfrak{b}_1)\right\rangle = -\left\langle x, {\mathrm{ad}^{2*}}_y\iota(\mathfrak{b}_1)\right\rangle.$$

Thus, ${\mathrm{ad}^{2*}}_x\iota(\mathfrak{b}_1) = 0$ if and only if

$$\left\langle x, {\mathrm{ad}^{2*}}_y\iota(\mathfrak{b}_1)\right\rangle = 0 \quad \text{for all} \quad y \in \mathfrak{b}_2^*,$$

which, by condition (4.18), is equivalent to $0 = \langle x, \iota(\mathfrak{b}_1)\rangle = \langle \iota^* x, \mathfrak{b}_1\rangle$, that is, $\iota^* x = 0$.

Next we prove that

$$\left\{x \in \mathfrak{b}_2^* \mid {\mathrm{ad}^{2*}}_x\iota(\mathfrak{b}_1) = 0\right\}$$

is an ideal. Indeed if x is in this subspace and $y, z \in \mathfrak{b}_2^*$ are arbitrary, then

$$\left\langle z, \operatorname{ad}^{2*}_{[x,y]}\iota(\mathfrak{b}_1)\right\rangle = \left\langle z, \operatorname{ad}^{2*}_{y}\operatorname{ad}^{2*}_{x}\iota(\mathfrak{b}_1)\right\rangle - \left\langle z, \operatorname{ad}^{2*}_{x}\operatorname{ad}^{2*}_{y}\iota(\mathfrak{b}_1)\right\rangle = 0$$

because in the second term $\operatorname{ad}^{2*}_{y}\iota(\mathfrak{b}_1) \subset \iota(\mathfrak{b}_1)$ by condition (4.18) and the element x satisfies $\operatorname{ad}^{2*}_{x}\iota(\mathfrak{b}_1) = 0$.

These two steps show that $\ker \iota^*$ is an ideal in the Banach Lie algebra $\mathfrak{b}_2^*$.

Therefore, if (4.18) holds, by Proposition 4.10, the space $\mathfrak{b}_1$ carries a unique Banach Lie-Poisson structure such that ι is a linear Poisson map. ■

The previous two propositions give an algebraic characterization of linear Poisson maps $\phi : \mathfrak{b}_1 \to \mathfrak{b}_2$ between Banach Lie-Poisson spaces analogous to that from linear algebra. We summarize this in the following theorem.

Theorem 4.12 *The linear continuous map $\phi : \mathfrak{b}_1 \to \mathfrak{b}_2$ between the Banach Lie-Poisson spaces $\mathfrak{b}_1$ and $\mathfrak{b}_2$, such that $\phi(\mathfrak{b}_1)$ is Banach subspace in $\mathfrak{b}_2$, is a linear Poisson map if and only if it has a decomposition $\phi = \iota \circ \pi$, where*

(i) *$\pi : \mathfrak{b}_1 \to \mathfrak{b}$ is a linear continuous surjective map of Banach spaces such that $\operatorname{im} \pi^* \subset \mathfrak{b}_1^*$ is closed with respect to Lie bracket of $\mathfrak{b}_1^*$;*

(ii) *$\iota : \mathfrak{b} \to \mathfrak{b}_2$ is a continuous injective linear map of Banach spaces with closed range such that $\ker \iota^*$ is an ideal in the Banach Lie algebra $\mathfrak{b}_2^*$.*

We will be interested now in finding the properties of an object in $\mathfrak{B}$ that correspond to the condition of being an ideal or a subalgebra of the related object in $\mathfrak{L}_0$. To do this we shall use Propositions 4.8 and 4.10.

Proposition 4.13 *Let $\mathfrak{b} \in \operatorname{Ob}(\mathfrak{B})$ be a Banach Lie-Poisson space and let $\mathfrak{g} \in \operatorname{Ob}(\mathfrak{L}_0)$ be a Banach Lie algebra such that $\mathfrak{b}^* = \mathfrak{g}$. Then:*

(i) *There exists a bijective correspondence between the coinduced Banach Lie-Poisson structures from $\mathfrak{b}$ and the Banach Lie subalgebras of $\mathfrak{g}$. If the surjective continuous linear map $\pi : \mathfrak{b} \to \mathfrak{c}$ coinduces a Banach Lie-Poisson structure on $\mathfrak{c}$, the Banach Lie subalgebra of $\mathfrak{g}$ given by this correspondence is $\pi^*(\mathfrak{c}^*)$.*

   Conversely, if $\mathfrak{k} \subset \mathfrak{g}$ is a Banach Lie subalgebra then the Banach Lie-Poisson space given by this correspondence is $\mathfrak{b}/\mathfrak{k}^\circ$, where $\mathfrak{k}^\circ$ is the annihilator of $\mathfrak{k}$ in $\mathfrak{b}$, and $\pi : \mathfrak{b} \to \mathfrak{b}/\mathfrak{k}^\circ$ is the quotient projection.

(ii) *There exists a bijective correspondence between the induced Banach Lie-Poisson structures in $\mathfrak{b}$ (i.e., the Banach Lie-Poisson subspaces of $\mathfrak{b}$) and the Banach ideals of $\mathfrak{g}$. If the injection $\iota : \mathfrak{c} \to \mathfrak{b}$ with closed range induces a Banach Lie-Poisson structure on $\mathfrak{c}$, then the ideal in $\mathfrak{g}$ given by this correspondence is $\ker \iota^*$.*

   Conversely, if $\mathfrak{i} \subset \mathfrak{g}$ is a Banach ideal, then the Banach Lie- Poisson subspace of $\mathfrak{b}$ given by this correspondence is $\mathfrak{i}^\circ$, where $\mathfrak{i}^\circ$ is the annihilator of $\mathfrak{i}$ in $\mathfrak{b}$ and $\iota : \mathfrak{i}^\circ \to \mathfrak{b}$ is the inclusion.

Proof. (i) If the surjective continuous linear map $\pi : \mathfrak{b} \to \mathfrak{c}$ coinduces a Banach Lie-Poisson structure on $\mathfrak{c}$, Proposition 4.8 states that $\pi^*(\mathfrak{c}^*)$ is a Banach Lie subalgebra of $\mathfrak{g}$. Conversely, if $\mathfrak{k} \subset \mathfrak{g}$ is a Banach Lie subalgebra then $\pi : \mathfrak{b} \to \mathfrak{b}/\mathfrak{k}^\circ$ is a surjective continuous linear map of Banach spaces. Consider the dual map $\pi^* : [\mathfrak{b}/\mathfrak{k}^\circ]^* \to \mathfrak{b}^*$. Since $\mathfrak{b}^* = \mathfrak{g}$ and since $[\mathfrak{b}/\mathfrak{k}^\circ]^* \cong \mathfrak{k}^{\circ\circ} = \mathfrak{k}$, it follows that $\operatorname{im}\pi^*$ is a Banach Lie subalgebra of $\mathfrak{g}$. Therefore, by Proposition 4.8, there is a unique coinduced Banach Lie-Poisson structure on $\mathfrak{b}/\mathfrak{k}^\circ$.

(ii) This is a direct consequence of Proposition 4.10. ■

Consider two Banach Lie-Poisson spaces $(\mathfrak{b}_1, \{\cdot,\cdot\}_1)$ and $(\mathfrak{b}_1, \{\cdot,\cdot\}_2)$. According to Theorem 2.2, the product $(\mathfrak{b}_1 \times \mathfrak{b}_2, \{\cdot,\cdot\}_{12})$ is a Banach Poisson manifold. The Banach space isomorphism $(\mathfrak{b}_1 \times \mathfrak{b}_2)^* \cong \mathfrak{b}_1^* \times \mathfrak{b}_2^*$ and formula (2.3) show that $(\mathfrak{b}_1 \times \mathfrak{b}_2)^*$ is closed under the product Poisson bracket $\{\cdot,\cdot\}_{12}$ which proves that $(\mathfrak{b}_1 \times \mathfrak{b}_2, \{\cdot,\cdot\}_{12})$ is also a Banach Lie-Poisson space.

As opposed to the general case of Poisson manifolds, the inclusions $i_k : \mathfrak{b}_k \to \mathfrak{b}_1 \times \mathfrak{b}_2$, $k = 1, 2$, defined by $i_1(b_1) := (b_1, 0)$ and $i_2(b_2) := (0, b_2)$ are Poisson maps. Indeed, by (2.3), we get

$$(i_1^*\{f,g\}_{12})(b_1) = \{f,g\}_{12}(b_1,0) = \{f_0, g_0\}_1(b_1) + \{f_{b_1}, g_{b_1}\}_2(0) = \{i_1^* f, i_1^* g\}_1(b_1),$$

where the term $\{f_{b_1}, g_{b_1}\}_2(0)$ vanishes because in this case the Poisson bracket is linear. The proof for i_2 is similar.

Regarding the product construction, the following question arises naturally. When does a Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot,\cdot\})$ allow a decomposition as a product of two Banach Lie-Poisson spaces $(\mathfrak{b}_1, \{\cdot,\cdot\}_1)$ and $(\mathfrak{b}_2, \{\cdot,\cdot\}_2)$?

In the category of Banach spaces this means that there is a splitting, i.e., $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$, for two Banach subspaces $\mathfrak{b}_1$ and $\mathfrak{b}_2$, which is equivalent to $\mathfrak{b} \cong \mathfrak{b}_1 \times \mathfrak{b}_2$. In view of the previous properties of the product of two Banach Lie-Poisson spaces, this suggests the following definition.

Definition 4.14 *Let $(\mathfrak{b}, \{\,,\})$ be a Banach Lie-Poisson space. The splitting $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ into two Banach subspaces $\mathfrak{b}_1$ and $\mathfrak{b}_2$ is called a* ***Poisson splitting*** *if*

(i) *$\mathfrak{b}_1$ and $\mathfrak{b}_2$ are Banach Lie Poisson spaces whose brackets shall be denoted by $\{\,,\}_1$ and $\{\,,\}_2$ respectively;*

(ii) *the projections $\pi_k : \mathfrak{b} \to \mathfrak{b}_k$ and the inclusions $i_k : \mathfrak{b}_k \to \mathfrak{b}$, $k = 1, 2$, consistent with the above splitting, are Poisson maps;*

(iii) *if $f \in \pi_1^*(C^\infty(P_1))$ and $g \in \pi_2^*(C^\infty(P_2))$, then $\{f, g\} = 0$.*

The following proposition gives equivalent conditions for the existence of Poisson splittings.

Proposition 4.15 *The following conditions are equivalent:*

(i) *the Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot,\cdot\})$ admits a Poisson splitting into the two Banach Lie-Poisson subspaces $(\mathfrak{b}_1, \{\cdot,\cdot\}_1)$ and $(\mathfrak{b}_2, \{\cdot,\cdot\}_2)$;*

(ii) *the Banach Lie-Poisson space* $(\mathfrak{b}, \{\cdot,\cdot\})$ *is isomorphic to the product Banach Lie-Poisson space* $(\mathfrak{b}_1 \times \mathfrak{b}_2, \{\cdot,\cdot\}_{12})$*;*

(iii) *the components* $\mathfrak{b}_1^*$ *and* $\mathfrak{b}_2^*$ *of the dual splitting* $\mathfrak{b}^* = \mathfrak{b}_1^* \oplus \mathfrak{b}_2^*$ *are ideals of the Banach Lie algebra* $\mathfrak{b}^*$*, where one identifies* $\mathfrak{b}_1^*$ *and* $\mathfrak{b}_2^*$ *with the annihilators of* $\mathfrak{b}_2$ *and* $\mathfrak{b}_1$ *in* $\mathfrak{b}^*$ *respectively.*

Proof. The equivalence of (i) and (ii) is a direct consequence of Theorem 2.2 and the subsequent comments. Conditions (i) and (iii) are equivalent by applying Propositions 4.8 and 4.10. ■

In this section we established an equivalence between the subcategory on Banach Lie algebras admitting a unique predual and the subcategory of Banach Lie-Poisson spaces that are unique preduals of their dual. The statements proved above give examples how this equivalence can be used in the study of these two categories. For example, the simplicity of a Banach Lie algebra from the category $\mathfrak{L}_{0u}$ is equivalent to the non existence of Banach Lie-Poisson subspaces of its predual.

5 Preduals of W^*-algebras as Banach Lie-Poisson spaces

In this section we shall consider the important class of Banach Lie-Poisson spaces related to the category of W^*-algebras.

Recall that a W^*-algebra is a C^*-algebra $\mathfrak{m}$ which posses a predual Banach space $\mathfrak{m}_*$, i.e. $\mathfrak{m} = (\mathfrak{m}_*)^*$; this predual is unique (Sakai [S]). Since $\mathfrak{m}^* = (\mathfrak{m}_*)^{**}$, the predual Banach space $\mathfrak{m}_*$ canonically embeds into the Banach space $\mathfrak{m}^*$ dual to $\mathfrak{m}$. Thus we shall always think of $\mathfrak{m}_*$ as a Banach subspace of $\mathfrak{m}^*$. The existence of $\mathfrak{m}_*$ allows the introduction of the $\sigma(\mathfrak{m}, \mathfrak{m}_*)$-topology on the W^*-algebra $\mathfrak{m}$; for simplicity we shall call it the σ-topology in the sequel. Recall that a net $\{x_\alpha\}_{\alpha\in A} \subset \mathfrak{m}$ converges to $x \in \mathfrak{m}$ in the σ-topology if, by definition, $\lim_{\alpha\in A}\langle x_\alpha, b\rangle = \langle x, b\rangle$ for all $b \in \mathfrak{m}_*$. The σ-topology is Hausdorff. Alaoglu's theorem states that the unit ball of $\mathfrak{m}$ is compact in the σ-topology. One can characterize the predual space $\mathfrak{m}_*$ as the subspace of $\mathfrak{m}^*$ consisting of all σ-continuous linear functionals, see Sakai [S]. A theorem of Diximier (see Sakai [S], §1.13) states that a positive linear functional $\nu \in \mathfrak{m}^*$ is σ-continuous if and only if it is normal, i.e. it satisfies

$$\langle \nu,\, l.u.b.x_\alpha\rangle = l.u.b.\langle \nu,\, x_\alpha\rangle$$

for every uniformly bounded increasing direct set $\{x_\alpha\}$ of positive elements in $\mathfrak{m}$. The normality is determined by the ordering on $\mathfrak{m}$ only. So, the predual space $\mathfrak{m}_*$ and thus the pairing

$$\mathfrak{m}_* \times \mathfrak{m} \ni (\nu, x) \mapsto \langle x, \nu\rangle := x(\nu) \in \mathbb{C}$$

are defined by the algebraic structure of $\mathfrak{m}$ in a unique way.

Theorem 5.1 *Let* $\mathfrak{m}$ *be a* W^**-algebra and* $\mathfrak{m}_*$ *be the predual of* $\mathfrak{m}$*. Then* $\mathfrak{m}_*$ *is a Banach Lie-Poisson space with the Poisson bracket* $\{f, g\}$ *of* $f, g \in C^\infty(\mathfrak{m}_*)$ *given by* (4.2)*. The Hamiltonian vector field* X_f *defined by the smooth function* $f \in C^\infty(\mathfrak{m}_*)$ *is given by* (4.3)*.*

Proof. We shall prove the theorem by checking the conditions of Theorem 4.2. Since the W^*-algebra $\mathfrak{m}$ is an associative Banach algebra we can define the Lie bracket in $\mathfrak{m}$ as the commutator

$$[x,y] = xy - yx$$

of $x, y \in \mathfrak{m}$. Left and right multiplication by $a \in \mathfrak{m}$ define uniformly and σ-continuous maps

$$L_a : \mathfrak{m} \ni x \mapsto ax \in \mathfrak{m}$$

$$R_a : \mathfrak{m} \ni x \mapsto xa \in \mathfrak{m},$$

see Sakai [S]. Let $L_a^* : \mathfrak{m}^* \to \mathfrak{m}^*$ and $R_a^* : \mathfrak{m}^* \to \mathfrak{m}^*$ denote the dual maps of L_a and R_a respectively. If $v \in \mathfrak{m}_*$, then $L_a^*(v)$ and $R_a^*(v)$ are σ-continuous functionals and therefore, by the characterization of the predual $\mathfrak{m}_*$ as the subspace of σ-continuous functionals in $\mathfrak{m}^*$, it follows that $L_a^*(v), R_a^*(v) \in \mathfrak{m}_*$. One has $\mathrm{ad}_a = [a, \cdot] = L_a - R_a$ and thus, $\mathrm{ad}_a^* = L_a^* - R_a^*$. We conclude from the above that $\mathfrak{m}$ is a Banach Lie algebra and $\mathrm{ad}_a^* \mathfrak{m}_* \subset \mathfrak{m}_*$ for each $a \in \mathfrak{m}$, which are the conditions of Theorem 4.2. ■

Corollary 5.2 *Let $\mathfrak{a}$ be a C^*-algebra. Then its dual $\mathfrak{a}^*$ is a Banach Lie-Poisson space.*

Proof. The bidual $\mathfrak{a}^{**}$ is isomorphic to the universal enveloping von Neumann algebra of $\mathfrak{a}$ and by the canonical inclusion $\mathfrak{a} \hookrightarrow \mathfrak{a}^{**}$ the C^*-algebra $\mathfrak{a}$ can be considered as a C^*-subalgebra of $\mathfrak{a}^{**}$ (see Sakai [S] §17.1, or Takesaki [T2]). Since $\mathfrak{a}^*$ is predual to $\mathfrak{a}^{**}$, Theorem 5.1 guarantees that it is a Banach Lie-Poisson space. ■

Any σ-closed C^*-subalgebra $\mathfrak{n} \subset \mathfrak{m}$ has the predual given by $\mathfrak{m}_*/\mathfrak{n}^\circ$, where $\mathfrak{n}^\circ$ is the annihilator of $\mathfrak{n}$ in $\mathfrak{m}_*$. Thus $\mathfrak{n}$ is a Banach Lie subalgebra of $\mathfrak{m}$ admitting a predual. By Proposition 4.13 (i), the quotient map $\pi : \mathfrak{m}_* \to \mathfrak{m}_*/\mathfrak{n}^\circ$ coinduces a Lie-Poisson structure on the quotient Banach space $\mathfrak{m}_*/\mathfrak{n}^\circ$. Therefore, there is a bijective correspondence between W^*-subalgebras of $\mathfrak{m}$ and a subclass of Banach Lie-Poisson spaces coinduced from $\mathfrak{m}_*$. It would be interesting to characterize this subclass in Poisson geometrical terms.

If $\mathfrak{n}$ is a hereditary subalgebra of $\mathfrak{m}$, then there exists a projector $p \in \mathfrak{m}$ such that $\mathfrak{n} = \operatorname{im} P$ (see Murphy [M]), where the map $P : \mathfrak{m} \to \mathfrak{m}$ is defined by

$$P(x) := pxp. \tag{5.1}$$

The map P is a σ-continuous projector with $\|P\| = 1$. Thus its dual $P^* : \mathfrak{m}^* \to \mathfrak{m}^*$ preserves $\mathfrak{m}_*$. We therefore conclude that $P_* := P^*|_{\mathfrak{m}_*} : \mathfrak{m}_* \to \mathfrak{m}_*$ is a projector with $\|P_*\| = 1$. Thus there is a splitting $\mathfrak{m}_* = \operatorname{im} P_* \oplus \ker P_*$ which allows one to canonically identify $\mathfrak{m}_*/\mathfrak{n}^\circ$ with $\operatorname{im} P_*$ since $\ker P_* = \mathfrak{n}^\circ$.

