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VARIATIONAL PRINCIPLES FOR LIE–POISSON AND HAMILTON–POINCARÉ EQUATIONS

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Dedicated to Vladimir Arnold on his 65th birthday

Abstract. As is well-known, there is a variational principle for the Euler–Poincaré equations on a Lie algebra $\mathfrak{g}$ of a Lie group G obtained by reducing Hamilton's principle on G by the action of G by, say, left multiplication. The purpose of this paper is to give a variational principle for the Lie–Poisson equations on $\mathfrak{g}^*$, the dual of $\mathfrak{g}$, and also to generalize this construction.

The more general situation is that in which the original configuration space is not a Lie group, but rather a configuration manifold Q on which a Lie group G acts freely and properly, so that $Q \to Q/G$ becomes a principal bundle. Starting with a Lagrangian system on TQ invariant under the tangent lifted action of G, the reduced equations on $(TQ)/G$, appropriately identified, are the Lagrange–Poincaré equations. Similarly, if we start with a Hamiltonian system on T^*Q, invariant under the cotangent lifted action of G, the resulting reduced equations on $(T^*Q)/G$ are called the Hamilton–Poincaré equations.

Amongst our new results, we derive a variational structure for the Hamilton–Poincaré equations, give a formula for the Poisson structure on these reduced spaces that simplifies previous formulas of Montgomery, and give a new representation for the symplectic structure on the associated symplectic leaves. We illustrate the formalism with a simple, but interesting example, that of a rigid body with internal rotors.

1. Introduction

This paper presents some advances in geometric mechanics and in particular variational principles and reduction for systems with symmetry. It is a great pleasure to dedicate this paper to Vladimir Arnold, since his pioneering paper [Arn1] has influenced the development of this theory in an absolutely fundamental way.

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The present paper assumes that the reader is familiar with geometric mechanics; the relevant background can be found in [Arn3] and [MR]. The paper [CMR2] is a basic ingredient for the present discussion. It and [CMR1] can be consulted for additional information and history.

The usual way in which Lagrangian reduction proceeds is to begin with Hamilton's principle for a system on a configuration manifold Q and with a symmetry group G acting on Q and then to drop this variational principle to the quotient Q/G to derive a reduced variational principle. This theory has its origins in specific examples such as fluid mechanics (see, for example, [Arn2], [Bre]), while the systematic theory of Lagrangian reduction was begun in [MS1], [MS2] and further developed in [CMR2]. In the case $Q = G$, the reduced equations are the Euler–Poincaré equations and the associated Euler–Poincaré reduction theorem is well-known (see, for example, [MR]). In the general case, the reduced equations associated to this construction are called the *Lagrange–Poincaré equations* and their geometry has been fairly well developed.

This reduction of Hamilton's principle is often regarded as a Lagrangian analog of Poisson reduction on the Hamiltonian side. However, a more faithful analog is the reduction of Hamilton's phase space principle. The development of this theory is one of the main objectives of the present paper. In the case $Q = G$, this collapses to a variational principle for the Lie–Poisson equations on $\mathfrak{g}^*$.

It should be stressed that in this paper we do not set any momentum maps equal to constants; that is, we are outside the realm of symplectic and Routh reduction. For variational principles in this context, we refer to [MRS].

2. The Lie–Poisson Case

It is well understood the sense in which Lie–Poisson dynamics on $\mathfrak{g}^*$, the dual of a Lie algebra $\mathfrak{g}$, are Hamiltonian relative to the Lie–Poisson bracket on functions on $\mathfrak{g}^*$. Here we show how these same equations can be derived from a variational principle for Lie–Poisson dynamics, which is a reduction of a certain form of Hamilton's phase space variational principle.

Notation and Setting. Let G be a Lie group and let $L\colon TG \to \mathbb{R}$ be a given Lagrangian. Let $\mathbb{F}L\colon TG \to T^*G$ be the fiber derivative, that is, the Legendre transformation. Assume that $\mathbb{F}L$ is a diffeomorphism, that is, L is hyperregular. Let $\mathfrak{g}$ be the Lie algebra of G, regarded as T_eG, the tangent space to G at the identity element e. Assume that L is invariant under the tangent lift of left translations and let $l\colon \mathfrak{g} \to \mathbb{R}$ be the reduced Lagrangian, given by $l = L|\mathfrak{g}$.

It is well-known that the reduced Euler–Lagrange equations, called the *Euler–Poincaré* equations, can be derived by a very simple and effective reduced variational method; see, for example, [MR]. To explain this reduced variational principle, let $g(t)$ be a curve in G with fixed endpoints $g_0 = g(t_0)$ and $g_1 = g(t_1)$, and let $v(t)$ be the body velocity defined by $v(t) = g^{-1}(t)\dot{g}(t)$, where the notation $g^{-1}(t)\dot{g}(t)$ stands for $TL_{g(t)^{-1}}\dot{g}(t)$.

Recall that Hamilton's principle is defined by the requirement that a curve $g(t)$ be a critical point of the action

$$\int_{t_0}^{t_1} L(g(t),\, \dot g(t))\, dt$$

for variations $\delta g(t)$ such that $\delta g(t_i) = 0$ for $i = 0,\, 1$. It is well-known that this is equivalent to the Euler–Lagrange equations, which we write symbolically as

$$\frac{\partial L}{\partial g} - \frac{d}{dt}\frac{\partial L}{\partial \dot g} = 0.$$

See the discussion later in Section 5 and, for example, [MR] for how to write these equations intrinsically on the second order bundle.

Because of invariance, Hamilton's principle holds if and only if $v(t)$ is a critical point of the reduced action

$$\int_{t_0}^{t_1} l(v(t))\, dt$$

for variations that are restricted (or, if one prefers, are *constrained*) to be of the form

$$\delta v(t) = \dot\eta(t) + [v(t),\, \eta(t)],$$

where $\eta(t)$ is a curve in $\mathfrak{g}$ such that $\eta(t_i) = 0$, for $i = 0,\, 1$. This is proved by tracing through the variations δv that are induced by variations δg. In fact, η is the body representation of the variations on the group, that is, $\eta = g^{-1}\delta g$ (see [MR] for details).

By applying the usual integration by parts argument one sees that the reduced Hamilton principle is equivalent to the *Euler–Poincaré equations*:

$$\frac{d}{dt}\frac{\delta l}{\delta v} = \mathrm{ad}_v^*\, \frac{\delta l}{\delta v}, \tag{1}$$

where $\delta l/\delta v \in \mathfrak{g}^*$ is the usual differential of l evaluated at v, that is, $\delta l/\delta v = \mathbf{d}l(v)$. As the next theorem describes, these equations, together with the *reconstruction equation*

$$v = g^{-1}\dot g \tag{2}$$

are equivalent to the Euler–Lagrange equations on G. The Euler–Poincaré equations (1) together with this reconstruction equations (2) are called the *Euler–Arnold* equations.

Since L is hyperregular and invariant, the Hamiltonian $H\colon T^*G \to \mathbb{R}$ which is obtained from the Legendre transformation by the usual formula

$$H(g,\, p) = p \cdot \dot g - L(g,\, \dot g),$$

where $p = \mathbb{F}L(g,\, \dot g)$, is well defined; in this expression for H, as usual, one uses hyperregularity to express $\dot g$ as a function of $(g,\, p)$. If L is hyperregular and invariant, so is H. That is, the inverse Legendre transformation $\mathbb{F}H = (\mathbb{F}L)^{-1}\colon T^*G \to TG$ is also a diffeomorphism and H is invariant under the left action of G. If we define $h\colon \mathfrak{g}^* \to \mathbb{R}$ by $h = H|\mathfrak{g}^*$, it is clear that

$$h(\mu) = \langle \mu,\, v\rangle - l(v),$$

which is called the *reduced Legendre transformation.*

It is well-known that Hamilton's equations are equivalent to the phase space version of Hamilton's principle, namely

$$\delta \int_{t_0}^{t_1} (p \cdot \dot{g} - H(g, p))\, dt = 0,$$

where variations are taken amongst curves in T^*G and where the variations δg, δp satisfy $\delta g(t_i) = 0$, for $i = 0, 1$, and δp arbitrary.

The Lie–Poisson equations for a Hamiltonian $h\colon \mathfrak{g}^* \to \mathbb{R}$, where $h = H|\mathfrak{g}^*$, are normally derived by using the method of Lie–Poisson reduction, that is, as a special case of Poisson reduction. In this context, the Lie–Poisson bracket is seen as the reduction of the canonical cotangent bracket on T^*G. See, for example, [MR] for these derivations. The Lie–Poisson equations are

$$\dot{\mu} = \mathrm{ad}^*_{\delta h/\delta \mu}\, \mu,$$

where $\delta h/\delta \mu \in \mathfrak{g}$ is defined by

$$\left\langle \frac{\delta h}{\delta \mu}, \nu \right\rangle = \mathbf{d}h(\mu) \cdot \nu$$

where $\mathbf{d}h(\mu)\colon \mathfrak{g}^* \to \mathbb{R}$ is the usual derivative of h.

Four Action Principles and Four Sets of Equations. Sticking with the special case of systems on Lie groups for the moment, one has the following elementary, but important result.

Theorem 2.1. *With the above notation and hypotheses of hyperregularity, the following conditions are equivalent.*

(i) Hamilton's Principle. *The curve $g(t) \in G$ is a critical point of the action*

$$\int_{t_0}^{t_1} L(g(t), \dot{g}(t))\, dt$$

for variations $\delta g(t)$ such that $\delta g(t_i) = 0$ for $i = 0, 1$.

(ii) The Euler–Poincaré Variational Principle. *The curve $v(t) \in \mathfrak{g}$ is a critical point of the reduced action*

$$\int_{t_0}^{t_1} l(v(t))\, dt$$

for variations of the form $\delta v(t) = \dot{\eta}(t) + [v(t), \eta(t)]$, where $\eta(t)$ is a curve in $\mathfrak{g}$ such that $\eta(t_i) = 0$, for $i = 1, 2$.

(iii) Hamilton's Phase Space Principle. *The curve $(g(t), p(t)) \in T^*G$ is a critical point of the action*

$$\int_{t_0}^{t_1} (p \cdot \dot{g} - H(g, p))\, dt,$$

where variations $(\delta g, \delta p)$ satisfy $\delta g(t_i) = 0$, for $i = 0, 1$, and $\delta p(t)$ is arbitrary.

(iv) The Lie–Poisson Variational Principle. *The curve $(v(t), \mu(t)) \in \mathfrak{g} \times \mathfrak{g}^*$ is a critical point of the action*

$$\int_{t_0}^{t_1} (\langle \mu(t), v(t) \rangle - h(\mu(t)))\, dt$$

for variations of the form

$$\delta v(t) = \dot{\eta}(t) + [v(t),\, \eta(t)],$$

where $\eta(t)$ *is a curve in* $\mathfrak{g}$ *such that* $\eta(t_i) = 0$, *for* $i = 0,\, 1$, *and where the variations* $\delta\mu$ *are arbitrary.*

(v) *The Euler–Lagrange equations on* G *hold*:

$$\frac{\partial L}{\partial g} - \frac{d}{dt}\frac{\partial L}{\partial \dot{g}} = 0$$

(vi) *The Euler–Poincaré equations on* $\mathfrak{g}^*$ *hold*:

$$\frac{d}{dt}\frac{\delta l}{\delta v} = \operatorname{ad}_v^* \frac{\delta l}{\delta v},$$

where $v(t) = g^{-1}(t)\dot{g}(t)$.

(vii) *Hamilton's equations on* T^*G *hold*:

$$(\dot{g}(t),\, \dot{p}(t)) = \left(\frac{\partial H}{\partial p},\, -\frac{\partial H}{\partial g}\right).$$

(viii) *The Lie–Poisson equations on* $\mathfrak{g}^*$ *hold*:

$$\dot{\mu} = \operatorname{ad}^*_{\delta h/\delta\mu}\, \mu.$$

The equivalence between statements (i), (ii), (v) *and* (vi) *holds for general Lagrangians, not necessarily nondegenerate. The equivalence between the statements* (iii), (iv), (vii) *and* (viii) *holds for general Hamiltonians, not necessarily nondegenerate.*

Proof. We have already remarked that Hamilton's principle (i) is equivalent to the Euler–Lagrange equations (v). We have also remarked that Euler–Poincaré reduction theory shows that (i) and (ii) are equivalent and that the standard arguments in the calculus of variations shows that (ii) and (vi) are equivalent. It is standard that the Euler–Lagrange equations (v), under the assumption of hyperregularity, are equivalent to Hamilton's equations (vii). It is also standard that Hamilton's phase space principle (iii) is equivalent to Hamilton's equations (vii). Another standard item is that under hyperregularity, the Euler–Poincaré equations (vi) are equivalent to the Lie–Poisson equations (viii). To complete the proof, one can show directly by the usual arguments in the calculus of variations that the variational principle (iv) is equivalent to the Lie–Poisson equations (viii). □

While this technically gives a complete proof of the theorem, it does not provide much insight into where the interesting Lie–Poisson variational principle, item (iv), comes from. Notice that this principle involves twice as many variables as does the Euler–Poincaré variational principle, just as Hamilton's phase space principle involves varying curves in T^*G, while Hamilton's principle involves curves varying in G, a space of half the dimension.

To get this insight, we first recall that Hamilton's phase space principle states that

$$\delta \int (p \cdot \dot{g} - H(g,\, p))\, dt = 0. \tag{3}$$

In this principle, note that the pointwise function in the integrand, namely

$$F(g,\, \dot g,\, p) = p \cdot \dot g - H(g,\, p)$$

is defined on $TG \oplus T^*G$, regarded as a bundle over G, the base space common to TG and T^*G. The group G acts on $TG \oplus T^*G$ by simultaneously left translating on each factor by the tangent and cotangent lift. Explicitly, the action of an element $h \in G$ is given by

$$h \cdot (g,\, \dot g,\, p) = (hg,\, T_g L_h \cdot \dot g,\, T^*_{hg} L_{h^{-1}} \cdot p)$$

where $T_g L_h \colon T_g G \to T_{hg} G$ is the tangent of the left translation map $L_h \colon g \in G \mapsto hg \in G$ at the point g and $T^*_{hg} L_{h^{-1}} \colon T^*_g G \to T^*_{hg} G$ is the dual of the map $T_{hg} L_{h^{-1}} \colon T_{hg} G \to T_g G$. The map F is invariant under this action of G as is easily checked, assuming invariance of H. Thus, the function F drops to the quotient, namely to the function $f \colon \mathfrak{g} \oplus \mathfrak{g}^* \to \mathbb{R}$ given by $f(v,\, \mu) = \langle \mu,\, v \rangle - h(\mu)$, where h is the reduction of H from T^*G to $\mathfrak{g}^*$.

