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Attracting curves on families of stationary solutions in two-dimensional Navier–Stokes and reduced magnetohydrodynamics

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Families of stable stationary solutions of the two-dimensional incompressible homogeneous Euler and ideal reduced magnetohydrodynamic equations are shown to be attracting for the corresponding viscous perturbations of these systems, i.e. for the Navier–Stokes and the reduced viscous MHD equations with magnetic diffusion. Each solution curve of the dissipative system starting in a cone around the family of stationary solutions of the unperturbed conservative system defines a shadowing curve which attracts the dissipative solution in an exponential manner. As a consequence, the specific exponential decay rates are also determined. The techniques to analyse these two equations can be applied to other dissipative perturbations of Hamiltonian systems. The method in its general setting is also presented.

Keywords: Navier–Stokes; magnetohydrodynamics; attractors;
stability; Euler's equations; shadowing

1. Introduction

There are several physical systems which can be modelled as a Hamiltonian system to which dissipation has been added. Examples of such systems considered in this paper are the reduced magnetohydrodynamics approximation of a two-dimensional charged homogeneous incompressible fluid with viscosity and resistivity and the two-dimensional Navier–Stokes equations. If the dissipation is ignored, then, in both examples, the purely Hamiltonian system possesses Casimir functionals. By using the energy-Casimir functional which is a linear combination of the Hamiltonian and the relevant Casimir functionals, we can give a variational description of stationary solutions of the Hamiltonian system. Usually, these stationary solutions are found in families. If the stationary solution is a minimum of the energy-Casimir functional, then it is conditionally Lyapunov stable (Holm *et al.* 1985).

In this paper we will build on the ideas of Holm *et al.* (1985) and use the energy-Casimir functional to show that certain families of such stable Hamiltonian stationary states are attractors for the dissipative system. More precisely, for every solution of

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the dissipative system attracted to these families, we define a *shadowing curve* on the family of stationary states. The rate of attraction of the solution curve to its shadowing curve is found by analysing the time derivative of a functional which measures a weighted distance between states of the physical system and the family of Hamiltonian stationary states. To be specific, this functional is the quotient of the specific energy-Casimir functional used in the stability analysis of the unperturbed Hamiltonian system and another positive definite conserved functional. It behaves like a scaled Lyapunov function and thus allows us to derive a sharp estimate for the deviation of the solution curve of the dissipative system from its attracting shadowing curve on the manifold of stationary solutions of the unperturbed Hamiltonian system.

This method extends the work in Derks *et al.* (1995) which presents a general theory for the approximation of the solutions of a finite dimensional dissipative system by shadowing curves of relative equilibria of the associated Hamiltonian system with symmetry. In general, there do not exist constants of the motion which are simultaneously norms for the states of the system. Thus, the general theory analyses the time derivative of only the energy-momentum functional, which is the analogue of the energy-Casimir functional. However, as is the case in our examples, if there are constants of the motion which are also positive definite, then the analysis of a specific quotient gives sharper estimates for the decay rate to the shadowing curve.

In this paper (and in Derks *et al.* (1995)) it is essential that we look at stationary solutions (relative equilibria) which are Lyapunov stable. As can be seen from Bloch *et al.* (1994, 1996), if a relative equilibrium is not a constrained minimum then a dissipation can induce instability. In Bloch *et al.* (1994, 1996), the analysis is restricted to Hamiltonian systems with a dissipation that respects the invariance of the constants of motion of the Hamiltonian system. The dissipations we look at do not have that property. Moreover, the technique in Bloch *et al.* (1994, 1996) implicitly requires that the stationary solution for the Hamiltonian system remains stationary even after the dissipation is added. This is precisely what does not happen in the case of the Navier–Stokes and reduced magnetohydrodynamics (RMHD) equations, so the results of this paper can be considered in some sense complementary to those in Bloch *et al.* (1994, 1996).

The main system in this paper is the RMHD approximation of a two-dimensional charged homogeneous incompressible fluid with viscosity and resistivity. Let $\mathcal{D}$ be a compact simply connected domain in the x–y-plane with a smooth boundary $\partial\mathcal{D}$. We shall study the motion of a two-dimensional charged homogeneous incompressible fluid in $\mathcal{D}$ in the RMHD approximation (Morrison & Hazeltine 1984; Morrison & Eliezer 1986). The Eulerian velocity field is denoted by $\boldsymbol{v}$ and the magnetic field by $\boldsymbol{B} = (\partial A/\partial y, -\partial A/\partial x)$, where A is the magnetic potential; both are vectors lying in the plane determined by $\mathcal{D}$.

Since div $\boldsymbol{v} = 0$ and $\mathcal{D}$ is simply connected, there is a *stream function* ψ for the velocity field $\boldsymbol{v}$, i.e. $\boldsymbol{v} = (\partial\psi/\partial y, -\partial\psi/\partial x)$. This function ψ, uniquely determined only up to a constant, is necessarily constant on $\partial\mathcal{D}$; thus one can fix this constant to be zero and then ψ is uniquely determined by $\boldsymbol{v}$.

The *vorticity* ω and the *electric current density* J are defined to equal the third (and only non-zero) components of curl $\boldsymbol{v}$ and of curl $\boldsymbol{B}$, respectively. Thus, $\omega = -\Delta\psi$ and $J = -\Delta A$, where $\Delta = \text{div} \circ \text{grad}$ is the Laplacian on the plane. The RMHD

equations with viscosity and resistivity are

$$\left.\begin{aligned}
\dot{\omega} &= -\boldsymbol{v} \cdot \nabla\omega + \boldsymbol{B} \cdot \nabla J + \nu\Delta\omega = \{\psi, \omega\} + \{J, A\} + \nu\Delta\omega, \\
\dot{A} &= -\boldsymbol{v} \cdot \nabla A + \eta\Delta A = \{\psi, A\} + \eta\Delta A.
\end{aligned}\right\} \tag{1.1}$$

In these equations $\{\,,\}$ is the usual Poisson bracket in the x–y-plane, ν is the fluid viscosity and η is the magnetic viscosity (also called resistivity or magnetic diffusivity).

We consider two types of boundary conditions. In the first set we shall impose the standard no-slip boundary condition on the velocity field $\boldsymbol{v}$, that is, $\boldsymbol{v} = 0$ on $\partial\mathcal{D}$ and the magnetic potential zero on the boundary, i.e.

$$\text{BC1} \quad \begin{cases} \boldsymbol{v} = 0, & \text{on } \partial\mathcal{D}, \\ A = 0, & \text{on } \partial\mathcal{D}. \end{cases}$$

Note that the no-slip boundary condition is equivalent to $\psi = 0$ and $\partial\psi/\partial n = 0$ on the boundary.

The second set does not impose the no-slip boundary condition, but replaces it with a condition on the vorticity: $\boldsymbol{\omega} \times \boldsymbol{n} = 0$ and the standard non-penetration boundary condition $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ (Temam 1988, §3.2.2). In our planar case this becomes

$$\text{BC2} \quad \begin{cases} \psi = 0, & \text{and} \quad \omega = 0, \quad \text{on } \partial\mathcal{D}, \\ A = 0, & \text{on } \partial\mathcal{D}. \end{cases}$$

The total energy of the system is

$$H = \frac{1}{2} \int_{\mathcal{D}} (|\boldsymbol{v}|^2 + |\boldsymbol{B}|^2) \, \mathrm{d}\boldsymbol{x} = \frac{1}{2} \int_{\mathcal{D}} (\omega\psi + JA) \, \mathrm{d}x. \tag{1.2}$$

If the initial conditions are sufficiently smooth, then there exist unique strong solutions of the dissipative RMHD equations (see, for example, Sermange & Temam (1983), and references therein). For ideal RMHD (i.e. without dissipation), short-time existence for classical solutions of planar MHD is proven in Kozono (1989). Given the existence and uniqueness of strong solutions, we shall use the term *stable* to denote Lyapunov stability of solutions or families of solutions, as opposed to the considerably weaker property of continuous dependence on initial conditions.

In this paper, we shall introduce a family of stable ground states for the RMHD equations without dissipation and define shadowing curves on this family for solutions of the RMHD equations with dissipation with appropriate initial conditions. By analysing a scaled Lyapunov function, we shall show how fast solutions of the dissipative equation are attracted to their shadowing curves. In Ghidaglia (1986) and Sermange & Temam (1983) it is shown that the flow of the RMHD equations is asymptotically described by a finite set of parameters that depend on the initial conditions. There remains the question of the relationship between the initial condition and the set of these parameters. In Hasegawa (1985), the asymptotic behaviour is related to self-organization in the RMHD equations. Self-organization is shown for Alfvén waves: the magnetic field $\boldsymbol{B}$ is parallel to the velocity $\boldsymbol{v}$ and the ratio of their lengths is prescribed. The question of what happens for more general initial conditions is explicitly raised. In this work we give quantitative precise answers to these questions in terms of the shadowing curves introduced above.

If no magnetic terms are present, the RMHD equations become the two-dimensional Navier–Stokes equations. The stable ground states are two-dimensional

versions of the Arnold–Beltrami–Childress flows (ABC) (cf. Dombre *et al.* 1986). In the case of the boundary conditions BC2, the definition and the analysis of the shadowing curves is straightforward. In order to illustrate our techniques more clearly, we first study the two-dimensional Navier–Stokes equations. In this way we recover and slightly improve some results known in literature. Our results on the attracting properties of the ABC flows are reminiscent of those obtained by Marchioro (1986) but the proof is closer in spirit to that in Constantin *et al.* (1988). The spectral-gap condition automatically holds and the cone condition guaranteeing the existence of the shadowing curve is similar to the one found in inertial manifold theorems. However, the papers above also permit forcing terms in both the Euler and Navier–Stokes equations and our techniques do not immediately generalize to this setting since they are heavily based on the Hamiltonian dynamics of the unperturbed system. Nevertheless, combining the procedure of this paper with the well-developed machinery of inertial manifolds will allow the treatment of the case with forcing too. Foias *et al.* (1989) is particularly relevant to this approach and we hope to address this in a future publication in which we will also tie the ideas of this paper with results on inertial manifolds.

It seems natural to ask what happens to our methods if we analyse the equations in a three-dimensional setting. There are several obstacles in the three-dimensional case that do not allow us to extend the methods of this paper. Firstly, there is the problem of the existence and uniqueness of long-time strong solutions. Only well posedness is known, and in particular, only short-time existence of solutions has been proved. Secondly, in three dimensions, the only known Casimir functional is the helicity (i.e. $\int_{\mathcal{D}} \boldsymbol{v} \cdot \boldsymbol{\omega} \, d\boldsymbol{x}$), if the domain is the entire space or if some very special boundary conditions hold. Thirdly, even if one would consider domains and boundary conditions for which the helicity is a Casimir, the relevant constrained critical points of the energy functional are the Beltrami flows which are formally unstable because they are not constrained maxima or minima on the level sets of the Casimir functional. Because of these problems our analysis cannot be extended to the three-dimensional situation.