Proposition 5.3 *Let $\mathfrak{n}$ be a hereditary W^*-subalgebra of $\mathfrak{m}$. Then the projector $P_* : \mathfrak{m}_* \to \mathfrak{m}_*$ coinduces a Banach Lie-Poisson structure on $\operatorname{im} P_*$.*

So, a necessary condition for $\mathfrak{n}$ to be a hereditary subalgebra is that the Lie-Poisson structure on $\mathfrak{m}_*$ coinduces one on a Banach subspace of $\mathfrak{m}_*$.

Let us recall that $\mathfrak{m}_*$ and $\mathfrak{m}^*$ have natural Banach Lie-Poisson structures according to Theorem 5.1 and Corollary 5.2 respectively.

Proposition 5.4 *Let $\mathfrak{m}_*$ be the predual of the W^*-algebra $\mathfrak{m}$ and $\iota : \mathfrak{m}_* \hookrightarrow \mathfrak{m}^*$ be the canonical inclusion. Then ι is an injective linear Poisson map and the Poisson structure induced by it from $\mathfrak{m}^*$ coincides with the original Lie-Poisson structure on $\mathfrak{m}_*$.*

Proof. Since $\|\iota(b)\| = \|b\|$ for $b \in \mathfrak{m}_*$, the range of $\iota : \mathfrak{m}_* \hookrightarrow \mathfrak{m}^*$ is closed in $\mathfrak{m}^*$. The dual map $\iota^* : \mathfrak{m}^{**} \to \mathfrak{m} = (\mathfrak{m}_*)^*$ is a projection of the universal enveloping W^*-algebra $\mathfrak{m}^{**}$ onto $\mathfrak{m}$ of norm one. One has the equality $\ker \iota^* = (\mathfrak{m}_*)^\circ$, where $(\mathfrak{m}_*)^\circ$ is the annihilator of $\mathfrak{m}_*$ in $\mathfrak{m}^{**}$. In addition, $L_x^*\mathfrak{m}_* \subset \mathfrak{m}_*$ and $R_x^*\mathfrak{m}_* \subset \mathfrak{m}_*$ for any $x \in \mathfrak{m}^{**}$ (see Sakai [S]). Thus, $\ker \iota^*$ is a $\sigma(\mathfrak{m}^{**}, \mathfrak{m}^*)$-closed ideal of $\mathfrak{m}^{**}$. Therefore, $\ker \iota^*$ is also an ideal in the Banach Lie algebra structure of $\mathfrak{m}^{**}$ defined by $[x, y] = xy - yx$. Proposition 4.10 implies that ι induces a Banach Lie-Poisson structure on $\mathfrak{m}_*$. Since $\mathfrak{m}^{**}/\ker \iota^*$ is isomorphic to $\mathfrak{m}$, this structure coincides with the original Banach Lie-Poisson structure of $\mathfrak{m}_*$ defined by Theorem 5.1. ■

Consider now a σ-closed two sided ideal $\mathfrak{i} \subset \mathfrak{m}$. Then it equals $\operatorname{im} P$, where P is given by (5.1) for p a central projector in $\mathfrak{m}$ (see Sakai [S] §1.10). The projector $P_\perp : \mathfrak{m} \to \mathfrak{m}$ defined by $P_\perp(x) := (1-p)x(1-p)$ is also a σ-continuous linear map with $\|P_\perp\| = 1$ and projects $\mathfrak{m}$ onto a σ-closed two sided ideal $\mathfrak{i}_\perp$. Since $P + P_\perp = I$, we have the splitting

$$\mathfrak{m} = \mathfrak{i} \oplus \mathfrak{i}_\perp \tag{5.2}$$

of $\mathfrak{m}$ into two sided ideals. The decomposition (5.2) is also a splitting into ideals in the category of Banach Lie algebras. By Proposition 4.15, the direct sum (5.2) induces a Poisson splitting

$$\mathfrak{m}_* = \mathfrak{i}^\circ \oplus \mathfrak{i}_\perp^\circ, \tag{5.3}$$

where $\mathfrak{i}^\circ$ and $\mathfrak{i}_\perp^\circ$ are the annihilators in $\mathfrak{m}_*$ of $\mathfrak{i}$ and $\mathfrak{i}_\perp$ respectively.

As a special case, one can consider the universal enveloping W^*-algebra $\mathfrak{m}^{**}$ of the W^*-algebra $\mathfrak{m}$ with predual $\mathfrak{m}_*$. Then $\mathfrak{m}^*$ is the predual to $\mathfrak{m}^{**}$ and $\mathfrak{m}_* \subset \mathfrak{m}^*$ is a $L_{\mathfrak{m}^{**}}^*$ and $R_{\mathfrak{m}^{**}}^*$ invariant Banach subspace. In this case, the splitting (5.3) gives the Poisson splitting

$$\mathfrak{m}^* = \mathfrak{m}_* \oplus \mathfrak{m}_*^\perp$$

of $\mathfrak{m}^*$ into the normal and singular functionals (see Takesaki [T2] for this terminology).

In order to illustrate Theorem 5.1 let us take a complex Hilbert space $\mathcal{M}$. By $L^1(\mathcal{M})$, $L^2(\mathcal{M})$, and $L^\infty(\mathcal{M})$ we shall denote the involutive Banach algebras of the trace class operators, the Hilbert-Schmidt operators, and the bounded operators on $\mathcal{M}$ respectively. Recall that $L^1(\mathcal{M})$ and $L^2(\mathcal{M})$ are self adjoint ideals in $L^\infty(\mathcal{M})$. Let $\mathcal{K}(\mathcal{M}) \subset L^\infty(\mathcal{M})$ denote the ideal of all compact operators on $\mathcal{M}$. Then

$$L^1(\mathcal{M}) \subset L^2(\mathcal{M}) \subset \mathcal{K}(\mathcal{M}) \subset L^\infty(\mathcal{M}) \tag{5.4}$$

and the following remarkable dualities hold (see e.g. Murphy [M]):

$$\mathcal{K}(\mathcal{M})^* \cong L^1(\mathcal{M}), \quad L^2(\mathcal{M})^* \cong L^2(\mathcal{M}), \quad \text{and} \quad L^1(\mathcal{M})^* \cong L^\infty(\mathcal{M}). \tag{5.5}$$

These are implemented by the strongly non-degenerate pairing

$$\langle x, \rho \rangle = \operatorname{tr}(x\rho) \tag{5.6}$$

where $x \in L^1(\mathcal{M})$, $\rho \in \mathcal{K}(\mathcal{M})$ for the first isomorphism, $\rho, x \in L^2(\mathcal{M})$ for the second isomorphism and $x \in L^\infty(\mathcal{M})$, $\rho \in L^1(\mathcal{M})$ for the third isomorphism. The isomorphism $L^1(\mathcal{M})^* \cong L^\infty(\mathcal{M})$ gives the crucial example of the W^*-algebra of bounded operators on the complex Hilbert space $\mathcal{M}$. So, we recover the result of Bona [2000] as a corollary of Theorem 5.1.

Corollary 5.5 *The Banach space $L^1(\mathcal{M})$ of trace class operators on the Hilbert space $\mathcal{M}$ is a Banach Lie-Poisson space relative to the Poisson bracket given by*

$$\{f, g\}(\rho) = \operatorname{tr}([Df(\rho), Dg(\rho)]\rho) \tag{5.7}$$

where $\rho \in L^1(\mathcal{M})$ and the bracket $[Df(\rho), Dg(\rho)]$ denotes the commutator of the bounded operators $Df(\rho), Dg(\rho) \in L^\infty(\mathcal{M}) \cong L^1(\mathcal{M})^*$. The Hamiltonian vector field associated to $f \in C^\infty(L^1(\mathcal{M}))$ is given by

$$X_f(\rho) = [Df(\rho), \rho]. \tag{5.8}$$

Proof. Formula (5.7) follows from (4.2) by using (5.6) for the pairing between $L^1(\mathcal{M})$ and $L^\infty(\mathcal{M})$. In order to obtain (5.8) from (4.3), let us notice that

$$\langle y, -\operatorname{ad}_x^* \rho\rangle = -\langle [x, y], \rho\rangle = -\operatorname{tr}([x, y]\rho) = \operatorname{tr}(y[x, \rho]) = \langle y, [x, \rho]\rangle \tag{5.9}$$

for $\rho \in L^1(\mathcal{M})$ and $x, y \in L^\infty(\mathcal{M})$. Thus $-\operatorname{ad}_x^* \rho = [x, \rho] \in L^1(\mathcal{M})$, since $L^1(\mathcal{M})$ is an ideal in $L^\infty(\mathcal{M})$. (We have identified here $\{\rho\} \times L^1(\mathcal{M})$ with the tangent space $T_\rho L^1(\mathcal{M})$.) ■

The other two isomorphisms in (5.5) also give rise to Banach Lie-Poisson spaces, but as a corollary of Theorem 4.2; Theorem 5.1 cannot be applied because $L^2(\mathcal{M})$ and $L^1(\mathcal{M})$ are not W^*-algebras.

Example 5.6 The Banach space $L^2(\mathcal{M})$ of Hilbert-Schmidt operators on the Hilbert space $\mathcal{M}$ is a Banach Lie-Poisson space. Indeed, we use the isomorphism $L^2(\mathcal{M})^* \cong L^2(\mathcal{M})$ given by the pairing (5.6) and notice that $L^2(\mathcal{M})$ is a reflexive (that is, $L^2(\mathcal{M})^{**} = L^2(\mathcal{M})$) Banach algebra. The formulas for the Poisson bracket and for the Hamiltonian vector field are (5.7) and (5.8) respectively with $\rho \in L^2(\mathcal{M})$. ♦

Example 5.7 The Banach space $\mathcal{K}(\mathcal{M})$ of compact operators on the Hilbert space $\mathcal{M}$, as a predual of $L^1(\mathcal{M})$, is a Banach Lie-Poisson space. The proof is identical to that of Corollary 5.5. The formulas for the Poisson bracket and for the Hamiltonian vector field are (5.7) and (5.8) respectively with $\rho \in \mathcal{K}(\mathcal{M})$. ♦

Example 5.8 Let $L^\infty(M, \mu)$ be the W^*-algebra of all essentially bounded μ-locally measurable functions on a localizable measure space M. Then its predual is the Banach space $L^1(M, \mu)$ of all μ-integrable functions on M. Since $L^\infty(M, \mu)$ is commutative, the Banach Lie-Poisson structure on $L^1(M, \mu)$ is trivial, that is, $\{f, g\} = 0$ for all $f, g \in C^\infty(L^1(M, \mu))$.

However, one can take the W^*-algebra tensor product $L^\infty(M, \mu)\bar{\otimes}\mathfrak{m}$, where $\mathfrak{m}$ is a W^*-algebra with predual $\mathfrak{m}_*$. Then (see, e.g. Sakai [S], §1.22) $L^\infty(M, \mu)\bar{\otimes}\mathfrak{m}$ is isomorphic with the Banach space $L^\infty(M, \mu, \mathfrak{m})$ of all $\mathfrak{m}$-valued essentially bounded weakly *

μ-locally measurable functions on M. Moreover, $L^\infty(M,\mu,\mathfrak{m})$ is a W^*-algebra under pointwise multiplication and its predual is the Banach space $L^1(M,\mu,\mathfrak{m}_*)$ of all $\mathfrak{m}_*$-valued Bochner μ-integrable functions on M. For details see Sakai [S], Takesaki [T2], or Bourbaki [Bo1] §2.6. The duality pairing between $b \in L^1(M,\mu,\mathfrak{m}_*)$ and $x \in L^\infty(M,\mu,\mathfrak{m})$ is given by

$$\langle b, x\rangle = \int_M \langle b(m), x(m)\rangle_{\mathfrak{m}} d\mu(m), \tag{5.10}$$

where $\langle\cdot,\cdot\rangle_{\mathfrak{m}}$ is the duality pairing between $\mathfrak{m}_*$ and $\mathfrak{m}$. Thus, by Theorem 5.1 and formula (4.2) the Lie-Poisson bracket of $f, g \in C^\infty(L^1(M,\mu,\mathfrak{m}_*))$ is given by

$$\{f,g\}(b) = \int_M \left\langle b(m), \left[\frac{\delta f}{\delta b}(m), \frac{\delta g}{\delta b}(m)\right]\right\rangle_{\mathfrak{m}} d\mu(m), \tag{5.11}$$

where $\delta f/\delta b, \delta g/\delta b \in L^\infty(M,\mu,\mathfrak{m})$ are the representatives via the paring (5.10) of the Fréchet derivatives $Df(b)$ and $Dg(b) \in L^1(M,\mu,\mathfrak{m}_*)^*$ respectively.

Applying to $L^1(M,\mu,\mathfrak{m}_*)$ the quantum reduction procedure (see Section 6), one obtains the Banach Lie-Poisson space $L^1(M,\mu,\mathfrak{g}_*)$, where $\mathfrak{g}_*$ is the predual space of the reduced Banach Lie algebra $\mathfrak{g} = R^*(\mathfrak{m})$. In the finite dimensional case, for example when $\mathfrak{m} = \mathfrak{gl}(N,\mathbb{C})$ and M is a smooth manifold, we will consider the Banach Lie algebra $L^\infty(M,\mu,\mathfrak{g})$ as the Lie algebra of the current group $C^\infty(M,G)$, where G is a Lie group with Lie algebra $\mathfrak{g}$ and the group structure on $C^\infty(M,G)$ is defined by pointwise multiplication of maps. Usually the Lie algebra of $C^\infty(M,G)$ is taken to be $C^\infty(M,\mathfrak{g})$ (see, e.g. Kirillov [K]); in our approach we shall work with the L^∞ completion of this Lie algebra. For $M = S^1$ one has the loop group case. So, we could consider the Banach Lie-Poisson space $L^1(M,\mu,\mathfrak{g}^*)$ with the bracket

$$\{f,g\}(\alpha) = \int_M C^i_{jk}\alpha_i(m)\frac{\delta f}{\delta \alpha_j}(m)\frac{\delta f}{\delta \alpha_k}(m) d\mu(m) \tag{5.12}$$

as one related to the current group. In order to clarify (5.12), let us mention that, since $\mathfrak{g}^{**} = \mathfrak{g}$, we identified $\mathfrak{g}_*$ with $\mathfrak{g}^*$. The scalar functions $\alpha_1,\ldots,\alpha_s$, where $s = \dim\mathfrak{g}$, denote the components of $\alpha \in L^1(M,\mu,\mathfrak{g})$ in a basis of $\mathfrak{g}^*$ dual to a given basis of $\mathfrak{g}$ relative to which the structure constants C^i_{jk}, $i,j,k = 1,\ldots,s$ are determined. ♦

Let us now discuss the realifications $\mathfrak{m}_{\mathbb{R}}$ and $\mathfrak{m}_{*\mathbb{R}}$ of the W^*-algebras $\mathfrak{m}$ and its predual $\mathfrak{m}_*$. As was mentioned in §4, $\mathfrak{m}_{*\mathbb{R}}$ has a real Banach Lie-Poisson structure. For a fixed Hermitian element $\eta \in \mathfrak{m}$ satisfying $\eta^2 = 1$, one defines the involutions

$$\sigma(b) = -\eta b^*\eta =: -\operatorname{Ad}^*_\eta(b^*), \qquad b \in \mathfrak{m}_{*\mathbb{R}} \tag{5.13}$$

$$\sigma^*(x) = -\eta x^*\eta =: -\operatorname{Ad}_\eta(x^*), \qquad x \in \mathfrak{m}_{\mathbb{R}} \tag{5.14}$$

in the sense of Example 4.9, i.e., they satisfy conditions (i), (ii), and (iii) given there. To check them, one uses the defining identity for the conjugation in the predual $\mathfrak{m}_*$:

$$\overline{\langle x, b\rangle} = \langle x^*, b^*\rangle, \tag{5.15}$$

where x^* and b^* are the conjugates of $x \in \mathfrak{m}$ and $b \in \mathfrak{m}_*$ respectively. The real Banach Lie algebra $\mathfrak{m}^\sigma := \{x \in \mathfrak{m} \mid \sigma^* x = x\} = \{x \in \mathfrak{m} \mid \eta x^* + x\eta = 0\}$ has underlying Banach Lie group

$$U(\mathfrak{m}, \eta) := \{g \in \mathfrak{m} \mid g^* \eta g = \eta\}$$

consisting of the set of pseudounitary elements (see Bourbaki [Bo3], Chapter 3, §3.10, Proposition 37). For $\eta = 1$ one obtains the group of unitary elements of $\mathfrak{m}$. From the considerations presented in Example 4.9, one can conclude that $\mathfrak{m}_*^\sigma := \{b \in \mathfrak{m}_* \mid \sigma(b) = b\}$ has the real Banach Lie-Poisson structure coinduced from $\mathfrak{m}_\mathbb{R}$ by the projector $R = (id + \sigma)/2$. This structure is given by (4.14) and is $\mathrm{Ad}^*_{U(\mathfrak{m},\eta)}$-invariant. In §7, we will discuss the orbits of this action.

The above more general constructions are of course valid if one considers the special case $\mathfrak{m} = L^\infty(\mathcal{M})$ and $\mathfrak{m}_* = L^1(\mathcal{M})$.

As we have seen, Poisson geometry naturally arises in the the theory of operator algebras. The links between these theories established above show the importance of the fact that the category of W^*-algebras can be considered as a subcategory of the category of Banach Lie-Poisson spaces. Finally, let us mention that Poisson structures that are fundamental for classical phase spaces appear in a natural way on quantum phase spaces, i.e. duals to C^*-algebras.

6 Quantum reduction

Recall that $L^1(\mathcal{M})$ contains the subset of mixed states ρ of the quantum mechanical physical system, i.e., $\rho^* = \rho \geq 0$ and $\operatorname{tr} \rho = \|\rho\|_1 = 1$. If the system under consideration is an isolated quantum system, its dynamics is reversible and is described by the Liouville-von Neumann equation

$$\dot{\rho} = [H, \rho] \tag{6.1}$$

which is a Hamiltonian equation on $(L^1(\mathcal{M}), \{\cdot,\cdot\})$ with Hamiltonian $\operatorname{tr}(H\rho)$. For simplicity, let us assume that $H^* = H \in L^\infty(\mathcal{M})$ is a given (ρ-independent) operator. Therefore the Schrödinger flow $U(t) = e^{itH}$ is a Poisson flow on the Banach Lie-Poisson space $(L^1(\mathcal{M}), \{\cdot,\cdot\})$.