When the function F is taken into the phase space variational principle, one is varying curves $(g(t),\, p(t))$ and one of course insists that the slot $\dot g$ actually is the time derivative of $g(t)$. This restriction induces in a natural way a restriction on the variations of $v = g^{-1}\dot g$ and these restrictions are computed exactly as in the Euler–Poincaré theory to be given by $\delta v(t) = \dot\eta(t) + [v(t),\, \eta(t)]$, where $\eta(t)$ is a curve in $\mathfrak{g}$ such that $\eta(t_i) = 0$, for $i = 0,\, 1$. In fact, this computation gives the relation $\eta = g^{-1}\delta g$.

In summary, the preceding argument gives a *direct* verification of the equivalence of (iii) and (iv). This point of view will be important in the generalizations to follow.

One of the main goals of the paper is to generalize the preceding theorem, replacing the spaces TG and T^*G with TQ and T^*Q for a general configuration space Q on which a Lie group G acts. This main result is given in Theorem 8.1 below.

3. Lagrange–Poincaré Bundles

Now we will generalize the preceding construction to the case in which we have an action of a Lie group G on a configuration manifold Q, in such a way that, with this action, Q becomes a principal bundle. As suggested by the preceding arguments, the bundle $T^*Q \oplus TQ$ as a bundle over Q should play a key role as this is the domain of definition of the function appearing in Hamilton's phase space action principle.

One of our first goals will be to recall some results on Lagrangian reduction from [CMR2], namely we shall describe the Lagrange–Poincaré bundles, the geometry of variations and the Lagrange–Poincaré equations. A key point in doing this is to choose a principal connection A on the bundle $\pi \colon Q \to Q/G$ and, using it, decompose arbitrary variations of curves in Q into vertical and horizontal components. This gives rise, correspondingly, to two reduced equations, namely, *vertical Lagrange–Poincaré equations*, corresponding to vertical variations, and *horizontal Lagrange–Poincaré equations*, corresponding to horizontal variations, which are Euler–Lagrange equations on Q/G with an additional term involving the curvature

B of A. Using this, the geometric description of reduced horizontal and reduced vertical variations, and Hamilton's phase space action principle, generalizing Theorem 2.1 to obtain the main result Theorem 8.1, is fairly straightforward.

In this section we recall some results from differential geometry on principal and associated bundles. We will need to do this mainly to establish our notations and conventions.

Horizontal and Vertical Spaces. As indicated above, we let $\pi\colon Q \to Q/G$ be a left principal bundle. Recall that a (*principal*) *connection* A on Q is a Lie algebra valued one form $A\colon TQ \to \mathfrak{g}$ with the properties

(i) $A(\xi q) = \xi$ for all $\xi \in \mathfrak{g}$; that is, A takes infinitesimal generators of a given Lie algebra element to that element; note that we denote the infinitesimal action of $\xi \in \mathfrak{g}$ on Q at $q \in Q$ by concatenation, as ξq, and

(ii) $A(g \cdot v) = \mathrm{Ad}_g(A(v))$, where Ad_g denotes the adjoint action of G on $\mathfrak{g}$ and gv denotes the lifted action of $g \in G$ on $v \in TG$.

The restriction of the connection A to the tangent space T_qQ is denoted A_q. Recall that connections may be characterized by giving their vertical and horizontal spaces defined at $q \in Q$ by

$$\mathrm{Ver}_q = \mathrm{Ker}\, T_q\pi, \qquad \mathrm{Hor}_q = \mathrm{Ker}\, A_q.$$

Thus, $\mathrm{Ver}(TQ) = \bigcup_{q\in Q} \mathrm{Ver}_q$ is the subbundle of vectors tangent to the group orbits. The *vertical and horizontal components* of a vector v_q will be denoted $\mathrm{Ver}(v_q)$ and $\mathrm{Hor}(v_q)$ respectively. By definition,

$$\mathrm{Ver}(v_q) = A(v_q)q \quad \text{and} \quad \mathrm{Hor}(v_q) = v_q - A(v_q)q.$$

This provides a Whitney sum decomposition $TQ = \mathrm{Hor}(TQ) \oplus \mathrm{Ver}(TQ)$ where $\mathrm{Hor}(TQ) = \bigcup_{q\in Q} \mathrm{Hor}_q$ and $\mathrm{Ver}(TQ)$ are the horizontal and vertical subbundles of TQ; both are invariant under the action of G. A vector is called *horizontal* if its vertical component is zero, i. e., if $A(v_q) = 0$, and it is called *vertical* if its horizontal component is zero, i. e., if $T_q\pi(v_q) = 0$. Note that $T_q\pi\colon \mathrm{Hor}_q \to T_{\pi(q)}(Q/G)$ is an isomorphism.

Curvature. The *curvature* of A will be denoted B^A or simply B. By definition, it is the Lie algebra valued two form on Q defined by

$$B(u_q, v_q) = \mathbf{d}A(\mathrm{Hor}_q(u_q), \mathrm{Hor}_q(v_q)),$$

where $\mathbf{d}$ denotes the exterior derivative.

Cartan Structure Equations. The *Cartan structure equations* state that

$$B(u, v) = \mathbf{d}A(u, v) - [A(u), A(v)], \tag{4}$$

for arbitrary vector fields u, v on Q (not necessarily horizontal), where the bracket on the right hand side is the Lie bracket in $\mathfrak{g}$. We write this equation for short as $B = \mathbf{d}A - [A, A]$.

Horizontal Lifts. Given a vector $X \in T_x(Q/G)$, and $q \in \pi^{-1}(x)$, the *horizontal lift* X^h_q of X at q is the unique horizontal vector in T_qQ that projects via $T\pi$ to the vector $X(x)$; that is, $X^h_q \in (T_q\pi)^{-1}(X)$. We denote by X^h the vector field along $\pi^{-1}(x)$ formed by all horizontal lifts of X at points of $\pi^{-1}(x)$.

Let $x(t)$ be a C^1 curve in Q/G, where $t \in [a, b]$. Given $q_0 \in \pi^{-1}(x_0)$, where $x_0 = x(t_0)$, for some $t_0 \in [a, b]$, the *horizontal lift* of $x(t)$, which at $t = t_0$ coincides with q_0, is uniquely determined by requiring all its tangent vectors to be horizontal. This curve is denoted $x^h_{q_0}$ and is defined on $[a, b]$.

Consider a curve $q(t)$, where $t \in [a, b]$, and choose $t_0 \in [a, b]$. Then there is a unique horizontal curve $q_h(t)$ such that $q_h(t_0) = q(t_0)$ and $\pi(q_h(t)) = \pi(q(t))$ for all $t \in [a, b]$. Therefore, we can define a curve $g_q(t) \in G$, for $t \in [a, b]$ by the decomposition

$$q(t) = g_q(t)q_h(t) \tag{5}$$

for all $t \in [a, b]$. Evidently $g_q(t_0)$ is the identity. Also, notice that if $x(t) = \pi(q(t))$ and $q_0 = q(t_0)$ then $q_h(t) = x^h_{q_0}(t)$. One can check (see, for example, [CMR2]) that for any curve $q(t)$, $t \in [a, b]$ in Q we have

$$A(q, \dot{q}) = \dot{g}_q g_q^{-1}. \tag{6}$$

Associated Bundles. Consider a left representation $\rho\colon G \times M \to M$ of the Lie group G on a vector space M. Recall that the *associated vector bundle* with standard fiber M is, by definition,

$$Q \times_G M = (Q \times M)/G,$$

where the action of G on $Q \times M$ is given by $g(q, m) = (gq, gm)$. The class (or orbit) of (q, m) is denoted $[q, m]_G$ or simply $[q, m]$. The *projection* $\pi_M : Q \times_G M \to Q/G$ is defined by $\pi_M([q, m]_G) = \pi(q)$ and it is easy to check that it is well defined and is a surjective submersion.

Parallel Transport in Associated Bundles. Let $[q_0, m_0]_G \in Q \times_G M$ and let $x_0 = \pi(q_0) \in Q/G$. Let $x(t)$, $t \in [a, b]$, be a curve in Q/G and let $t_0 \in [a, b]$ be such that $x(t_0) = x_0$. The *parallel transport* of this element $[q_0, m_0]_G$ along the curve $x(t)$ is defined to be the curve

$$[q, m]_G(t) = \left[x^h_{q_0}(t), m_0\right]_G.$$

For t, $t + s \in [a, b]$, we adopt the notation

$$\tau^t_{t+s} : \pi_M^{-1}(x(t)) \to \pi_M^{-1}(x(t + s))$$

for the parallel transport map along the curve $x(s)$ of any point

$$[q(t), m(t)]_G \in \pi_M^{-1}(x(t))$$

to the corresponding point

$$\tau^t_{t+s}[q(t), m(t)]_G \in \pi_M^{-1}(x(t + s)).$$

Thus,

$$\tau^t_{t+s}[q(t), m(t)]_G = \left[x^h_{q(t)}(t + s), m(t)\right]_G.$$

We shall sometimes use the notation $\rho'(\xi)$ for the second component of the infinitesimal generator of an element $\xi \in \mathfrak{g}$, that is, $\xi m = (m, \rho'(\xi)m)$. Here we are using the identification $TM = M \times M$, appropriate for vector spaces. Thus, we are thinking of the infinitesimal generator as a map $\rho' \colon \mathfrak{g} \to \mathrm{End}(M)$ (the linear vector fields on M are identified with the space of linear maps of M to itself). Thus, we have a linear representation of the Lie algebra $\mathfrak{g}$ on the vector space M. Let $[q(t), m(t)]_G$, $t \in [a, b]$, be a curve in $Q \times_G M$, denote by

$$x(t) = \pi_M([q(t), m(t)]_G) = \pi(q(t))$$

its projection on the base Q/G, and let, as above, τ^t_{t+s}, where $t, t+s \in [a, b]$, denote parallel transport along $x(t)$ from time t to time $t+s$.

The Covariant Derivative in Associated Bundles. The *covariant derivative* of $[q(t), m(t)]_G$ along $x(t)$ is defined as follows

$$\frac{D[q(t), m(t)]_G}{Dt} = \lim_{s\to 0} \frac{\tau_t^{t+s}([q(t+s), m(t+s)]_G) - [q(t), m(t)]_G}{s} \in \pi_M^{-1}(x(t)).$$

Notice that if $[q(t), m(t)]_G$ is a vertical curve, then its base point is constant; that is, for each $t \in [a, b]$,

$$x(t+s) = \pi_M([q(t+s), m(t+s)]_G) = x(t),$$

so that $x^h_{q(t)}(t+s) = q(t)$ for all s. Therefore,

$$\tau_t^{t+s}[q(t+s), m(t+s)]_G = \left[x^h_{q(t+s)}(t), m(t+s)\right]_G = [q(t), m(t+s)]_G$$

and so we get the well-known fact that *the covariant derivative of a vertical curve in the associated bundle is just the fiber derivative*, that is,

$$\frac{D[q(t), m(t)]_G}{Dt} = [q(t), m'(t)]_G,$$

where $m'(t)$ is the time derivative of m.

Affine Connections in Vector Bundles. Recall from, for example, [KN], that the covariant derivative for curves in a given vector bundle $\tau \colon V \to S$ is related to the notion of an *affine connection* ∇ by

$$\nabla_X v(s_0) = \left.\frac{D}{Dt} v(t)\right|_{t=t_0},$$

where, for each $s_0 \in S$, each $X \in \mathfrak{X}^\infty(S)$ (the smooth vector fields on S), and each $v \in \Gamma(V)$ (the space of sections of V), and $s(t)$ is any curve in S such that $\dot{s}(t_0) = X(s_0)$ and $v(t) = v(s(t))$ for all t.

Affine Connections on Associated Bundles. The following formula gives the relation between the covariant derivative of the affine connection and the principal connection.

$$\frac{D[q(t), m(t)]_G}{Dt} = \left[q(t), -\rho'(A(q(t), \dot{q}(t)))m(t) + \dot{m}(t)\right]_G.$$

The previous definition of the covariant derivative of a curve in the associated vector bundle $Q \times_G M$ thus leads to an affine connection on $Q \times_G M$. Let us call this connection $\tilde{\nabla}^A$ or simply $\tilde{\nabla}$. Let $\varphi \colon Q/G \to Q \times_G M$ be a section of the associated bundle and let $X(x) \in T_x(Q/G)$ be a given vector tangent to Q/G at x. Let $x(t)$ be a curve in Q/G such that $\dot{x}(0) = X(x)$; thus, $\varphi(x(t))$ is a curve in

$Q \times_G M$. The *covariant derivative* of the section φ with respect to X at x is, by definition,

$$\tilde{\nabla}^A_{X(x)} \varphi = \left. \frac{D\varphi(x(t))}{Dt} \right|_{t=0} . \tag{7}$$

Notice that we only need to know φ along the curve $x(t)$ in order to calculate the covariant derivative.

The notion of a *horizontal curve* $[q(t), m(t)]_G$ on $Q \times_G M$ is defined by the condition that its covariant derivative vanishes. A vector tangent to $Q \times_G M$ is called *horizontal* if it is tangent to a horizontal curve. Correspondingly, the *horizontal space* at a point $[q, m]_G \in Q \times_G M$ is the space of all horizontal vectors at $[q, m]_G$.

The Adjoint Bundle. The associated bundle with standard fiber $\mathfrak{g}$, where the action of G on $\mathfrak{g}$ is the adjoint action, is called the *adjoint bundle*, and is denoted $\tilde{\mathfrak{g}} := \mathrm{Ad}(Q)$. We let $\tilde{\pi}_G \colon \tilde{\mathfrak{g}} \to Q/G$ denote the projection given by $\tilde{\pi}_G([q, \xi]_G) = [q]_G$.

Let $[q(s), \xi(s)]_G$ be any curve in $\tilde{\mathfrak{g}}$. Then one checks that (again see [CMR2])

$$\frac{D[q(s), \xi(s)]_G}{Ds} = \big[q(s), -[A(q(s), \dot{q}(s)), \xi(s)] + \dot{\xi}(s)\big]_G. \tag{8}$$

In addition, the adjoint bundle is a *Lie algebra bundle*; that is, each fiber $\tilde{\mathfrak{g}}_x$, $x \in Q/G$, of $\tilde{\mathfrak{g}}$ carries a natural Lie algebra structure defined by

$$[[q, \xi]_G, [q, \eta]_G] = [q, [\xi, \eta]]_G. \tag{9}$$

The Bundle TQ/G. The tangent lift of the action of G on Q defines an action of G on TQ and so we can form the quotient $(TQ)/G =: TQ/G$. There is a well defined map $\tau_{Q/G} \colon TQ/G \to Q/G$ induced by the tangent of the projection map $\pi \colon Q \to Q/G$ and given by $[v_q]_G \mapsto [q]_Q$. The vector bundle structure of TQ is inherited by this bundle.