A special class of these Beltrami flows are the ABC flows. Arnold (1972) showed that in the Euler equations some three-dimensional steady flows for which the vorticity and the velocity are collinear can exhibit exponential stretching of particle paths, which induces exponential instability. He conjectured that the ABC flows also have exponential stretching. For some subclasses of ABC flows this was proved in Dombre *et al.* (1986) and Friedlander *et al.* (1993). Furthermore, in Galloway & Frisch (1987) it is numerically suggested that the ABC flows are exponentially unstable in the three-dimensional Navier–Stokes equations if the Reynolds number (i.e. ν^{-1}) passes a certain threshold. On the other hand, inertial manifold theory shows that there is an attractor which consists of special Beltrami flows (see Foias & Saut 1984; Foias *et al.* 1989).

2. The main ideas in the analysis

Some of the expositions and proofs that follow are technically quite intricate, although the ideas behind them are relatively simple. In order to facilitate the reading, we sketch the main ideas underlying the definition of the shadowing curves as well as the main steps of the proof of their existence and the estimates to the solution of the dissipative system. This rough outline is valid both in the case of the two-

dimensional Navier–Stokes equation as well as in the case of the RMHD equations, where some technical issues appear which will complicate and slightly obscure the analysis.

(1) First we look at the conservative system. In the case of the Navier–Stokes equations, the conservative system is formed by the Euler equations. In the magnetohydrodynamical case, the conservative system is given by the ideal RMHD equations. To define the ground states, we use the Hamiltonian and some Casimirs. In the case of the Euler equations the relevant Casimir is the enstrophy $W = \frac{1}{2}\int_{\mathcal{D}}\omega^2\,\mathrm{d}\boldsymbol{x}$. For the ideal RMHD equations we use the following two Casimirs:

$$I_1 = \int_{\mathcal{D}}\omega A\,\mathrm{d}\boldsymbol{x} \quad\text{and}\quad I_2 = \frac{1}{2}\int_{\mathcal{D}}\omega^2\,\mathrm{d}\boldsymbol{x}.$$

By looking at constrained minima of the Hamiltonian on level sets of the Casimir(s), we get the ground-state solutions. We define a Lyapunov functional $L(u)$ based on the Euler–Lagrange equation of the critical-point problem. This Lyapunov functional is a linear combination of the Hamiltonian and the Casimir(s). For the conservative systems, the Lyapunov functional gives (constrained) stability of the ground states.

(2) The next step is to look at the dissipative system. The ground states form an invariant manifold for the dissipative system. To analyse if and how they attract other solutions, we first define a projection of a solution onto the invariant manifold of ground states. This projection will form a *shadowing curve* if the solution is attracted to the ground states manifold. We denote this projected curve by $\bar{u}(t)$. The Lyapunov functional $L(u(t))$ is a measure of the distance between the solution $u(t)$ of the dissipative system and the projected curve $\bar{u}(t)$.

If $C(u)$ is a positive definite conserved quantity, which can be used as a (semi)-norm for u, then the scaled Lyapunov functional $\hat{L}(u(t)) = L(u(t))/C(u(t))$ is a normalized measure for the distance between the solution $u(t)$ and the projected curve $\bar{u}(t)$. For the Navier–Stokes equations we use $C(u) = H(u)$ and for the RMHD equations we use $C(u) = I_2(u)$.

(3) The final and most elaborate step is to derive an estimate for the time behaviour of the scaled Lyapunov functional $\hat{L}(u(t))$. For both the Navier–Stokes equations and the RMHD equations we will show that for appropriate initial conditions we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}(u(t)) \leqslant -2\nu d\hat{L}(u(t)) + \text{remainder}.$$

Here $d > 0$ is some constant that 'measures' the dissipative behaviour of the scaled Lyapunov functional. The remainder is of order $\hat{L}(u(t))^2$ or can be written as $f(t)\hat{L}(u(t))$, with $f(t)$ some integrable function. This estimate on the time derivative implies that $\hat{L}(u(t))$ decays like $\mathrm{e}^{-2\nu dt}$.

In the case of the Navier–Stokes equations it is straightforward to carry out the steps above. This analysis can be found in § 3.

In the case of the RMHD equations, there is a restriction on the values of the Casimirs I_1 and I_2 in order to get constrained minima. This is shown in § 4 which deals with the analysis of the ideal RMHD equations. In § 5 we show that the set of stable ground states form an invariant manifold for the dissipative RMHD equations. Before we can start the analysis of the time behaviour of the scaled Lyapunov functional, we have to make sure that our solutions satisfy the restriction on the value of the Casimirs I_1 and I_2 for all time. So in § 6 we make some *a priori* estimates to verify that for appropriate initial conditions the restrictions are satisfied for all time.

For these initial conditions, we analyse the time behaviour of the scaled Lyapunov functional $\hat{L}(u(t))$ in §7. A nice (coincidental) property of the *a priori* estimates in §6 is that they also provide a first estimate for the decay of $\hat{L}(u(t))$.

3. Attracting ABC-flows in two-dimensional Navier–Stokes

By ignoring the influence of the magnetic field in the RMHD equations, we recover the Navier–Stokes equations in two dimensions. The goal of this section is to illustrate our method for this simpler system which was already studied in Foias & Saut (1984) and in van Groesen (1988) by different techniques. We shall use the same conserved functionals but exploit the fact that a judicious combination thereof is a Lyapunov functional for the ideal homogeneous two-dimensional Euler equations. This then enables us to slightly improve some estimates in Foias & Saut (1984) and van Groesen (1988).

Specifically, in this section we consider

$$\left.\begin{aligned}\dot{\omega} &= -\boldsymbol{v}\cdot\nabla\omega + \nu\Delta\omega = \{\psi,\omega\} + \nu\Delta\omega, \quad \text{in } \mathcal{D},\\ \psi &= 0 \text{ and } \omega = 0, \quad \text{on } \partial\mathcal{D},\end{aligned}\right\} \tag{3.1}$$

where $\mathcal{D}$ is any compact simply connected domain in the x–y-plane with smooth boundary $\partial\mathcal{D}$.

(*a*) *Ground states in the Euler equations*

If we ignore the viscosity, i.e. $\nu = 0$, then the Navier–Stokes equations become the Euler equations, which are a Hamiltonian system relative to the total energy

$$H(\omega) = \frac{1}{2}\int_{\mathcal{D}}|\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_{\mathcal{D}}\omega\psi\,\mathrm{d}x.$$

A family of stationary solutions of Euler's equation is given by

$$-\Delta\omega_\lambda = \lambda\omega_\lambda, \quad \text{implying } \omega_\lambda = \lambda\psi_\lambda,$$

where λ is a positive eigenvalue of $(-\Delta)$ with zero boundary conditions. These solutions are a two-dimensional version of the ABC flows (cf. Dombre *et al.* 1986). In particular, if $\mathcal{D}$ is a disc, then these solutions are necessarily axisymmetric. In literature, sometimes the ABC flows are called Stokes flows, since they are solutions of the Stokes operator.

We define the enstrophy

$$W(\omega) = \frac{1}{2}\int_{\mathcal{D}}\omega^2\,\mathrm{d}\boldsymbol{x}.$$

It is easily verified that the critical points of $W - \lambda H$ are the stationary solutions considered above. Moreover, let $0 < \lambda_0 < \lambda_1 \leqslant \cdots$ be the eigenvalues of $(-\Delta)$ with zero boundary conditions; λ_0 is strictly positive and simple (Aubin 1982, ch. 4). Let $\mathcal{E}_0$ be the eigenspace with the eigenvalue λ_0, let Π_0 be the L^2-orthogonal projection onto $\mathcal{E}_0$, and let $\Pi_0^\perp$ be the projection on the orthogonal complement $\mathcal{E}_0^\perp$ of $\mathcal{E}_0$. Then the ABC flows ω_{λ_0} are *global* constrained minima of the enstrophy W on level sets of the Hamiltonian. To see this, fix $\omega_{\lambda_0} = -\Delta\psi_{\lambda_0} = \lambda_0\psi_{\lambda_0}$, define $h_{\lambda_0} := H(\omega_{\lambda_0})$ and let $\xi = -\Delta\varphi$, $\xi|_{\partial\mathcal{D}} = 0$, $\varphi|_{\partial\mathcal{D}} = 0$, be arbitrary but such that $H(\omega_{\lambda_0} + \xi) = h_{\lambda_0}$,

which is equivalent to

$$\int_{\mathcal{D}} \psi_{\lambda_0} \xi \, \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_{\mathcal{D}} \xi \varphi \, \mathrm{d}\boldsymbol{x} = 0.$$

Multiplying this identity by λ_0 yields therefore

$$\int_{\mathcal{D}} \omega_{\lambda_0} \xi \, \mathrm{d}\boldsymbol{x} = -\frac{1}{2} \lambda_0 \int_{\mathcal{D}} \xi \varphi \, \mathrm{d}\boldsymbol{x},$$

so by denoting the L^2-norm by $\|.\|$ we get

$$W(\omega_{\lambda_0} + \xi) - W(\omega_{\lambda_0}) = \frac{1}{2} \int_{\mathcal{D}} \xi(\xi - \lambda_0 \varphi) \, \mathrm{d}\boldsymbol{x} \geqslant \frac{\lambda_1 - \lambda_0}{2\lambda_1} \|\Pi_0^\perp \xi\|^2, \qquad (3.2)$$

the last inequality being obtained by a straightforward eigenfunction expansion. Thus $W(\omega_{\lambda_0} + \xi) \geqslant W(\omega_{\lambda_0})$ for any ξ in the h_{λ_0}-level set of H, i.e.

$$W(\omega_{\lambda_0}) = \min_\omega \{ W(\omega) \mid H(\omega) = h_{\lambda_0} \}.$$

The inequality $W(\omega_{\lambda_0} + \xi) \geqslant W(\omega_{\lambda_0})$ becomes an equality only if $\xi \in \mathcal{E}_0$. Since λ_0 is simple, the conditions $\xi \in \mathcal{E}_0$ and $H(\omega_{\lambda_0} + \xi) = h_{\lambda_0}$ imply that $\xi = 0$ or $\xi = -2\omega_{\lambda_0}$. Thus the constrained minimum is unique up to a sign.