Let us now apply a measurement operation to the system corresponding to the discrete orthonormal decomposition of the unit

$$P_n P_m = \delta_{nm} P_n, \qquad \sum_{n=1}^{\infty} P_n = 1. \tag{6.2}$$

For example, this is the case when one measures the physical quantity given by the operator $X = \sum_{n=1}^{\infty} x_n P_n$, with $x_n \in \mathbb{R}$. Then, according to the well known von Neumann projection postulate, the density operator ρ of the state before measurement is transformed by the measurement to the density operator $R(\rho)$ given by

$$R(\rho) := \sum_{n=1}^{\infty} P_n \rho P_n \tag{6.3}$$

Proposition 6.1 *The measurement operator $R : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ has the following properties:*

(i) *R is a continuous norm one projector, i.e., $R^2 = R$ and $\|R\| = 1$;*

(ii) *it preserves the space of states, i.e., if $\rho^* = \rho > 0$, then $R(\rho)^* = R(\rho^*) = R(\rho) > 0$;*

(iii) *the range $\operatorname{im} R^*$ of its dual $R^* : L^\infty(\mathcal{M}) \to L^\infty(\mathcal{M})$ is a Banach Lie subalgebra of $L^\infty(\mathcal{M})$.*

Proof. Using the natural pairing between $L^1(\mathcal{M})$ and $L^\infty(\mathcal{M})$ given by the trace of the product, it follows that R^* is given by

$$R^*(X) := \sum_{n=1}^{\infty} P_n X P_n \tag{6.4}$$

for $X \in L^\infty(\mathcal{M})$. Then, for $v \in \mathcal{M}$ one concludes

$$\|R^*(X)v\|^2 = \sum_{n=1}^{\infty} \|P_n X P_n v\|^2 \leq \|X\|^2 \sum_{n=1}^{\infty} \|P_n v\|^2 = \|X\|^2 \|v\|^2$$

which proves that $\|R^*\| \leq 1$. Since $\|R^*\| = \|R^{*2}\| \leq \|R^*\|^2$ it follows that $\|R^*\| = 1$. Now, using the defining identity $\operatorname{Tr} R(\rho)X = \operatorname{Tr} \rho R^*(X)$ of R^* it follows that $\|R\| = 1$. This proves (i). Property (ii) follows directly from (6.4). Finally, in order to prove (iii) it is enough to remark that

$$R^*(X)R^*(Y) = R^*(R^*(X)R^*(Y)). \qquad \blacksquare \tag{6.5}$$

We conclude from Propositions 6.1 and 4.8 that the quantum measurement procedure gives a Poisson projection $R : L^1(\mathcal{M}) \to \operatorname{im} R$ of $L^1(\mathcal{M})$ on the Banach subspace $\operatorname{im} R = \ker(1 - R)$ endowed with the Poisson bracket $\{\cdot, \cdot\}_{\operatorname{im} R}$ coinduced from $L^1(\mathcal{M})$. Clearly, opposite to the U-procedure, i.e., the unitary time evolution $U(t)$, $t \in \mathbb{R}$, the R-procedure is not reversible. However, both the U-procedure and the R-procedure share an essential common feature: they are linear Poisson maps.

After this physical introduction, let us now come back to the case when the Banach Lie-Poisson space is the predual space $\mathfrak{m}_*$ of a W^*-algebra $\mathfrak{m}$. In the theory of quantum physical systems (including statistical physics) the W^*-algebra is the algebra of observables and the norm one positive elements of $\mathfrak{m}_* \subset \mathfrak{m}^*$ are the normal states of the considered system, see e.g. Bratteli and Robinson [B-R1, B-R2], or Emch [Em].

The norm one map $E : \mathfrak{m} \to \mathfrak{m}$ which is idempotent ($E^2 = E$) and maps $\mathfrak{m}$ onto a C^*-subalgebra $\mathfrak{n}$ is called ***conditional expectation***. If E is σ-continuous then $\mathfrak{n}$ is a W^*-subalgebra of $\mathfrak{m}$. In that case, the adjoint map $E^* : \mathfrak{m}^* \to \mathfrak{m}^*$ preserves $\mathfrak{m}_* \subset \mathfrak{m}^*$ and maps $\mathfrak{m}_*$ onto $\mathfrak{n}_*$. The conditional expectation is said to be ***compatible with the state*** $\mu \in \mathfrak{m}_*$ if $E^*(\mu) = \mu$.

The concept of conditional expectation comes from probability theory where it is very important in martingale theory. The definition of the conditional expectation as the linear map $E : \mathfrak{m} \to \mathfrak{m}$ on the W^*-algebra $\mathfrak{m}$ with the properties mentioned above is the generalization of the conditional expectation concept in non-commutative probability theory. The role of conditional expectation in the theory of quantum measurement theory and in quantum statistical physics and their remarkable mathematical properties was elucidated in many remarkable publications such as Takesaki [T1], Accardi, Frigerio, and Gorini [A-F-G], and Accardi and von Waldenfels [A-vW]. See Holevo [H] for an extended list of references to publications concerning conditional expectations.

Resuming, we see that the restriction $R := E^*|_{\mathfrak{m}_*} : \mathfrak{m}_* \to \mathfrak{m}_*$ of the map dual to a conditional expectation $E : \mathfrak{m} \to \mathfrak{m}$ is a continuous projector. Since $\operatorname{im} R^* = \operatorname{im} E = \mathfrak{n}$, the range of the projector $R^* : \mathfrak{m} \to \mathfrak{m}$ is a Banach Lie subalgebra $(\mathfrak{n}, [\cdot,\cdot])$ of $(\mathfrak{m}, [\cdot,\cdot])$. So, like in the case of the measurement map (6.3), one can apply Proposition 4.8 in order to coinduce a Banach Lie-Poisson structure on $\operatorname{im} R$.

Motivated by the above two examples, we introduce the following definition.

Definition 6.2 *A* ***quantum reduction map*** *is a continuous projector $R : \mathfrak{b} \to \mathfrak{b}$ on a Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot,\cdot\})$ such that the range $\operatorname{im} R^*$ of the dual map $R^* : \mathfrak{b}^* \to \mathfrak{b}^*$ is a Banach Lie subalgebra of $\mathfrak{b}^*$.*

This immediately implies that R coinduces a Poisson structure on $\operatorname{im} R$ (see Proposition 4.8) and, in particular, $R : (\mathfrak{b}, \{\cdot,\cdot\}) \to (\operatorname{im} R, \{\cdot,\cdot\}_{\operatorname{im} R})$ is a Poisson map.

Let us now give some important examples of quantum reduction.

Example 6.3 Every self-adjoint projector p in the W^*-algebra $\mathfrak{m}$ defines a uniformly and σ-continuous projector

$$\mathfrak{m} \ni x \mapsto P(x) := pxp \in \mathfrak{m} \tag{6.6}$$

of $\mathfrak{m}$, see Sakai [S] or Takesaki [T2]. Let $P^* : \mathfrak{m}^* \to \mathfrak{m}^*$ be the projector dual to P, i.e.

$$\langle P^*\mu, x\rangle = \langle \mu, Px\rangle$$

for any $\mu \in \mathfrak{m}^*$ and $x \in \mathfrak{m}$, where $\langle \mu, x\rangle := \mu(x)$. Since P is σ-continuous, the predual space $\mathfrak{m}_* \subset \mathfrak{m}^*$ is preserved by P^*. Let P_* be the restriction of P^* to $\mathfrak{m}_*$. The dual $(P_*)^*$ of the projector P_* is equal to P. The range $\operatorname{im} P$ of the projector P is a W^*-subalgebra of $\mathfrak{m}$ (see Sakai [S]). Recalling that $\operatorname{ad}^*_x \mathfrak{m}_* \subset \mathfrak{m}_*$ for $x \in \mathfrak{m}$, we see that $\operatorname{ad}_x \operatorname{im} P_* \subset \operatorname{im} P_*$, for $x \in \operatorname{im} P$.

Summarizing, we have proved the following.

Proposition 6.4 *The projector $P_* : \mathfrak{m}_* \to \mathfrak{m}_*$ has the following properties:*

(i) $\|P_*\| = 1$*;*

(ii) $\operatorname{im}(P_*)^*$ *is a Banach-Lie algebra;*

(iii) $\operatorname{ad}_x \operatorname{im} P_* \subset \operatorname{im} P_*$*, for* $x \in \operatorname{im} P$.

Therefore $P_* : \mathfrak{m}_* \to \mathfrak{m}_*$ is a quantum reduction map.

If $\mathfrak{m} = L^\infty(\mathcal{M})$ and $\mathfrak{m}_* = L^1(\mathcal{M})$ the projector $P_* : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ reduces the mixed state ρ of the quantum system to the state $p\rho p = P_*\rho$ localized on the subspace $L^1(p\mathcal{M}) \subset L^1(\mathcal{M})$. In the quantum mechanical formalism the projector $p : \mathcal{M} \to \mathcal{M}$ represents the so called elementary observable "proposition" (or "question") which can have only two alternative outcomes: "yes" or "no". The measurement of the "proposition" p reduces the state ρ to the state $P_*\rho$ and the non-negative number $\mathrm{tr}(P_*\rho)$ is the probability of the yes-answer. Since P_* is a projector, the repetition of the measurement does not change the state $P_*\rho$. This is the mathematical expression of the von Neumann reproducing postulate (von Neumann [vN]). ♦

Example 6.5 Let $\mathfrak{m}$ be a W^*-algebra and $\{p_\alpha\}_{\alpha\in I} \subset \mathfrak{m}$, be a family of self-adjoint mutually orthogonal projectors (i.e., $p_\alpha p_\beta = \delta_{\alpha\beta} p_\alpha$, and $p_\alpha^* = p_\alpha$) such that $\sum_{\alpha\in I} p_\alpha = 1$; the index set I is not necessary countable. Define the map $R^* : \mathfrak{m} \to \mathfrak{m}$ by

$$R^*(x) := \sum_{\alpha\in I} p_\alpha x p_\alpha \tag{6.7}$$

for $x \in \mathfrak{m}$, where the summation is taken in the sense of the σ-topology.

Proposition 6.6 *The map $R^* : \mathfrak{m} \to \mathfrak{m}$ is a σ-continuous linear projector with $\|R^*\| = 1$. Moreover, $\operatorname{im} R^*$ is a W^*-subalgebra of $\mathfrak{m}$ and hence a Banach Lie subalgebra of $(\mathfrak{m}, [\cdot,\cdot])$. Additionally, one has*

$$R^*(R^*(x)R^*(y)) = R^*(x)R^*(y) \quad \textit{and} \quad R^*(x^*) = (R^*(x))^* \tag{6.8}$$

for all $x, y, \in \mathfrak{m}$.

Proof. We can always consider $\mathfrak{m}$ as a von Neumann algebra of operators on the Hilbert space $\mathcal{M}$. If $v \in \mathcal{M}$, then

$$\|R^*(x)v\|^2 = \Big\| \sum_{\alpha\in I} p_\alpha x p_\alpha \Big\|^2 = \sum_{\alpha\in I} \|p_\alpha x p_\alpha v\|^2 \le \|x\|^2 \sum_{\alpha\in I} \|p_\alpha v\|^2 = \|x\|^2\|v\|^2$$

which shows that $\|R^*(x)\| \le 1$. From (6.7) we have

$$R^{*2}(x) = \sum_{\beta\in I} p_\beta \left(\sum_{\alpha\in I} p_\alpha x p_\alpha \right) p_\beta = \sum_{\beta\in I}\sum_{\alpha\in I} \delta_{\alpha\beta} p_\alpha x p_\alpha \delta_{\alpha\beta} = \sum_{\alpha\in I} p_\alpha x p_\alpha = R^*(x);$$

in this computation the σ-continuity of left and right multiplication with an element p_β was used. Thus $R^{*2} = R^*$ and $\|R^*\| = 1$.

For any $b \in \mathfrak{m}_*$, there is an element $\rho \in L^1(\mathcal{M})$ such that $\langle x, b\rangle = \mathrm{tr}(x\rho)$. Thus

$$\langle R^*(x), b\rangle = \mathrm{tr}(R^*(x)\rho) = \sum_{\alpha\in I} \mathrm{tr}(p_\alpha x p_\alpha \rho) = \sum_{\alpha\in I} \mathrm{tr}(x p_\alpha \rho p_\alpha) = \mathrm{tr}\left(x \sum_{\alpha\in I} p_\alpha x p_\alpha \right).$$

We want to check that $x_i \xrightarrow{\sigma} x$ implies that $R^*(x_i) \xrightarrow{\sigma} R^*(x)$. To do this, substitute x_i in the previous identity to get

$$\langle R^*(x_i), b\rangle = \mathrm{tr}\left(x_i \sum_{\alpha\in I} p_\alpha x p_\alpha \right) \xrightarrow{\sigma} \mathrm{tr}\left(x \sum_{\alpha\in I} p_\alpha x p_\alpha \right) = \mathrm{tr}(R^*(x)b) = \langle R^*(x), b\rangle$$

for any $b \in \mathfrak{m}_*$. So R^* is a σ-continuous linear map.

The defining formula for R^* shows that for $x, y \in \mathfrak{m}$ one has $R^*(R^*(x)R^*(y)) = R^*(x)R^*(y)$ and $R^*(x^*) = (R^*(x))^*$. Thus $\operatorname{im} R^*$ is a W^*-subalgebra of $\mathfrak{m}$ which implies that it is also a Banach Lie subalgebra of $(\mathfrak{m}, [\cdot,\cdot])$. ■

We conclude from Proposition 6.6 that $(R^*)^* : \mathfrak{m}^* \to \mathfrak{m}^*$ preserves the predual subspace $\mathfrak{m}_* \subset (\mathfrak{m}_*)^{**} = \mathfrak{m}^*$ and hence $R := (R^*)^*|_{\mathfrak{m}_*}$ is a quantum reduction. Note that one has the splitting $\mathfrak{m} = \operatorname{im} R^* \oplus \ker R^*$. ♦

Example 6.7 Take the decomposition of the unit (6.2) and define the map $R_- : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ by

$$R_-(\rho) := \sum_{n=1}^{\infty} \sum_{m=1}^{n} p_n \rho p_m = \sum_{n=1}^{\infty} p_n \rho q_n \tag{6.9}$$

where $q_n := \sum_{m=1}^{n} p_m$. It is clear that R_- is a linear projector on $L^1(\mathcal{M})$ whose range is the linear subspace of all "lower triangular" trace class operators $L^1(\mathcal{M})_-$. From

$$R_-(\rho)^* R_-(\rho) = \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} q_\ell \rho^* p_\ell p_n \rho q_n = \sum_{n=1}^{\infty} q_n \rho^* p_n \rho q_n \leq \sum_{n=1}^{\infty} \rho^* p_n \rho = \rho^* \rho$$

we have

$$\operatorname{tr} \sqrt{R(\rho)^* R(\rho)} \leq \operatorname{tr} \sqrt{\rho^* \rho}$$

which shows that $\|R_-(\rho)\|_1 \leq \|\rho\|_1$. Thus, $R_- : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ is a continuous projector with $\|R_-\| = 1$. So, the dual map $R_-^* : L^\infty(\mathcal{M}) \to L^\infty(\mathcal{M})$ defined by

$$R_-^*(x) := \sum_{n=1}^{\infty} q_n x p_n$$

also satisfies $\|R_-^*\| = 1$ and projects $L^\infty(\mathcal{M})$ onto the "upper triangular" Banach Lie subalgebra $L^\infty(\mathcal{M})_+ \subset L^\infty(\mathcal{M})$. In this way, $R_- : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ is a quantum reduction. Note that R_-^* satisfies (6.8). In §8 we will use the quantum reduction R_- in the description of the Toda lattice. ♦

The discussion below uses the standard notion of Banach Lie group and its associated Banach Lie algebra. Recall that a (real or complex) ***Banach Lie group*** is a (real or complex) smooth Banach manifold G with a group structure such that the multiplication and inversion are smooth maps. As in the finite dimensional case, the tangent space at the identity, T_eG, carries a Lie algebra structure which makes it isomorphic to the Lie algebra of left invariant vector fields on G endowed with the usual bracket operation on vector fields. Due to the smoothness of the group operations, this Lie bracket is a continuous bilinear map on the Banach space T_eG. Thus T_eG is a ***Banach Lie algebra*** which will be denoted, as customary, by $\mathfrak{g}$; it is called the Lie algebra of G.

Let $G(\mathfrak{m})$ be the group of invertible elements of a W^*-algebra $\mathfrak{m}$; it is an open subset (in the norm topology) of $\mathfrak{m}$ and is therefore a (real or complex) Banach Lie group whose Lie algebra is $\mathfrak{m}$ relative to the commutator bracket $[\cdot,\cdot]$ (Bourbaki [Bo3], Chapter III, §3.9). Its exponential map is the usual exponential function on $\mathfrak{m}$.

Proposition 6.8 *Let $R : \mathfrak{m}_* \to \mathfrak{m}_*$ be a quantum reduction as given in Definition 6.2 that also satisfies properties* (6.8) *and $R^*(1) = 1$. Then the set $G(\mathfrak{m}) \cap \operatorname{im} R^* = G(\operatorname{im} R^*)$ is a closed Banach Lie subgroup of $G(\mathfrak{m})$ whose Lie algebra is the Banach Lie subalgebra $\operatorname{im} R^*$.*

Proof. We prove that $G(\operatorname{im} R^*)$ is a subgroup of $G(\mathfrak{m})$.

From (6.8) and the fact that R^* maps the identity to the identity, it follows that $G(\operatorname{im} R^*)$ is closed under multiplication in $\mathfrak{m}$ and that it contains the identity element. We shall prove now that if $R^*(x)$ is invertible, then its inverse is also an element of $G(\operatorname{im} R^*)$. To see this, we assume without loss of generality that $\|1 - x\| < 1/\|R^*\|$. Since $R^*(1) = 1$ one has $\|R^*(1 - x)\| < 1$ and therefore

$$
\begin{aligned}
(R^*(x))^{-1} &= (1 - R^*(1 - x))^{-1} = \sum_{k=0}^{\infty} [R^*(1 - x)]^k \\
&= \sum_{k=0}^{\infty} R^* \left([R^*(1 - x)]^k \right) = R^* \left(\sum_{k=0}^{\infty} [R^*(1 - x)]^k \right) = R^* \left((R^*(x))^{-1} \right).
\end{aligned}
$$

Thus $G(\operatorname{im} R^*)$ is also closed under inversion and is therefore a subgroup of $G(\mathfrak{m})$. This argument also proves the equality in the statement.

Since $\operatorname{im} R^*$ is closed in $\mathfrak{m}$, it follows that $G(\operatorname{im} R^*)$ is a closed subgroup of $G(\mathfrak{m})$. As the group of invertible elements of the W^*-algebra $\operatorname{im} R^*$, $G(\operatorname{im} R^*)$ is a Lie group in its own right whose Lie algebra equals $\operatorname{im} R^*$. Because $\operatorname{im} R^*$ splits in $\mathfrak{m}$ it follows that the inclusion of $G(\operatorname{im} R^*)$ into $G(\mathfrak{m})$ is an immersion. However, the topologies on $G(\operatorname{im} R^*)$ and $G(\mathfrak{m})$ are both induced by the norm topology of $\mathfrak{m}$ and thus the inclusion $G(\operatorname{im} R^*) \hookrightarrow G(\mathfrak{m})$ is a homeomorphism onto its image which shows that this inclusion is an embedding and hence $G(\operatorname{im} R^*)$ is a Lie subgroup of $G(\mathfrak{m})$. ■

We shall return to this proposition in §7. Note that both Examples 6.5 and 6.7 satisfy the hypotheses of Proposition 6.8.

7 Symplectic leaves and coadjoint orbits

A smooth map $f : M \to N$ between *finite* dimensional manifolds is called an immersion, if for every $m \in M$ the derivative $T_m f : T_m M \to T_{f(m)} N$ is injective. In infinite dimensions there are various notions generalizing this concept.