One can express reduced variational principles in a natural way in terms of this bundle without any reference to a connection on Q. It is, however, also interesting to introduce an (arbitrarily chosen) connection on Q relative to which one can more concretely realize the space TQ/G. This is also useful for writing the reduced Euler–Lagrange equations, called Lagrange–Poincaré equations.

One of the main tools needed for realizing the structure of the bundle is the map $\alpha_A \colon TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$ defined by

$$\alpha_A([q, \dot{q}]_G) = T\pi(q, \dot{q}) \oplus [q, A(q, \dot{q})]_G \tag{10}$$

In fact (see [CMR2]), α_A is a well defined vector bundle isomorphism with inverse given by

$$\alpha_A^{-1}((x, \dot{x}) \oplus [q, \xi]_G) = [(x, \dot{x})^h_q + \xi q]_G.$$

4. The Geometry of Variations

Spaces of Curves. The space of all (smooth) curves from a fixed time interval $I = [t_0, t_1]$ to Q will be denoted $\Omega(Q)$. Given a map $f\colon Q_1 \to Q_2$, the map $\Omega(f)\colon \Omega(Q_1) \to \Omega(Q_2)$ is defined by

$$\Omega(f)(q)(t) = f(q(t)),$$

for $q(t) \in \Omega(Q_1)$. For given $q_i \in Q$, $i = 0, 1$, by definition, $\Omega(Q;\, q_0)$ and $\Omega(Q;\, q_0,\, q_1)$ are, respectively, the spaces of curves $q(t)$ on Q such that $q(t_0) = q_0$ and $q(t_i) = q_i$, $i = 0, 1$. If $\pi\colon Q \to S$ is a bundle, $q_0 \in Q$, and $\pi(q_0) = x_0$, then $\Omega(Q;\, x_0)$ denotes the space of all curves in $\Omega(Q)$ such that $\pi(q(t_0)) = x_0$. The space $\Omega(Q;\, x_1)$ is defined in an analogous way. Similarly, $\Omega(Q;\, x_0,\, q_1)$ is the space of all curves in $\Omega(Q)$ such that $\pi(q(t_0)) = x_0$ and $q(t_1) = q_1$. The spaces of curves $\Omega(Q;\, q_0,\, x_1)$, $\Omega(Q;\, x_0,\, x_1)$, etc. are defined in a similar way.

If $V \to Q$ and $W \to Q$ are vector bundles then $\Omega(V) \to \Omega(Q)$ and $\Omega(W) \to \Omega(Q)$ are vector bundles in a natural way and there is a natural identification $\Omega(V \oplus W) \equiv \Omega(V) \oplus \Omega(W)$.

Deformations of Curves. A *deformation* of a curve $q(t)$ on a manifold Q is a (smooth) function $q(t,\, \lambda)$ such that $q(t,\, 0) = q(t)$ for all t. The corresponding *variation* is defined by

$$\delta q(t) = \left.\frac{\partial q(t,\, \lambda)}{\partial \lambda}\right|_{\lambda=0}.$$

Variations of curves $q(t)$ belonging to $\Omega(Q;\, q_0)$ or $\Omega(Q;\, q_0,\, q_1)$ satisfy the corresponding fixed endpoints conditions, namely, $\delta q(t_0) = 0$ or $\delta q(t_i) = 0$ for $i = 0, 1$, respectively.

Let $\tau\colon V \to Q$ be a vector bundle and let $v(t,\, \lambda)$ be a deformation in V of a curve $v(t)$ in V. If $\tau(v(t,\, \lambda)) = q(t)$ does not depend on λ we will call $v(t,\, \lambda)$ a *V-fiber deformation* of $v(t)$, or simply, a *fiber deformation* of $v(t)$. For each t, the variation

$$\delta v(t) = \left.\frac{\partial v(t,\, \lambda)}{\partial \lambda}\right|_{\lambda=0}$$

may be naturally identified with an element, also called $\delta v(t)$, of $\tau^{-1}(q(t))$. In this case, the curve δv in V is, by definition, a *V-fiber variation* of the curve v, or, simply, a *fiber variation* of the curve v.

Horizontal and Vertical Variations. Consider a curve $q \in \Omega(Q;\, q_0)$. A *vertical variation* δq of q satisfies, by definition, the condition $\delta q(t) = \mathrm{Ver}(\delta q(t))$ for all t. Similarly, a *horizontal variation* satisfies $\delta q(t) = \mathrm{Hor}(\delta q(t))$ for all t.

Clearly, any variation δq can be uniquely decomposed as follows:

$$\delta q(t) = \mathrm{Hor}(\delta q(t)) + \mathrm{Ver}(\delta q(t))$$

for all t, where $\mathrm{Ver}(\delta q(t)) = A(q(t),\, \delta q(t))q(t)$ and $\mathrm{Hor}(\delta q(t)) = \delta q(t) - \mathrm{Ver}(\delta q(t))$.

The Structure of Vertical Variations. Given a curve $q \in \Omega(Q; q_0, q_1)$, let $v = A(q, \dot{q}) \in \mathfrak{g}$. Variations δq of $q(t)$ induce corresponding variations $\delta v \in \mathfrak{g}$ in the obvious way:

$$\delta v = \left. \frac{\partial A(q(t, \lambda), \dot{q}(t, \lambda))}{\partial \lambda} \right|_{\lambda=0}.$$

Consider the decomposition $q = g_q q_h$ introduced in (5). A *vertical deformation* $q(t, \lambda)$ can be written as $q(t, \lambda) = g_q(t, \lambda) q_h(t)$. The corresponding variation $\delta q(t) = \delta g_q(t) q_h(t)$ is of course also vertical.

Now we introduce some important notation. Define the curve

$$\eta(t) = \delta g_q(t) g_q(t)^{-1}$$

in $\mathfrak{g}$. The fixed endpoint condition gives $\eta(t_i) = 0$, $i = 1, 2$.

Notice that, by construction,

$$\delta q(t) = \delta g_q(t) q_h(t) = \eta(t) g_q(t) q_h(t) = \eta(t) q(t).$$

Lemma 4.1. *For any vertical variation $\delta q = \eta q$ of a curve $q \in \Omega(Q; q_0, q_1)$ the corresponding variation δv of $v = A(q, \dot{q})$ is given by $\delta v = \dot{\eta} + [\eta, v]$ with $\eta_i = 0$, $i = 0, 1$.*

Proof. For completeness, we will give the proof in the case that G is a matrix group. The more general case can be treated using the appendix to [BKMR][1]. By (6), we have $v = \dot{g}_q g_q^{-1}$. Then

$$\begin{aligned} \delta v &= (\delta \dot{g}_q) g_q^{-1} - \dot{g}_q g_q^{-1} \delta g_q g_q^{-1} \\ &= \dot{(\delta g_q)} g_q^{-1} - v\eta \\ &= (\dot{\eta} g_q + \eta \dot{g}_q) g_q^{-1} - v\eta \\ &= \dot{\eta} + [\eta, v]. \end{aligned}$$ □

As we saw in Theorem 2.1, one uses the constrained variations $\delta \xi = \dot{\eta} + [\xi, \eta]$ for computing the corresponding variational principle. The above construction of v, η is *not* computing the same objects. These constrained variations are, instead, special instances of the construction of covariant variations, to be introduced shortly in Definition 4.3. In the second remark following Lemma 4.4, we shall explicitly remark on how the constructions of variations for the Euler–Poincaré equations and those for the Lagrange–Poincaré case are related.

The Structure of Horizontal Variations. Using the relation

$$v = A(q, \dot{q})$$

and differentiating with the help of the Cartan structure equations, one finds that variations δv corresponding to horizontal variations δq of a curve $q \in \Omega(Q; q_0, q_1)$ are given as follows.

Lemma 4.2. *Let δq be a horizontal variation of a curve $q \in \Omega(Q; q_0, q_1)$. Then the corresponding variation δv of $v = A(q, \dot{q})$ satisfies*

$$\delta v = B(q)(\delta q, \dot{q}).$$

[1]Another direct proof in the general case was told to us by Marco Castrillon.

The Covariant Variation on the Adjoint Bundle. Any curve in Q, $q \in \Omega(Q;\, q_0,\, q_1)$ induces a curve in $\tilde{\mathfrak{g}}$ in a natural way, namely,

$$[q,\, v]_G(t) = [q(t),\, v(t)]_G,$$

where $v(t) = A(q,\, \dot{q})$. Observe that, for each t, $[q,\, v]_G(t) \in \tilde{\mathfrak{g}}_{x(t)}$ (the fiber over $x(t)$), where $x(t) = \pi(q(t))$ for all t. We want to study variations $\delta[q,\, v]_G$ corresponding to vertical and also to horizontal variations δq of q.

While vertical variations δq give rise to vertical variations $\delta[q,\, v]_G$, horizontal variations δq need not give rise to horizontal variations $\delta[q,\, v]_G$. The deviation of any variation $\delta[q,\, v]_G$ from being horizontal is measured by the *covariant variation* $\delta^A[q,\, v]_G(t)$, defined as follows.

Definition 4.3. For any given deformation $q(t,\, \lambda)$ of $q(t)$, the *covariant variation* $\delta^A[q,\, v]_G(t)$ is defined by

$$\delta^A[q,\, v]_G(t) = \left.\frac{D[q(t,\, \lambda),\, v(t,\, \lambda)]_G}{D\lambda}\right|_{\lambda=0}.$$

Vertical Variations and the Adjoint Bundle. We first consider the case of vertical variations. Using Lemma 4.1 and (8), one finds the following.

Lemma 4.4. *The covariant variation $\delta^A[q,\, v]_G(t)$ corresponding to a vertical variation $\delta q = \eta q$ is given by*

$$\delta^A[q,\, v]_G(t) = \frac{D[q,\, \eta]_G}{Dt} + [q,\, [v,\, \eta]]_G.$$

Remarks. 1. In view of (9), we can write

$$[q,\, [v,\, \eta]]_G = [[q,\, v]_G,\, [q,\, \eta]_G].$$

2. Let us now show that the formula $\delta v = \dot{\eta} + [v,\, \eta]$ for the constrained variations for the Euler–Poincaré equations coincides with the construction of the covariant variation given in Definition 4.3. Given a Lie group G, we regard it as a principal bundle over a point, that is, we take $G = Q$. The identification of $\mathfrak{g}$ with $T(Q/G) \oplus \tilde{\mathfrak{g}}$ in this case is given by $v \mapsto [e,\, v]_G$. This equivalence defines $\delta v \equiv \delta^A[e,\, v]_G$ and the preceding lemma shows that $\delta v = \dot{\eta} + [v,\, \eta]$, which is the same type of variation one has for the Euler–Poincaré equations.

The Reduced Curvature Form. In preparation for the consideration of variations $\delta^A[q,\, v]_G(t)$ corresponding to horizontal variations, one has the following.

Lemma 4.5. *The curvature 2-form $B \equiv B^A$ of the connection A induces a $\tilde{\mathfrak{g}}$-valued 2-form $\tilde{B} \equiv \tilde{B}^A$ on Q/G called the* reduced curvature form *given by*

$$\tilde{B}(x)(\delta x,\, \dot{x}) = [q,\, B(q)(\delta q,\, \dot{q})]_G, \tag{11}$$

where for each $(x,\, \dot{x})$ and $(x,\, \delta x)$ in $T_x(Q/G)$, $(q,\, \dot{q})$ and $(q,\, \delta q)$ are any elements of T_qQ such that $\pi(q) = x$, $T\pi(q,\, \dot{q}) = (x,\, \dot{x})$ and $T\pi(q,\, \delta q) = (x,\, \delta x)$.

This is proved readily by showing that the right hand side does not depend on the choice of $(q,\, \dot{q})$ and $(q,\, \delta q)$ using equivariance properties of the curvature.

Horizontal Variations and the Adjoint Bundle. Now we are ready to describe covariant variations $\delta^A[q, v]_G(t)$ corresponding to horizontal variations δq. By (8), we have $\delta^A[q, v]_G(t) = [q, -[A(q, \delta q), v] + \delta v]_G$. Since δq is horizontal, we have $A(q, \delta q) = 0$. Using this and Lemmas 4.2 and 4.5, we obtain the following result.

Lemma 4.6. *Variations $\delta^A[q, v]_G(t)$ corresponding to horizontal variations δq are given by*

$$\delta^A[q, v]_G(t) = \tilde{B}(x)(\delta x, \dot{x})(t),$$

where $T\pi(q, \dot{q}) = (x, \dot{x})$, $T\pi(q, \delta q) = (x, \delta x)$, and $v = A(q, \dot{q})$.

5. The Euler–Lagrange and Euler–Poincaré Operators

Now we have the tools needed to carry out the reduction of the Euler–Lagrange equations by means of reduction of Hamilton's principle.

Reduced Spaces of Curves. In what follows we often identify the bundles TQ/G and $T(Q/G) \oplus \tilde{\mathfrak{g}}$, using the isomorphism α_A (see (10)). This leads to other natural identifications as well. For instance, the reduced set of curves $[\Omega(Q; q_0, q_1)]_G$ is the set of curves $[q]_G(t) = [q(t)]_G$ on Q/G such that the curve $q(t)$ belongs to $\Omega(Q; q_0, q_1)$. This reduced set of curves is naturally identified with the set of curves $[q(t), \dot{q}(t)]_G$ in TQ/G such that $q(t_i) = q_i$, for $i = 0, 1$, and in turn, this is identified, via the map

$$\Omega(\alpha_A) \colon [\Omega(Q; q_0, q_1)]_G \to \Omega(T(Q/G) \oplus \tilde{\mathfrak{g}}),$$

with the set of curves

$$T\pi(q(t), \dot{q}(t)) \oplus [q(t), A(q(t), \dot{q}(t))]_G$$

in $T(Q/G) \oplus \tilde{\mathfrak{g}}$, such that $q(t_i) = q_i$, for $i = 0, 1$. The image of this reduced set of curves will be denoted $\Omega(\alpha_A)([\Omega(Q; q_0, q_1)]_G)$.