Remark 1. A more general result in the non-smooth case was obtained by Burton & McLeod (1991) in any dimension. Consider the weak closure of the set $\mathcal{O}_{\omega_0}$ of all the not necessarily invertible measure-preserving rearrangements of a given 'vorticity function' ω_0. (It is known that this equals the norm closed convex hull of $\mathcal{O}_{\omega_0}$.) Then it is shown that the infimum of the 'energy function' $\frac{1}{2} \int_{\mathcal{D}} \omega(-\Delta)^{-1}\omega \, \mathrm{d}x$ over this weak closure is always attained, is unique and is one-signed almost everywhere. In addition, the minimum $\omega_{\min}$ is a decreasing function of its 'stream function' $(-\Delta)^{-1}\omega_{\min}$. Moreover, if ω_0 is one-signed almost everywhere, then the minimum $\omega_{\min}$ is a rearrangement of ω_0. In Burton (1987) it is shown that the maximum $\omega_{\max}$ of the 'energy function' over just $\mathcal{O}_{\omega_0}$ is always attained but that it is not unique in general. (Of course, it coincides with the maximum over the weak closure of $\mathcal{O}_{\omega_0}$.) Any maximum $\omega_{\max}$ is an increasing function of its 'stream function' $(-\Delta)^{-1}\omega_{\max}$. If the domain is a ball and ω_0 is one-signed almost everywhere, then the maximum $\omega_{\max}$ is unique (Burton & McLeod 1991).

The lower bound (3.2) is part of the string of inequalities

$$\tfrac{1}{2}\|\Pi_0^\perp \xi\|^2 \geqslant (W - \lambda_0 H)(\omega_{\lambda_0} + \xi) = \frac{1}{2} \int_{\mathcal{D}} \xi(\xi - \lambda_0 \varphi) \, \mathrm{d}\boldsymbol{x} \geqslant \frac{\lambda_1 - \lambda_0}{2\lambda_1} \|\Pi_0^\perp \xi\|^2, \quad (3.3)$$

which will be used several times throughout this section. Together with conservation of H and W and the long-time existence theorem for classical solutions for the two-dimensional homogeneous incompressible Euler equations (see Kato 1967), it implies that the solutions ω_{λ_0} are Lyapunov stable in the L^2 semi-norm on the finite perturbations ξ given by $[(W - \lambda_0 H)(\omega_{\lambda_0} + \xi)]^{1/2}$. The direction of degeneracy of this semi-norm is $\mathcal{E}_0$, i.e. precisely the family of ABC flows. Thus the ABC flows are Lyapunov stable as a *family*; the direction of the first eigenfunction cannot be controlled by the semi-norm $[(W - \lambda_0 H)(\omega_{\lambda_0} + \xi)]^{1/2}$.

(*b*) *The Navier–Stokes equations*

Now we return to the full Navier–Stokes equations. Let $\mathcal{E}_k$ denote the λ_k-eigenspace of $(-\Delta)$ with zero boundary conditions. It is straightforward to verify that for any

$k \in \mathbb{N}$ and any $\omega_{\lambda_k} \in \mathcal{E}_k$,

$$\hat{\omega}(t) = \mathrm{e}^{-\nu\lambda_k t}\omega_{\lambda_k}, \quad t \geqslant 0$$

determines a solution of the Navier–Stokes equations. By global existence and unique-ness of the classical solutions for the Navier–Stokes equations (see Temam 1988, §2.2), these are all the solutions of the Navier–Stokes equations which at some t_0 (and hence for all t) lie in $\mathcal{E}_k$. Thus if $\hat{\omega}(0) = \omega_{\lambda_k}$, some ABC flow, at any other subsequent time according to the dynamics of the Navier–Stokes equations, $\hat{\omega}(t)$ is proportional to ω_{λ_k}, the proportionality constant $\mathrm{e}^{-\nu\lambda_k t}$ depending on time. Thus the ABC flows form invariant families under the dynamics of the Navier–Stokes equations.

Note also that by uniqueness of classical solutions, if $\omega_0 \neq 0$, then the solution $\omega(t)$ of the Navier–Stokes equations with initial condition $\omega(0) = \omega_0$ also satisfies $\omega(t) \neq 0$ for all $t \geqslant 0$ (as a map from $[0,\infty)$ to the space of functions on the domain $\mathcal{D}$, *not* as a function of $(t, \boldsymbol{x})$). In particular, $H(\omega(t)) \neq 0$ and $W(\omega(t)) \neq 0$.

We shall show below that the ABC flows in $\mathcal{E}_0$ form an attractor for the solutions of the Navier–Stokes equations with a cone-like-shaped basin of attraction. The domain of attraction cannot be an unscaled ball, because, as we have seen above, the spaces $\mathcal{E}_k$ are invariant under the dynamics of the Navier–Stokes equations. To show that $\mathcal{E}_0$ is an attractor, we analyse the time behaviour of the scaled Lyapunov function

$$\hat{L}(\omega) = \frac{W(\omega) - \lambda_0 H(\omega)}{H(\omega)} \geqslant \frac{(\lambda_1 - \lambda_0)\|\Pi_0^{\perp}\omega\|^2}{2\lambda_1 H(\omega)} \geqslant 0. \tag{3.4}$$

The lower bound is obtained from a standard Poincaré inequality as in (3.3).

Theorem 3.1. *Assume that $\omega_0 \neq 0$ does not belong to $\mathcal{E}_0$ and is such that $W(\omega_0) < \lambda_1 H(\omega_0)$ (i.e. $\hat{L}(\omega_0) < \lambda_1 - \lambda_0$). If $\omega(t)$ is a solution of the Navier–Stokes equations with initial condition ω_0, then*

$$\hat{L}(\omega(t)) \leqslant \mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t}\hat{L}(\omega_0)\left(1 - \frac{\hat{L}(\omega_0)}{\lambda_1 - \lambda_0}(1 - \mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t})\right)^{-1}, \tag{3.5}$$

implying that

$$\hat{L}(\omega(t)) \leqslant \mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t}\frac{\hat{L}(\omega_0)(\lambda_1 - \lambda_0)}{\lambda_1 - \lambda_0 - \hat{L}(\omega_0)}. \tag{3.6}$$

Proof. Using the boundary condition $\omega|_{\partial\mathcal{D}} = 0$, the time derivatives of the Hamil-tonian and the enstrophy functional are given by

$$\left. \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}H(\omega(t)) &= -\nu\int_{\mathcal{D}}\omega^2\,\mathrm{d}\boldsymbol{x} - \nu\int_{\partial D}\omega\frac{\partial\psi}{\partial n} = -2\nu W(\omega(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t}W(\omega(t)) &= -\nu\int_{\mathcal{D}}\omega(-\Delta\omega)\,\mathrm{d}\boldsymbol{x} \\ &= -\nu\int_{\mathcal{D}}|\nabla\omega|^2\,\mathrm{d}\boldsymbol{x} - \nu\int_{\partial D}\omega\frac{\partial\omega}{\partial n} = -\nu\int_{\mathcal{D}}|\nabla\omega|^2\,\mathrm{d}\boldsymbol{x}. \end{aligned} \right\} \tag{3.7}$$

From this, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}(\omega(t)) = \frac{\dot{W}H - W\dot{H}}{H^2} = -\frac{\nu}{H^2}\left[H\int_{\mathcal{D}}|\nabla\omega|^2\,\mathrm{d}\boldsymbol{x} - 2W^2\right]$$

$$= \frac{2\nu(W - \lambda_0 H)^2}{H^2} - \frac{\nu}{H}\left[\int_{\mathcal{D}}(-\Delta\omega)\omega\,\mathrm{d}\boldsymbol{x} - 2\lambda_0 W - 2\lambda_0(W - \lambda_0 H)\right]$$

$$= 2\nu\hat{L}^2 + 2\nu\lambda_0\,\hat{L} - \frac{\nu}{H}\int_{\mathcal{D}}\omega(-\Delta - \lambda_0)\omega\,\mathrm{d}\boldsymbol{x}.$$

Poincaré inequalities show $\int_{\mathcal{D}}\omega(-\Delta - \lambda_0)\omega\,\mathrm{d}\boldsymbol{x} \geqslant \lambda_1\int_{\mathcal{D}}\omega(\omega - \lambda_0\psi) = 2\lambda_1\,(W - \lambda_0 H)$. Substitution in the expression for the time derivative of $\hat{L}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}(\omega(t)) \leqslant 2\nu\hat{L}^2 + 2\nu\lambda_0\hat{L} - \frac{2\lambda_1\nu}{H}(W - \lambda_0 H) = 2\nu\hat{L}^2 - 2\nu(\lambda_1 - \lambda_0)\hat{L}.$$

Since ω_0 is not an element of $\mathcal{E}_0$, it follows that $\hat{L}(\omega(t))\mathrm{e}^{2\nu(\lambda_1 - \lambda_0)t} > 0$, so we can rewrite the above inequality as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\mathrm{e}^{2\nu(\lambda_1 - \lambda_0)t}\hat{L}(\omega(t))}\right] \geqslant -2\nu\mathrm{e}^{-2\nu(\lambda_1 - \lambda_0)t}.$$

A straightforward integration gives that

$$\frac{1}{\mathrm{e}^{2\nu(\lambda_1 - \lambda_0)t}\hat{L}(\omega(t))} \geqslant \frac{1}{\hat{L}(\omega_0)}\left(1 - \frac{\hat{L}(\omega_0)}{\lambda_1 - \lambda_0}(1 - \mathrm{e}^{-2\nu(\lambda_1 - \lambda_0)t})\right).$$

The expression on the right-hand side is positive for all $t \geqslant 0$, if $\hat{L}(\omega_0) \leqslant \lambda_1 - \lambda_0$. Hence for ω_0 such that $\hat{L}(\omega_0) \leqslant \lambda_1 - \lambda_0$ the inequality above can be rewritten as (3.5). ∎

(c) Consequences

We shall draw now several consequences of theorem 3.1. The consequences (1)-(6) give several interpretations about how solutions with initial conditions ω_0 such that $W(\omega_0) < \lambda_1 H(\omega_0)$ are attracted to the family of ABC flows and how theorem 3.1 compares to results known in literature. In consequence (7) we show that theorem 3.1 implies that the invariant manifold $\mathcal{E}_1$ is unstable.