Definition 7.1 *A smooth map $f : M \to N$ between Banach manifolds is called a*

(i) ***immersion*** *if for every $m \in M$ the tangent map $T_m f : T_m M \to T_{f(m)} N$ is injective with closed split range;*

(ii) ***quasi immersion*** *if for every $m \in M$ the tangent map $T_m f : T_m M \to T_{f(m)} N$ is injective with closed range;*

(iii) ***weak immersion*** *if for every $m \in M$ the tangent map $T_m f : T_m M \to T_{f(m)} N$ is injective.*

An immersion between Banach manifolds has the same properties as an immersion between finite dimensional manifolds. For example, it is characterized by the property that locally it is given by a map of the form $u \mapsto (u, 0)$, where the model space of the chart on N necessarily splits. This is the concept widely used in the literature; see e.g. Abraham, Marsden, and Ratiu [A-M-R], or Bourbaki [Bo2]. The notion of quasi immersion is modeled on the concept of quasi regular submanifold introduced in Bourbaki [Bo2]. Unfortunately, in the study of Banach Poisson manifolds not even this weaker concept of quasi immersion is satisfactory and one is forced to work with genuine weak immersions, as we shall see in this section.

If $(P, \{\cdot,\cdot\}_P)$ is a Banach Poisson manifold (in the sense of Definition 2.1), the vector subspace $S_p := \{X_f(p) \mid f \in C^\infty(P)\}$ of T_pP is called the ***characteristic subspace*** at p. Note that S_p is, in general, not a closed subspace of the Banach space T_pP. The union $S := \cup_{p\in P} S_p \subset TP$ is called the ***characteristic distribution*** of the Poisson structure on P. Note that even if S_p were closed and split in T_pP for every $p \in P$, S would not necessarily be a subbundle of TP. However, the characteristic distribution S is always ***smooth***, in the sense that for every $v_p \in S_p \subset T_pP$ there is a locally defined smooth vector field (namely some X_f) whose value at p is v_p. Assume that the characteristic distribution is integrable. For finite dimensional manifolds this is automatic by the Stefan-Sussmann theorem (see, e.g. Libermann and Marle [L-M], Appendix 3, Theorem 3.9) which, to our knowledge, is not available in infinite dimensions.

Let $\mathcal{L}$ be a leaf of the characteristic distribution, that is,

- $\mathcal{L}$ is a connected smooth Banach manifold,
- the inclusion $\iota : \mathcal{L} \hookrightarrow P$ is a weak injective immersion,
- $T_q\iota(T_q\mathcal{L}) = S_q$ for each $q \in \mathcal{L}$,
- if the inclusion $\iota' : \mathcal{L}' \hookrightarrow P$ is another weak injective immersion satisfying the three conditions above and $\mathcal{L} \subset \mathcal{L}'$, then necessarily $\mathcal{L}' = \mathcal{L}$, that is, $\mathcal{L}$ is maximal.

If we assume in addition that on the leaf $\mathcal{L}$

- there is a weak symplectic form $\omega_{\mathcal{L}}$ consistent with the Poisson structure on P,

then $\mathcal{L}$ will be called a ***symplectic leaf***.

In order to explain what this consistency means, consider from Definition 2.1 the bundle map $\sharp : T^*P \to TP$ associated to the Poisson tensor ϖ on P and note that for each $p \in P$, the linear continuous map $\sharp_p : T_p^*P \to T_pP$ induces a bijective continuous map $[\sharp_p] : T_p^*P/\ker\sharp_p \to S_p$. By definition, $\omega_{\mathcal{L}}$ is ***consistent*** with the Poisson structure on P if

$$\omega_{\mathcal{L}}(q)(u_q, v_q) = \varpi(\iota(q))\left(([\sharp_{\iota(q)}]^{-1} \circ T_q\iota)(u_q), ([\sharp_{\iota(q)}]^{-1} \circ T_q\iota)(v_q)\right). \tag{7.1}$$

This shows that the weak symplectic form $\omega_{\mathcal{L}}$ consistent with the Poisson structure on P is unique.

For finite dimensional Poisson manifolds, it is known that all leaves are symplectic (see Weinstein [W1, W2]) and so the last assumption above is not necessary. In the infinite dimensional case this question is open, even in the case of a Banach Lie group G

whose Lie algebra $\mathfrak{g}$ has a predual $\mathfrak{g}_*$ invariant under the coadjoint action. In this case, $\mathfrak{g}_*$ is a Banach Lie-Poisson space and we will characterize a large class of points in $\mathfrak{g}_*$ whose coadjoint orbits are all weak symplectic manifolds. Their connected components are therefore symplectic leaves. Among these, we will also identify a class of coadjoint orbits that are strong symplectic manifolds.

Proposition 7.2 *Let $\iota : (\mathcal{L}, \omega_{\mathcal{L}}) \hookrightarrow P$ be a symplectic leaf of the characteristic distribution of P. Then*

(i) *for any $f \in C^\infty(U)$, $q \in \iota^{-1}(U) \cap \mathcal{L}$, where U is an open subset of P, one has*

$$d\left((f \circ \iota)|_{\iota^{-1}(U)}\right)(q) = \omega_{\mathcal{L}}(q)\left((T_q\iota)^{-1}\left(X_f(\iota(q))\right), \cdot\right). \tag{7.2}$$

(ii) *the subspace $\iota^*(C^\infty(P))$ of $C^\infty(\mathcal{L})$ consisting of functions that are obtained as restrictions of smooth functions from P is a Poisson algebra relative to the bracket $\{\cdot,\cdot\}_{\mathcal{L}}$ given by*

$$\{f \circ \iota, g \circ \iota\}_{\mathcal{L}}(q) := \omega_{\mathcal{L}}(q)\left((T_q\iota)^{-1}\left(X_f(\iota(q))\right), (T_q\iota)^{-1}\left(X_g(\iota(q))\right)\right); \tag{7.3}$$

(iii) *$\iota^* : C^\infty(P) \to \iota^*(C^\infty(P))$ is a homomorphism of Poisson algebras, that is,*

$$\{f \circ \iota, g \circ \iota\}_{\mathcal{L}} = \{f, g\}_P \circ \iota \tag{7.4}$$

for any $f, g \in C^\infty(P)$.

Proof. We begin with the proof of formula (7.2). For any $q \in \mathcal{L} \cap \iota^{-1}(U)$, $v_q \in T_q\mathcal{L}$, $f \in C^\infty(U)$, U open in P, we have by (7.1) and the definition of $\sharp$

$$\begin{aligned}\omega_{\mathcal{L}}(q)\left((T_q\iota)^{-1}\left(X_f(\iota(q))\right), v_q\right) &= \varpi(\iota(q))\left(df(\iota(q)), ([\sharp_{\iota(q)}]^{-1} \circ T_q\iota)(v_q)\right)\\ &= \langle df(\iota(q)), T_q\iota(v_q)\rangle = d(f \circ \iota)(q)(v_q),\end{aligned}$$

which proves (7.2). Now replace in the above computation v_q by $(T_q\iota)^{-1}\left(X_g(\iota(q))\right)$ and get

$$\omega_{\mathcal{L}}(q)\left((T_q\iota)^{-1}\left(X_f(\iota(q))\right), (T_q\iota)^{-1}\left(X_g(\iota(q))\right)\right) = \langle df(\iota(q)), X_g(\iota(q))\rangle = \{f, g\}_P(\iota(q)),$$

which, in view of (7.3), proves (7.4). Finally, (7.4) shows that (7.3) defines a Poisson bracket on $\mathcal{L}$. ■

Formula (7.2) is remarkable since it guarantees the existence of Hamiltonian vector fields on the weak symplectic manifold $\mathcal{L}$ for a large class of functions, namely those that are pull backs to the symplectic leaf.

We shall give below a class of Banach Poisson manifolds for which some of the symplectic leaves can be explicitly determined. In what follows G denotes a (real or complex) Banach Lie group, L_g and R_g denote the diffeomorphisms of G given by left and right translations by $g \in G$, and $\mathfrak{g}$ denotes the (left) Lie algebra of G.

Theorem 7.3 *Let G be a (real or complex) Banach Lie group with Lie algebra $\mathfrak{g}$. Assume that:*

(i) *$\mathfrak{g}$ admits a predual $\mathfrak{g}_*$, that is, $\mathfrak{g}_*$ is a Banach space whose dual is $\mathfrak{g}$;*

(ii) *the coadjoint action of G on the dual $\mathfrak{g}^*$ leaves the predual $\mathfrak{g}_*$ invariant, that is, $\mathrm{Ad}_g^*(\mathfrak{g}_*) \subset \mathfrak{g}_*$, for any $g \in G$;*

(iii) *for a fixed $\rho \in \mathfrak{g}_*$ the coadjoint isotropy subgroup $G_\rho := \{g \in G \mid \mathrm{Ad}_g^* \rho = \rho\}$, which is closed in G, is a Lie subgroup of G (in the sense that it is a submanifold of G and not just injectively immersed).*

Then the Lie algebra of G_ρ equals $\mathfrak{g}_\rho := \{\xi \in \mathfrak{g} \mid \mathrm{ad}_\xi^* \rho = 0\}$ and the quotient topological space $G/G_\rho := \{gG_\rho \mid g \in G\}$ admits a unique smooth (real or complex) Banach manifold structure making the canonical projection $\pi : G \to G/G_\rho$ a surjective submersion. The manifold G/G_ρ is weakly symplectic relative to the two-form ω_ρ given by

$$\omega_\rho([g])(T_g\pi(T_eL_g\xi), T_g\pi(T_eL_g\eta)) := \langle \rho, [\xi, \eta] \rangle \,, \tag{7.5}$$

where $\xi, \eta \in \mathfrak{g}$, $g \in G$, $[g] := \pi(g) = gG_\rho$, and $\langle \cdot, \cdot \rangle : \mathfrak{g}_* \times \mathfrak{g} \to \mathbb{R}$ (or $\mathbb{C}$) is the canonical pairing between $\mathfrak{g}_*$ and $\mathfrak{g}$. Alternatively, this form can be expressed as

$$\omega_\rho([g])(T_g\pi(T_eR_g\xi), T_g\pi(T_eR_g\eta)) := \left\langle \mathrm{Ad}_{g^{-1}}^* \rho, [\xi, \eta] \right\rangle . \tag{7.6}$$

The two-form ω_ρ is invariant under the left action of G on G/G_ρ given by $g \cdot [h] := [gh]$, for $g, h \in G$.

Proof. The subgroup G_ρ is clearly closed. For Banach Lie groups it is no longer true that closed subgroups are Lie subgroups (for a counterexample see Bourbaki [Bo3], Chapter III, Exercise 8.2). However, as in the finite dimensional case, if G_ρ is assumed to be a Lie subgroup of G, then (Bourbaki [Bo3], Chapter III, §6.4, Corollary 1) $\xi \in \mathfrak{g}$ is an element of the Lie algebra of G_ρ if and only if $\exp t\xi \in G_\rho$ for all $t \in \mathbb{R}$ (or $\mathbb{C}$ depending on whether G is a real or complex Banach Lie group). Thus, since (see, e.g. Marsden and Ratiu [M-R2], Chapter 9)

$$\frac{d}{dt} \mathrm{Ad}_{\exp t\xi}^* \rho = \mathrm{Ad}_{\exp t\xi}^* \mathrm{ad}_\xi^* \rho,$$

it follows that

$$\begin{aligned} \exp t\xi \in G_\rho \iff \mathrm{Ad}_{\exp t\xi}^* \rho = \rho \iff 0 = \frac{d}{dt} \mathrm{Ad}_{\exp t\xi}^* \rho = \mathrm{Ad}_{\exp t\xi}^* \mathrm{ad}_\xi^* \rho \\ \iff \mathrm{ad}_\xi^* \rho = 0 \iff \xi \in \mathfrak{g}_\rho, \end{aligned}$$

which shows that the Lie algebra of G_ρ is $\mathfrak{g}_\rho$.

Since G_ρ is assumed to be a Lie subgroup of G, the set G/G_ρ has a unique smooth manifold structure such that the canonical projection $\pi : G \to G/G_\rho$ is a submersion. The underlying manifold topology of G/G_ρ is the quotient topology. The left action $(g, [h]) \in G \times G/G_\rho \mapsto g \cdot [h] := [gh]$ is smooth (see Bourbaki [Bo3], Chapter III, §1.6, Proposition 11, for a proof of these statements).

In what follows we shall need the following observation. Condition (ii) implies that $\operatorname{ad}^*_\xi(\mathfrak{g}_*) \subset \mathfrak{g}_*$ for any $\xi \in \mathfrak{g}$.

The two-forms defined in formulas (7.5) and (7.6) are equal. Indeed, taking (7.5) as the definition but applying it to tangent vectors of the form $T_g\pi(T_eR_g\xi)$, $T_g\pi(T_eR_g\eta)$, we get

$$\begin{aligned}
\omega_\rho([g])(T_g\pi(T_eR_g\xi), T_g\pi(T_eR_g\eta)) &= \omega_\rho([g])\left(T_g\pi(T_eL_g(\operatorname{Ad}_{g^{-1}}\xi)), T_g\pi(T_eL_g(\operatorname{Ad}_{g^{-1}}\xi))\right) \\
&= \langle \rho, [\operatorname{Ad}_{g^{-1}}\xi, \operatorname{Ad}_{g^{-1}}\eta]\rangle = \langle \rho, \operatorname{Ad}_{g^{-1}}[\xi,\eta]\rangle = \left\langle \operatorname{Ad}^*_{g^{-1}}\rho, [\xi,\eta]\right\rangle
\end{aligned}$$

which gives formula (7.6).

We shall prove now that the two-form (7.6) is well defined. Indeed, if $[g] = [g']$ and $T_g\pi(T_eR_g\xi) = T_{g'}\pi(T_eR_{g'}\xi')$, then there is some $h \in G_\rho$ such that $g' = gh$ and hence

$$T_g\pi(T_eR_g\xi) = T_{g'}\pi(T_eR_{g'}\xi') = T_{gh}\pi(T_eR_{gh}\xi') = T_g(\pi \circ R_h)(T_eR_g\xi') = T_g\pi(T_eR_g\xi'),$$

which means that $T_g\pi(T_eR_g(\xi - \xi')) = 0$. Due to the fact that the fibers of π are of the form gG_ρ, this is equivalent to $T_eR_g(\xi - \xi') \in T_eL_g(\mathfrak{g}_\rho)$, that is, $\xi - \xi' \in \operatorname{Ad}_g(\mathfrak{g}_\rho)$. Thus there is some $\zeta \in \mathfrak{g}_\rho$ such that $\xi' = \xi + \operatorname{Ad}_g\zeta$. Similarly, if $T_g\pi(T_eR_g\eta) = T_{g'}\pi(T_eR_{g'}\eta')$ there is some $\zeta' \in \mathfrak{g}_\rho$ such that $\eta' = \eta + \operatorname{Ad}_g\zeta'$. Therefore, since $\operatorname{ad}^*_\zeta\rho = \operatorname{ad}^*_{\zeta'}\rho = 0$, it follows that

$$\begin{aligned}
\omega_\rho([g'])(T_{g'}\pi(T_eR_{g'}\xi'), T_{g'}\pi(T_eR_{g'}\eta')) &= \left\langle \operatorname{Ad}^*_{g'^{-1}}\rho, [\xi',\eta']\right\rangle \\
&= \left\langle \operatorname{Ad}^*_{(gh)^{-1}}\rho,\ [\xi + \operatorname{Ad}_g\zeta,\ \eta + \operatorname{Ad}_g\zeta']\right\rangle \\
&= \left\langle \operatorname{Ad}^*_{g^{-1}}\rho,\ [\xi + \operatorname{Ad}_g\zeta,\ \eta + \operatorname{Ad}_g\zeta']\right\rangle \\
&= \left\langle \operatorname{Ad}^*_{g^{-1}}\rho,\ [\xi,\eta]\right\rangle + \left\langle \operatorname{Ad}^*_{g^{-1}}\rho,\ [\operatorname{Ad}_g\zeta,\ \eta]\right\rangle \\
&\qquad + \left\langle \operatorname{Ad}^*_{g^{-1}}\rho,\ [\xi, \operatorname{Ad}_g\zeta']\right\rangle + \left\langle \operatorname{Ad}^*_{g^{-1}}\rho,\ \operatorname{Ad}_g[\zeta,\zeta']\right\rangle \\
&= \left\langle \operatorname{Ad}^*_{g^{-1}}\rho,\ [\xi,\eta]\right\rangle + \langle \rho,\ [\zeta,\ \operatorname{Ad}_{g^{-1}}\eta]\rangle + \langle \rho,\ [\operatorname{Ad}_{g^{-1}}\xi, \zeta']\rangle + \langle \rho,\ [\zeta,\zeta']\rangle \\
&= \left\langle \operatorname{Ad}^*_{g^{-1}}\rho,\ [\xi,\eta]\right\rangle + \left\langle \operatorname{ad}^*_\zeta\rho,\ \operatorname{Ad}_{g^{-1}}\eta\right\rangle - \left\langle \operatorname{ad}^*_{\zeta'}\rho,\ \operatorname{Ad}_{g^{-1}}\xi\right\rangle + \left\langle \operatorname{ad}^*_\zeta\rho,\ \zeta'\right\rangle \\
&= \left\langle \operatorname{Ad}^*_{g^{-1}}\rho,\ [\xi,\eta]\right\rangle = \omega_\rho([g])(T_g\pi(T_eR_g\xi), T_g\pi(T_eR_g\eta)).
\end{aligned}$$

The two-form ω_ρ is weakly non degenerate. Indeed if $[g] \in G/G_\rho$ is given and if

$$\omega_\rho([g])(T_g\pi(T_eL_g\xi), T_g\pi(T_eL_g\eta)) = 0$$

for all $T_g\pi(T_eL_g\eta)$ then, by (7.5), $\langle \operatorname{ad}^*_\xi\rho, \eta\rangle = 0$ for all $\eta \in \mathfrak{g}$ and thus $\xi \in \mathfrak{g}_\rho$ (since $\operatorname{ad}^*_\xi\rho \in \mathfrak{g}_*$) which is equivalent to $T_g\pi(T_eL_g\xi) = 0$.

We shall prove that ω_ρ is a smooth closed two-form on G/G_ρ by showing that the smooth one-form on G given by $\nu_\rho(g)(T_eL_g\xi) := -\langle \rho, \xi\rangle$ satisfies $d\nu_\rho = \pi^*\omega_\rho$. Since π is a surjective submersion this immediately implies that ω_ρ is smooth and closed. To compute the exterior derivative of ν_ρ, we denote by X, Y the vector fields on G given by $X(g) = T_eL_g\xi$ and $Y(g) = T_eL_g\eta$ and note that $\nu_\rho(X)(g) = -\langle \rho, \xi\rangle$ is constant and $[X,Y](g) = T_eL_g[\xi,\eta]$. Therefore, by Cartan's formula,

$$\begin{aligned}
d\nu_\rho(g)(T_eL_g\xi, T_eL_g\eta) &= d\nu_\rho(X,Y)(g) = X[\nu_\rho(Y)](g) - Y[\nu_\rho(\mathcal{X})](g) - \nu_\rho([X,Y])(g) \\
&= -\nu_\rho(g)(T_eL_g[\xi,\eta]) = \langle \rho, [\xi,\eta]\rangle = (\pi^*\omega_\rho)(g)(T_eL_g\xi, T_eL_g\eta).
\end{aligned}$$

To show that ω_ρ is G-invariant, we note that π is G-equivariant, $\omega_\rho = \pi^* \nu_\rho$, and that ν_ρ is G-invariant. ■

We shall see a concrete example of a symplectic form ω_ρ that is weak and not strong after Example 7.9.