The Reduced Lagrangian. Let $L \colon TQ \to \mathbb{R}$ be an invariant Lagrangian, that is, $L(g(q, \dot{q})) = L(q, \dot{q})$ for all $(q, \dot{q}) \in TQ$ and all $g \in G$. Because of this invariance, we get a well defined *reduced Lagrangian* $l \colon TQ/G \to \mathbb{R}$ satisfying

$$l([q, \dot{q}]_G) = L(q, \dot{q}).$$

As we will see in detail in this section, the evolution of the reduced system will be a critical point, say a curve $[q]_G$ in the reduced set of curves $[\Omega(Q; q_0, q_1)]_G$, of the reduced action

$$\int_{t_0}^{t_1} l([q, \dot{q}]_G)\, dt$$

for suitable types of variations.

However, variations of curves in the reduced family of curves are not of the usual sort found in Hamilton's principle, and so the equations of motion in the bundle TQ/G cannot be written in a direct way.

We will use the description of vertical and horizontal variations given in the preceding section to derive equations of motion in a suitably reduced bundle. Equations corresponding to vertical variations will be called the *vertical Lagrange–Poincaré* *equations*, and equations corresponding to horizontal variations will be called the *horizontal Lagrange–Poincaré equations.*

Identification of Bundles. We shall allow a slight abuse of notation, namely we will consider l as a function defined on $T(Q/G)\oplus\tilde{\mathfrak{g}}$ or TQ/G interchangeably, using the isomorphism α_A. Also, we often use a slight abuse of the variable-notation for a function, namely we will write $l(x,\,\dot{x},\,\bar{v})$ to emphasize the dependence of l on $(x,\,\dot{x})\in T(Q/G)$ and $\bar{v}\in\tilde{\mathfrak{g}}$.

The Second Order Tangent Bundle $T^{(2)}Q$. As will be clear below, the second order bundle plays a fundamental role in the study of the Euler–Lagrange equations. For $\bar{q}\in Q$, elements of $T^{(2)}_{\bar{q}}Q$ are equivalence classes of curves in Q; namely, two given curves $q_i(t)$, $i=1,\,2$, such that $q_1(\bar{t}_1)=q_2(\bar{t}_2)=\bar{q}$ are *equivalent* if in any local chart q_1 and q_2 agree up to and including their second derivatives. The equivalence class of the curve $q(t)$ at $\bar{q}=q(\bar{t})$ will be denoted $[q]^{(2)}_{\bar{q}}$ or sometimes[2] by $(q,\,\dot{q},\,\ddot{q})$. There are natural fiber bundle structures $T^{(2)}Q\to TQ\to Q$. The bundle $T^{(2)}Q\to TQ$ is, in fact, a vector bundle in a natural way.

Using the projection $\pi_G(Q)\colon Q\to Q/G$, we obtain a bundle map

$$T^{(2)}\pi_G(Q)\colon T^{(2)}Q\to T^{(2)}(Q/G).$$

This bundle map induces a bundle map

$$T^{(2)}Q/G\to T^{(2)}(Q/G)\quad\text{given by}\quad \big[[q]^{(2)}_{\bar{q}}\big]_G\mapsto T^{(2)}\pi_G(Q)\big([q]^{(2)}_{\bar{q}}\big).$$

The class of the element $[q]^{(2)}_{\bar{q}}$ in the quotient $T^{(2)}Q/G$ will be denoted $\big[[q]^{(2)}_{\bar{q}}\big]_G$.

More generally, it is easy to see that for any map $f\colon M\to N$ we have a naturally induced map

$$T^{(2)}f\colon T^{(2)}M\to T^{(2)}N\quad\text{given by}\quad T^{(2)}f\big([q]^{(2)}_{\bar{q}}\big)=[f\circ q]^{(2)}_{f(\bar{q})}.$$

In particular, a group action $\rho\colon G\times Q\to Q$ can be naturally lifted to a group action

$$\rho^{(2)}\colon G\times T^{(2)}Q\to T^{(2)}Q\quad\text{given by}\quad \rho^{(2)}_g\big([q]^{(2)}_{\bar{q}}\big)=[\rho_g\circ q]^{(2)}_{\rho(g,\bar{q})}.$$

We will often write $\rho^{(2)}_g\big([q]^{(2)}_{\bar{q}}\big)=\rho^{(2)}\big(g,\,[q]^{(2)}_{\bar{q}}\big)=g[q]^{(2)}_{\bar{q}}$.

Let $\bar{q}\in Q$, denote $\pi(\bar{q})=[\bar{q}]_G=\bar{x}$, and let $[x]^{(2)}_{\bar{x}}\in T^{(2)}(Q/G)$ be given. Let $x(t)$ be any curve belonging to the class $[x]^{(2)}_{\bar{x}}$. Then there is a unique horizontal lift $x^h_{\bar{q}}$ of $x(t)$. We define the horizontal lift of $[x]^{(2)}_{\bar{x}}$ at $\bar{q}$ by

$$[x]^{(2),h}_{\bar{x},\bar{q}}:=[x^h_{\bar{q}}]^{(2)}_{\bar{q}}.$$

We remark as an aside that $T^{(2)}G$ carries a natural Lie group structure.[3] If $[g]^{(2)}_{\bar{g}}$, and $[h]^{(2)}_{\bar{h}}$ are classes of curves g and h in G, we define the product $[g]^{(2)}_{\bar{g}}[h]^{(2)}_{\bar{h}}$ as being the class $[gh]^{(2)}_{\bar{g}\bar{h}}$ at the point $\bar{g}\bar{h}$ of the curve gh. The Lie algebra $T_eT^{(2)}G$ of

[2] The second order bundle $T^{(2)}Q$ is also denoted $\ddot{Q}$ by some authors (see, for example, [MPS], [MR] and references therein).

[3] Recall that $T^{(1)}G=TG$ is the semidirect product group $G\circledS\mathfrak{g}$ whose Lie algebra is the semidirect product $\mathfrak{g}\circledS\mathfrak{g}$, where the second factor is regarded as the representation space of the adjoint action. This semidirect product Lie algebra is, as a vector space, equal to $\mathfrak{g}\oplus\mathfrak{g}$.

$T^{(2)}G$ can be naturally identified, as a vector space, with $\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}$, which, therefore, carries a unique Lie algebra structure such that this identification becomes a Lie algebra isomorphism.

There is also a natural identification of $T_e^{(2)}G$ with $\mathfrak{g}\oplus\mathfrak{g}$, which is useful in what follows. Let us assume that G is a group of matrices. Let $g(t)$ be any curve such that $g(0)=e$ and let $\xi(t)=\dot g(t)g^{-1}(t)$. Then $\dot\xi(t)=-\dot g(t)g^{-1}(t)\dot g(t)g^{-1}(t)+\ddot g(t)g^{-1}(t)$. In particular, we have $\xi(0)=\dot g(0)$ and $\dot\xi(0)=-\dot g(t)^2+\ddot g(t)$. Let $\xi_1\oplus\xi_2=\xi(0)\oplus\dot\xi(0)$. Then the identification $T_e^{(2)}G\equiv\mathfrak{g}\oplus\mathfrak{g}$ is given by $[g]_e^{(2)}\equiv\xi_1\oplus\xi_2$.

Also, $T^{(2)}Q$ is a principal bundle with structure group $T^{(2)}G$ in a natural way. More precisely, if $[g]_{\bar g}^{(2)}\in T^{(2)}G$ is the class of a curve g in G and $[q]_{\bar q}^{(2)}\in T_{\bar q}^{(2)}(Q)$ is the class of a curve q in Q we let $[g]_{\bar g}^{(2)}[q]_{\bar q}^{(2)}\in T_{\bar g\bar q}^{(2)}Q$ denote the class $[gq]_{\bar g\bar q}^{(2)}$ of the curve gq at the point $\bar g\bar q$. In particular, if $\xi\in\mathfrak{g}\oplus\mathfrak{g}$ and $[q]_{\bar q}^{(2)}\in T^{(2)}Q$ are given, there is a well defined element $\xi[q]_{\bar q}^{(2)}\in T^{(2)}Q$. More precisely, let $\xi=\xi_1\oplus\xi_2$, and let $g(t)$ be any curve in G such that $\xi=[g]_e^{(2)}$, according with the identification described above. Then, we have

$$\xi[q]_{\bar q}^{(2)}=\left.\frac{dg(t)\bar q}{dt}\right|_{t=0}.$$

Given a curve q in Q with $q(\bar t)=\bar q$, we get a curve $[q(t),\,v(t)]_G$ in $\tilde{\mathfrak{g}}$, where

$$v(t)=A(q(t),\,\dot q(t))\in\mathfrak{g}.$$

As with the map $A\colon TQ\to\mathfrak{g}$, we let $A_2\colon T^{(2)}Q\to\mathfrak{g}\oplus\mathfrak{g}$ be defined by

$$A_2\big([q]_{\bar q}^{(2)}\big)=v(\bar t)\oplus\dot v(\bar t).\tag{12}$$

For any given $\xi=\xi_1\oplus\xi_2\in\mathfrak{g}\oplus\mathfrak{g}$ and $q\in Q,$ we can easily see, using the definition, that $A_2(\xi q)=\xi$. Similarly, as with α_A, we get an isomorphism

$$\alpha_{A_2}\colon T^{(2)}Q/G\to T^{(2)}(Q/G)\times_{Q/G}\big(\tilde{\mathfrak{g}}\oplus\tilde{\mathfrak{g}}\big)\tag{13}$$

defined by

$$\alpha_{A_2}\big(\big[[q]_{\bar q}^{(2)}\big]_G\big)=T^{(2)}\pi_G(Q)\big([q]_{\bar q}^{(2)}\big)\times_{Q/G}[\bar q,\,v(\bar t)\oplus\dot v(\bar t)]_G.$$

It is a well defined bundle isomorphism, because

$$\operatorname{Ad}_g(\dot v(t))=\frac{d}{dt}\operatorname{Ad}_g v(t).$$

We have also used the natural isomorphism $\tilde{\mathfrak{g}}\oplus\tilde{\mathfrak{g}}\equiv\widetilde{\mathfrak{g}\oplus\mathfrak{g}}$, given by the natural identification

$$\operatorname{Ad}_g\xi_1\oplus\operatorname{Ad}_g\xi_2\equiv\operatorname{Ad}_g(\xi_1\oplus\xi_2).$$

The inverse of α_{A_2} is given by

$$\alpha_{A_2}^{-1}\big([x]_{\bar x}^{(2)}\times_{Q/G}[\bar q,\,\xi]_G\big)=\xi[x]_{\bar x,\bar q}^{(2),h},$$

where the meaning of $[x]_{\bar x,\bar q}^{(2),h}$ has been explained above.

We refer to [CMR2] for additional properties and discussion of higher order bundles.

Euler–Lagrange Operator. Next we introduce some notation and recall some basic results concerning Euler–Lagrange operators. The fundamental relationship between the variational and differential-equation description of the evolution of a given system is given by the following well-known result.

Theorem 5.1 (Euler–Lagrange)**.** *Let $L\colon TQ \to \mathbb{R}$ be a given Lagrangian on a manifold Q and let*

$$\mathfrak{S}(L)(q) = \int_{t_0}^{t_1} L(q,\, \dot{q})\, dt$$

be the action of L defined on $\Omega(Q;\, q_0,\, q_1)$. Let $q(t,\, \lambda)$ be a deformation of a curve $q(t)$ in $\Omega(Q;\, q_0,\, q_1)$ and let $\delta q(t)$ be the corresponding variation. Then, by definition, $\delta q(t_i) = 0$ for $i = 0,\, 1$.

There is a unique bundle map

$$\mathscr{EL}(L)\colon T^{(2)}Q \to T^*Q$$

such that, for any deformation $q(t,\, \lambda)$, keeping the endpoints fixed, we have

$$\mathbf{d}\mathfrak{S}(L)(q) \cdot \delta q = \int_{t_0}^{t_1} \mathscr{EL}(L)(q,\, \dot{q},\, \ddot{q}) \cdot \delta q,$$

where, as usual,

$$\mathbf{d}\mathfrak{S}(L)(q) \cdot \delta q = \frac{d}{d\lambda}\mathfrak{S}(L)\left(q(t,\, \lambda)\right)\bigg|_{\lambda=0}$$

with

$$\delta q(t) = \frac{\partial q(t,\, \lambda)}{\partial \lambda}\bigg|_{\lambda=0}.$$

The 1-form bundle-valued map $\mathscr{EL}(L)$ is called the Euler–Lagrange operator*.*

In local coordinates $\mathscr{EL}(L)$ has the following classical expression:

$$\mathscr{EL}(L)_i(q,\, \dot{q},\, \ddot{q})\, dq^i = \left(\frac{\partial L}{\partial q^i}(q,\, \dot{q}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(q,\, \dot{q})\right) dq^i$$

in which it is understood that one regards the second term on the right hand side as a function on the second order tangent bundle by formally applying the chain rule and then replacing everywhere dq/dt by $\dot{q}$ and $d\dot{q}/dt$ by $\ddot{q}$. The Euler–Lagrange equations can, of course, be written simply as $\mathscr{EL}(L)(q,\, \dot{q},\, \ddot{q}) = 0$.

Euler–Poincaré Operator. Analogous to the Euler–Lagrange operator, the Euler–Poincaré theorem (see Theorem 2.1) induces an operator, called the *Euler–Poincaré operator.*

Theorem 5.2. *Let G be a Lie group, $L\colon TG \to \mathbb{R}$ a left G-invariant Lagrangian and $l\colon \mathfrak{g} \to \mathbb{R}$ its reduction. There is a unique bundle map*

$$\mathscr{EP}(l)\colon \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}^*$$

such that, for any deformation $v(t,\, \lambda) = g(t,\, \lambda)^{-1}\dot{g}(t,\, \lambda) \in \mathfrak{g}$ induced on $\mathfrak{g}$ by a deformation $g(t,\, \lambda) \in G$ of $g(t) \in \Omega(G;\, g_0,\, g_1)$ keeping the endpoints fixed, and *thus* $\delta g(t_i) = 0$, *for* $i = 0, 1$, *we have*

$$\mathbf{d}\mathfrak{S}_{\mathrm{red}}(l)(v) \cdot \delta v = \int_{t_0}^{t_1} \mathscr{E}\mathscr{P}(l)(v, \dot{v}) \cdot \eta \, dt,$$

where, as usual,

$$\mathbf{d}\mathfrak{S}_{\mathrm{red}}(l)(v) \cdot \delta v = \frac{d}{d\lambda} \mathfrak{S}_{\mathrm{red}}(l)(v(t, \lambda)) \bigg|_{\lambda=0}$$

and

$$\delta v(t) = \frac{\partial v(t, \lambda)}{\partial t} \bigg|_{\lambda=0} = \dot{\eta}(t) + [v(t), \eta(t)]. \tag{14}$$

The map $\mathscr{E}\mathscr{P}(l)$ *is called the* Euler–Poincaré operator *and its expression is given by*

$$\mathscr{E}\mathscr{P}(l)(v, \dot{v}) = \mathrm{ad}_v^* \frac{\delta l}{\delta v} - \frac{d}{dt} \frac{\delta l}{\delta v}$$

where, as before, it is to be understood that the time derivative on the second term is performed formally using the chain rule and that the expression dv/dt *is replaced throughout by* $\dot{v}$.