(1) The statement can be interpreted in terms of shadowing curves on the manifold of ABC flows. We define a shadowing curve $\bar{\omega}_{\lambda_0}(t)$ on the manifold of ABC flows in the following way. Let χ_0 to be a λ_0-eigenfunction of $(-\Delta)$ whose L^2-norm equals 1. Define

$$\bar{\omega}_{\lambda_0}(t) = \sqrt{2\lambda_0}\,\mathrm{sgn}\left(\int_{\mathcal{D}}\omega(t)\chi_0\,\mathrm{d}\boldsymbol{x}\right)\sqrt{H(\omega(t))}\chi_0.$$

This definition implies $H(\bar{\omega}_{\lambda_0}(t)) = H(\omega(t))$ for all t and $\bar{\omega}_{\lambda_0}(t)$ is a minimiser of W on the level set $\{\omega \mid H(\omega) = H(\omega(t))\}$. From $\frac{1}{2}\|\bar{\omega}_{\lambda_0}\|^2 = \lambda_0 H(\bar{\omega}_{\lambda_0})$ and formulae (3.4) and (3.6) we get

$$\frac{\|\Pi_0^{\perp}\omega(t)\|^2}{\|\bar{\omega}_{\lambda_0}(t)\|^2} \leqslant \frac{\lambda_1(W(\omega_0) - \lambda_0 H(\omega_0))}{\lambda_0(\lambda_1 H(\omega_0) - W(\omega_0))}\mathrm{e}^{-2\nu(\lambda_1 - \lambda_0)t}. \tag{3.8}$$

This estimate gives the relative error from the significant part of the solution of the Navier–Stokes equations to the shadowing curve at every instant of time.

(2) A second interpretation of theorem 3.1 is in terms of the decay properties of the significant part of the solutions of the Navier–Stokes equations. Since $(\mathrm{d}/\mathrm{d}t)H(\omega(t)) = -2\nu W(\omega(t)) \leqslant -2\nu\lambda_0 H(\omega(t))$, we have $H(\omega(t)) \leqslant H(\omega_0)\mathrm{e}^{-2\nu\lambda_0 t}$ and therefore

$$\|\Pi_0^{\perp}\omega(t)\|^2 \leqslant \frac{2\lambda_1(W(\omega_0) - \lambda_0 H(\omega_0))}{\lambda_1 H(\omega_0) - W(\omega_0)}H(\omega_0)\mathrm{e}^{-2\nu\lambda_1 t}. \tag{3.9}$$

This last estimate rephrases the attractor properties of the ABC-manifold (for initial conditions as specified in theorem 3.1) in terms of the familiar L^2 norm on the significant part of the solutions of the Navier–Stokes equations instead of the more precise but less familiar scaled Lyapunov function used before.

(3) In Foias & Saut (1984) it is shown that for any regular solution $\omega(t)$, there exists some Λ in the spectrum of $(-\Delta)$, such that $L(\omega(t)) \to \Lambda - \lambda_0$, for $t \to \infty$. Theorem 3.1 implies that $\Lambda = \lambda_0$, if $W(\omega(0)) < \lambda_1 H(\omega(0))$.

Also, from theorem 2 of Foias & Saut (1984), it follows that if $\Lambda = \lambda_0$ and if $\lambda_1 < 2\lambda_0$, then $\lim_{t\to\infty} e^{\lambda_1 \nu t} \Pi_1 u(t)$ exists. Here Π_1 is the projection onto the eigenspace of the eigenvalue λ_1 of $(-\Delta)$. Theorem 3.1 generalizes this to $\lim_{t\to\infty} e^{\lambda_1 \nu t} \Pi_0^{\perp} u(t)$ exists, without any condition on λ_1.

(4) Theorem 3.1 slightly improves the following estimate for large $t > 0$ due to van Groesen (1988):

$$\frac{\|\Pi_0^{\perp}\omega(t)\|}{\|\Pi_0\omega(t)\|} \leqslant C e^{-\nu\sigma t}, \tag{3.10}$$

where $C, \sigma > 0$ are constants subject to the condition $\sigma \leqslant (\lambda_1 - \lambda_0)^2/\lambda_1$. To deduce it, we substitute the estimate $2\lambda_0 H(\omega) \leqslant 2W(\omega) = \|\Pi_0\omega\|^2 + \|\Pi_0^{\perp}\omega\|^2$ into (3.4) and get

$$\hat{L}(\omega) \geqslant \frac{\lambda_0(\lambda_1 - \lambda_0)}{\lambda_1} \frac{\|\Pi_0^{\perp}\omega\|^2}{\|\Pi_0\omega\|^2 + \|\Pi_0^{\perp}\omega\|^2}.$$

Substituting this inequality into (3.5) and recombining terms gives

$$[\lambda_0(\lambda_1 H(\omega_0) - W(\omega_0)) - e^{-2\nu(\lambda_1 - \lambda_0)t}(\lambda_1 - \lambda_0)(W(\omega_0) - \lambda_0 H(\omega_0))]\|\Pi_0^{\perp}\omega(t)\|^2$$
$$\leqslant e^{-2\nu(\lambda_1 - \lambda_0)t}\lambda_1(W(\omega_0) - \lambda_0 H(\omega_0))\|\Pi_0\omega(t)\|^2. \tag{3.11}$$

Because of the hypotheses of theorem 3.1 we have $\lambda_1 H(\omega_0) - W(\omega_0) > 0$. If in addition ω_0 is such that

$$W(\omega_0) < \frac{\lambda_0(2\lambda_1 - \lambda_0)}{\lambda_1} H(\omega_0),$$

then the expression on the left-hand side in front of $\|\Pi_0^{\perp}\omega(t)\|^2$ is positive for all $t \geqslant 0$. If not, then this is true for

$$t > \frac{1}{2\nu(\lambda_1 - \lambda_0)} \ln\left(\frac{(\lambda_1 - \lambda_0)(W(\omega_0) - \lambda_0 H(\omega_0))}{\lambda_0(\lambda_1 H(\omega_0) - W(\omega_0))}\right).$$

In conclusion, there is some $T_0 \geqslant 0$, depending on ω_0, such that for $t > T_0$

$$\frac{\|\Pi_0^{\perp}\omega(t)\|^2}{\|\Pi_0\omega(t)\|^2} \leqslant \frac{\lambda_1(W(\omega_0) - \lambda_0 H(\omega_0))e^{-2\nu(\lambda_1 - \lambda_0)t}}{\lambda_0(\lambda_1 H(\omega_0) - W(\omega_0)) - e^{-2\nu(\lambda_1 - \lambda_0)t}(\lambda_1 - \lambda_0)(W(\omega_0) - \lambda_0 H(\omega_0))}. \tag{3.12}$$

Since $(\lambda_1 - \lambda_0)^2/\lambda_1 = (\lambda_1 - \lambda_0)(1 - \lambda_0/\lambda_1) < \lambda_1 - \lambda_0$, this result slightly improves (3.10).

Note that, in general, the projection curve $\Pi_0\omega(t)$ differs from the shadowing curve $\bar{\omega}_{\lambda_0}(t)$, although both are in the manifold of ABC flows.

(5) From (3.9) and (3.12) we can infer that $\omega(t)/\|\omega(t)\|$ approaches a specific normalized element of $\mathcal{E}_0$ as $t \to \infty$. To see this, note that since $\Pi_0\omega(0) \in \mathcal{E}_0$ and $\mathcal{E}_0$ is one dimensional, there exists some function $f(t)$ such that $\Pi_0\omega(t) = f(t)\Pi_0\omega(0)$. The function $f(t)$ is continuous because $\omega(t)$ is continuous and $\Pi_0\omega(0)$ never vanishes

(the eigenfunction of the first eigenvalue of the Laplacian is never zero). Note that if $f(t_0) = 0$ for some $t_0 \geqslant 0$, then $\Pi_0 \omega(t_0) = 0$.

However, this cannot happen. To see this, we first observe that if $\Pi_0 \omega(t_0) = 0$, then necessarily $\Pi_0 \psi(t_0) = 0$, so that $\|\Pi_0^{\perp} \omega(t_0)\|^2 \geqslant 2\lambda_1 H(\omega(t_0))$ by a standard Poincaré inequality. Thus (3.4) implies that $L(\omega(t_0)) \geqslant \lambda_1 - \lambda_0$. But by (3.5) in theorem 3.1 we always have $\hat{L}(\omega(t)) < \lambda_1 - \lambda_0$, for all $t \geqslant 0$. Hence $\Pi_0 \omega(t_0) \neq 0$.

Thus $f(t)$ will have a definite sign, which is positive since $f(0) = 1$. Therefore (3.12) implies that

$$\lim_{t \to \infty} \frac{\omega(t)}{\|\omega(t)\|} = \lim_{t \to \infty} \frac{\Pi_0 \omega(t)}{\|\omega(t)\|} + \lim_{t \to \infty} \frac{\Pi_0^{\perp} \omega(t)}{\|\omega(t)\|} = \frac{\Pi_0 \omega(0)}{\|\Pi_0 \omega(0)\|}.$$

In other words, in the hypotheses of theorem 3.1, the flow of the Navier–Stokes equations 'self-organizes' itself at infinity.

(6) If $\omega(t)$ is a solution of the Navier–Stokes equation which satisfies the conditions of theorem 3.1, then we can show that the shadowing curve $\bar{\omega}_{\lambda_0}(t)$ and the projected curve $\Pi_0 \omega(t)$ decay exactly like $\mathrm{e}^{-\nu\lambda_0 t}$. To be precise, we have

$$\|\bar{\omega}_{\lambda_0}(t)\|\mathrm{e}^{\nu\lambda_0 t} = \mathcal{O}(1) = \frac{\mathrm{e}^{-\nu\lambda_0 t}}{\|\bar{\omega}_{\lambda_0}(t)\|} \quad \text{and} \quad \|\Pi_0 \omega(t)\|\mathrm{e}^{\nu\lambda_0 t} = \mathcal{O}(1) = \frac{\mathrm{e}^{-\nu\lambda_0 t}}{\|\Pi_0 \omega(t)\|}.$$

Using these decay rates we will show that the solutions on the ground-state manifold $\mathcal{E}_0$ are stable. To prove these decay rates we first show that $H(\omega(t))$ decays exactly like $\mathrm{e}^{-2\nu\lambda_0 t}$.