Next we study the coadjoint orbits of G through points of $\mathfrak{g}_*$.

Theorem 7.4 *Let the Banach Lie group G and the element $\rho \in \mathfrak{g}_*$ satisfy the hypotheses of Theorem 7.3. Then the map*

$$\iota : [g] \in G/G_\rho \mapsto \operatorname{Ad}^*_{g^{-1}} \rho \in \mathfrak{g}_* \tag{7.7}$$

is an injective weak immersion of the quotient manifold G/G_ρ into the predual space $\mathfrak{g}_*$. Endow the coadjoint orbit $\mathcal{O} := \{\operatorname{Ad}^*_g \rho \mid g \in G\}$ with the smooth manifold structure making ι into a diffeomorphism. The push forward $\iota_*(\omega_\rho)$ of the weak symplectic form $\omega_\rho \in \Omega^2(G/G_\rho)$ to $\mathcal{O}$ has the expression

$$\omega_{\mathcal{O}}(\operatorname{Ad}^*_{g^{-1}} \rho)\left(\operatorname{ad}^*_{\operatorname{Ad}_g \xi} \operatorname{Ad}^*_{g^{-1}} \rho, \operatorname{ad}^*_{\operatorname{Ad}_g \eta} \operatorname{Ad}^*_{g^{-1}} \rho\right) = \langle \rho, [\xi, \eta] \rangle, \tag{7.8}$$

for $g \in G$, $\xi, \eta \in \mathfrak{g}$, and $\rho \in \mathfrak{g}_*$. Relative to this symplectic form the connected components of the coadjoint orbit $\mathcal{O}$ are symplectic leaves of the Banach Lie-Poisson space $\mathfrak{g}_*$.

Proof. By Theorem 4.6 the predual $\mathfrak{g}_*$ is a (real or holomorphic) Banach Lie-Poisson space whose (real or complex) Poisson bracket is given by (4.2). For each $\rho \in \mathfrak{g}_*$ its characteristic subspace is therefore given by $S_\rho = \{\operatorname{ad}^*_\xi \rho \mid \xi \in \mathfrak{g}\} \subset \mathfrak{g}_*$ since $\operatorname{ad}^*_\xi(\mathfrak{g}_*) \subset \mathfrak{g}_*$ for any $\xi \in \mathfrak{g}$.

The map $\iota : [g] \in G/G_\rho \mapsto \operatorname{Ad}^*_{g^{-1}} \rho \in \mathcal{O}$ is a bijection, so we put on $\mathcal{O}$ the smooth Banach manifold structure making this bijection into a diffeomorphism. Since the map $g \in G \mapsto \operatorname{Ad}^*_{g^{-1}} \rho \in \mathfrak{g}_*$ is continuous, it thus follows that the inclusion $\mathcal{O} \subset \mathfrak{g}_*$ is also continuous.

We shall prove now that the map $g \in G \mapsto \operatorname{Ad}^*_{g^{-1}} \rho \in \mathfrak{g}_*$ is smooth. Indeed its derivative

$$T_e L_g \xi \in T_g G \mapsto -\operatorname{Ad}^*_{g^{-1}}\left(\operatorname{ad}^*_\xi \rho\right) = -\operatorname{ad}^*_{\operatorname{Ad}_g \xi} \operatorname{Ad}^*_{g^{-1}} \rho \in \mathfrak{g}_* \tag{7.9}$$

is a continuous linear map from $T_g G$ to $\mathfrak{g}_*$, that is, the map $g \in G \mapsto \operatorname{Ad}^*_{g^{-1}} \rho \in \mathfrak{g}_*$ is C^1. Inductively, it follows that it is C^∞. In addition, the range of the derivative at g is the characteristic subspace $S_{\operatorname{Ad}^*_{g^{-1}} \rho}$ at $\operatorname{Ad}^*_{g^{-1}} \rho$.

Since the map $g \in G \mapsto \operatorname{Ad}^*_{g^{-1}} \rho \in \mathfrak{g}_*$ is G_ρ-invariant, it follows that $\iota : [g] \in G/G_\rho \mapsto \operatorname{Ad}^*_{g^{-1}} \rho \in \mathfrak{g}_*$ is smooth and that the range of its derivative at $[g]$, given by $T_{[g]}\iota : T_g\pi(T_e L_g \xi) \in T_{[g]}(G/G_\rho) \mapsto -\operatorname{ad}^*_{\operatorname{Ad}_g \xi} \operatorname{Ad}^*_{g^{-1}} \rho \in \mathfrak{g}_*$, equals $S_{\operatorname{Ad}^*_{g^{-1}} \rho}$. The map $T_{[g]}\iota$ is injective. Indeed, if

$$0 = T_{[g]}\iota(T_g\pi(T_e L_g \xi)) = -\operatorname{Ad}^*_{g^{-1}}\left(\operatorname{ad}^*_\xi \rho\right)$$

then $\xi \in \mathfrak{g}_\rho$ and hence $T_g\pi(T_e L_g \xi) = 0$. This shows that ι is an injective weak immersion.

Let us endow the manifold $\mathcal{O} \subset \mathfrak{g}_*$ with the push forward weak symplectic form $\omega_{\mathcal{O}}$ given by the diffeomorphism $[g] \in G/G_\rho \mapsto \operatorname{Ad}^*_{g^{-1}} \rho \in \mathcal{O} \subset \mathfrak{g}_*$. From the formula for its derivative given by (7.9) and (7.5), it immediately follows that this weak symplectic form on $\mathcal{O} \subset \mathfrak{g}_*$ has the expression

$$\omega_{\mathcal{O}}(\operatorname{Ad}^*_{g^{-1}} \rho) \left(\operatorname{ad}^*_{\operatorname{Ad}_g \xi} \operatorname{Ad}^*_{g^{-1}} \rho, \operatorname{ad}^*_{\operatorname{Ad}_g \eta} \operatorname{Ad}^*_{g^{-1}} \rho \right) = \langle \rho, [\xi, \eta] \rangle. \tag{7.10}$$

Let now $f \in C^\infty(\mathfrak{g}_*)$ and $\rho \in \mathfrak{g}_*$. Since

$$X_f(\operatorname{Ad}^*_{g^{-1}} \rho) = -\operatorname{ad}^*_{Df(\operatorname{Ad}^*_{g^{-1}} \rho)} \left(\operatorname{Ad}^*_{g^{-1}} \rho \right) \in S_{\operatorname{Ad}^*_{g^{-1}} \rho},$$

and $S_{\operatorname{Ad}^*_{g^{-1}} \rho}$ is the tangent space at $\operatorname{Ad}^*_{g^{-1}} \rho$ to the orbit $\mathcal{O}$, it follows that this orbit is a Poisson submanifold of $\mathfrak{g}_*$. Since

$$Df(\operatorname{Ad}^*_{g^{-1}} \rho) = \operatorname{Ad}_g \left(D(f \circ \operatorname{Ad}^*_{g^{-1}})(\rho) \right)$$

for $g \in G$ and $\rho \in \mathfrak{g}_*$, it follows that

$$X_f(\operatorname{Ad}^*_{g^{-1}} \rho) = -\operatorname{ad}^*_{\operatorname{Ad}_g \left(D(f \circ \operatorname{Ad}^*_{g^{-1}})(\rho) \right)} \left(\operatorname{Ad}^*_{g^{-1}} \rho \right)$$

and hence for any $\eta \in \mathfrak{g}$ we have by (7.8)

$$\begin{aligned}
&\omega_{\mathcal{O}}(\operatorname{Ad}^*_{g^{-1}} \rho) \left(X_f(\operatorname{Ad}^*_{g^{-1}} \rho), -\operatorname{ad}^*_{\operatorname{Ad}_g \eta} \left(\operatorname{Ad}^*_{g^{-1}} \rho \right) \right) \\
&\quad = \omega_{\mathcal{O}}(\operatorname{Ad}^*_{g^{-1}} \rho) \left(-\operatorname{ad}^*_{\operatorname{Ad}_g \left(D(f \circ \operatorname{Ad}^*_{g^{-1}})(\rho) \right)} \left(\operatorname{Ad}^*_{g^{-1}} \rho \right), -\operatorname{ad}^*_{\operatorname{Ad}_g \eta} \left(\operatorname{Ad}^*_{g^{-1}} \rho \right) \right) \\
&\quad = \langle \rho, [D(f \circ \operatorname{Ad}^*_{g^{-1}})(\rho), \eta] \rangle = -\langle \operatorname{ad}^*_\eta \rho, D(f \circ \operatorname{Ad}^*_{g^{-1}})(\rho) \rangle \\
&\quad = -Df(\operatorname{Ad}^*_{g^{-1}} \rho) \left(\operatorname{Ad}^*_{g^{-1}} \left(\operatorname{ad}^*_\eta \rho \right) \right) = -Df(\operatorname{Ad}^*_{g^{-1}} \rho) \left(\operatorname{ad}^*_{\operatorname{Ad}_g \eta} \left(\operatorname{Ad}^*_{g^{-1}} \rho \right) \right),
\end{aligned}$$

which shows that the Hamiltonian vector field X_f relative to the Lie-Poisson structure (4.2) computed at a point of the orbit $\mathcal{O}$ is also Hamiltonian relative to the weak symplectic form (7.8). Thus the Lie-Poisson structure on $\mathfrak{g}_*$ and the weak symplectic form on the orbit are compatible, i.e., the Lie-Poisson structure (4.2) induces the weak symplectic form (7.8) on the orbit.

Summarizing, we have shown that each connected component of a coadjoint orbit is a connected smooth Banach manifold, that the inclusion of the orbit in $\mathfrak{g}_*$ is a weak injective immersion such that its tangent map has at each point as range the characteristic subspace, and that the Lie-Poisson structure induces the weak symplectic form on the orbit given by the canonical diffeomorphism of the orbit with the quotient G/G_ρ. In addition, since the orbits are a partition of $\mathfrak{g}_*$, the maximality condition holds automatically. ■

Next we analyze a remarkable particular situation that will give much stronger conclusions.

Theorem 7.5 *Let the Banach Lie group G and the element $\rho \in \mathfrak{g}_*$ satisfy the hypotheses of Theorem 7.3. The following conditions are equivalent:*

(i) $\iota : G/G_\rho \to \mathfrak{g}_*$ *is an injective immersion;*

(ii) *the characteristic subspace* $S_\rho := \{\operatorname{ad}^*_\xi \rho \mid \xi \in \mathfrak{g}\}$ *is closed in* $\mathfrak{g}_*$*;*

(iii) $S_\rho = \mathfrak{g}_\rho^\circ$*, where* $\mathfrak{g}_\rho^\circ$ *is the annihilator of* $\mathfrak{g}_\rho$ *in* $\mathfrak{g}_*$.

Endow the coadjoint orbit $\mathcal{O} := \{\operatorname{Ad}^*_g \rho \mid g \in G\}$ *with the manifold structure making* ι *a diffeomorphism. Then, under any of the hypotheses* (i)–(iii)*, the two-form given by* (7.8) *is a strong symplectic form.*

Proof. It is a general fact that any set is included in its double annihilator, so $S_\rho \subset S_\rho^{\circ\circ}$. We shall prove now that if S_ρ is closed, then this inclusion is actually an equality (this is also a general fact for closed subspaces in any Banach space). Indeed, if $S_\rho \neq S_\rho^{\circ\circ} \subset \mathfrak{g}^*$, then closedness of S_ρ guarantees by the Hahn-Banach theorem that there exists $0 \neq \varphi \in \mathfrak{g}^{**}$ such that $\varphi \in S_\rho^\circ$ and $\varphi \notin S_\rho^{\circ\circ\circ}$. The inclusion $S_\rho \subset S_\rho^{\circ\circ}$ implies $S_\rho^{\circ\circ\circ} \subset S_\rho^\circ$. Since it is in general true that $S_\rho^\circ \subset S_\rho^{\circ\circ\circ}$ we get $S_\rho^\circ = S_\rho^{\circ\circ\circ}$, which contradicts the existence of φ. Thus if S_ρ is closed, then $S_\rho = S_\rho^{\circ\circ}$.

Using the identity $\langle \eta, \operatorname{ad}^*_\xi \rho\rangle = -\langle \xi, \operatorname{ad}^*_\eta \rho\rangle$ for any $\xi, \eta \in \mathfrak{g}$, it follows that $S_\rho^\circ = \mathfrak{g}_\rho$. Taking the annihilator in this relation and using the equality $S_\rho = S_\rho^{\circ\circ}$, for S_ρ closed, yields $S_\rho = \mathfrak{g}_\rho^\circ$. Thus S_ρ is closed if and only if $S_\rho = \mathfrak{g}_\rho^\circ$. This proves the equivalence of (ii) and (iii).

Assume now that (iii) holds. Since G_ρ is a Banach Lie subgroup one has the splitting $\mathfrak{g} = \mathfrak{g}_\rho \oplus \mathfrak{g}_\rho^c$, where $\mathfrak{g}_\rho^c$ is a closed subspace. This induces the splitting of the dual space $\mathfrak{g}^* = \mathfrak{g}_\rho^\circ \oplus \left(\mathfrak{g}_\rho^c\right)^\circ = S_\rho \oplus \left(\mathfrak{g}_\rho^c\right)^\circ$. Thus, using the inclusion $S_\rho \subset \mathfrak{g}_*$ we obtain the splitting

$$\mathfrak{g}_* = S_\rho \oplus \left[\left(\mathfrak{g}_\rho^c\right)^\circ \cap \mathfrak{g}_*\right].$$

The identity

$$T_{[g]}\iota\left(T_{[g]}(G/G_\rho)\right) = S_{\operatorname{Ad}^*_{g^{-1}}\rho} = \operatorname{Ad}^*_{g^{-1}} S_\rho, \tag{7.11}$$

and the fact that $\operatorname{Ad}^*_{g^{-1}}$ is a Banach space isomorphism show that ι is a immersion. So (i) holds. Conversely, if (i) is valid then S_ρ is closed by definition, so (ii) is satisfied.

In order to prove that (7.8) is a strong symplectic form, let us notice that since S_ρ is a closed subspace of $\mathfrak{g}_*$, by the Hahn-Banach Theorem, for any $f \in S_\rho^*$ there exists an element $\eta \in (\mathfrak{g}_*)^* = \mathfrak{g}$ such that

$$f(\operatorname{ad}^*_\xi \rho) = \langle \operatorname{ad}^*_\xi \rho, \eta\rangle = \langle \rho, [\xi, \eta]\rangle$$

for any $\xi \in \mathfrak{g}$. So the linear map $\operatorname{ad}^*_\eta \rho \in S_\rho \mapsto \omega_{\mathcal{O}}(\rho)(\cdot, \operatorname{ad}^*_\eta \rho) \in S_\rho^*$ is surjective. Due to left invariance of $\omega_{\mathcal{O}}$ this surjectivity will hold at all points of the orbit $\mathcal{O}$. Since $\omega_{\mathcal{O}}$ is in general a weak symplectic form according to Theorem 7.4, it follows that $\omega_{\mathcal{O}}$ is a strong symplectic form. ■

Corollary 7.6 *Let the Banach Lie group* G *and the element* $\rho \in \mathfrak{g}_*$ *satisfy the hypotheses of Theorem 7.3. Then* $\iota : G/G_\rho \to \mathfrak{g}_*$ *is a quasi immersion if and only if it is an immersion.*

We now apply the above theorems to the important class of W^*-algebras. From Theorem 5.1 one knows that the predual $\mathfrak{m}_*$ of the (complex) W^*-algebra $\mathfrak{m}$ is a holomorphic Banach Lie-Poisson space. Recall that the set $G(\mathfrak{m})$ of invertible elements of $\mathfrak{m}$ is a Banach Lie group who acts on $\mathfrak{m}_*$ by the coadjoint action.

Corollary 7.7 *Let $\rho \in \mathfrak{m}_*$ be such that $G(\mathfrak{m})_\rho = \operatorname{im} R^* \cap G(\mathfrak{m})$, where R^* is given by* (6.7) *(so $R := (R^*)^*|_{\mathfrak{m}_*}$ is a quantum reduction). Then the connected components of the coadjoint orbit through ρ are weakly immersed weak symplectic manifolds that are symplectic leaves of the Banach Lie-Poisson space $\mathfrak{m}_*$.*

Proof. By Proposition 6.8, $G(\mathfrak{m})_\rho$ is a Lie subgroup of $G(\mathfrak{m})$, so the hypotheses of Theorems 7.3 and 7.4 hold and the conclusion follows. ■

Corollary 7.8 *Let $\rho \in \mathfrak{m}_*$ be such that $G(\mathfrak{m})_\rho = \operatorname{im} R^* \cap G(\mathfrak{m})$, where R^* is given by* (6.7)*. Then the following conditions are equivalent:*

(i) $S_\rho = \ker R$;

(ii) *the map $\iota : G(\mathfrak{m})/G(\mathfrak{m})_\rho \to \mathfrak{m}_*$ is an injective immersion.*

Under any of these conditions, the coadjoint orbit $\mathcal{O}$ endowed with the smooth manifold structure making ι a diffeomorphism onto its image is strong symplectic.

Proof. If $S_\rho = \ker R$, then S_ρ is closed and Theorem 7.5 applies thus guaranteeing that ι is an injective immersion.

Conversely, if ι is an immersion, Theorem 7.5 states that $S_\rho = \mathfrak{m}_\rho^\circ$. However, the hypothesis and Proposition 6.8 guarantee that G_ρ is a Lie subgroup of $G(\mathfrak{m})$ whose Lie algebra is $\operatorname{im} R^*$. On the other hand it is clear that the Lie algebra of $G(\mathfrak{m})_\rho$ is $\mathfrak{m}_\rho$ since $\exp(\lambda x) \in G(\mathfrak{m})_\rho$ for all $x \in \mathfrak{m}_\rho$ and all $\lambda \in \mathbb{C}$ (see Bourbaki [Bo3], Chapter III, §6.4, Corollary 1). Therefore $\mathfrak{m}_\rho = \operatorname{im} R^*$.

The decomposition $\mathfrak{m} = \operatorname{im} R^* \oplus \ker R^*$ and the one induced on the dual imply the general identities

$$(\operatorname{im} R^*)^\circ = \ker R^{**} \quad \text{and} \quad (\ker R^*)^\circ = \operatorname{im} R^{**},$$

where the annihilators are taken in $\mathfrak{m}^*$. Therefore, using $\mathfrak{m}_\rho = \operatorname{im} R^*$, we get $\ker R = \ker R^{**} \cap \mathfrak{m}_* = (\operatorname{im} R^*)^\circ \cap \mathfrak{m}_* = \mathfrak{m}_\rho^\circ = S_\rho$ and the equivalence of (i) and (ii) is proved.