The Euler–Poincaré equations can be written simply as $\mathscr{E}\mathscr{P}(l)(v, \dot{v}) = 0$. Formula (14) represents the most general variation δv of v induced by an arbitrary variation δg via left translation. As in Theorem 2.1, $\eta = g^{-1}\delta g$ and so the condition $\delta g = 0$ at the endpoints is equivalent to the condition $\eta = 0$ at the endpoints.

6. The Lagrange–Poincaré Operator

In this section we introduce the Lagrange–Poincaré operator using the same type of technique of reduction of variational principles that was used in the preceding section to define the Euler–Lagrange and the Euler–Poincaré operators.

Reducing the Euler–Lagrange Operator. The map $\mathscr{E}\mathscr{L}(L) \colon T^{(2)}Q \to T^*Q$, being G-equivariant, induces a quotient map

$$[\mathscr{E}\mathscr{L}(L)]_G \colon T^{(2)}Q/G \to T^*Q/G,$$

which depends only on the reduced Lagrangian $l \colon TQ/G \to \mathbb{R}$; that is, we can identify $[\mathscr{E}\mathscr{L}(L)]_G$ with an operator $\mathscr{E}\mathscr{L}(l)$. This is called the *reduced Euler–Lagrange operator* and it does not depend on any extra structure on the principal bundle Q. However, to write the explicit expressions, which are also physically meaningful, we use the additional structure of a principal connection A on the principal bundle $Q \to Q/G$ to identify the quotient bundle

$$T^{(2)}Q/G \quad \text{with} \quad T^{(2)}(Q/G) \times_{Q/G} (\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}})$$

and

$$T^*Q/G \quad \text{with} \quad T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

using the bundle isomorphisms α_{A_2} from Lemma 13 and α_A from (10), and also a connection ∇ on Q/G to concretely realize the reduced Euler–Lagrange operator; this will naturally lead us to the Lagrange–Poincaré operator.

Geometry of Reduced Variations. A general variation $\delta\bar{v}(t)$ of a given curve $\bar{v}(t)$ in $\tilde{\mathfrak{g}}$ is constructed as follows: choose a family of curves $\bar{v}(t, s)$ in $\tilde{\mathfrak{g}}$ such that $\bar{v}(t, 0) = \bar{v}(t)$ and define

$$\delta\bar{v}(t) = \left.\frac{\partial\bar{v}(t, s)}{\partial s}\right|_{s=0}.$$

This $\delta\bar{v}(t)$ is, for each t, an element of $T\tilde{\mathfrak{g}}$. However, it turns out that we will not need these kinds of general variations $\delta\bar{v}$ subsequently. Instead, we are interested in the special kind of deformations $\bar{v}(t, s)$ of the curve $\bar{v}(t)$ for which the projection $\tilde{\pi}_G(\bar{v}(t, s)) = x(t, s)$ does not depend on s, that is, deformations that take place only in the fiber of $\tilde{\mathfrak{g}}$ over $x(t) = \tilde{\pi}_G(\bar{v}(t))$; thus, for each fixed t, the curve $s \mapsto \bar{v}(t, s)$ is a curve in the fiber over $x(t)$. Then, since $\tilde{\mathfrak{g}}$ is a vector bundle, the variation $\delta\bar{v}(t)$ induced by such a deformation $\bar{v}(t, s)$, is naturally identified with a curve in $\tilde{\mathfrak{g}}$, also called $\delta\bar{v}(t)$; this is a $\tilde{\mathfrak{g}}$-*fiber variation*, according to the notation introduced in Section 4. Below, $\delta\bar{v}$ will always mean a $\tilde{\mathfrak{g}}$-variation, unless explicitly stated otherwise.

Examples of $\tilde{\mathfrak{g}}$-fiber variations $\delta\bar{v}$ are the covariant variations $\delta^A\bar{v}$ considered in Definition 4.3, but, of course, there are more general variations of this type. We encountered such an example when reviewing the Euler–Poincaré equations. In that case, $Q = G$, the connection $A\colon TG \to \mathfrak{g}$ is given by right translation, and $\delta^A v(t) = \dot{\eta}(t) + [v(t), \eta(t)]$, for $\eta(t)$ a curve in $\mathfrak{g}$ vanishing at the endpoints.

In the Euler–Poincaré case, *any* deformation of a curve $v(t)$ is a deformation along the fiber, because in this situation the base of $\tilde{\mathfrak{g}}$ is a point. However, it is *not* true that *any* curve in $\mathfrak{g}$ is induced by a variation δg that vanishes at the endpoints; the latter are only the curves of the type $\dot{\eta}(t) + [v(t), \eta(t)]$, for $\eta(t)$ an arbitrary curve in $\mathfrak{g}$ vanishing at the endpoints.

In the study of the Lagrange–Poincaré operator, we will use variations of curves in $Q/G \oplus \tilde{\mathfrak{g}}$ (the first summand means the vector bundle over Q/G with zero dimensional fiber). For a given curve $x(t) \oplus \bar{v}(t)$ in $Q/G \oplus \tilde{\mathfrak{g}}$, and a given arbitrary deformation $x(t, \lambda) \oplus \bar{v}(t, \lambda)$, with $x(t, 0) \oplus \bar{v}(t, 0) = x(t) \oplus \bar{v}(t)$, the corresponding *covariant variation* $\delta x(t) \oplus \delta^A\bar{v}(t)$ is, by definition,

$$\delta x(t) \oplus \delta^A\bar{v}(t) = \left.\frac{\partial x(t, s)}{\partial s}\right|_{s=0} \oplus \left.\frac{D\bar{v}(t, s)}{Ds}\right|_{s=0}.$$

It is clear that $\delta^A\bar{v}$ is a $\tilde{\mathfrak{g}}$-fiber variation of $\bar{v}$.

The most important example of a covariant variation $\delta x(t) \oplus \delta^A\bar{v}(t)$ is the one to be described next. Let $q(t, s)$ be a deformation of a curve $q(t) = q(t, 0)$ in Q. This induces a deformation $x(t, s) \oplus \bar{v}(t, s)$ of the curve $x(t) \oplus \bar{v}(t)$ by taking $x(t, s) = [q(t, s)]_G$ and $\bar{v}(t, s) = [q(t, s), A(q(t, s), \dot{q}(t, s))]_G$, where $\dot{q}(t, s)$ represents the derivative with respect to t. Using (8) and Definition 4.3, it follows that the covariant variation corresponding to this deformation of $x(t) \oplus \bar{v}(t)$ is given by $\delta x(t) \oplus \delta^A\bar{v}(t)$, where

$$\delta^A\bar{v}(t) = \frac{D[q(t), \eta(t)]_G}{Dt} + [q(t), [A(q(t), \dot{q}(t)), \eta(t)]]_G + \tilde{B}(\delta x(t), \dot{x}(t)),$$

is an element of $\tilde{\mathfrak{g}}$ for each t, with $\eta(t) \in \mathfrak{g}$ an arbitrary curve vanishing at the endpoints. This is a special kind of covariant variation. It is precisely to these kinds of variations that we will apply the usual techniques of the calculus of variations in the next theorems to derive the Lagrange–Poincaré operator and equation. The previous formula may be rewritten as

$$\delta^A \bar{v}(t) = \frac{D\bar{\eta}}{Dt}(t) + [\bar{v}(t), \bar{\eta}(t)] + \tilde{B}(\delta x(t), \dot{x}(t)),$$

where $\bar{\eta} = [q(t), \eta(t)]_G$, which emphasizes the similarity with the Euler–Poincaré case.

Lagrange–Poincaré Operator. We are now ready to state a theorem that introduces the Lagrange–Poincaré operator. Its proof will be contained in the proof of Theorem 6.4.

Theorem 6.1. *Let $L\colon TQ \to \mathbb{R}$ be an invariant Lagrangian on the principal bundle Q. Choose a principal connection A on $Q \to Q/G$ and identify the bundles TQ/G and $T(Q/G) \oplus \tilde{\mathfrak{g}}$ using the isomorphism α_A and also the bundles $T^{(2)}Q/G$ and $T^{(2)}(Q/G) \times_{Q/G} (\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}})$ using the isomorphism α_{A_2}. Thus, an element $[q, \dot{q}]_G$ of TQ/G can be written, equivalently, as an element $(x, \dot{x}, \bar{v}) \in T(Q/G) \oplus \tilde{\mathfrak{g}}$. Let $l\colon T(Q/G) \oplus \tilde{\mathfrak{g}} \to \mathbb{R}$ be the reduced Lagrangian. Then there is a unique bundle map*

$$\mathscr{LP}(l)\colon T^{(2)}(Q/G) \times_{Q/G} (\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}) \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

such that for any curve $q \in \Omega(Q; q_0, q_1)$ and any variation δq of q vanishing at the endpoints, the corresponding reduced curve $[q, \dot{q}]_G = (x, \dot{x}, \bar{v})$, where $\bar{v} = [q, A(q, \dot{q})]_G$, and covariant variation $\delta x \oplus \delta^A \bar{v}$, where

$$\delta^A \bar{v}(t) = \frac{D\bar{\eta}}{Dt}(t) + [\bar{v}(t), \bar{\eta}(t)] + \tilde{B}(\delta x(t), \dot{x}(t)),$$

with $\bar{\eta}(t) = [q(t), \eta(t)]_G$ and

$$\delta x(t) = T\pi(\delta q(t)),$$

satisfy

$$\mathscr{EL}(L)(q(t), \dot{q}(t), \ddot{q}(t)) \cdot \delta q(t) = \mathscr{LP}(l)(x(t), \dot{x}(t), \bar{v}(t)) \cdot (\delta x(t) \oplus \bar{\eta}(t)).$$

Notice that, after all the identifications described at the beginning of the present paragraph, the operator $\mathscr{LP}(l)$ coincides with the operator $[\mathscr{EL}(L)]_G$.

Definition 6.2. The 1-form valued bundle map

$$\mathscr{LP}(l)\colon T^{(2)}(Q/G) \times_{Q/G} (\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}) \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

defined in the preceding theorem will be called the *Lagrange–Poincaré operator.* The decomposition of the range space for $\mathscr{LP}(l)$ as a direct sum naturally induces a decomposition of the Lagrange–Poincaré operator

$$\mathscr{LP}(l) = \mathrm{Hor}(\mathscr{LP})(l) \oplus \mathrm{Ver}(\mathscr{LP})(l)$$

which define the *horizontal Lagrange–Poincaré operator* and the *vertical Lagrange–Poincaré operator.*

The *Lagrange–Poincaré equations* are, by definition, the equations $\mathscr{LP}(l) = 0$. The *horizontal Lagrange–Poincaré equation* is the equation $\mathrm{Hor}(\mathscr{LP})(l) = 0$ and the *vertical Lagrange–Poincaré equation* is the equation $\mathrm{Ver}(\mathscr{LP})(l) = 0$.

In the following paragraph we introduce some additional structure, namely, an arbitrary connection ∇ on the manifold Q/G. This will also help us write explicit expressions of $\mathrm{Hor}(\mathscr{LP})(l)$ and $\mathrm{Ver}(\mathscr{LP})(l)$. For simplicity we shall assume, as in [CMR2], that this connection is torsion free; however, as in [GSS], it is easy to relax this condition.

The problem of computing the Lagrange–Poincaré equations can be done using any connection, as we remarked earlier and, in addition, the problem can be localized to any local trivialization of the bundle $Q \to Q/G$. Because of this, one may choose the vector space or trivial connection associated with such a local trivialization of the bundle. Of course we are not assuming that the bundle has a global flat connection.

Explicit formulas for $\mathrm{Hor}(\mathscr{LP})(l)$ and $\mathrm{Ver}(\mathscr{LP})(l)$ in coordinates using any connection can be calculated from what we have developed and are given in [CMR2]. Doing so, one arrives at the coordinate formulas given in [MS2]. We also mention that it is possible to derive these equations from [CIM] in a straightforward way.

Reduced Covariant Derivatives. Calculating intrinsic formulas for the horizontal and vertical Lagrange–Poincaré operators $\mathrm{Hor}(\mathscr{LP})(l)$ and $\mathrm{Ver}(\mathscr{LP})(l)$ can be made more explicit by giving meaning to the partial derivatives

$$\frac{\partial l}{\partial x}, \quad \frac{\partial l}{\partial \dot{x}}, \quad \text{and} \quad \frac{\partial l}{\partial \bar{v}}.$$

Since $\tilde{\mathfrak{g}}$ and $T(Q/G)$ are vector bundles, we may interpret the last two derivatives in a standard (fiber derivative) way as being elements of the dual bundles $T^*(Q/G)$ and $\tilde{\mathfrak{g}}^*$ respectively, for each choice of $(x, \dot{x}, \bar{v})$ in $T(Q/G) \oplus \tilde{\mathfrak{g}}$. In other words, for given $(x_0, \dot{x}_0, \bar{v}_0)$ and $(x_0, x', \bar{v}')$ we define

$$\frac{\partial l}{\partial \dot{x}}(x_0, \dot{x}_0, \bar{v}_0) \cdot x' = \left.\frac{d}{ds}\right|_{s=0} l(x_0, \dot{x}_0 + sx', \bar{v}_0)$$

and

$$\frac{\partial l}{\partial \bar{v}}(x_0, \dot{x}_0, \bar{v}_0) \cdot \bar{v}' = \left.\frac{d}{ds}\right|_{s=0} l(x_0, \dot{x}_0, \bar{v}_0 + s\bar{v}').$$

To define the derivative $\partial l/\partial x$, one uses the chosen connection ∇ on the manifold Q/G, as we will explain next. Let $(x_0, \dot{x}_0, \bar{v}_0)$ be a given element of $T(Q/G) \oplus \tilde{\mathfrak{g}}$. For any given curve $x(s)$ on Q/G, let $(x(s), \bar{v}(s))$ be the horizontal lift of $x(s)$ with respect to the connection $\tilde{\nabla}^A$ on $\tilde{\mathfrak{g}}$ (see (7)) such that $(x(0), \bar{v}(0)) = (x_0, \bar{v}_0)$ and let $(x(s), u(s))$ be the horizontal lift of $x(s)$ with respect to the connection ∇ such that $(x(0), u(0)) = (x_0, \dot{x}_0)$. (Notice that, in general, $(x(s), u(s))$ is *not* the tangent vector $(x(s), \dot{x}(s))$ to $x(s)$.)