Let $\omega(t)$ be a solution of the Navier–Stokes equation which satisfies the conditions of theorem 3.1. Denoting by $\|\cdot\|$ the L^2-norm, equation (3.7) states that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\omega(t)) = -2\nu W(\omega(t)) = -\nu\|\omega(t)\|^2 = -\nu(\|\Pi_0 \omega(t)\|^2 + \|\Pi_0^{\perp}\omega(t)\|^2). \quad (3.13)$$

Using the inequality $W(\omega(t)) \geqslant \lambda_0 H(\omega(t))$, we conclude that

$$H(\omega(t)) \leqslant H(\omega_0)\mathrm{e}^{-2\nu\lambda_0 t}. \quad (3.14)$$

Substituting (3.8) into (3.13) and recalling that $H(\bar{\omega}_{\lambda_0}(t)) = H(\omega(t)) \geqslant H(\Pi_0\omega(t))$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\omega(t)) \geqslant -\nu\|\Pi_0\omega(t)\|^2 - \nu C(\omega_0)\mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t}\|\bar{\omega}_{\lambda_0}(t)\|^2$$

$$= -2\nu\lambda_0[H(\Pi_0\omega(t)) + C(\omega_0)\mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t}H(\bar{\omega}_{\lambda_0}(t))]$$

$$\geqslant -2\nu\lambda_0[1 + C(\omega_0)\mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t}]H(\omega(t)),$$

where

$$C(\omega_0) = \frac{\lambda_1(W(\omega_0) - \lambda_0 H(\omega_0))}{\lambda_0(\lambda_1 H(\omega_0) - W(\omega_0))}.$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln H(\omega(t)) \geqslant -2\nu\lambda_0[1 + C(\omega_0)\mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t}].$$

Integration of this inequality gives

$$H(\omega(t)) \geqslant H(\omega_0)\mathrm{e}^{-2\nu\lambda_0 t}\exp\left(C(\omega_0)\frac{\lambda_0}{\lambda_1-\lambda_0}[\mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t} - 1]\right).$$

Define

$$\tilde{C}(\omega_0) = C(\omega_0)\frac{\lambda_0}{\lambda_1 - \lambda_0}.$$

Then

$$\exp\left(C(\omega_0)\frac{\lambda_0}{\lambda_1 - \lambda_0}[\mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t} - 1]\right) \geqslant \mathrm{e}^{-\tilde{C}(\omega_0)},$$

so we have

$$\mathrm{e}^{-\tilde{C}(\omega_0)}\, H(\omega_0) \leqslant H(\omega(t))\mathrm{e}^{2\nu\lambda_0 t} \leqslant H(\omega_0) \tag{3.15}$$

by (3.14). This proves that $H(\omega(t))$ decays exactly like $\mathrm{e}^{-2\nu\lambda_0 t}$.

Since by definition $H(\bar{\omega}_{\lambda_0}(t)) = H(\omega(t))$, we get

$$H(\omega_0)\mathrm{e}^{-\tilde{C}(\omega_0)} \leqslant \frac{1}{2\lambda_0}\|\bar{\omega}_{\lambda_0}(t)\|^2 \mathrm{e}^{2\nu\lambda_0 t} \leqslant H(\omega_0),$$

which implies that $\|\bar{\omega}_{\lambda_0}(t)\| \sim \mathrm{e}^{-\nu\lambda_0 t}$.

Furthermore, using (3.12) we get for $t > T_0$

$$2\lambda_0 H(\omega(t)) \leqslant 2W(\omega(t)) = \|\Pi_0\omega(t)\|^2 + \|\Pi_0^\perp\omega(t)\|^2$$
$$\leqslant (1 + C_1(\omega_0, t)\mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t})\|\Pi_0\omega(t)\|^2,$$

where $C_1(\omega_0, t)$ is the expression on the right-hand side of (3.12), divided by $\mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t}$. Note that $C_1(\omega_0, t) < C_1(\omega_0, T_0)$ for $t > T_0$. With the estimate (3.15) on $H(\omega(t))$ this gives

$$\|\Pi_0\omega(t)\|^2 \geqslant \frac{2\lambda_0}{1 + C_1(\omega_0, T_0)}H(\omega_0)\mathrm{e}^{-\tilde{C}(\omega_0)}\mathrm{e}^{-2\nu\lambda_0 t}.$$

Also, using $\|\Pi_0\omega(t)\|^2 \leqslant 2\lambda_0 H(\omega(t)) \leqslant 2\lambda_0 H(\omega_0)\mathrm{e}^{-2\nu\lambda_0 t}$, we see that

$$\frac{2\lambda_0}{1 + C_1(\omega_0, T_0)}H(\omega_0)\mathrm{e}^{-\tilde{C}(\omega_0)}\mathrm{e}^{-2\nu\lambda_0 t} \leqslant \|\Pi_0\omega(t)\|^2 \leqslant 2\lambda_0 H(\omega_0)\mathrm{e}^{-2\nu\lambda_0 t}, \tag{3.16}$$

hence $\|\Pi_0\omega(t)\| \sim \mathrm{e}^{-\nu\lambda_0 t}$ for t large.

After determining these decay rates, we are ready to prove that the solutions on the ground-state manifold $\mathcal{E}_0$ are stable. Let $\hat{\omega}(t)$ be a solution in $\mathcal{E}_0$, hence $\hat{\omega}(t) = \hat{\omega}(0)\mathrm{e}^{-\nu\lambda_0 t}$. Let ω_0 be some initial condition near $\hat{\omega}(0)$ and let $\omega(t)$ be the solution of the Navier–Stokes equations which starts at $\omega(0) = \omega_0$. Define $\varepsilon = \|\omega_0 - \hat{\omega}(0)\|/\|\hat{\omega}(0)\|$. If ε is sufficiently small, then

$$2(W(\omega_0) - \lambda_0 H(\omega_0)) \leqslant \|\Pi_0^\perp\omega_0\|^2 = \|\Pi_0^\perp(\omega_0 - \hat{\omega}(0))\|^2 \leqslant \|\omega_0 - \hat{\omega}(0)\|^2 = \varepsilon^2\|\hat{\omega}(0)\|^2.$$

Hence ω_0 satisfies the conditions of theorem 3.1 and $T_0 = 0$ (T_0 is defined in consequence (4)). Furthermore, $\tilde{C}(\omega_0) = \mathcal{O}(\varepsilon^2)$ and $C_1(\omega_0, 0) = \mathcal{O}(\varepsilon^2)$.

From conclusion (4) it follows that $\|\Pi_0^\perp\omega(t)\|^2 = \mathcal{O}(\varepsilon^2\mathrm{e}^{-2\nu(\lambda_1-\lambda_0)t}\|\Pi_0\omega(t)\|^2)$. Also, since $\mathcal{E}_0$ is one dimensional, we can define some function $f(t)$ such that

$$\Pi_0\omega(t) = f(t)\hat{\omega}(t) = f(t)\mathrm{e}^{-\nu\lambda_0 t}\hat{\omega}(0).$$

From (3.16) it follows that

$$\frac{2\lambda_0}{1 + C_1(\omega_0, 0)}H(\omega_0)\mathrm{e}^{-\tilde{C}(\omega_0)}\mathrm{e}^{-2\nu\lambda_0 t} \leqslant |f(t)|^2\mathrm{e}^{-2\nu\lambda_0 t}\|\hat{\omega}(0)\|^2 \leqslant 2\lambda_0 H(\omega_0)\mathrm{e}^{-2\nu\lambda_0 t}.$$

Since $\tilde{C}(\omega_0) = \mathcal{O}(\varepsilon^2)$, $C_1(\omega_0, 0) = \mathcal{O}(\varepsilon^2)$, and $\lambda_0 H(\omega_0) \leqslant W(\omega_0) = W(\hat{\omega}(0))(1 + \mathcal{O}(\varepsilon))$, this implies

$$|f(t)|^2 = \frac{\lambda_0 H(\omega_0)}{W(\hat{\omega}(0))}(1 - \mathcal{O}(\varepsilon^2))^2 = (1 + \mathcal{O}(\varepsilon))(1 - \mathcal{O}(\varepsilon^2))^2 = (1 + \mathcal{O}(\varepsilon))^2.$$

So we can conclude $|f(t)| = 1 + \mathcal{O}(\varepsilon)$ and therefore

$$\begin{aligned}
\|\omega(t) - \hat{\omega}(t)\|^2 &= \|\Pi_0\omega(t) - \hat{\omega}(t)\|^2 + \|\Pi_0^\perp\omega(t)\|^2 \\
&= |f(t) - 1|^2\|\hat{\omega}(t)\|^2 + \mathcal{O}(\varepsilon^2 e^{-2\nu(\lambda_1-\lambda_0)t}\|\Pi_0\omega(t)\|^2) \\
&= \mathcal{O}(\varepsilon^2)\|\hat{\omega}(t)\|^2 + \mathcal{O}(\varepsilon^2|f(t)|^2)\|\hat{\omega}(t)\|^2 = \mathcal{O}(\varepsilon^2)\|\hat{\omega}(t)\|^2.
\end{aligned}$$

So we see that the solutions in $\mathcal{E}_0$ are stable, i.e. if initially $\|\omega(0) - \hat{\omega}(0)\|/\|\hat{\omega}(0)\| = \mathcal{O}(\varepsilon)$, then $\|\omega(t) - \hat{\omega}(t)\|/\|\hat{\omega}(t)\| = \mathcal{O}(\varepsilon)$, for all time.

(7) As a final consequence of theorem 3.1 we will show that it implies that the solutions on the manifold $\mathcal{E}_1$ (eigenfunctions with eigenvalue λ_1) are unstable under perturbations in the direction $\mathcal{E}_0$. Let $\hat{\omega}_1 \in \mathcal{E}_1\backslash\{0\}$ and $\hat{\omega}_0 \in \mathcal{E}_0\backslash\{0\}$. Define $\omega_\varepsilon = \hat{\omega}_1 + \varepsilon\hat{\omega}_0$. This implies

$$W(\omega_\varepsilon) = \lambda_1 H(\hat{\omega}_1) + \varepsilon^2\lambda_0 H(\hat{\omega}_0) \quad \text{and} \quad H(\omega_\varepsilon) = H(\hat{\omega}_1) + \varepsilon^2 H(\hat{\omega}_0).$$

Write $\alpha = H(\hat{\omega}_0)/H(\hat{\omega}_1) > 0$, $\mu_1 = \lambda_0/\lambda_1 < 1$. Then

$$\frac{W(\omega_\varepsilon)}{H(\omega_\varepsilon)} = \lambda_1\frac{1 + \varepsilon^2\mu_1\alpha}{1 + \varepsilon^2\alpha}.$$

Since $\mu_1 < 1$, we see immediately that $1 + \varepsilon^2\mu_1\alpha \leqslant 1 + \varepsilon^2\alpha$ with equality only if $\varepsilon = 0$. Hence $W(\omega_\varepsilon)/H(\omega_\varepsilon) < \lambda_1$, for $\varepsilon \neq 0$. This implies that $\hat{L}(\omega_\varepsilon) < \lambda_1 - \lambda_0$. Thus theorem 3.1 implies that a solution which starts at ω_ε with $\varepsilon \neq 0$ is attracted by the set $\mathcal{E}_0$. However, if $\varepsilon = 0$, the solutions stays in the set $\mathcal{E}_1$.