The last statement follows by directly applying Theorem 7.5. ■

Example 7.9 Take in the previous considerations $\mathfrak{m} = L^\infty(\mathcal{M})$, $\mathfrak{m}_* = L^1(\mathcal{M})$, and the quantum reduction map $R : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ defined by (6.3), where $\sum_{k=1}^N P_k = 1$, with $N \in \mathbb{N}$ or $N = \infty$. If $N = \infty$, let

$$\rho = \sum_{k=1}^{\infty} \lambda_k P_k, \quad \lambda_k \in \mathbb{C}, \quad \lambda_k \neq \lambda_\ell \neq 0 \ \text{ for } \ k \neq \ell, \quad \operatorname{rank} P_k < \infty \ \text{ for } k \geq 1 \tag{7.12}$$

and if $N < \infty$ let

$$\rho = \sum_{k=1}^{N} \lambda_k P_k,\ \lambda_k \in \mathbb{C},\ \lambda_k \neq \lambda_\ell \ \text{ for } \ k \neq \ell, \quad \lambda_1 = 0, \quad \operatorname{rank} P_k < \infty \ \text{ for } k \geq 2. \tag{7.13}$$

Thus $\rho \in L^1(\mathcal{M})$. It is easy to check that

$$L^\infty(\mathcal{M})_\rho = \operatorname{im} R^* = \left\{ \sum_{k=1}^{N} P_k X P_k \mid X \in L^\infty(\mathcal{M}) \right\}.$$

So all conclusions of Theorem 7.4 hold, that is, the coadjoint orbit $\mathcal{O} := \{g\rho g^{-1} \mid g \in GL^\infty(\mathcal{M})\} \subset L^1(\mathcal{M})$ through ρ is weakly immersed in $L^1(\mathcal{M})$ and is weakly symplectic relative to the two-form

$$\omega_{\mathcal{O}}(g\rho g^{-1})([gXg^{-1}, g\rho g^{-1}], [gYg^{-1}, g\rho g^{-1}]) = \operatorname{tr}(\rho[X,Y]) \tag{7.14}$$

for $\rho \in L^1(\mathcal{M})$ given by (7.12) or (7.13), $g \in GL^\infty(\mathcal{M})$, $X, Y \in L^\infty(\mathcal{M})$. In addition, for $\mathcal{M}$ a complex separable Hilbert space, recall that the group $GL^\infty(\mathcal{M})$ is path connected (see e.g. Boos and Bleecker [B-B] §I.6) and hence the coadjoint orbit is also connected; thus it is a symplectic leaf of the Banach Lie-Poisson space $L^1(\mathcal{M})$.

The characteristic subspace $S_\rho = \{[X, \rho] \mid X \in L^\infty(\mathcal{M})\}$ is contained in

$$\ker R \subsetneq \left\{ \sum_{k \neq \ell} P_k X P_\ell \mid X \in L^\infty(\mathcal{M}) \right\} = \ker R^*$$

and if $N \in \mathbb{N}$ one has

$$\sum_{k \neq \ell} P_k X P_\ell = [\rho, Y]$$

for some $Y \in L^\infty(\mathcal{M})$ which is related to X through the system of equations

$$P_k X P_\ell = (\lambda_k - \lambda_\ell) P_k Y P_\ell$$

for all $k \neq \ell$. Note that if $N = \infty$, the above system is not solvable for some $Y \in L^\infty(\mathcal{M})$. Therefore, if $N \in \mathbb{N}$, $S_\rho = \ker R$ and by Theorem 7.5 one concludes that the connected coadjoint orbit is immersed in $L^1(\mathcal{M})$ and that it is strongly symplectic. ♦

Remark. The weak symplectic form ω_ρ given by (7.5) or (7.6) is, in general, not strong since $\omega_\rho([e])$ is, in general, not a strong bilinear form on $T_{[e]}(G/G_\rho)$. To begin with, one notices that $T_e\pi(\xi) \in T_{[e]}(G/G_\rho) \mapsto [\xi] \in \mathfrak{g}/\mathfrak{g}_\rho$ is a linear continuous bijective map and hence a Banach space isomorphism. Thus $\omega_\rho([e])$ can be viewed as a bilinear continuous map $\omega_\rho([e]) : \mathfrak{g}/\mathfrak{g}_\rho \times \mathfrak{g}/\mathfrak{g}_\rho \to \mathbb{R}$ given by $\omega_\rho([e])([\xi], [\eta]) = \langle \rho, [\xi, \eta] \rangle$. The map $[\xi] \in \mathfrak{g}/\mathfrak{g}_\rho \mapsto \omega_\rho([e])([\xi], [\cdot]) = \langle \operatorname{ad}^*_\xi \rho, \cdot \rangle \in (\mathfrak{g}/\mathfrak{g}_\rho)^*$ is clearly linear continuous and injective. Thus, if the symplectic form ω_ρ were strong, then the Banach spaces $\mathfrak{g}/\mathfrak{g}_\rho$ and $(\mathfrak{g}/\mathfrak{g}_\rho)^*$ would necessarily be isomorphic. Here is a concrete situation in which this cannot occur.

Consider the case described in Example 7.9 for the trace class operator ρ given by (7.12). Then $L^\infty(\mathcal{M})/L^\infty(\mathcal{M})_\rho \cong \ker R^*$ since $L^\infty(\mathcal{M})_\rho = \operatorname{im} R^*$. The map

$$\flat : \ker R^* \ni X \mapsto \operatorname{Tr}(\rho[X, \cdot]) = \operatorname{Tr}([\rho, X]\cdot) \in (\ker R^*)^*$$

given by the defining formula (7.14) of the symplectic form $\omega_{\mathcal{O}}$, has the following explicit form

$$\flat : \ker R^* \ni X \mapsto \sum_{l \neq m}^{\infty} (\lambda_l - \lambda_m) P_l X P_m \in \ker R \subsetneq (\ker R^*)^* = (\operatorname{im} R^*)^\circ,$$

where the annihilator is taken in $(L^\infty(\mathcal{M}))^*$. This formula shows that $\flat : \ker R^* \to (\ker R^*)^*$ is not surjective if ρ is given by (7.12). Hence, in this case, the orbit symplectic form $\omega_{\mathcal{O}}$ given by (7.14) is not strong. ♦

It was shown in Bona [2000] that unitary group coadjoint orbits through Hermitian finite rank operators are always strong symplectic manifolds. We present this case below.

Example 7.10 We apply the considerations of this section to the real closed Banach Lie subgroup $U^\infty(\mathcal{M}) := \{U \in L^\infty(\mathcal{M}) \mid UU^* = U^*U = I\}$ of unitary elements of $GL^\infty(\mathcal{M})$ (Bourbaki [Bo3], Chapter III, §3.10, Corollary 2). Its Lie algebra consists of the skew Hermitian bounded operators $\mathfrak{u}^\infty(\mathcal{M}) := \{X \in L^\infty(\mathcal{M}) \mid X + X^* = 0\}$; this is a closed split real Banach Lie subalgebra of $L^\infty(\mathcal{M})$. To study this case, we place ourselves in the context of Example 4.9, that is, we take $\mathfrak{g} = L^\infty(\mathcal{M})$, $\mathfrak{g}_* = L^1(\mathcal{M})$, $\mathfrak{g}_{*\mathbb{R}} = L^1(\mathcal{M})_\mathbb{R}$, $\mathfrak{g}_\mathbb{R} = L^\infty(\mathcal{M})_\mathbb{R}$ (in other words, the Banach spaces $L^1(\mathcal{M})$ and $L^\infty(\mathcal{M})$ thought of as a real Banach spaces), the continuous $\mathbb{R}$-linear involution $\sigma : L^1(\mathcal{M})_\mathbb{R} \to L^1(\mathcal{M})_\mathbb{R}$ given by $\sigma\rho = -\rho^*$, for $\rho \in L^1(\mathcal{M})$, and the complex structure I given by $I\rho = i\rho$. It is easily verified that the involution σ satisfies the conditions (i), (ii), and (iii) of Example 4.9. Then, by construction, $\mathfrak{g}_*^\sigma = \{\rho \in L^1(\mathcal{M}) \mid \rho + \rho^* = 0\} =: \mathfrak{u}^1(\mathcal{M})$ and, as was shown in Example 4.9, $\mathfrak{u}^1(\mathcal{M})$ is a real Banach Lie-Poisson space and the map $R : L^1(\mathcal{M})_\mathbb{R} \to \mathfrak{u}^1(\mathcal{M})$ given by $R = (id + \sigma)/2$ is a linear Poisson map.

The same type of argument as in Example 7.9 shows that one can directly apply Theorems 7.4 and 7.5 to $G = U^\infty(\mathcal{M})$ and $\mathfrak{g}_* = \mathfrak{u}^1(\mathcal{M})$. The symplectic leaves in this case correspond to the infinite dimensional flag manifolds and the strong symplectic form given by (7.14) (ρ of finite rank and the arguments in the correct spaces) coincides with the imaginary part of the natural Kähler metric on these manifolds. A particular example of such an infinite dimensional flag manifold is the projectivized Hilbert space $\mathbb{CP}(\mathcal{M})$ thought of as immersed in $L^1(\mathcal{M})$ as the coadjoint orbit through $\rho := |\psi\rangle\langle\psi|/\langle\psi|\psi\rangle$ for any $|\psi\rangle \in \mathcal{M}$. ♦

We next discuss the cotangent bundle of a Banach Lie group and introduce a remarkable submanifold, called in the sequel the precotangent bundle. Consider a Banach Lie group G with Banach Lie algebra $\mathfrak{g}$ admitting a predual $\mathfrak{g}_*$ and assume that $\operatorname{Ad}_G^*(\mathfrak{g}_*) \subset \mathfrak{g}_*$. If L_g and R_g denote the left and right translations by $g \in G$ respectively, it follows that $T_g L_{g^{-1}} : T_g G \to T_e G = \mathfrak{g}$ and $T_g R_{g^{-1}} : T_g G \to T_e G = \mathfrak{g}$ are a Banach space isomorphisms. Their duals $T_g^* L_{g^{-1}} : \mathfrak{g}^* \to T_g^* G$ and $T_g^* R_{g^{-1}} : \mathfrak{g}^* \to T_g^* G$ are therefore also Banach space isomorphisms. Define $T_{g_*}G := T_g^* L_{g^{-1}} \mathfrak{g}_*$, $T_* G := \cup_{g \in G} T_{g_*} G$, and conclude, as usual, that T_*G is a vector bundle over G which is also a subbundle of T^*G (see, e.g. Abraham, Marsden, Ratiu [A-M-R] for such an argument); it will be called the ***precotangent bundle*** of G. This construction could have been equally well done using right translations since $T_g^* R_{g^{-1}} \operatorname{Ad}_{g^{-1}}^* \mathfrak{g}_* = T_g^* L_{g^{-1}} \mathfrak{g}_*$ and, by hypothesis, $\operatorname{Ad}_g^* \mathfrak{g}_* = \mathfrak{g}_*$ for any $g \in G$. The precotangent bundle T_*G has been constructed using the left trivialization $L : T_*G \to G \times \mathfrak{g}_*$, $L(\rho_g) := (g, T_e^* L_g \rho_g)$ with inverse $L^{-1}(g, \rho) = T_g^* L_{g^{-1}} \rho$, for $\rho_g \in T_{g_*}G$ and $\rho \in \mathfrak{g}_*$. Completely analogous formulas hold for the right trivialization $R : T_*G \to G \times \mathfrak{g}_*$, $R(\rho_g) := (g, T_g^* R_{g^{-1}} \rho_g)$, $R^{-1}(g, \rho) = T_e^* R_{g^{-1}} \rho$; L and R are vector bundle isomorphisms covering the identity of G. Denote by $\pi : T^*G \to G$ the cotangent bundle projection and use the same letter to denote its restriction to T_*G.

The usual construction of the canonical one-form on the cotangent bundle T^*G works also in the case of the precotangent bundle T_*G. Indeed, define the one form Θ on T^*G or on T_*G by

$$\Theta(\rho_g)(v) := \langle \rho_g, T_{\rho_g} \pi(v) \rangle \tag{7.15}$$

for any $\rho_g \in T_g^*G$ (respectively $T_{g_*}G$), $v \in T_{\rho_g}(T^*G)$ (respectively $T_{\rho_g}(T_*G)$) and where the pairing is between T_g^*G and T_gG (respectively $T_{g_*}G$ and T_gG). Left trivialized, this formula reads

$$\Theta_L(g, \rho)(u_g, \mu) := (L_*\Theta)(g, \rho)(u_g, \upsilon) = \langle \rho, T_g L_{g^{-1}} u_g \rangle \tag{7.16}$$

for $g \in G$, $u_g \in T_gG$, and $\rho, \mu \in \mathfrak{g}^*$ (respectively $\mathfrak{g}_*$), where the pairing is now between $\mathfrak{g}^*$ (respectively $\mathfrak{g}_*$) and $\mathfrak{g}$. Define the canonical symplectic form on T^*G or T_*G by $\Omega := -d\Theta$ and let $\Omega_L := L_*\Omega$. A computation identical to the one in the finite dimensional case using the identity $\Omega_L = -\mathbf{d}\Theta_L$ (see Abraham and Marsden [A-M], §4.4), leads to the expression of the canonical two-form in the left trivialization

$$\begin{aligned} \Omega_L(g, \rho)\left((u_g, \mu), (v_g, \nu)\right) &:= (L_*\Omega)(g, \rho)\left((u_g, \mu), (v_g, \nu)\right) \\ &= \langle \nu, T_g L_{g^{-1}} u_g \rangle - \langle \mu, T_g L_{g^{-1}} v_g \rangle + \langle \rho, [T_g L_{g^{-1}} u_g, T_g L_{g^{-1}} v_g] \rangle, \end{aligned} \tag{7.17}$$

where $g \in G$, $u_g, v_g \in T_gG$, and $\rho, \mu, \nu \in \mathfrak{g}^*$ (respectively $\mathfrak{g}_*$). This formula immediately shows that Ω_L and hence Ω is a weak symplectic form on both T^*G and T_*G. We shall see below that it is not strong in general, for different reasons, on both T^*G and T_*G.

To show that Ω_L is strong on $G \times \mathfrak{g}^*$, one needs to prove that for fixed $(g, \rho) \in G \times \mathfrak{g}^*$ the linear continuous map $(u_g, \mu) \in T_gG \times \mathfrak{g}^* \mapsto \Omega_L(g, \rho)\left((u_g, \mu), (\cdot, \cdot)\right) \in (T_gG \times \mathfrak{g}^*)^* \cong T_g^*G \times \mathfrak{g}^{**}$ is surjective, that is, given $\alpha_g \in T_g^*G$ and $\Gamma \in \mathfrak{g}^{**}$ one can find $u_g \in T_gG$ and $\mu \in \mathfrak{g}^*$ such that

$$\langle \nu, T_g L_{g^{-1}} u_g \rangle - \langle \mu + \operatorname{ad}^*_{T_g L_{g^{-1}} u_g} \rho, T_g L_{g^{-1}} v_g \rangle = \langle \alpha_g, v_g \rangle + \langle \Gamma, \nu \rangle$$

for all $v_g \in T_gG$, $\nu \in \mathfrak{g}^*$. If this were possible, a necessary condition is that $\Gamma = T_g L_{g^{-1}} u_g$ which is not the case if $\mathfrak{g}$ is strictly included in $\mathfrak{g}^{**}$. If the Banach space $\mathfrak{g}$ is reflexive then $\Gamma \in \mathfrak{g}$ and one can choose $\mu = -\operatorname{ad}^*_\Gamma \rho - T_e^* L_g \alpha_g$. Thus, *if $\mathfrak{g}$ is reflexive, the canonical weak symplectic form on T^*G is strong.*

Next we analyze Ω_L on $G \times \mathfrak{g}_*$. As before, we fix $(g, \rho) \in G \times \mathfrak{g}_*$ and study the linear continuous map $(u_g, \mu) \in T_gG \times \mathfrak{g}_* \mapsto \Omega_L(g, \rho)\left((u_g, \mu), (\cdot, \cdot)\right) \in (T_gG \times \mathfrak{g}_*)^* \cong T_g^*G \times \mathfrak{g}$.

To prove its surjectivity one needs to find for given $\alpha_g \in T_g^*G$ and $\xi \in \mathfrak{g}$ a vector $u_g \in T_gG$ and a form $\mu \in \mathfrak{g}^*$ such that

$$\langle \nu, T_gL_{g^{-1}}u_g\rangle - \langle \mu + \operatorname{ad}^*_{T_gL_{g^{-1}}u_g}\rho, T_gL_{g^{-1}}v_g\rangle = \langle \alpha_g, v_g\rangle + \langle \nu, \xi\rangle$$

for all $v_g \in T_gG$, $\nu \in \mathfrak{g}_*$. This identity implies that $u_g = T_eL_g\xi$, which, unlike the previous case, is possible. However, this identity also requires that $\mu = -\operatorname{ad}^*_\xi \rho - T_e^*L_g\alpha_g$ which is, in general, impossible to achieve since $T_e^*L_g\alpha_g \in \mathfrak{g}^*$ but is not necessarily an element of $\mathfrak{g}_*$.

In spite of this obstruction, there is a Poisson bracket on $G \times \mathfrak{g}_*$. Given $f, h \in C^\infty(G \times \mathfrak{g}_*)$ their Poisson bracket is given by

$$\begin{aligned}\{f,h\}(g,\rho) = \langle d_1f(g,\rho), T_eL_gd_2h(g,\rho)\rangle - \langle d_1h(g,\rho), T_eL_gd_2f(g,\rho)\rangle \\ - \langle \rho, [d_2f(g,\rho), d_2h(g,\rho)]\rangle\,, \end{aligned} \tag{7.18}$$

where $d_1f(g,\rho) \in T_g^*G$ and $d_2f(g,\rho) \in (\mathfrak{g}_*)^* = \mathfrak{g}$ are the first and second partial derivatives of f, the pairing in the first two terms is between T_g^*G and T_gG, whereas in the third term it is between $\mathfrak{g}_*$ and $\mathfrak{g}$. Conditions (i) and (ii) in Definition 2.1 are satisfied. The proof of the Jacobi identity is a tedious direct verification. However, condition (iii) does not hold. Indeed formula (7.18) shows that the Hamiltonian vector field of h, if well defined, must have the expression

$$X_h(g,\rho) = \left(T_eL_gd_2h(g,\rho), \operatorname{ad}^*_{d_2h(g,\rho)}\rho - T_e^*L_gd_1h(g,\rho)\right) \tag{7.19}$$

The same obstruction encountered in the attempted proof of the strongness of Ω_L on $G \times \mathfrak{g}_*$ appears here in the second summand of the second component: the term $T_e^*L_gd_1h(g,\rho)$ is, in general, not an element of $\mathfrak{g}_*$.

Thus, *unlike T^*G, the precotangent bundle T_*G is naturally endowed with a Poisson bracket, but is not a Poisson manifold in the sense of Definition* 2.1. However, before Lie-Poisson reduction, the unreduced space $G \times \mathfrak{g}_*$ is only weakly symplectic, admits a Poisson bracket, but has no Hamiltonian vector fields. For functions admitting Hamiltonian vector fields, the Poisson bracket is naturally induced by the weak symplectic form which is the pull back of the canonical symplectic form on the cotangent bundle of the group. Finally, the projection $G \times \mathfrak{g}_* \to \mathfrak{g}_*$ preserves the Poisson brackets, if one changes the sign of the Lie-Poisson bracket on $\mathfrak{g}_*$. Similar considerations can be carried out for right translations and one obtains, as in finite dimensions, a dual pair (Weinstein [W1], Vaisman [V])

$$\mathfrak{g}_{*+} \longleftarrow T_*G \longrightarrow \mathfrak{g}_{*-}$$

where the signs refer to the Lie-Poisson bracket on $\mathfrak{g}_*$; the two arrows are the momentum maps for left and right translations (see §8 for a presentation of momentum maps in our setting and Marsden and Ratiu [M-R2] for more information in the finite dimensional case).