Thus, $(x(s), u(s), \bar{v}(s))$ is a horizontal curve with respect to the connection $C = \nabla \oplus \tilde{\nabla}^A$ naturally defined on $T(Q/G) \oplus \tilde{\mathfrak{g}}$ in terms of the connection ∇ on $T(Q/G)$ and the connection $\tilde{\nabla}^A$ on $\tilde{\mathfrak{g}}$.

Definition 6.3. The *covariant derivative* of l with respect to x at $(x_0, \dot{x}_0, \bar{v}_0)$ in the direction of $(x(0), \dot{x}(0))$ is defined by

$$\frac{\partial^C l}{\partial x}(x_0, \dot{x}_0, \bar{v}_0)(x(0), \dot{x}(0)) = \left.\frac{d}{ds}\right|_{s=0} l(x(s), u(s), \bar{v}(s)).$$

We often write

$$\frac{\partial^C l}{\partial x} \equiv \frac{\partial l}{\partial x},$$

whenever there is no danger of confusion.

The covariant derivative on a given vector bundle, for instance $\tilde{\mathfrak{g}}$, induces a corresponding covariant derivative on the dual bundle, in our case $\tilde{\mathfrak{g}}^*$. More precisely, let $\alpha(t)$ be a curve in $\tilde{\mathfrak{g}}^*$. We define the covariant derivative of $\alpha(t)$ in such a way that for any curve $\bar{v}(t)$ on $\tilde{\mathfrak{g}}$, such that both $\alpha(t)$ and $\bar{v}(t)$ project on the same curve $x(t)$ on Q/G, we have

$$\frac{d}{dt}\left\langle \alpha(t),\, \bar{v}(t)\right\rangle = \left\langle \frac{D\alpha(t)}{Dt},\, \bar{v}(t)\right\rangle + \left\langle \alpha(t),\, \frac{D\bar{v}(t)}{Dt}\right\rangle .$$

Likewise, we can define the covariant derivative in the vector bundle $T^*(Q/G)$. Then we obtain a covariant derivative on the vector bundle $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$.

It is in the sense of this definition that terms like

$$\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}$$

in the second equation (which defines the horizontal Lagrange–Poincaré operator) and

$$\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}$$

in the first equation (which defines the vertical Lagrange–Poincaré equation) of the following theorem should be interpreted. In this case D/Dt means the covariant derivative in the bundle $T^*(Q/G)$. In the first equation D/Dt is the covariant derivative in the bundle $\tilde{\mathfrak{g}}^*$.

Reduced Variational Principles and the Lagrange–Poincaré Equations. The main result in this section is the following theorem. Its proof also contains the proof of the preceding theorem.

Theorem 6.4. *Under the hypothesis of Theorem 6.1, the* vertical Lagrange–Poincaré operator *is given by*

$$\operatorname{Ver}(\mathscr{LP})(l) \cdot \bar{\eta} = \left(-\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v}) + \operatorname{ad}^*_{\bar{v}} \frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v})\right) \cdot \bar{\eta}$$

or simply,

$$\operatorname{Ver}(\mathscr{LP})(l) = \left(-\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v}) + \operatorname{ad}^*_{\bar{v}} \frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v})\right)$$

and the horizontal Lagrange–Poincaré operator *is given by*

$$\operatorname{Hor}(\mathscr{LP})(l) \cdot \delta x = \left(\frac{\partial l}{\partial x}(x,\, \dot{x},\, \bar{v}) - \frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}(x,\, \dot{x},\, \bar{v})\right) \delta x - \frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v})\tilde{B}(x)(\dot{x},\, \delta x),$$

or simply,

$$\operatorname{Hor}(\mathscr{LP})(l) = \left(\frac{\partial l}{\partial x}(x,\, \dot{x},\, \bar{v}) - \frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}(x,\, \dot{x},\, \bar{v})\right) - \frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v})\tilde{B}(x)(\dot{x},\, .\,).$$

Proof. To compute the vertical and horizontal Lagrange–Poincaré operator, it suffices to consider variations $\delta^A \bar{v}$ of a curve $x(t) \oplus \bar{v}(t)$ corresponding to vertical and horizontal variations δq of a curve $q \in \Omega(Q, q_0, q_1)$. The computations below will show that these variations suffice to give us the variational principle in the directions of the two summands in $\delta x \oplus \bar{\eta} \in T(Q/G) \oplus \tilde{\mathfrak{g}}$.

First, consider variations $\delta^A \bar{v}$ of a curve $x(t) \oplus \bar{v}(t)$ corresponding to vertical variations δq of a curve q. We have

$$0 = \delta \int_{t_0}^{t_1} l(x, \dot{x}, \bar{v})\, dt = \int_{t_0}^{t_1} \frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v}) \delta^A \bar{v}\, dt.$$

According to Lemma 4.4 with $\bar{v} = [q, v]_G$ we obtain, for all curves $\eta(t) \in \mathfrak{g}$ such that $\eta(t_i) = 0$ for $i = 1, 2$, the equation

$$\begin{aligned} 0 &= \int_{t_0}^{t_1} \left\langle \frac{\partial l}{\partial \bar{v}}, \frac{D[q, \eta]_G}{Dt} + [q, [v, \eta]]_G \right\rangle dt \\ &= \int_{t_0}^{t_1} \left\langle -\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}} + \operatorname{ad}^*_{\bar{v}} \frac{\partial l}{\partial \bar{v}}, [q, \eta]_G \right\rangle dt. \end{aligned}$$

Arbitrariness of η then yields arbitrariness of $\bar{\eta} = [q, \eta]_G$, so we get

$$\operatorname{Ver}(\mathscr{L}\mathscr{P})(l) = -\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v}) + \operatorname{ad}^*_{\bar{v}} \frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v}).$$

Now consider variations $\delta x \oplus \delta^A \bar{v}$ corresponding to horizontal variations δq. Then we have, for all δx with $\delta x(t_i) = 0$, for $i = 0, 1$

$$\delta \int_{t_0}^{t_1} l(x, \dot{x}, \bar{v}) dt = \int_{t_0}^{t_1} \left(\frac{\partial l}{\partial x}\delta x + \frac{\partial l}{\partial \dot{x}}\delta \dot{x} + \frac{\partial l}{\partial \bar{v}}\delta^A \bar{v} \right) dt.$$

Integration by parts and Lemma 4.6 with $\bar{v} = [q, v]_G$ gives

$$\delta \int_{t_0}^{t_1} l(x, \dot{x}, \bar{v})\, dt = \int_{t_0}^{t_1} \left[\left(\frac{\partial l}{\partial x} - \frac{D}{Dt}\frac{\partial l}{\partial \dot{x}} \right)(x, \dot{x}, \bar{v})\delta x - \frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v})\tilde{B}(x)(\dot{x}, \delta x) \right] dt.$$

Integration by parts of the term $(\partial l/\partial \dot{x})\delta \dot{x}$ is justified by showing that

$$\delta \dot{x} = \frac{D}{D\lambda}\frac{\partial x}{\partial t} = \frac{D}{Dt}\frac{\partial x}{\partial \lambda},$$

which can be done, for example, by using Gaussian coordinates relative to the connection ∇ at each point $x(t)$. Arbitrariness of δx then yields

$$\operatorname{Hor}(\mathscr{L}\mathscr{P})(l)(x, \dot{x}, \bar{v}) = \frac{\partial l}{\partial x}(x, \dot{x}, \bar{v}) - \frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}(x, \dot{x}, \bar{v}) - \frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v})\mathbf{i}_{\dot{x}}\tilde{B}(x). \quad \square$$

Remarks. 1. The operators $\mathscr{E}\mathscr{L}(l)$, $\operatorname{Hor}(\mathscr{L}\mathscr{P})(l)$ and $\operatorname{Ver}(\mathscr{L}\mathscr{P})(l)$ depend on the (principal) connection A on the principal bundle Q but not on the connection ∇ on Q/G. It is only the explicit expressions of $\operatorname{Hor}(\mathscr{L}\mathscr{P})(l)$ and $\operatorname{Ver}(\mathscr{L}\mathscr{P})(l)$ that appear in Theorem 6.4 that depend on ∇. As we have remarked previously, in local coordinates it is often convenient to choose ∇ to be simply the usual Euclidean, or vector space, connection.

2. Important particular cases of Theorems 6.1 and 6.4 occur when $G = Q$ and also when $G = \{e\}$. If $G = Q$ then the operator $\operatorname{Hor}(\mathscr{L}\mathscr{P})(l)$ is 0 and $\operatorname{Ver}(\mathscr{L}\mathscr{P})(l)$

is the Euler–Poincaré operator, as we saw before. Thus, in a sense, the vertical Lagrange–Poincaré operator in the bundle $\tilde{\mathfrak{g}}$ is a covariant version of the usual Euler–Poincaré operator on a Lie algebra. If $G = \{e\}$ then $\mathrm{Ver}(\mathscr{LP})(l)$ is 0, $L = l$ and $\mathrm{Hor}(\mathscr{LP})(l) = \mathscr{EL}(L)$ is the usual Euler–Lagrange operator.

Explicit Covariant Description of the Lagrange–Poincaré Operator. We begin by describing the identification of $T^{(2)}Q$ with $TQ \oplus TQ$ using a connection on Q, whose proof is straightforward.

Lemma 6.5. *Let ∇ be a given connection in Q. Then there is a natural diffeomorphism γ^2, depending on ∇,*

$$\gamma^2 \colon T^{(2)}Q \to TQ \oplus TQ,$$

given in local coordinates by the expression

$$\gamma^2(q, \dot{q}, \ddot{q}) = (q, \dot{q}) \oplus (q, \ddot{q} + \Gamma \dot{q}\dot{q}),$$

where Γ is the Christoffel symbol of ∇. Let

$$\gamma^2 = \gamma_1^2 \oplus \gamma_2^2,$$

be the decomposition of γ^2, given in local coordinates by

$$\gamma_1^2(q, \dot{q}, \ddot{q}) = (q, \dot{q})$$

and

$$\gamma_2^2(q, \dot{q}, \ddot{q}) = (\ddot{q} + \Gamma \dot{q}\dot{q}).$$

The map γ_1^2 coincides with the natural vector bundle projection $T^{(2)}Q \to TQ$ and the map γ_2^2 is an affine map on each fiber $T^{(2)}_{(q,\dot{q})}Q$.

Using the diffeomorphism γ^2 as an identification, for a given Lagrangian $L \colon TQ \to \mathbb{R}$, the Euler–Lagrange operator

$$\mathscr{EL} \colon T^{(2)}Q \to T^*Q,$$

becomes a map

$$\mathscr{EL} \colon TQ \oplus TQ \to T^*Q.$$

Now we shall use this description of $\mathscr{EL}$ to write a more explicit covariant expression of the Euler–Lagrange operator

$$\frac{\partial^C L}{\partial q} - \frac{D}{Dt}\frac{\partial L}{\partial \dot{q}}. \tag{15}$$

Let $\tau \colon V \to Q$ and $\varphi \colon F \to Q$ be given vector bundles and denote by Λ and Σ the Christoffel symbols of the given connections in the bundles V and F, respectively. We shall call

$$\frac{D}{Dt}$$

the corresponding covariant derivative in any of the bundles V or F. Recall that

$$\frac{\partial^C f}{\partial x} \quad \text{and} \quad \frac{\partial^C v}{\partial x}$$

are to be understood in the covariant sense, as explained before. Also recall that, if there is no danger of confusion, we often omit the superscript C, and write the previous covariant derivatives simply as

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial x}.$$

We need the following general lemma.

Lemma 6.6. *Let $f\colon V \to F$ be a given fiber-preserving map. Let $v(t)$ be any given curve in V. Then we have*

$$\frac{Df(v(t))}{Dt} = \frac{\partial^C f}{\partial x}(v(t))\dot{x} + \frac{\partial f}{\partial v}(v(t))\frac{Dv(t)}{Dt}.$$

Proof. Let us choose arbitrary local trivializations of V and F. Then we have

$$\left.\frac{Df(v(t))}{Dt}\right|_{t=t_0} = \left.\frac{df(v(t))}{dt}\right|_{t=t_0} + \Sigma\dot{x}(t_0)f(v(t_0)).$$

But

$$\left.\frac{df(v(t))}{dt}\right|_{t=t_0} = \frac{\partial f}{\partial v}(v(t_0))\dot{v}(t_0) + \frac{\partial f}{\partial x}(v(t_0))\dot{x}(t_0),$$

where, in this case, since we are working in a local trivialization, the meaning of $\frac{\partial f}{\partial x}$ is the usual one. Since

$$\dot{v}(t_0) = \left.\frac{Dv(t)}{Dt}\right|_{t=t_0} - \Lambda\dot{x}(t_0)v(t_0),$$

we obtain

$$\begin{aligned}\left.\frac{Df(v(t))}{Dt}\right|_{t=t_0} = \frac{\partial f}{\partial v}(v(t))\left.\frac{Dv(t)}{Dt}\right|_{t=t_0} &- \left.\frac{\partial f}{\partial v}(v(t))\right|_{t=t_0}\Lambda\dot{x}(t_0)v(t_0)\\ &+ \frac{\partial f}{\partial x}(v(t_0))\dot{x}(t_0) + \Sigma\dot{x}(t_0)f(v(t_0)).\end{aligned}$$

Now, let $v_h(t)$ be the curve in V such that $v_h(t_0) = v(t_0)$, $v_h(t)$ is horizontal, and $\tau(v(t)) = \tau(v_h(t))$ for all t. Then, by definition, we have

$$\frac{\partial^C f}{\partial x}(v(t_0))\dot{x}(t_0) = \left.\frac{Df(v_h(t))}{Dt}\right|_{t=t_0}.$$

On the other hand, since v_h is horizontal, its covariant derivative is 0, so we have

$$\dot{v}_h(t_0) + \Lambda\dot{x}(t_0)v(t_0) = 0.$$

We also have

$$\left.\frac{Df(v_h(t))}{Dt}\right|_{t=t_0} = \frac{\partial f}{\partial x}(v(t_0))\dot{x}(t_0) + \left.\frac{\partial f}{\partial v}(v(t))\right|_{t=t_0}\dot{v}_h(t_0) + \Sigma\dot{x}(t_0)f(v(t_0)).$$

By using the previous expressions we easily obtain the assertion of the lemma. □

Now let us apply the previous lemmas to find expressions for the Euler–Lagrange operator. Take $V = TQ$, $v = (q,\, \dot{q})$, $F = T^*Q$, and

$$f(q,\, \dot{q}) = \frac{\partial L}{\partial \dot{q}}.$$

Then, using the previous lemmas, the Euler–Lagrange operator (15) at a point

$$(q,\, \dot{q}) \oplus \left(q,\, \frac{D\dot{q}}{Dt}\right) \in TQ \oplus TQ,$$

has the expression

$$\mathscr{E}\mathscr{L}\left((q,\, \dot{q}) \oplus \left(q,\, \frac{D\dot{q}}{Dt}\right)\right) = \frac{\partial L}{\partial q}(q,\, \dot{q}) - \frac{\partial^2 L}{\partial q \partial \dot{q}}(q,\, \dot{q})\dot{q} - \frac{\partial^2 L}{\partial \dot{q} \partial \dot{q}}(q,\, \dot{q})\frac{D\dot{q}}{Dt},$$

where, this time, as we often do, we have eliminated the superscript C.