To be explicit, let $\omega_\varepsilon(t)$ be the solution of Navier–Stokes with initial condition ω_ε and $\hat{\omega}_1(t) = \hat{\omega}_1 e^{-\nu\lambda_1 t}$ is the solution of Navier–Stokes with initial condition $\hat{\omega}_1$. Then

$$\|\omega_\varepsilon(t) - \hat{\omega}_1(t)\|^2 = \|\Pi_0(\omega_\varepsilon(t) - \hat{\omega}_1(t))\|^2 + \|\Pi_0^\perp(\omega_\varepsilon(t) - \hat{\omega}_1(t))\|^2 \geqslant \|\Pi_0\omega_\varepsilon(t)\|^2.$$

From (3.16) we get that

$$\|\Pi_0\omega_\varepsilon(t)\|^2 \geqslant \frac{2\lambda_0}{1 + C_1(\omega_\varepsilon, T_0)}H(\omega_\varepsilon)e^{-\tilde{C}(\omega_\varepsilon)}e^{-2\nu\lambda_0 t},$$

for $t > T_0$. So we can conclude that

$$\frac{\|\omega_\varepsilon(t) - \hat{\omega}_1(t)\|^2}{\|\hat{\omega}_1(t))\|^2} \geqslant \frac{2\lambda_0}{1 + C_1(\omega_\varepsilon, T_0)}\frac{H(\omega_\varepsilon)}{\|\hat{\omega}_1\|^2}e^{-\tilde{C}(\omega_\varepsilon)}e^{2\nu(\lambda_1-\lambda_0)t},$$

for t large. This implies an exponential growth in the scaled distance between the solutions.

(d) *Navier–Stokes on a sphere*

Many statements proved before go through if we consider a sphere in $\mathbb{R}^3$ instead of a compact domain in the plane. To be specific, we consider the Navier–Stokes equations on a sphere $\mathcal{S}$ of radius R in $\mathbb{R}^3$, i.e.

$$\dot{\omega} = -\boldsymbol{v}\cdot\nabla\omega + \nu\Delta\omega = \{\psi, \omega\} + \nu\Delta\omega \quad \text{on } \mathcal{S}.$$

If $\nu = 0$, these equations become the Euler equations and a family of stationary solutions are given by

$$-\Delta\omega_\lambda = \lambda\omega_\lambda.$$

These are zonal flows. There are no harmonic functions on a sphere and the smallest eigenvalue of $(-\Delta)$ is $\lambda_0 = 2/R^2$. Again, the next eigenvalue is called λ_1. The dimension of the eigenspace of the eigenvalue λ_0 is three; the space is

$$\mathcal{V}_{\lambda_0} = \mathrm{span}\{\sin\varphi\sin\theta, \sin\varphi\cos\theta, \cos\varphi\},$$

where φ, θ are the standard spherical coordinates. The solutions in $\mathcal{V}_{\lambda_0}$ are called spherical ABC flows. In Chern & Marsden (1990) it is shown that they are Lyapunov stable.

The family $\mathcal{V}_{\lambda_0}$ is an attractor for the solutions of Navier–Stokes equation, with a cone-shaped basin of attraction. Indeed, we can just quote theorem 3.1 and prove it in exactly the same way as we did before. The consequences (2)–(4), (7) and the first part of (6) go through for the spherical case as well. However, for the other consequences we have used the one dimensionality of the family of ground states $(\mathcal{E}_0)$, allowing us to define a *unique* shadowing curve, which at each time lies on the same H-level set as the solution. In the spherical case, however, the family of ground states $(\mathcal{V}_{\lambda_0})$ is three-dimensional. Now there is a family of shadowing curves, which are all equivalent for our purposes.

To be specific, let $\omega(t)$ be a solution of the Navier–Stokes equations with initial condition $\omega(0)$ such that $W(\omega(0)) < \lambda_1 H(\omega(0))$. Any curve

$$\bar{\omega}(t) = \alpha_1(t)\sin\varphi\sin\theta + \alpha_2(t)\sin\varphi\cos\theta + \alpha_3(t)\cos\varphi,$$

with

$$\alpha_1^2(t) + \alpha_2^2(t) + \alpha_3^2(t) = \frac{3\lambda_0}{\pi R^3}H(\omega(t)),$$

can act as a shadowing curve, since they are on the same H-level set as the solution $\omega(t)$.

4. Ideal homogeneous incompressible RMHD

After this analysis of the Navier–Stokes equations to illustrate our method, we will concentrate on the main system, the RMHD equations with dissipation. From now on, the shorthand notation $u = (\omega, A)$ will be used. If we ignore the viscosity and resistance, i.e. $\nu = \eta = 0$, then the RMHD equations form a Hamiltonian system on the dual of the semidirect product Lie algebra consisting of functions on $\mathcal{D}$ acting on themselves by Poisson bracket (Marsden & Morison 1984). This Lie–Poisson space admits an infinite number of Casimir functions of the form $\int_\mathcal{D} \omega F(A)\,\mathrm{d}\boldsymbol{x}$ and $\int_\mathcal{D} G(A)\,\mathrm{d}\boldsymbol{x}$ with F and G arbitrary real-valued functions of a real variable (Holm *et al.* 1985). The Hamiltonian function H for this system is the total energy (1.2).

In this work we shall consider the family of stationary solutions given by

$$-\Delta A_{c,\lambda} = \lambda A_{c,\lambda}, \quad \psi_{c,\lambda} = -cA_{c,\lambda}, \tag{4.1}$$

where $\omega_{c,\lambda} = -\Delta\psi_{c,\lambda}$, $c \in \mathbb{R}$ is an arbitrary number and λ is a positive eigenvalue of $(-\Delta)$ with zero boundary conditions. We will use the two Casimir functions

$$I_1(u) = \int_\mathcal{D} \omega A\,\mathrm{d}\boldsymbol{x} \quad \text{and} \quad I_2(u) = \frac{1}{2}\int_\mathcal{D} A^2\,\mathrm{d}\boldsymbol{x} \tag{4.2}$$

in order to variationally characterize these stationary solutions. It is easily verified that the critical points of $H + c\,I_1(u) - (1 - c^2)\,\lambda\,I_2(u)$ are the stationary solutions defined in (4.1). Equivalently, the stationary solutions in (4.1) are constrained critical points of the Hamiltonian on level sets of I_1 and I_2. If $\tilde{u} = (\tilde{\omega}, \tilde{A})$ denotes a stationary solution as described by (4.1) with $I_1(u) = \gamma_1$ and $I_2(u) = \gamma_2$, where $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\tilde{A}$ is an eigenfunction of $-\Delta$ with eigenvalue λ_k (or, equivalently, a constrained critical point of H on the level set $\{u \mid I_1(u) = \gamma_1,\ I_2(u) = \gamma_2\}$), then the parameter c is given by

$$c = -\frac{\gamma_1}{2\lambda_k\gamma_2}.$$

Theorem 4.1. *Let $C_{c,\lambda}(u) = cI_1(u) - (1 - c^2)\lambda I_2(u)$. For every $c \in \mathbb{R}$ we have the following relations:*

$$(H + C_{c,\lambda})(\omega, A) = \frac{1}{2}\int_{\mathcal{D}}(|\nabla(\psi + cA)|^2 + (1 - c^2)[|\nabla A|^2 - \lambda A^2])\,\mathrm{d}\boldsymbol{x}$$

$$= \frac{1}{2}\int_{\mathcal{D}}(|\nabla(\psi + cA)|^2 + (1 - c^2)A(J - \lambda A))\,\mathrm{d}\boldsymbol{x},$$

$$(H + C_{c,\lambda})(\omega_{c,\lambda}, A_{c,\lambda}) = 0,$$

$$(H + C_{c,\lambda})(\omega_{c,\lambda} - \Delta\xi, A_{c,\lambda} + \alpha) = \frac{1}{2}\int_{\mathcal{D}}(|\nabla(\xi + c\alpha)|^2 + (1 - c^2)[|\nabla\alpha|^2 - \lambda\alpha^2])\,\mathrm{d}\boldsymbol{x}.$$

Let λ_0 denote the first positive eigenvalue of $(-\Delta)$ with zero boundary conditions. If $|c| < 1$, the relations above imply that the stationary solutions corresponding to $\lambda = \lambda_0$ are minima of $H + C_{c,\lambda_0}$. In other words, these stationary solutions are conditionally Lyapunov stable (that is, Lyapunov stable as long as classical solutions exist) in the semi-norm on the finite perturbations $(-\Delta\xi, \alpha)$ given by $[(H + C_{c,\lambda_0})(\omega_{c,\lambda_0} - \Delta\xi, A_{c,\lambda_0} + \alpha)]^{1/2}$.

Proof. The proof is standard and follows the method in Holm *et al.* (1985). One directly verifies the first relation which in turn implies the second using the definition of the stationary solutions considered. The third relation follows by subtracting from the left-hand side two terms that are zero, $(H + C_{c,\lambda})(\omega_{c,\lambda}, A_{c,\lambda})$ and $D(H + C_{c,\lambda})(\omega_{c,\lambda}, A_{c,\lambda}) \cdot (-\Delta\xi, \alpha)$, regrouping the summands in the integrand and integrating by parts, taking into account that $\psi_{c,\lambda}, A_{c,\lambda}, \xi$, and α all vanish on the boundary.

It is clear that $[(H + C_{c,\lambda_0})(\omega_{c,\lambda_0} - \Delta\xi, A_{c,\lambda_0} + \alpha)]^{1/2}$ defines a semi-norm whose null space is given by $\{(-\Delta\xi, \alpha) \mid \alpha \in \mathcal{E}_0, \xi = -c\alpha\}$, where $\mathcal{E}_0$ is the λ_0-eigenspace of $(-\Delta)$. Conditional Lyapunov stability follows now from the third relation by conservation of H and C_{c,λ_0} and by invoking the short-time existence theorem for classical solutions of planar MHD due to Kozono (1989). These Lyapunov stable stationary solutions are minima since the second variation of $H + C_{c,\lambda_0}$ has the same expression as the third identity above. ∎

As in the case of the ABC flows for the Navier–Stokes equations, we see that the stationary solutions considered here are conditionally Lyapunov stable as a *family* and not individually. The proof shows that the null space of the stability semi-norm consists precisely of the stationary solutions (4.1).