8 Momentum maps and reduction

In this section we shall explore the relationship between the classical theory of reduction for Poisson manifolds discussed in §3 and that of quantum reduction presented in §6. We shall show that this link will be crucial for the integration and quantization of Hamiltonian systems.

We shall introduce a definition of the momentum map which is a direct generalization of this concept from finite dimensional Poisson geometry.

Definition 8.1 *A **momentum map** is a Poisson map $J : P \to \mathfrak{b}$ from a Banach Poisson manifold P to a Banach Lie-Poisson space $\mathfrak{b}$.*

Recall that $\mathfrak{b}$ is a Banach space, that $C^\infty(\mathfrak{b})$ is endowed with a Poisson bracket $\{\cdot,\cdot\}$, that $\mathfrak{b}^*$ is closed under this bracket, and that $\operatorname{ad}^*_{\mathfrak{b}^*} \mathfrak{b} \subset \mathfrak{b}$. Thus $\mathfrak{b}^*$ is a Lie algebra and the prescription $\xi \in \mathfrak{b}^* \mapsto \xi_P := X_{\xi\circ J}$ defines a (left) Lie algebra action on P, that is, $[\xi_P, \eta_P] = -[\xi,\eta]_P$ for any $\xi, \eta \in \mathfrak{b}^*$. Indeed, recalling that the Hamiltonian vector field defined by $h \in C^\infty(P)$ is defined by $df(X_h) = \{f, h\}$, the Jacobi identity for the Poisson bracket is equivalent to $[X_f, X_g] = -X_{\{f,g\}}$. Using this relation and the fact that J is a Poisson map, we conclude

$$[\xi_P, \eta_P] = [X_{\xi\circ J}, X_{\eta\circ J}] = -X_{\{\xi\circ J,\eta\circ J\}} = -X_{\{\xi,\eta\}\circ J} = -[\xi,\eta]_P$$

which proves that $\xi \mapsto \xi_P$ is indeed a (left) Lie algebra action.

Theorem 8.2 *(Noether's Theorem) If $h \in C^\infty(P)$ is $\mathfrak{b}^*$-invariant, that is, $\langle dh, \xi_P\rangle = 0$ for all $\xi \in \mathfrak{b}^*$, then J is conserved along the flow of the Hamiltonian vector field X_h.*

Proof. The condition of invariance states that

$$0 = \langle dh, X_{\xi\circ J}\rangle = \{h, \xi\circ J\} = -\langle d(\xi\circ J), X_h\rangle$$

for every $\xi \in \mathfrak{b}^*$, which is equivalent to

$$\frac{d}{dt}\sigma_h(t)^*(\xi\circ J) = 0$$

for every $\xi \in \mathfrak{b}^*$, where $\sigma_h(t)$ is the flow of X_h. This in turn means that $\xi\circ J\circ\sigma_h(t) = \xi\circ J$, which says that $\langle \xi, (J\circ\sigma_h(t) - J)(b)\rangle = 0$ for all $b \in \mathfrak{b}$. Since ξ is arbitrary, one concludes that $(J\circ\sigma_h(t) - J)(b) = 0$ for every $b \in \mathfrak{b}$, that is, $J\circ\sigma_h(t) = J$ for all t. ■

A Hamiltonian system $(P, \{\cdot,\cdot\}_P, h)$ is called ***collective*** if there is a momentum map $J : P \to \mathfrak{b}$ and a function $H \in C^\infty(\mathfrak{b})$ such that $h = H\circ J$. Therefore, the corresponding Hamiltonian vector fields X_h^P and $X_H^{\mathfrak{b}}$ on P and $\mathfrak{b}$ respectively are J-related, that is,

$$TJ\circ X_h^P = X_H^{\mathfrak{b}}\circ J$$

which is equivalent to the commutation of the respective flows $\sigma_h(t)$ and $\sigma_H(t)$ of X_h^P and $X_H^{\mathfrak{b}}$ respectively, that is,

$$\sigma_H(t)\circ J = J\circ\sigma_h(t).$$

Thus the integration of Hamilton's equations on $\mathfrak{b}$ given by

$$\dot{b} = -\operatorname{ad}^*_{DH(b)} b, \tag{8.1}$$

where $b \in \mathfrak{b}$, leads to the partial integration of Hamilton's equations

$$\dot{f} = \{f, h\}_P \tag{8.2}$$

on P, where $f \in C^\infty(P)$. If $J : P \hookrightarrow \mathfrak{b}$ is an embedding, then solving (8.1) is equivalent to solving (8.2). In the other extreme case, namely when the trajectory $\{\sigma_H(t)(b) \mid t \in \mathbb{R}\}$ is a point $b \in \mathfrak{b}$, any trajectory $\sigma_H(t)(p)$ with $p \in J^{-1}(b)$ remains in the level set $J^{-1}(b)$. If there is a distribution E covering this level set satisfying the hypotheses of the classical reduction theorem, then the trajectories above drop to the quotient and one is led to the problem of solving a reduced system. Provided this can be integrated, for example, if the reduced system is integrable, then the standard reconstruction method (Abraham and Marsden [A-M], §4.3) gives the solution of the original system on the level set of the momentum map.

In the special case when $\mathfrak{b} = L^1(\mathcal{M})$, equation (8.1) assumes the form of the nonlinear Liouville-von Neumann equation

$$\dot{\rho} = [DH(\rho), \rho] \tag{8.3}$$

for $\rho \in L^1(\mathcal{M})$. The search for a collective Hamiltonian system on P is equivalent to finding a "Lax representation" on $L^1(\mathcal{M})$. The functions $T_k := (\operatorname{tr} \rho^k)/k$, $k \in \mathbb{N}$ are Casimir functions on $L^1(\mathcal{M})$. Therefore the functions $t_k := T_k \circ J$, $k \in \mathbb{N}$, are, in general, integrals of motion in involution for the system (8.2). If $J : P \hookrightarrow \mathfrak{b}$ is an embedding, then they are Casimirs of the system given by the phase space P. So, the problem of integration of an equation having Lax representation in $L^1(\mathcal{M})$ reduces to a large extent to the integration of the equation (8.3).

For direct investigations of the nonlinear Liouville-von Neumann equation in the physics literature, see for example Leble and Czachor [L-C] and references therein.

We shall illustrate the above considerations with the example of the infinite Toda lattice system associated to the Banach Lie-Poisson space $L^1(\mathcal{M})$.

Example 8.3 ***The infinite Toda lattice system***. The details for this example can be found in Odzijewicz and Ratiu [2003]. On the weak symplectic Banach space $\ell^\infty \times \ell^1 = (\ell^1)^* \times \ell^1$ define the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) := \frac{1}{2} \sum_{k=1}^{\infty} p_k^2 + \sum_{k=1}^{\infty} \alpha_k \lambda_k \exp(q_k - q_{k+1}), \tag{8.4}$$

where $\lambda_k \neq 0$, $\{\alpha_k\}, \{\lambda_k\} \in \ell^1$. Since the Banach space on which this Hamiltonian is defined is only weak symplectic, the existence of the Hamiltonian vector field associated to H is not guaranteed. Formally, this Hamiltonian is that of the Toda lattice. We will also assume that $\sum_{k=1}^{\infty} p_k = 0$, which means that the velocity of the center of mass is zero. We also observe that H is invariant relative to the action of $\mathbb{R}$ on the $\mathbf{q}$-space by translation. Thus we shall consider H defined on the weak symplectic Banach space

$(\ell^\infty/\mathbb{R}\mathbf{q}_0) \times \ell^1_0$, where $\ell^\infty/\mathbb{R}\mathbf{q}_0$ is the quotient Banach space by the closed subspace $\mathbb{R}\mathbf{q}_0$, where $q_{0k} = 1$ for all $k \in \mathbb{N}$, and $\ell^1_0 := \ker \mathbf{q}_0$. Relative to the canonical coordinates $x_k := q_k - q_{k+1}$ on $\ell^\infty/\mathbb{R}\mathbf{q}_0$ and p_k on ℓ^1_0 the weak symplectic form has the expression

$$\omega = -d\left(\sum_{k=1}^{\infty} p_k dx_k\right) = \sum_{k=1}^{\infty} dx_k \wedge dp_k. \tag{8.5}$$

Let $\mathcal{M}$ be a separable Hilbert space. Let $P_n := |n\rangle\langle n| : \mathcal{M} \to \mathcal{M}$ be the rank one projection onto the span of $|n\rangle$. If $\rho \in L^1(\mathcal{M})$ and $X \in L^\infty(\mathcal{M})$ write

$$\rho = \sum_{n,m=1}^{\infty} \rho_{nm}|n\rangle\langle m| \qquad \text{and} \qquad X = \sum_{n,m=1}^{\infty} X_{nm}|n\rangle\langle m|$$

where $\rho_{nm} := \langle n|\rho|m\rangle$ and $X_{nm} := \langle n|X|m\rangle$. From Example 6.7 we know that $L^1(\mathcal{M})_-$ is a Banach Lie-Poisson space whose dual is the Banach Lie algebra $L^\infty(\mathcal{M})_+$. The paring between these two spaces is given by

$$\langle \rho_-, X_+\rangle := \operatorname{tr}(\rho_- X_+) = \sum_{n\geq m} \rho_{nm} X_{mn},$$

where $\rho_- \in L^1(\mathcal{M})_-$ and $X_+ \in L^\infty(\mathcal{M})_+$. Using this pairing, a direct verification shows that the coinduced Poisson bracket of $L^1(\mathcal{M})_-$ has the expression

$$\begin{aligned} \{f, g\}_{L^1(\mathcal{M})_-}(\rho_-) &= \operatorname{tr}([(\pi^1_-)^*(df(\rho_-)), (\pi^1_-)^*(dg(\rho_-))]\rho_-) \\ &= \sum_{n\geq m\geq \ell} \left(\frac{\partial f}{\partial \rho_{nm}}\frac{\partial g}{\partial \rho_{m\ell}} - \frac{\partial g}{\partial \rho_{nm}}\frac{\partial f}{\partial \rho_{m\ell}}\right)\rho_{n\ell}, \end{aligned} \tag{8.6}$$

where $\pi^1_- : L^1(\mathcal{M}) \to L^1(\mathcal{M})_-$ is the projector given by

$$\pi^1_-(\rho) := \sum_{n\geq m} \rho_{nm}|n\rangle\langle m|,$$

$(\pi^1_-)^* : (L^1(\mathcal{M})_-)^* \cong L^\infty(\mathcal{M})_+ \to L^\infty(\mathcal{M})$ is its dual, and the formula

$$(\pi^1_-)^*(df(\rho_-))_{mn} = \frac{\partial f}{\partial \rho_{nm}}(\rho_-)$$

was used in the proof of the second equality.

Define the Flaschka transformation $J : (\ell^\infty/\mathbb{R}\mathbf{q}_0) \times \ell^1_0 \to L^1(\mathcal{M})_-$ by

$$J(\mathbf{z}) = \sum_{k=1}^{\infty} \left(p_k|k\rangle\langle k| + \lambda_k e^{x_k}|k+1\rangle\langle k|\right). \tag{8.7}$$

One verifies that J is a smooth injective map whose tangent map at every point is a continuous injection.

Since the closed two-form ω given by (8.5) is only weak, the Poisson bracket of two functions $\varphi, \psi \in C^\infty(\ell^\infty/\mathbb{R}\mathbf{q}_0 \times \ell^1_0)$ cannot be defined, in general. However, for collective functions $f \circ J$ and $g \circ J$, where $f, g \in C^\infty(L^1(\mathcal{M})_-)$, one has

$$\langle \mathbf{d}(f \circ J), X_{g\circ J}\rangle = -X_{f\circ J}[g \circ J] = -\{f, g\}_{L^1(\mathcal{M})_-} \circ J$$

since in this case, the Hamiltonian vector field

$$X_{f\circ J} = \sum_{k=1}^{\infty} \left(\frac{\partial(f \circ J)}{\partial p_k} \frac{\partial}{\partial x_k} - \frac{\partial(f \circ J)}{\partial x_k} \frac{\partial}{\partial p_k} \right)$$

on the weak symplectic manifold $(\ell^\infty/\mathbb{R}\mathbf{q}_0 \times \ell^1_0, \omega)$ makes sense, as a derivation, if it acts on collective functions $g \circ J$. This is so because the sequence $\{\rho_{k+1,k}\}_{k\in\mathbb{N}} = \{\lambda_k e^{x_k}\}_{k\in\mathbb{N}}$ belongs to ℓ^1 and the four sequences $\{\partial f/\partial\rho_{k+1,k}\}_{k\in\mathbb{N}}$, $\{\partial g/\partial\rho_{k+1,k}\}_{k\in\mathbb{N}}$, $\{\partial f/\partial\rho_{k,k}\}_{k\in\mathbb{N}}$, $\{\partial g/\partial\rho_{k,k}\}_{k\in\mathbb{N}}$ all belong to ℓ^∞. Thus, the Flaschka transformation is a momentum map in a generalized sense.

Additionally, let us mention that the Toda lattice Hamiltonian (8.4) is of the form $H = h \circ J$, for

$$h(\rho_-) = \operatorname{tr}(\rho_- + a)^2, \tag{8.8}$$

where $a := \sum_{n=1}^\infty \alpha_k |k\rangle\langle k+1| \in L^1(\mathcal{M})$, that is, $\sum_{n=1}^\infty |\alpha_k| < \infty$. Thus, the Toda lattice is a Hamiltonian system on a Poisson submanifold of $L^1(\mathcal{M})_-$ endowed with the bracket (8.6) and relative to the Hamiltonian function (8.8).

We shall prove below a version of an involution theorem combining the Kostant-Symes and the Mischchenko-Fomenko involution theorems for $L^1(\mathcal{M})_-$ with the goal to show that the functions $h_k(\rho_-) := \operatorname{tr}(\rho_- + a)^k/k$, $k \in \mathbb{N}$, are in involution relative to the Poisson bracket $\{\cdot,\cdot\}_{L^1(\mathcal{M})_-}$. The proof turns out to follow the finite dimensional one (see Kostant [Ko3] or Ratiu [R]). Note that f is a Casimir function on $L^1(\mathcal{M})$ if and only if $X_f = 0$, which by (5.8) is equivalent to $[df(\rho), \rho] = 0$ for every $\rho \in L^1(\mathcal{M})$. Decompose $L^1(\mathcal{M}) = L^1(\mathcal{M})_- \oplus L^1(\mathcal{M})^+$ and $L^\infty(\mathcal{M}) = L^\infty(\mathcal{M})_+ \oplus L^\infty(\mathcal{M})^-$ where $L^1(\mathcal{M})^+ := \{\rho \in L^1(\mathcal{M}) \mid \rho_{nm} = 0 \text{ for } n \geq m\}$ are the strictly upper triangular linear trace class operators (no diagonal) and $L^\infty(\mathcal{M})^- := \{X \in L^\infty(\mathcal{M}) \mid X_{nm} = 0 \text{ for } m \geq n\}$ are the strictly lower triangular bounded linear operators (no diagonal). Let $\pi^1_- : L^1(\mathcal{M}) \to L^1(\mathcal{M})_-$, $\pi^{1^+} : L^1(\mathcal{M}) \to L^1(\mathcal{M})^+$, $\pi^\infty_+ : L^\infty(\mathcal{M}) \to L^\infty(\mathcal{M})_+$, and $\pi^{\infty-} : L^\infty(\mathcal{M}) \to L^\infty(\mathcal{M})^-$ be the projections associated to the Banach space direct sums $L^1(\mathcal{M}) = L^1(\mathcal{M})_- \oplus L^1(\mathcal{M})^+$ and $L^\infty(\mathcal{M}) = L^\infty(\mathcal{M})_+ \oplus L^\infty(\mathcal{M})^-$ respectively. With this notation the Poisson bracket (8.6) becomes

$$\{f, g\}_{L^1(\mathcal{M})_-}(\rho_-) = \operatorname{tr}\left(\left[\pi^\infty_+(d\tilde{f}(\rho_-)), \pi^\infty_+(d\tilde{g}(\rho_-))\right] \rho_- \right), \tag{8.9}$$

where on the right hand side, $\tilde{f}$ and $\tilde{g}$ are arbitrary extensions of $f, g : L^1(\mathcal{M})_- \to \mathbb{R}$ to $L^1(\mathcal{M})$ respectively. Thus $d\tilde{f}(\rho_-) \in L^\infty(\mathcal{M})$ and $\pi^\infty_+(d\tilde{f}(\rho_-)) \in L^\infty(\mathcal{M})_+$ and similarly for g.

Proposition 8.4 *Let $a \in L^1(\mathcal{M})$ be a given element satisfying*

$$\operatorname{tr}(a[L^\infty(\mathcal{M})_+, L^\infty(\mathcal{M})_+]) = 0 \quad \text{and} \quad \operatorname{tr}(a[L^\infty(\mathcal{M})^-, L^\infty(\mathcal{M})^-]) = 0.$$

For any two Casimir functions f, g on $L^1(\mathcal{M})$ the functions $f_a(\rho_-) := f(\rho_- + a)$, $g_a(\rho_-) := g(\rho_- + a)$ are in involution on $L^1(\mathcal{M})_-$.