Now, we shall show how to write the Lagrange–Poincaré operator in an explicit covariant way. Using the connection in the bundle $T(Q/G)$ we have a natural identification

$$T^{(2)}(Q/G) = T(Q/G) \oplus T(Q/G)$$

and therefore, also an identification

$$\begin{aligned} T^{(2)}(Q/G) \times_{Q/G} \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}} &= T(Q/G) \oplus T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}} \\ &= T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus T(Q/G) \oplus \tilde{\mathfrak{g}}. \end{aligned}$$

To use the previous lemmas, take $V = T(Q/G) \oplus \tilde{\mathfrak{g}}$, $F = T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, and

$$f(x,\, \dot{x},\, \bar{v}) = \frac{\partial l}{\partial \dot{x}} \oplus \frac{\partial l}{\partial \bar{v}},$$

where l is the reduced Lagrangian. Recall that since we have connections in the vector bundles $T(Q/G)$ and $\tilde{\mathfrak{g}}$, we have the Whitney sum connection in $T(Q/G) \oplus \tilde{\mathfrak{g}}$ and also the dual connection in $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$. Then the term

$$\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v})$$

that appears in the Vertical Lagrange–Poincaré operator can be written as

$$\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v}) = \frac{\partial^2 l}{\partial x \partial \bar{v}}(x,\, \dot{x},\, \bar{v})\dot{x} + \frac{\partial^2 l}{\partial \dot{x} \partial \bar{v}}(x,\, \dot{x},\, \bar{v})\frac{D\dot{x}}{Dt} + \frac{\partial^2 l}{\partial \bar{v} \partial \bar{v}}(x,\, \dot{x},\, \bar{v})\frac{D\bar{v}}{Dt}.$$

The term

$$\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}$$

that appears in the Horizontal Lagrange–Poincaré operator is written as

$$\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}(x,\, \dot{x},\, \bar{v}) = \frac{\partial^2 l}{\partial x \partial \dot{x}}(x,\, \dot{x},\, \bar{v})\dot{x} + \frac{\partial^2 l}{\partial \dot{x} \partial \dot{x}}(x,\, \dot{x},\, \bar{v})\frac{D\dot{x}}{Dt} + \frac{\partial^2 l}{\partial \bar{v} \partial \dot{x}}(x,\, \dot{x},\, \bar{v})\frac{D\bar{v}}{Dt}.$$

Collecting formulas, we have proved the following theorem.

Theorem 6.7. *Using the above notations, the Lagrange–Poincaré operator*

$$\mathscr{L}\mathscr{P} \colon T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus T(Q/G) \oplus \tilde{\mathfrak{g}} \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*,$$

is given by

$$
\begin{aligned}
&\mathscr{L}\mathscr{P}\left((x,\, \dot{x},\, \bar{v}) \oplus \left(x,\, \frac{D\dot{x}}{Dt},\, \frac{D\bar{v}}{Dt}\right)\right)\\
&\qquad = \frac{\partial l}{\partial x}(x,\, \dot{x},\, \bar{v}) - \frac{\partial^2 l}{\partial x \partial \dot{x}}(x,\, \dot{x},\, \bar{v})\dot{x} - \frac{\partial^2 l}{\partial \dot{x} \partial \dot{x}}(x,\, \dot{x},\, \bar{v})\frac{D\dot{x}}{Dt}\\
&\qquad\quad - \frac{\partial^2 l}{\partial \bar{v} \partial \dot{x}}(x,\, \dot{x},\, \bar{v})\frac{D\bar{v}}{Dt} - \frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v})\tilde{B}((x,\, \dot{x}),\ \ )\\
&\qquad\quad - \frac{\partial^2 l}{\partial x \partial \bar{v}}(x,\, \dot{x},\, \bar{v})\dot{x} - \frac{\partial^2 l}{\partial \dot{x} \partial \bar{v}}(x,\, \dot{x},\, \bar{v})\frac{D\dot{x}}{Dt} - \frac{\partial^2 l}{\partial \bar{v} \partial \bar{v}}(x,\, \dot{x},\, \bar{v})\frac{D\bar{v}}{Dt}\\
&\qquad\quad - \operatorname{ad}^*_{\bar{v}} \frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v}).
\end{aligned}
$$

7. The Hamilton–Poincaré Equations

Using the conventions and notations of the previous sections, it is clear that the dual $(TQ/G)^*$ of the quotient bundle TQ/G is canonically identified with the quotient bundle T^*Q/G, where the action of G on T^*Q is the cotangent lift of the action of G on Q. Choosing a principal connection A on the principal bundle $\pi\colon Q \to Q/G$ as before, the vector bundle isomorphism α_A defines, by duality, a vector bundle isomorphism

$$\alpha_{A*}\colon T^*Q/G \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*.$$

We often consider α_A and α_{A*} as being identifications.

A Formula for α_{A*}. We now give an interesting description of the projections $T^*Q/G \to \tilde{\mathfrak{g}}^*$ and $T^*Q/G \to T^*(Q/G)$. Recall that the momentum mapping is the map $\mathbf{J}\colon T^*Q \to \mathfrak{g}^*$ given by $\mathbf{J}(\alpha_q)(\xi) = \alpha_q(\xi_Q(q))$, where $\xi_Q(q)$ is the infinitesimal generator of ξ. Equivariance of the momentum mapping implies that its graph $\operatorname{Graph}(\mathbf{J}) \subset T^*Q \times \mathfrak{g}^*$ is an invariant subset under the action of G on $T^*Q \times \mathfrak{g}^*$ given by $g \cdot (q,\, \mu) = (gq,\, \operatorname{Ad}^*_{g^{-1}}(\mu))$. One can easily see that $\operatorname{Graph}(\mathbf{J})/G$ is simply the graph of a vector bundle projection $T^*Q/G \to \tilde{\mathfrak{g}}^*$; therefore, it is justified to call this projection the *reduced momentum mapping*, denoted $[\mathbf{J}]_G$. On the other hand, the connection A induces in a natural way an invariant map $H_{A*}\colon T^*Q \to T^*(Q/G)$; its quotient map

$$[H_{A*}]_G\colon T^*Q/G \to T^*X,$$

is thus well defined.

Using this setting, one can prove without difficulty that

$$\alpha_{A*} = [H_{A*}]_G \oplus [\mathbf{J}]_G.$$

One can also prove the following formula, which shows that the natural contraction between tangent vectors $(q,\, \dot{q})$ and covectors γ_q is preserved under reduction and the identifications α_A and α_{A*}. Namely, we have,

$$\begin{aligned}\langle \gamma_q, (q, \dot{q}) \rangle &= \langle [\gamma_q]_G, [(q, \dot{q})]_G \rangle \\ &= \langle \alpha_{A*}[\gamma_q]_G, \alpha_A[(q, \dot{q})]_G \rangle \\ &= \langle H_{A*}\gamma_q, T\pi(q, \dot{q}) \rangle + \langle \mathbf{J}(\gamma_q), A(q, \dot{q}) \rangle\end{aligned}$$

Reducing Hamilton's Principle. For a given Hamiltonian $H\colon T^*Q \to \mathbb{R}$ Hamilton's phase space principle states that

$$\delta \int (p \cdot \dot{q} - H(q, p))\, dt = 0$$

with suitable boundary conditions, which is equivalent to Hamilton's equations of motion. In this principle, note that the pointwise function in the integrand, namely

$$F(q, \dot{q}, p) = p \cdot \dot{q} - H(q, p)$$

is defined on $TQ \oplus T^*Q$, regarded as a bundle over Q, the base space common to T^*Q and TQ. The group G acts on $TQ \oplus T^*Q$ by simultaneously left translating on each factor by the tangent and cotangent lift. Explicitly, the action of an element $h \in G$ is given by

$$h \cdot (q, \dot{q}, p) = (hq, T_qL_h \cdot \dot{q}, T^*_{hq}L_{h^{-1}} \cdot p)$$

where $T_qL_h\colon T_qQ \to T_{hq}Q$ is the tangent of the left action $L_h\colon q \in Q \mapsto hq \in Q$ at the point q and $T^*_{hq}L_{h^{-1}}\colon T^*_qQ \to T^*_{hq}Q$ is the dual of the map $T_{hq}L_{h^{-1}}\colon T_{hq}Q \to T_qQ$.

The map F is invariant under this action of G as is easily checked, assuming invariance of H. Thus, the function F drops to the quotient, namely to the function $f\colon TQ/G \oplus T^*Q/G \to \mathbb{R}$, or, equivalently, $f\colon T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus T^*(Q/G) \oplus \tilde{\mathfrak{g}}^* \to \mathbb{R}$ given by $f(x, \dot{x}, \bar{v}) \oplus (x, y, \bar{\mu}) = \langle y, \dot{x} \rangle + \langle \bar{\mu}, \bar{v} \rangle - h(x, y, \bar{\mu})$, where h is the reduction of H from T^*Q to $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$.

When the function F is used in the phase space variational principle, one is varying curves $(q(t), p(t))$ and one of course insists that the slot $\dot{q}$ actually is the time derivative of $q(t)$. This restriction induces in a natural way a restriction on the variations of the curve $[q(t), \dot{q}(t)]_G = (x(t), \dot{x}(t), \bar{v}(t))$ and these restrictions are computed as in Theorem 6.4. Thus,

$$\delta x \oplus \delta^A \bar{v} = \delta x \oplus \frac{D\bar{\eta}}{Dt} + [\bar{v}, \bar{\eta}] + \tilde{B}(\delta x, \dot{x}),$$

with the boundary conditions $\delta x(t_i) = 0$ and $\bar{\eta}(t_i) = 0$, for $i = 0, 1$. On the other hand, arbitrary variations δp induce arbitrary fiber variations δy and $\delta\bar{\mu}$.

Using the same kind of argument, based on reducing the action and the variations, that we have used to derive the Lagrange–Poincaré reduced variational principle and equations, we can easily show that Hamilton's phase space variational principle can also be reduced. In fact, we obtain the reduced equations of motion by applying the usual integration by parts argument to the action

$$\int_{t_0}^{t_1} (\langle y, \dot{x} \rangle + \langle \bar{\mu}, \bar{v} \rangle - h(x, y, \bar{\mu}))\, dt,$$

with variations

$$(\delta x \oplus \delta^A \bar{v}) \oplus (\delta y \oplus \delta \bar{\mu}) = \left(\delta x \oplus \frac{D\bar{\eta}}{Dt} + [\bar{v}, \bar{\eta}] + \tilde{B}(\delta x, \dot{x}) \right) \oplus (\delta y \oplus \delta \bar{\mu}),$$

with the restrictions explained above. In this way, we obtain the following equations of motion, called *Hamilton–Poincaré* equations:

$$\frac{Dy}{Dt} = -\frac{\partial h}{\partial x} - \langle \bar{\mu}, \tilde{B}(\dot{x}, \,.\,) \rangle,$$
$$\dot{x} = \frac{\partial h}{\partial y},$$
$$\bar{v} = \frac{\partial h}{\partial \bar{\mu}},$$
$$\frac{D\bar{\mu}}{Dt} = \mathrm{ad}^*_{\bar{v}} \bar{\mu}.$$

One should be aware that the derivative $\partial h/\partial x$ must be interpreted in a covariant sense, similar to the partial derivative $\partial l/\partial x$ as defined in the previous section; this time one uses the covariant derivatives in the bundles $T^*(Q/G)$ and $\tilde{\mathfrak{g}}^*$ induced by duality by the corresponding covariant derivatives in $T(Q/G)$ and $\tilde{\mathfrak{g}}$.

Equivalence with the Lagrange–Poincaré Equations. Let $L(q, \dot{q})$ be an invariant hyperregular Lagrangian and let $l(x, \dot{x}, \bar{v})$ be the reduced Lagrangian. Let

$$h(x, y, \bar{\mu}) = \langle y, \dot{x} \rangle + \langle \bar{\mu}, \bar{v} \rangle - l(x, \dot{x}, \bar{v}).$$

If we write

$$y = \frac{\partial l}{\partial \dot{x}} \quad \text{and} \quad \bar{\mu} = \frac{\partial l}{\partial \bar{v}},$$

and take into account the equality

$$\frac{\partial l}{\partial x} = -\frac{\partial h}{\partial x},$$

then the Hamilton–Poincaré equations become the Lagrange–Poincaré equations.

8. The Hamilton–Poincaré Variational Principle

We are now in a position to summarize what we have obtained in the following result that generalizes Theorem 2.1.

Theorem 8.1. *With the above notation and hypotheses of hyperregularity, the following conditions are equivalent.*

(i) Hamilton's Principle. *The curve $q(t)$ in Q is a critical point of the action*

$$\int_{t_0}^{t_1} L(q(t), \dot{q}(t))\, dt$$

for variations $\delta q(t)$ such that $\delta q(t_i) = 0$ for $i = 0, 1$.