In what follows, our interest is mainly in the conditionally Lyapunov stable sta-

tionary solutions, called *ground states*. Define

$$A_c = A_{c,\lambda_0}, \quad \psi_c = \psi_{c,\lambda_0}, \quad \omega_c = -\Delta\psi_c \quad \text{and} \quad C_c = C_{c,\lambda_0}.$$

If $a = \|A_c\|_{L^2}$, the values of the Casimirs I_1, I_2, C_c and the Hamiltonian H at the equilibrium (ω_c, A_c) are given by:

$$I_1(u_c) = -c\lambda_0 a^2, \quad I_2(u_c) = \tfrac{1}{2}a^2, \quad H(u_c) = \tfrac{1}{2}\lambda_0(1+c^2)a^2 = -C_c(u_c).$$

The eigenvalue λ_0 is simple (Aubin 1982, ch. 4); hence the family of ground states considered is a two-dimensional submanifold of the space of all (ω, A)'s parametrized by $a \in \mathbb{R}$ and $|c| < 1$.

A more convenient way to parametrize the ground states for our purpose is by using the integrals I_1 and I_2. In all that follows we shall fix χ_0 to be a λ_0-eigenfunction of $(-\Delta)$ whose L^2-norm equals 1. We define

$$\bar{A}_\pm(\gamma_2) = \pm\sqrt{2\gamma_2}\chi_0 \quad \text{and} \quad \bar{\omega}_\pm(\gamma_1, \gamma_2) = -c\lambda_0\bar{A}_\pm, \quad \text{where} \quad c = -\frac{\gamma_1}{2\lambda_0\gamma_2}.$$

Thus γ_1 and γ_2 determine the I_1- and I_2-level sets, respectively. The family of ground states is

$$\mathcal{M} = \{(\bar{\omega}_\pm(\gamma_1, \gamma_2), \bar{A}_\pm(\gamma_2)) \mid \gamma_2 \geqslant 0, |\gamma_1| < 2\lambda_0\gamma_2\}.$$

5. Dissipative solutions

Under the dynamics of the RMHD equations with dissipation, the stationary solutions defined by equation (4.1) form invariant families. Let $\mathcal{E}_k$ denote the λ_k-eigenspace of $(-\Delta)$ with zero boundary conditions. Define the families $\mathcal{S}_k \subset \mathcal{E}_k \times \mathcal{E}_k$ by $\mathcal{S}_k := \{(-c\lambda_k \Upsilon_k, \Upsilon_k) \mid c \in \mathbb{R}, \Upsilon_k \in \mathcal{E}_k\}$.

Theorem 5.1. *For $k \in \mathbb{N}_0$, $c \in \mathbb{R}$ and $\Upsilon_k \in \mathcal{E}_k$, define*

$$\hat{A}(t) = \mathrm{e}^{-\eta\lambda_k t}\Upsilon_k, \quad \hat{\omega}(t) = -c\lambda_k\mathrm{e}^{-\nu\lambda_k t}\Upsilon_k.$$

The pair $(\hat{\omega}(t), \hat{A}(t))$ determines a solution of the dissipative RMHD equations in $\mathcal{S}_k$ for $t \in \mathbb{R}$. Variations of c and Υ_k give all the solutions of the dissipative RMHD equations which at some t_0 (and hence for all t) lie in $\mathcal{S}_k$.

Proof. A direct verification shows that the formulae above satisfy the dissipative RMHD equations. By global existence and uniqueness of the classical solutions for the dissipative RMHD equations (Temam 1988, §3.3), these are the only solutions that have initial conditions in $\mathcal{S}_k$. Conversely, if a solution $(\tilde{\omega}(t), \tilde{A}(t))$ is such that $(\tilde{\omega}(t_0), \tilde{A}(t_0)) \in \mathcal{S}_k$ for some t_0, then $(\tilde{\omega}(t), \tilde{A}(t)) = (\tilde{\omega}(t_0)\mathrm{e}^{-\nu\lambda_k(t-t_0)}, \tilde{A}(t_0)\mathrm{e}^{-\eta\lambda_k(t-t_0)})$ for all $t \in \mathbb{R}$, i.e. this solution is necessarily in $\mathcal{S}_k$. ∎

Theorem 5.2. *If $\nu \geqslant \eta$ then the family of ground states $\mathcal{M}$ is invariant under the dynamics of the RMHD equations with dissipation.*

Proof. It is easy to see that $\mathcal{M} \subset \mathcal{S}_0$. Hence solutions starting in $\mathcal{M}$ look like

$$(\omega(t), A(t)) = (-ac\lambda_0\mathrm{e}^{-\nu\lambda_0 t}\chi_0, a\mathrm{e}^{-\eta\lambda_0 t}\chi_0),$$

for some $a \in \mathbb{R}$ and $|c| < 1$. Since $\nu \geqslant \eta$ the quotient $|c\mathrm{e}^{-\nu\lambda_0 t}/\mathrm{e}^{-\eta\lambda_0 t}| = |c|\,|\mathrm{e}^{-(\nu-\eta)\lambda_0 t}| < 1$ and thus the above solution is in $\mathcal{M}$ for all time. ∎

In order to show how solutions of the RMHD equations with dissipation are attracted by shadowing curves of ground states in $\mathcal{M}$, we consider the time behaviour of the relevant 'conserved quantities'. For later analysis, it is convenient to introduce the semi-definite combination

$$K(u) = H(u) - \lambda_0 I_2(u) = \frac{1}{2} \int_{\mathcal{D}} \omega\psi \, \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_{\mathcal{D}} A(J - \lambda_0 A) \, \mathrm{d}\boldsymbol{x} \geqslant 0. \qquad (5.1)$$

Theorem 5.3. *For any solution $u(t)$ of the RMHD equations with dissipation satisfying the boundary conditions BC1 or BC2, the time behaviour of the Hamiltonian and the Casimir functionals is given by*

$$\left.\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}H(u(t)) &= -\nu \int_{\mathcal{D}} \omega^2 \, \mathrm{d}\boldsymbol{x} - \eta \int_{\mathcal{D}} J^2 \, \mathrm{d}\boldsymbol{x} \leqslant -2\min(\nu,\eta)\lambda_0 H(u), \\
\frac{\mathrm{d}}{\mathrm{d}t}I_2(u(t)) &= -\eta \int_{\mathcal{D}} AJ \, \mathrm{d}\boldsymbol{x} \leqslant -2\eta\lambda_0 I_2(u), \\
\frac{\mathrm{d}}{\mathrm{d}t}K(u(t)) &= -\nu \int_{\mathcal{D}} \omega^2 \, \mathrm{d}\boldsymbol{x} - \eta \int_{\mathcal{D}} J(J - \lambda_0 A) \, \mathrm{d}\boldsymbol{x} \leqslant -2\min(\nu\lambda_0, \eta\lambda_1)K(u).
\end{aligned}\right\}$$
$$(5.2)$$

With the boundary conditions BC2, we have in addition

$$\frac{\mathrm{d}}{\mathrm{d}t}I_1(u(t)) = -(\nu + \eta) \int_{\mathcal{D}} \omega J \, \mathrm{d}\boldsymbol{x}.$$

Proof. Indeed, differentiation of the expressions for the Hamiltonian and Casimirs gives

$$\frac{\mathrm{d}}{\mathrm{d}t}H(u(t)) = -\nu \int_{\mathcal{D}} \omega^2 \, \mathrm{d}\boldsymbol{x} - \eta \int_{\mathcal{D}} J^2 \, \mathrm{d}\boldsymbol{x} - \nu \int_{\partial D} \omega \frac{\partial\psi}{\partial n}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}I_2(u(t)) = -\eta \int_{\mathcal{D}} AJ \, \mathrm{d}\boldsymbol{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}K(u(t)) = -\nu \int_{\mathcal{D}} \omega^2 \, \mathrm{d}\boldsymbol{x} - \eta \int_{\mathcal{D}} J(J - \lambda_0 A) \, \mathrm{d}\boldsymbol{x} - \nu \int_{\partial D} \omega \frac{\partial\psi}{\partial n}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}I_1(u(t)) = -(\nu + \eta) \int_{\mathcal{D}} \omega J \, \mathrm{d}\boldsymbol{x} - \nu \int_{\partial D} \omega \frac{\partial A}{\partial n}.$$

Both boundary conditions imply that the boundary integrals in the time derivatives of H and K vanish. The boundary integral in the time derivative of I_1 vanishes only with the boundary conditions BC2.

The inequalities in the theorem follow from Poincaré-like inequalities. ∎

6. Shadowing curves on ground-state solutions

In this section we are using either the boundary conditions BC1 or BC2. We are interested in the deviation of a solution of the RMHD equations with dissipation from the family of ground states. To find this deviation, we will define a shadowing curve on the manifold of ground states uniquely determined by the given solution $u(t)$.

In what follows it is useful to introduce the quantity

$$c(u) = -\frac{I_1(u)}{2\lambda_0 I_2(u)}. \tag{6.1}$$

We project every $u = (\omega, A)$ with $\int_{\mathcal{D}} A\chi_0 \, d\boldsymbol{x} \neq 0$ and $|c(u)| < 1$ onto the family of ground states in the following way. Define

$$\bar{A}(u) = \mathrm{sgn}\left(\int_{\mathcal{D}} A\chi_0 \, d\boldsymbol{x}\right)\sqrt{2I_2}\chi_0, \quad \bar{\omega}(u) = -c(u)\lambda_0\bar{A}(u), \quad \bar{u}(u) = (\bar{\omega}(u), \bar{A}(u)).$$

Then $\bar{u}(u)$ is a projection of u on the ground states and

$$I_1(u) = I_1(\bar{u}) \quad \text{and} \quad I_2(u) = I_2(\bar{u}).$$

In order to measure the distance between u and $\bar{u}(u)$ we look at the Lyapunov functional $L(u) = H(u) + C_{c(u)}$ used in theorem 4.1. This functional can be written as

$$L(u) = K(u) + c(u)I_1(u) + c(u)^2\lambda_0 I_2(u). \tag{6.2}$$

Next, we write $u = (\omega, A)$ as

$$A = \bar{A}(u) + \alpha \quad \text{and} \quad \omega = \bar{\omega}(u) - c(u)j - \Delta\varphi, \quad \text{where} \quad j = -\Delta\alpha, \tag{6.3}$$

and α is a function on $\mathcal{D}$ vanishing on the boundary. The Lyapunov functional $L(u)$ is a norm for (α, φ), if $\int_{\mathcal{D}} A\chi_0 \, d\boldsymbol{x} \neq 0$ and $|c(u)| < 1$. We make this precise in the following lemma.