Proof. Note that in (8.9) one can take $\tilde{f}_a(\rho) = f(\rho + a)$ for any $\rho \in L^1(\mathcal{M})$. Since $\operatorname{tr}(a[L^\infty(\mathcal{M})_+, L^\infty(\mathcal{M})_+]) = 0$, it follows

$$\begin{aligned}
\{f_a, g_a\}_{L^1(\mathcal{M})_-}(\rho_-) &= \operatorname{tr}\left(\left[\pi_+^\infty(d\tilde{f}_a(\rho_-)), \pi_+^\infty(d\tilde{g}_a(\rho_-))\right]\rho_-\right)\\
&= \operatorname{tr}\left(\left[\pi_+^\infty(df(\rho_- + a)), \pi_+^\infty(dg(\rho_- + a))\right](\rho_- + a)\right)\\
&= \operatorname{tr}\left(\pi_+^\infty(df(\rho_- + a))\left[dg(\rho_- + a) - \pi^{\infty -}(dg(\rho_- + a)), \rho_- + a\right]\right)\\
&= -\operatorname{tr}\left(\pi_+^\infty(df(\rho_- + a))\left[\pi^{\infty -}(dg(\rho_- + a)), \rho_- + a\right]\right)\\
&= -\operatorname{tr}\left(\left[\pi_+^\infty(df(\rho_- + a)), \pi^{\infty -}(dg(\rho_- + a))\right](\rho_- + a)\right)\\
&= \operatorname{tr}\left(\left[\pi^{\infty -}(dg(\rho_- + a)), \pi_+^\infty(df(\rho_- + a))\right](\rho_- + a)\right)\\
&= \operatorname{tr}\left(\pi^{\infty -}(dg(\rho_- + a))\left[\pi_+^\infty(df(\rho_- + a)), \rho_- + a\right]\right)\\
&= \operatorname{tr}\left(\pi^{\infty -}(dg(\rho_- + a))\left[df(\rho_- + a) - \pi^{\infty -}(df(\rho_- + a)), \rho_- + a\right]\right)\\
&= -\operatorname{tr}\left(\pi^{\infty -}(dg(\rho_- + a))\left[\pi^{\infty -}(df(\rho_- + a)), \rho_- + a\right]\right)\\
&= -\operatorname{tr}\left(\left[\pi^{\infty -}(dg(\rho_- + a)), \pi^{\infty -}(df(\rho_- + a))\right](\rho_- + a)\right) = 0
\end{aligned}$$

because $\operatorname{tr}(\rho_-[L^\infty(\mathcal{M})^-, L^\infty(\mathcal{M})^-]) = 0$ and $\operatorname{tr}(a[L^\infty(\mathcal{M})^-, L^\infty(\mathcal{M})^-]) = 0$. ■

In the case of the Toda lattice one takes $a = \sum_{n=1}^\infty \alpha_k |k+1\rangle\langle k| \in L^1(\mathcal{M})$ with $\sum_{n=1}^\infty |\alpha_k| < \infty$ and then it immediately follows that $\operatorname{tr}(a[L^\infty(\mathcal{M})^-, L^\infty(\mathcal{M})^-]) = 0$ and $\operatorname{tr}(a[L^\infty(\mathcal{M})_+, L^\infty(\mathcal{M})_+]) = 0$. Thus, the hypotheses of Proposition 8.4 are satisfied and we conclude that all the functions $h_k(\rho_-) := \operatorname{tr}(\rho_- + a)^k/k$, $k \in \mathbb{N}$, are in involution relative to the Poisson bracket $\{\cdot,\cdot\}_{L^1(\mathcal{M})_-}$ and hence the relation $J^*\omega_{\mathcal{O}_\Lambda} = \omega$ shows that $h_k \circ J$ are commuting conserved quantities for the Toda Hamiltonian $H := h_2 \circ J$. ♦

Next we discuss the Poisson reduction in a Banach Lie-Poisson space $\mathfrak{b}$ associated to a quantum reduction operator $R : \mathfrak{b} \to \mathfrak{b}$. Assume that $i : N \hookrightarrow \mathfrak{b}$ is a (locally closed) submanifold. Since $T\mathfrak{b}|_N = N \times \mathfrak{b}$, define the subbundle $E \subset T\mathfrak{b}|_N$ by $E_b := \{b\} \times \ker R$. Next, make the topological assumption that $E \cap TN$ is the tangent bundle to a regular foliation $\mathcal{F}$ and that the space of leaves $M := N/\mathcal{F}$ is a smooth manifold with the projection $\pi : N \to M$ a submersion.

Lemma 8.5 *The subbundle E is compatible with the Poisson structure of $\mathfrak{b}$.*

Proof. Let U be an open subset of $\mathfrak{b}$ and $f, g \in C^\infty(U, \mathbb{C})$ have the property that df, dg vanish on E, that is, $\langle df(b), \ker R\rangle = 0$ and $\langle dg(b), \ker R\rangle = 0$ for $b \in U \cap N$. Therefore, there exist functions $\tilde{f}, \tilde{g} \in C^\infty(R(U), \mathbb{C})$ such that $f = \tilde{f} \circ R$ and $g = \tilde{g} \circ R$. Recall from §7 that the quantum reduction $R : (\mathfrak{b}, \{\cdot,\cdot\}) \to (\operatorname{im} R, \{\cdot,\cdot\}_R)$ is a Poisson map. Thus, $\{f, g\} = \{\tilde{f} \circ R, \tilde{g} \circ R\} = \{\tilde{f}, \tilde{g}\}_R \circ R$ whence $d\{f, g\}(b) = d\{\tilde{f}, \tilde{g}\}_R(R(b)) \circ R$ which implies that $d\{f, g\}(b)$ vanishes on $E_b = \ker R$. ▼

The following commutative diagram

$$\begin{tikzcd} N \arrow[r, "i"] \arrow[d, "\pi"'] & \mathfrak{b} \arrow[d, "R"] \\ N/\mathcal{F} \arrow[r, "J"] & \operatorname{im} R \end{tikzcd}$$

summarizes the maps involved in the theorem below.

Theorem 8.6 *Let $i : N \hookrightarrow \mathfrak{b}$ be a submanifold, $R : \mathfrak{b} \to \mathfrak{b}$ be a quantum reduction, and E be the distribution on $\mathfrak{b}$ given at every point by $\ker R$. Assume that:*

(i) *$E \cap TN$ is the tangent bundle of a regularfoliation $\mathcal{F}$ on N and the projection $\pi : N \to M := N/\mathcal{F}$ is a submersion;*

(ii) *$\sharp(\ker R)^\circ \subset \overline{\ker R + T_n N}$ for every $n \in N$.*

Then M is the reduction of $\mathfrak{b}$ by (N, E) and is thus a Banach Poisson manifold. The map $J : M \to \operatorname{im} R$ defined by $J([n]) := (R \circ i)(n)$ is Poisson, that is, J is a momentum map.

Proof. In view of the previous lemma, by Theorem 3.1, the two hypotheses guarantee that the triple $(\mathfrak{b}, N, E)$ is reducible. Thus M is a Banach Poisson manifold.

Since $\operatorname{im} R$ can be regarded as the quotient manifold obtained by collapsing the fibers of R, that is, by dividing with $\ker R$, the inclusion map i is obviously compatible with the equivalence relations on N and on $\mathfrak{b}$. Therefore, i induces a smooth map $J : M \to \operatorname{im} R$ on the quotients (see, for example, Abraham, Marsden, Ratiu [A-M-R] or Bourbaki [Bo2]) given by $J([b]) := R(i(b))$. The diagram above commutes by construction. It remains to be shown that J is a Poisson map.

Let $f, g \in C^\infty(\operatorname{im} R, \mathbb{C})$. Then $f \circ R \in C^\infty(\mathfrak{b}, \mathbb{C})$ is an extension of $f \circ J \circ \pi \in C^\infty(N, \mathbb{C})$ and similarly for g. Therefore, by the definition of the reduced bracket on M, since R is a Poisson map, we get

$$\{f \circ J, g \circ J\}_M \circ \pi = \{f \circ R, g \circ R\} \circ i = \{f, g\}_R \circ R \circ i = \{f, g\}_R \circ J \circ \pi.$$

Since π is a surjective map, this implies that $J : M \to \operatorname{im} R$ is a Poisson map. ■

Example 8.7 ***Averaging***. Let $i : N \hookrightarrow L^1(\mathcal{M})$ be the inclusion map of a smooth (regular) Banach submanifold in $L^1(\mathcal{M})$. Let G be a compact Lie group and denote by $\mu(g)$ the normalized Haar measure on G. Given are:

- a smooth left action $\sigma : G \to \operatorname{Diff}(N)$ of G on N,
- a smooth Lie group homomorphism $U : G \to GL^\infty(\mathcal{M})$ such that $U(g)$ is unitary for each $g \in G$.

Assume also that the inclusion $i : N \hookrightarrow L^1(\mathcal{M})$ is equivariant, that is, $i(\sigma(g)(n)) = U(g)i(n)U(g)^{-1}$, for all $n \in N$ and all $g \in G$.

The homomorphism U defines the operator $R : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ by

$$R(\rho) := \int_G U(g)\rho U(g)^* d\mu(g). \tag{8.10}$$

We shall prove below that this R is a quantum reduction operator.

We begin by showing that R is a projector. By invariance of the Haar measure under translations, we have for $\rho \in L^1(\mathcal{M})$,

$$\begin{aligned} R^2(\rho) &= \int_G U(h) \left(\int_G U(g)\rho U(g)^* d\mu(g) \right) U(h)^* d\mu(h) \\ &= \int_G \left(\int_G U(hg)\rho U(hg)^* d\mu(g) \right) d\mu(h) \\ &= \int_G R(\rho) d\mu(g) = R(\rho). \end{aligned}$$

Next, we show that $\|R\| = 1$. Since R is a projector we have $\|R\| \geq 1$. To prove equality, note first that if $\rho \geq 0$, that is, $\langle \psi | \rho | \psi \rangle \geq 0$ for all $|\psi\rangle \in \mathcal{M}$, then we also have $\langle \psi | U(g)\rho U(g)^* | \psi \rangle \geq 0$ for all $|\psi\rangle \in \mathcal{M}$ and integration over G yields $R(\rho) \geq 0$. Thus we showed that $\rho \geq 0$ implies $R(\rho) \geq 0$. Continuity of the trace in the $\| \cdot \|_1$–norm gives then for $\rho \geq 0$,

$$\|R(\rho)\|_1 = \operatorname{tr} R(\rho) = \int_G \operatorname{tr}\left(U(g)\rho U(g)^*\right) d\mu(g) = \int_G \operatorname{tr} \rho \, d\mu(g) = \operatorname{tr} \rho = \|\rho\|_1,$$

which proves that $\|R\| = 1$.

Finally, we need to show that $\operatorname{im} R^*$ is a Banach Lie subalgebra of $L^\infty(\mathcal{M})$. It is easy to see that for any $X \in L^\infty(\mathcal{M})$

$$R^*(X) = \int_G U(g)^* X U(g) d\mu(g).$$

Using this formula we find

$$R^*(X)R^*(Y) = R^*(XR^*(Y)) = R^*(R^*(X)R^*(Y))$$

and the condition that $\operatorname{im} R^*$ is a Lie subalgebra of $L^\infty(\mathcal{M})$ follows immediately.

Thus all conditions in the definition of a quantum reduction map are satisfied and hence R given by (8.10) is a quantum reduction operator. One can regard $R : L^1(\mathcal{M}) \to \operatorname{im} R$ as a momentum map. Consider the distribution E on N given at every point $n \in N$ by $E_n = \{n\} \times \ker R$. Assume that $E \cap TN$ is the tangent bundle of a regular foliation $\mathcal{F}$ on N and that the projection $\pi : N \to N/\mathcal{F}$ is a submersion. If $\sharp(\ker R)^\circ \subset \overline{\ker R + T_n N}$ for every $n \in N$, the conditions of Theorem 8.6 are satisfied and we obtain a momentum map $J : N/\mathcal{F} \to \operatorname{im} R$.

Note that the tanget spaces to the G-orbits determine a distribution on N that is included in the distribution $E \cap TN$. Supposing that the quotient by the group action is regular, that is, that N/G is a smooth manifold with the canonical projection $N \to N/G$ a surjective submersion, it follows that there is smooth surjective map between quotient spaces $\Sigma : N/G \to N/\mathcal{F}$ and hence a map $J \circ \Sigma : N/G \to \operatorname{im} R$. If, in addition, N/G is a Poisson manifold and Σ is a Poisson map, then $J \circ \Sigma : N/G \to \operatorname{im} R$ is a momentum map. This is satisfied, for example, if $\Sigma = identity$. ♦

Certain momentum maps play a special role in the physical description of various systems. An important class of such momentum maps are the coherent states maps. We shall introduce this notion in the context of Banach Poisson manifolds, modeling it on the definition introduced by Odzijewicz [O2] for the case of a canonical map between a finite dimensional symplectic manifold and the projectivization of a complex Hilbert space.

Definition 8.8 *Let P be a Banach Poisson manifold and $\mathfrak{b}$ be a Banach Lie-Poisson space. A* ***coherent states map*** *of P into $\mathfrak{b}$ is a Poisson embedding $\mathcal{K} : P \to \mathfrak{b}$ with linearly dense range, that is, the closure of the span of $\operatorname{im} \mathcal{K}$ equals $\mathfrak{b}$.*

The situation investigated by Odzijewicz [O2] is the case when P is a finite dimensional Poisson manifold, $\mathfrak{b} = \mathfrak{h}^1(\mathcal{M})$ is the Banach space of Hermitian trace class operators on a separable complex Hilbert space $\mathcal{M}$, and $\mathcal{K}(p)$ is a rank one orthogonal projector for every $p \in P$. In this case, the range of $\mathcal{K}$ lies in the projectivization $\mathbb{CP}(\mathcal{M})$ of $\mathcal{M}$ by identifying a rank one projector with the point in projective space determined by its image. To illustrate this situation, let us recall how $\mathcal{K}$ is used in the quantization of a physical system. For the Poisson diffeomorphism $\sigma : P \to P$, we assume that there is a linear Poisson automorphism $\Sigma : \mathfrak{h}^1(\mathcal{M}) \to \mathfrak{h}^1(\mathcal{M})$, such that the diagram of canonical maps

$$
\begin{tikzcd}
P \arrow[r, "\mathcal{K}"] \arrow[d, "\sigma"'] & \mathfrak{h}^1(\mathcal{M}) \arrow[d, "\Sigma"] \\
P \arrow[r, "\mathcal{K}"] & \mathfrak{h}^1(\mathcal{M})
\end{tikzcd}
$$

commutes. By a theorem of Wigner, the automorphism Σ is of the form $\Sigma(\rho) = U\rho U^*$, where U is a unitary or anti-unitary operator on $\mathcal{M}$. Due to the hypothesis that $\operatorname{im} \mathcal{K}$ is linearly dense in $\mathfrak{b}$, if such an automorphism Σ exists, it is necessarily unique. It is natural to interpret Σ as the quantization of σ. Denote by $\operatorname{Aut}(\mathfrak{h}^1(\mathcal{M}))$ the linear Poisson isomorphisms of $\mathfrak{h}^1(\mathcal{M})$. The set of all Poisson diffeomorphisms σ for which a Σ as above exists, forms a subgroup $\operatorname{Diff}_{\mathcal{K}}(P, \{\cdot,\cdot\})$ of the Poisson diffeomorphism group $\operatorname{Diff}(P, \{\cdot,\cdot\})$ of P. The map

$$\mathcal{E} : \operatorname{Diff}_{\mathcal{K}}(P, \{\cdot,\cdot\}) \to \operatorname{Aut}(\mathfrak{h}^1(\mathcal{M}))$$

so defined, is a group homomorphism which will be called, according to Odzijewicz [O2], ***Ehrenfest quantization***.

Consider now the flow σ_t of the Hamiltonian vector field X_h on M and assume that $\sigma_t \in \mathrm{Diff}_{\mathcal{K}}(P, \{\cdot,\cdot\})$ for all t. It is known that the set of all Hamiltonian functions satisfying this condition form a Poisson subalgebra in the Poisson algebra of all smooth functions on P. Then

$$\mathcal{E}(\sigma_t)(\rho) = \exp(itH)\rho\exp(-itH) \tag{8.11}$$

for H a self-adjoint operator (unbounded, in general) whose domain includes the linear span of the set $\mathcal{K}(P)(\mathcal{M})$. The correspondence

$$\mathcal{Q} : h \mapsto H$$

defined above is linear and satisfies the relation

$$\mathcal{Q}(\{h_1, h_2\}) = i[\mathcal{Q}(h_1), \mathcal{Q}(h_2)],$$

that is, $\mathcal{Q}$ is the Lie algebra homomorphism induced by $\mathcal{E}$. For more details on the precise relationship between the coherent states map quantization and the Kostant-Souriau geometric quantization as well as $*$-product quantization, we refer to Odzijewicz [O1, O2, O4].

The traditional coherent states map sends classical states to pure quantum states. Definition 8.8 generalizes this idea by letting the coherent states map to send classical states to mixed quantum states. Mathematically, mixed quantum states are in $L^1(\mathcal{M})$, or, more generally, in a Banach Lie-Poisson space. Definition 8.8 further generalizes the usual approach by also allowing in this scheme infinite dimensional classical systems.

Example 8.9 (See Odzijewicz [O3]) Consider the coherent states map $\mathcal{K} : P \to \mathfrak{h}^1(\mathcal{M})$ from a finite dimensional Poisson manifold $(P, \{\cdot,\cdot\})$ into the Banach Lie-Poisson space $\mathfrak{h}^1(\mathcal{M})$ of Hermitian trace class operators on the separable complex Hilbert space $\mathcal{M}$. Assume that the functions $f_1, \ldots, f_k \in C^\infty(P)$ are in involution, that is, $\{f_i, f_j\} = 0$, for all $i, j = 1, \ldots, k$, and that their differentials $df_1(p), \ldots, df_k(p)$ are linearly independent for all $p \in N$, where $\iota : N \hookrightarrow P$ is a given submanifold of P, invariant under the Hamiltonian flows $\sigma_1(t), \ldots, \sigma_k(t)$, $t \in \mathbb{R}$, generated by $f_1, \ldots, f_k$ respectively.

Quantize the flows $\sigma_1(t), \ldots, \sigma_k(t)$ using the Ehrenfest quantization procedure (8.11), $\mathcal{E} : \sigma_i(t) \mapsto \Sigma_i(t)$, where it is assumed that the generators F_i, $i = 1, \ldots, k$ of the quantum flows $\Sigma_i(t)$ are all self adjoint operators with discrete spectrum, i.e.,

$$F_i = \sum_{n=1}^{\infty} \lambda_n^i P_n^i \in \mathfrak{h}^\infty(\mathcal{M}) \tag{8.12}$$

where $\{P_n^i\}_{n=1}^\infty$ is an orthonormal decomposition of the unit related to F_i.

Consider the orthogonal projector

$$P_{\lambda^1_{i_1} \ldots \lambda^k_{i_k}} := P^1_{\lambda^1_{i_1}} \ldots P^k_{\lambda^k_{i_k}}$$

of $\mathcal{M}$ onto the common eigensubspace

$$\mathcal{M}_{\lambda^1_{i_1} \ldots \lambda^k_{i_k}} := \{v \in \mathcal{M} \mid F_1 v = \lambda^1_{i_1} v, \ldots F_k v = \lambda^k_{i_k} v\}$$

of the generators $F_1, \ldots, F_k$. According to Example 6.3, the map $R_{\lambda^1_{i_1}\ldots\lambda^k_{i_k}} : \mathfrak{h}^1(\mathcal{M}) \to \mathfrak{h}^1(\mathcal{M})$ defined by

$$R_{\lambda^1_{i_1}\ldots\lambda^k_{i_k}}(\rho) := P_{\lambda^1_{i_1}\ldots\lambda^k_{i_k}} \rho P_{\lambda^1_{i_1}\ldots\lambda^k_{i_k}} \tag{8.13}$$

is a quantum reduction.

Assume now that conditions (i) and (ii) of Theorem 8.6 hold. In addition, assume that:

(iii) the foliation $\mathcal{F}$ is given by the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_k}$.

Then the quotient manifold $N/\mathcal{F} =: M$ is a Poisson manifold and the map

$$\mathcal{I}_{\lambda^1_{i_1}\ldots\lambda^k_{i_k}} := R_{\lambda^1_{i_1}\ldots\lambda^k_{i_k}} \circ \iota \tag{8.14}$$

is a momentum map of M with values in $\operatorname{im} R_{\lambda^1_{i_1}\ldots\lambda^k_{i_k}}$.

In the special case when $N = \mathbf{f}^{-1}(\mu)$, $\mu \in \mathbb{R}^k$, is the level set of the map $\mathbf{f} := (f_1, \ldots, f_k)$, conditions (i), (ii), and (iii) imply certain restrictions on μ. For example, it can happen that $\mu_1 = \lambda^1_{i_1}, \ldots, \mu_k = \lambda^k_{i_k}$. In this case, the existence of the momentum map (8.14), i.e., the existence of the Ehrenfest quantization for the quotient system $N/\mathcal{F} = M$, leads to a discretization (quantization) of $\mathbf{f} : P \to \mathbb{R}^k$. The above method has been applied to the quantization of the MIC-Kepler system in Odzijewicz and Swietochowski [O-S]. ♦

The above examples show the importance of the relation between the classical and quantum reduction procedures for the quantization and the integration of Hamiltonian systems. The present paper raises several important questions regarding these connections to which we shall return in future publications.

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