(ii) The Lagrange–Poincaré Variational Principle. *The curve $x(t) \oplus \bar{v}(t)$ is a critical point of the action*

$$\int_{t_0}^{t_1} l(x(t), \dot{x}(t), \bar{v}(t))\, dt$$

on the reduced family of curves $\alpha_A([\Omega(Q;\, q_0,\, q_1)]_G)$*; that is,*

$$\delta \int_{t_0}^{t_1} l(x(t),\, \dot{x}(t),\, \bar{v}(t))\, dt = 0,$$

for variations $\delta x \oplus \delta^A \bar{v}$ *of the curve* $x(t) \oplus \bar{v}(t)$*, where* $\delta^A \bar{v}$ *has the form*

$$\delta^A \bar{v} = \frac{D\bar{\eta}}{Dt} + [\bar{v},\, \bar{\eta}] + \tilde{B}(\delta x,\, \dot{x}),$$

with the boundary conditions $\delta x(t_i) = 0$ *and* $\bar{\eta}(t_i) = 0$*, for* $i = 0,\, 1$*.*

(iii) Hamilton's Phase Space Principle. *The curve* $(q(t),\, p(t))$ *is a critical point of the action*

$$\int_{t_0}^{t_1} (p \cdot \dot{q} - H(q,\, p))\, dt,$$

where the variations $(\delta q,\, \delta p)$ *satisfy* $\delta q(t_i) = 0,$ *for* $i = 0,\, 1,$ *and* $\delta p(t)$ *is arbitrary.*

(iv) The Hamilton–Poincaré Variational Principle. *The reduced curve*

$$[(q,\, \dot{q}) \oplus (q,\, p)]_G = (\dot{x}(t) \oplus \bar{v}(t)) \oplus (y(t) \oplus \bar{\mu}(t))$$

is a critical point of the action

$$\int_{t_0}^{t_1} (\langle y,\, \dot{x}\rangle + \langle \bar{\mu},\, \bar{v}\rangle - h(x,\, y,\, \bar{\mu}))\, dt,$$

with variations

$$(\delta x \oplus \delta^A \bar{v}) \oplus (\delta y \oplus \delta \bar{\mu}) = \left(\delta x \oplus \frac{D\bar{\eta}}{Dt} + [\bar{v},\, \bar{\eta}] + \tilde{B}(\delta x,\, \dot{x})\right) \oplus (\delta y \oplus \delta \bar{\mu}),$$

where $\delta x(t)$ *and* $\bar{\eta}(t)$ *satisfy the same conditions as in* (ii) *and* δy *and* $\delta \bar{\mu}$ *are arbitrary fiber variations.*

(v) *The Euler–Lagrange equations hold*:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0.$$

(vi) *The vertical Lagrange–Poincaré equations, corresponding to vertical variations, hold*:

$$\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v}) = \mathrm{ad}^*_{\bar{v}}\, \frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v})$$

and the horizontal Lagrange–Poincaré equations, corresponding to horizontal variations, hold:

$$\frac{\partial l}{\partial x}(x,\, \dot{x},\, \bar{v}) - \frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}(x,\, \dot{x},\, \bar{v}) = \left\langle \frac{\partial l}{\partial \bar{v}}(x,\, \dot{x},\, \bar{v}),\, \mathbf{i}_{\dot{x}} \tilde{B}(x) \right\rangle.$$

(vii) *Hamilton's equations hold*:

$$(\dot{q}(t),\, \dot{p}(t)) = \left(\frac{\partial H}{\partial p},\, -\frac{\partial H}{\partial q}\right).$$

(viii) *The Hamilton–Poincaré equations hold*:

$$\frac{Dy}{Dt} = -\frac{\partial h}{\partial x} - \langle \bar{\mu},\, \tilde{B}(\dot{x},\, .\,)\rangle, \tag{16}$$

$$\dot{x} = \frac{\partial h}{\partial y}, \tag{17}$$

$$\bar{v} = \frac{\partial h}{\partial \bar{\mu}}, \tag{18}$$

$$\frac{D\bar{\mu}}{Dt} = \operatorname{ad}^*_{\bar{v}} \bar{\mu}. \tag{19}$$

The equivalence between statements (i), (ii), (v) *and* (vi) *holds for general Lagrangians, not necessarily nondegenerate. The equivalence between the statements* (iii), (iv), (vii) *and* (viii) *holds for general Hamiltonians, not necessarily nondegenerate.*

Remark. In (ii), if $\bar{v} = [q,\, v]_G$ with $v = A(q,\, \dot{q})$, then $\bar{\eta}$ can be always written $\bar{\eta} = [q,\, \eta]_G$, and the condition $\bar{\eta}(t_i) = 0$ for $i = 0,\, 1$ is equivalent to the condition $\eta(t_i) = 0$ for $i = 0,\, 1$. Also, if $x(t) = [q]_G$ and $\bar{v} = [q,\, v]_G$, where $v = A(q,\, \dot{q})$, then variations $\delta x \oplus \delta^A \bar{v}$ such that

$$\delta^A \bar{v} = \frac{D\bar{\eta}}{Dt} + [\bar{v},\, \bar{\eta}] \equiv \frac{D[q,\, \eta]_G}{Dt} + [q,\, [v,\, \eta]]_G$$

with $\bar{\eta}(t_i) = 0$ (or, equivalently, $\eta(t_i) = 0$) for $i = 0,\, 1$ correspond exactly to vertical variations δq of the curve q such that $\delta q(t_i) = 0$ for $i = 0,\, 1$, while variations $\delta x \oplus \delta^A \bar{v}$ such that

$$\delta^A \bar{v} = \tilde{B}(\delta x,\, \dot{x})$$

with $\delta x(t_i) = 0$ for $i = 0,\, 1$, correspond exactly to horizontal variations δq of the curve q such that $\delta q(t_i) = 0$.

The Reduced Poisson Structure. With the Hamilton–Poincaré equations established, it is not difficult to obtain the reduced bracket in the Poisson manifold $T^*Q/G = T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, in terms of the structure of Lagrange–Poincaré bundles.

Theorem 8.2. *The reduced Poisson bracket in the quotient Poisson manifold*

$$T^*Q/G = T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

is given by

$$\{f,\, h\} = \frac{\partial f}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial h}{\partial x} + \left\langle \bar{\mu},\, \tilde{B}\left(\frac{\partial f}{\partial y},\, \frac{\partial h}{\partial y}\right)\right\rangle + \left\langle \bar{\mu},\, \left[\frac{\partial h}{\partial \bar{\mu}},\, \frac{\partial f}{\partial \bar{\mu}}\right]\right\rangle$$

Proof. The evolution of the function f is given by

$$\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\frac{Dy}{Dt} + \frac{\partial f}{\partial \bar{\mu}}\frac{D\bar{\mu}}{Dt}.$$

Using this and the Hamilton–Poincaré equations, the formula for the Poisson bracket follows easily. □

The proof of this result is considerably simpler than that given in [MMR] and [Mon] (see also [LMMR]).

The Symplectic Leaves of $T^*(Q/G)\oplus\tilde{\mathfrak{g}}$. For each $\mu \in \mathfrak{g}^*$, let $\mathscr{O}_\mu$ be the coadjoint orbit through μ in $\mathfrak{g}^*$. Let us define the subset $S_\mu \subset T^*Q$ by $S_\mu = \{\alpha_q \colon \mathbf{J}(\alpha_q) \in \mathscr{O}_\mu\}$, where $\mathbf{J}\colon T^*Q \to \mathfrak{g}^*$ is the standard cotangent bundle momentum mapping for the action of G on Q. It is clear that S_μ is an invariant subset under the cotangent action of G on T^*Q. By the well-known equivalence of point and orbit reduction (see [OR] for the case when the coadjoint orbit is not an embedded submanifold of $\mathfrak{g}^*$), the reduced spaces $\tilde{S}_\mu = S_\mu/G$ are the symplectic leaves of the Poisson manifold $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$. To describe $\tilde{S}_\mu$ more precisely, let us define, for each $\mu \in \mathfrak{g}^*$, the bundle $\tilde{\mathscr{O}}_\mu \subset \tilde{\mathfrak{g}}^*$, by $\tilde{\mathscr{O}}_\mu = (Q \times O_\mu)/G$. Then, as explained for instance in [MP], it is easy to prove using the previous definitions, that $\tilde{S}_\mu = T^*(Q/G) \times_X \tilde{\mathscr{O}}_\mu$.

We now show how to write a covariant formula for the symplectic structure ω_μ of $\tilde{S}_\mu$, using the Hamilton–Poincaré equations. A generic tangent vector to $\tilde{S}_\mu$ at a point $(x,\, y) \oplus \bar{\nu}$ is given, according to the Hamilton–Poincaré equations, by

$$\left(\dot{x},\, \frac{Dy}{Dt},\, \mathrm{ad}_{\bar{v}}\, \bar{\nu}\right) \equiv (\dot{x},\, p,\, \mathrm{ad}_{\bar{v}}\, \bar{\nu})\, .$$

Proposition 8.3. *The symplectic form ω_μ is given by*

$$\begin{aligned}\omega_\mu((x,\, y) \oplus \bar{\nu})\big((\dot{x}_1,\, p^1,\, \mathrm{ad}_{\bar{v}_1}\, \bar{\nu}),\, (\dot{x}_2,\, p^2,\, \mathrm{ad}_{\bar{v}_2}\, \bar{\nu})\big) \\ = \langle p^2,\, \dot{x}_1\rangle - \langle p^1,\, \dot{x}_2\rangle + \langle \bar{\nu},\, \tilde{B}(x)(\dot{x}_1,\, \dot{x}_2)\rangle + \langle \bar{\nu},\, [\bar{v}_2,\, \bar{v}_1]\rangle.\end{aligned}$$

Proof. In order to prove that the preceding expression is, in fact, the symplectic form, one can simply observe that

$$\omega_\mu((x,\, y)\oplus\bar{\nu}) \left(\left(\frac{\partial f_1}{\partial y},\, \frac{\partial f_1}{\partial x},\, \mathrm{ad}_{\bar{v}_1}\, \bar{\nu}\right),\, \left(\frac{\partial f_2}{\partial y},\, \frac{\partial f_2}{\partial x},\, \mathrm{ad}_{\bar{v}_2}\, \bar{\nu}\right)\right) = \{f_1,\, f_2\}((x,\, y)\oplus\bar{\nu}),$$

where

$$\bar{v}_i = \frac{\partial f_i}{\partial \bar{\nu}},$$

for $i = 1,\, 2$. Closedness and nondegeneracy of ω_μ are proved as follows. To prove nondegeneracy, note that, by construction,

$$\omega_\mu((x,\, y) \oplus \bar{\nu})(X_g,\, X_f) = \mathbf{d}g \cdot X_f$$

where $f,\, g\colon T^*X \oplus \tilde{\mathfrak{g}}^* \to \mathbb{R}$ are arbitrary functions, so that at a given point $((x,\, y)\oplus\bar{\nu})$, X_f represents an arbitrary tangent vector to the symplectic leaf. Since we can choose g such that $\mathbf{d}g \cdot X_f$ is nonzero at the given point whenever X_f is nonzero, we see that ω_μ is nondegenerate. It is standard that the Jacobi identity for the Poisson bracket gives closedness of ω_μ. □

Using the previous expressions one sees that ω_μ is the sum of three 2-forms, corresponding to the three terms of the reduced Poisson bracket. The one corresponding to the third term is the usual Kostant–Kirillov–Souriau symplectic form in the fiber $\tilde{\mathfrak{g}}^*_x$ of $\tilde{\mathfrak{g}}^*$ at $x \in Q/G$, the one corresponding to the first term is the canonical symplectic form on $T^*(Q/G)$, and the one corresponding to the second term is a "magnetic term" involving the curvature.

9. Example: The Rigid Body with Rotors

Following [MS2], the configuration space of a rigid body with rotors whose axes are parallel to the three principal axes of inertia of the rigid body, is the principal bundle $\pi\colon \mathrm{SO}(3)\times S^1\times S^1\times S^1 \to S^1\times S^1\times S^1,$ with structure group SO(3) acting on the left on the first factor. Since it is a trivial bundle, we can choose the trivial principal connection A. The reduced cotangent bundle is easily seen to be

$$\mathfrak{so}(3)^* \times \mathbb{R}^{3*} \times S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1.$$

The reduced Lagrangian of the system is given by

$$l(\Omega,\, \dot\theta) = \frac{1}{2}\sum_{r=1}^{3} I_r\Omega_r^2 + \frac{1}{2}\sum_{r=1}^{3} K_r(\Omega_r + \dot\theta)^2.$$

Let

$$\nu_r = \frac{\partial l}{\partial \Omega_r}$$

and

$$y_r = \frac{\partial l}{\partial \dot\theta_r}$$

The Hamiltonian of the system is obtained by the Legendre transformation and the reduced Hamiltonian is given by

$$h(\nu,\, y) = \frac{1}{2}\sum_{r=1}^{3} \frac{(\nu_r - y_r)^2}{I_r} + \frac{1}{2}\sum_{r=1}^{3} \frac{y_r^2}{K_r}.$$

This reduced Hamiltonian is in agreement with the reduced Lagrangian, as given in [MS2], via the Legendre transformation, after one identifies vectors and covectors in $\mathfrak{so}(3) \equiv \mathbb{R}^3$ via the Euclidean metric. Since the connection is trivial, covariant derivatives are the same as usual derivatives. We have

$$\frac{\partial h}{\partial \theta_r} = 0, \tag{20}$$

$$\frac{\partial h}{\partial y_r} = \left(\frac{1}{K_r} + \frac{1}{I_r}\right) y_r - \frac{\nu_r}{I_r}, \tag{21}$$

$$\frac{\partial h}{\partial \nu_r} = \frac{\nu_r - y_r}{I_r}. \tag{22}$$

Then, using the Hamilton–Poincaré equations we can conclude that y is a constant of the motion. We observe that the vector whose components are

$$\frac{\nu_r - y_r}{I_r}$$

is Ω. Thus, the Hamilton–Poincaré equations can be written using the standard identification $\mathfrak{so}(3) \equiv \mathbb{R}^3$, as follows:

$$\dot{y} = 0, \tag{23}$$

$$\dot{\theta}_r = \left(\frac{1}{K_r} + \frac{1}{I_r} \right) y_r - \frac{\nu_r}{I_r}, \tag{24}$$

$$\frac{\partial h}{\partial \nu_r} = \Omega, \tag{25}$$

$$\dot{\nu} = \nu \times \Omega. \tag{26}$$

These equations are equivalent to the Lagrange–Poincaré equations given in [MS2] via the reduced Legendre transformation, as is easy to check. The variational description of the Hamilton–Poincaré equations in this example is given by

$$\int_{t_0}^{t_1} \left(\langle y, \dot{\theta} \rangle + \langle \nu . \Omega \rangle - h(\nu, y) \right) dt,$$

with variations restricted by $\delta\Omega = \dot{\eta} + [\Omega, \eta]$, $\delta\eta(t_i) = 0$, $\delta\theta(t_i) = 0$, for $i = 0, 1$, and $\delta\nu$, δy arbitrary.

Future Work. We plan to develop the geometry of the bundle $TQ \oplus T^*Q$, how it fits into the spaces T^*TQ, TT^*Q, and the generalized Legendre transform (allowing one to treat degenerate Lagrangians) introduced by [Tul].

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