Lemma 6.1. *Let $u = (\omega, A)$ be such that $\int_{\mathcal{D}} A\chi_0 \, d\boldsymbol{x} \neq 0$ and $|c(u)| < 1$. Define α and φ as described in (6.3). Then the Lyapunov functional $L(u)$ measures the distance between u and its projection $\bar{u}(u)$, since $\sqrt{L(u)}$ is equivalent to the norm whose square is given by*

$$[\![\varphi]\!]_1^2 + (1 - c(u)^2)[\![\Pi_0^\perp \alpha]\!]_1^2,$$

where $\Pi_0^\perp$ is the projection onto the space L^2-orthogonal to $\mathcal{E}_0$ and $[\![f]\!]_1^2 = \|\nabla f\|_{L^2}^2$.

Proof. Since $I_2(u) = I_2(\bar{u})$, we have

$$2\int_{\mathcal{D}} \alpha\bar{A} \, d\boldsymbol{x} = -\int_{\mathcal{D}} \alpha^2 \, d\boldsymbol{x}. \tag{6.4}$$

Furthermore, $I_1(u) = I_1(\bar{u})$ and (6.4) imply

$$\lambda_0 \int_{\mathcal{D}} \varphi\bar{A} \, d\boldsymbol{x} = -\int_{\mathcal{D}} (-\Delta\varphi)\alpha \, d\boldsymbol{x} + c\int_{\mathcal{D}} \alpha(j - \lambda_0\alpha) \, d\boldsymbol{x}. \tag{6.5}$$

With the notation above, the functional $L(u)$ in (6.2) can be written as

$$L(u) = \frac{1}{2}\int_{\mathcal{D}}[|\nabla\varphi|^2 + (1 - c(u)^2)\alpha(j - \lambda_0\alpha)] \, d\boldsymbol{x}.$$

This expression shows that if $|c(u)| < 1$, then $\sqrt{L(u)}$ is equivalent to the semi-norm whose square is given by

$$[\![\varphi]\!]_1^2 + (1 - c(u)^2)[\![\Pi_0^\perp \alpha]\!]_1^2,$$

Note that $[\![\cdot]\!]_1$ is a true norm on the space of H^2 functions which vanish on the

boundary. Hence $L(u) = 0$ if and only if $\varphi = 0$ and $\alpha = b\chi_0$, for any $b \in \mathbb{R}$. Using the definition of $\bar{A}$ and substituting $\alpha = b\chi_0$ in equation (6.4) implies that

$$b = 0 \quad \text{or} \quad b = -2\sqrt{2I_2(u)}\,\text{sgn}\left(\int_{\mathcal{D}} A\chi_0\,\mathrm{d}\boldsymbol{x}\right).$$

Also, substitution of $\alpha = b\chi_0$ in the definition of α gives

$$A = \left(\text{sgn}\left(\int_{\mathcal{D}} A\chi_0\,\mathrm{d}\boldsymbol{x}\right)\sqrt{2I_2(u)} + b\right)\chi_0.$$

Thus $b = 0$ gives

$$A = \text{sgn}\left(\int_{\mathcal{D}} A\chi_0\,\mathrm{d}\boldsymbol{x}\right)\sqrt{2I_2(u)}\chi_0 = \bar{A}(u).$$

However, if

$$\int_{\mathcal{D}} A\chi_0\,\mathrm{d}\boldsymbol{x} \neq 0 \quad \text{and} \quad b = -2\sqrt{2I_2(u)}\,\text{sgn}\left(\int_{\mathcal{D}} A\chi_0\,\mathrm{d}\boldsymbol{x}\right),$$

then

$$A = -\text{sgn}\left(\int_{\mathcal{D}} A\chi_0\,\mathrm{d}\boldsymbol{x}\right)\sqrt{2I_2(u)}\chi_0,$$

so integrating against χ_0 yields the nonsensical expression

$$\int_{\mathcal{D}} A\chi_0\,\mathrm{d}\boldsymbol{x} = -\text{sgn}\left(\int_{\mathcal{D}} A\chi_0\,\mathrm{d}\boldsymbol{x}\right)\sqrt{2I_2(u)},$$

thus showing that this solution for b is not allowed. We conclude thus that $L(u) = 0$ if and only if $\varphi = 0$ and $\alpha = 0$ which shows that $L(u)$ measures the distance between u and the projection $\bar{u}(u)$. ∎

If $u(t)$ is a solution of the RMHD equations such that $|c(u(t))| < 1$ and $\int_{\mathcal{D}} A(t)\chi_0\,\mathrm{d}\boldsymbol{x} \neq 0$ for all $t \geqslant 0$, then we can define the ground-state projection curve $\bar{u}(u(t))$. To study the deviation from this projection curve, we define a scaled Lyapunov functional

$$\hat{L}(u) = \frac{L(u)}{I_2(u)} = \frac{K(u)}{I_2(u)} - \lambda_0 c(u)^2 \tag{6.6}$$

that depends on $c(u)$ and the quotient

$$Q(u) = \frac{K(u)}{I_2(u)}, \quad \text{i.e.} \quad Q(u)\int_{\mathcal{D}} A^2\,\mathrm{d}\boldsymbol{x} = \int_{\mathcal{D}} \omega\psi\,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{D}} A(J - \lambda_0 A)\,\mathrm{d}\boldsymbol{x}. \tag{6.7}$$

There are some immediate *a priori* estimates on $Q(u)$ and $c(u)$ that will be used later.

Lemma 6.2. *For any state $u = (\omega, A)$ with $A \neq 0$ we have $Q(u) \geqslant 0$. Furthermore,*
- *if $Q(u) < \lambda_0$ then $|c(u)| < 1$;*
- *if $Q(u) < \lambda_1 - \lambda_0$ then $\int_{\mathcal{D}} A\chi_0\,\mathrm{d}\boldsymbol{x} \neq 0$.*

Proof. In (5.1) it is shown that $K(u) \geqslant 0$, which implies $Q(u) \geqslant 0$ for any state u with $A \neq 0$.

To prove the next statement, we observe that for any state u with $A \neq 0$

$$|I_1(u)| = \text{sgn}\left(\int_{\mathcal{D}} \omega A \, \mathrm{d}\boldsymbol{x}\right) \int_{\mathcal{D}} \omega A \, \mathrm{d}\boldsymbol{x} = \text{sgn}\left(\int_{\mathcal{D}} \omega A \, \mathrm{d}\boldsymbol{x}\right) \int_{\mathcal{D}} \nabla\psi \cdot \nabla A \, \mathrm{d}\boldsymbol{x}$$

$$= \frac{1}{2} \int_{\mathcal{D}} |\nabla\psi|^2 \, \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_{\mathcal{D}} |\nabla A|^2 \, \mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_{\mathcal{D}} \left|\nabla\psi - \text{sgn}\left(\int_{\mathcal{D}} \omega A \, \mathrm{d}\boldsymbol{x}\right)\nabla A\right|^2 \, \mathrm{d}\boldsymbol{x}$$

$$= K(u) + \lambda_0 I_2(u) - \frac{1}{2} \int_{\mathcal{D}} \left|\nabla\psi - \text{sgn}\left(\int_{\mathcal{D}} \omega A \, \mathrm{d}\boldsymbol{x}\right)\nabla A\right|^2 \, \mathrm{d}\boldsymbol{x}$$

$$= I_2(u)(Q(u) + \lambda_0) - \frac{1}{2} \int_{\mathcal{D}} \left|\nabla\psi - \text{sgn}\left(\int_{\mathcal{D}} \omega A \, \mathrm{d}\boldsymbol{x}\right)\nabla A\right|^2 \, \mathrm{d}\boldsymbol{x}. \tag{6.8}$$

Now recall from (6.1) that $c(u) = I_1(u)/2\lambda_0 I_2(u)$, so that (6.8) and the condition $Q(u) < \lambda_0$ imply that $|c(u)| < 1$.

Finally, if $\int_{\mathcal{D}} A\chi_0 \, \mathrm{d}\boldsymbol{x} = 0$, then

$$\int_{\mathcal{D}} A(J - \lambda_0 A) \, \mathrm{d}\boldsymbol{x} \geqslant (\lambda_1 - \lambda_0) \int_{\mathcal{D}} A^2 \, \mathrm{d}\boldsymbol{x}.$$

Hence

$$K(u) \geqslant \frac{1}{2} \int_{\mathcal{D}} \omega\psi \, \mathrm{d}\boldsymbol{x} + (\lambda_1 - \lambda_0) I_2(u)$$

and therefore $Q(u) \geqslant \lambda_1 - \lambda_0$, contradicting the hypothesis. ∎

From this lemma it follows that we can define the projection curve $\bar{u}(u(t))$ for a solution $u(t)$ if $Q(u(t)) < \min(\lambda_0, \lambda_1 - \lambda_0)$, for all $t \geqslant 0$. So we want to have an estimate on $Q(u(t))$.

Lemma 6.3. *Assume that either the boundary conditions BC1 or BC2 hold and that $\eta \leqslant \nu \leqslant (\lambda_1/\lambda_0)\eta$. For all $t \geqslant 0$, every solution $u(t)$ of the dissipative RMHD equations with initial condition $u(0) = u_0$ such that $Q(u_0) < \lambda_1 - \lambda_0$ satisfies*

$$Q(u(t)) \leqslant Q(u_0) \quad \text{and} \quad Q(u(t)) \leqslant C(u_0)Q(u_0)\mathrm{e}^{-2(\nu-\eta)\lambda_0 t}, \tag{6.9}$$

where $C(u_0)$ is given by

$$C(u_0) = \max\left(1, \exp\left(\frac{\lambda_0(\nu - \eta)(Q(u_0) - \lambda_1 + \lambda_0(\nu/\eta))}{(\lambda_1\eta - \lambda_0\nu)(\lambda_1 - \lambda_0 - Q(u_0))}\right)\right).$$

This implies

$$K(u(t)) \leqslant K(u_0) \, C(u_0)\mathrm{e}^{-2\lambda_0\nu t}, \tag{6.10}$$

$$\hat{L}(u(t)) \leqslant (\hat{L}(u_0) + \lambda_0 c(u_0)^2)C(u_0)\mathrm{e}^{-2(\nu-\eta)\lambda_0 t}. \tag{6.11}$$

The proof of this lemma is done by analysing the time derivative of $Q(u(t))$. It is quite technical and does not give much insight. It can be found in appendix A.

With lemma 6.3 we can show that $|c(u(t))| < 1$ and $\int_{\mathcal{D}} A(t)\chi_0 \, \mathrm{d}\boldsymbol{x} \neq 0$ for appropriate initial conditions. Hence for solutions with these initial conditions the projection curve $\bar{u}(u(t))$ exists.

Lemma 6.4. *Assume that either the boundary conditions BC1 or BC2 hold and that $\eta \leqslant \nu \leqslant (\lambda_1/\lambda_0)\eta$. Every solution $u(t)$ of the dissipative RMHD equations with*

initial conditions $u(0) = u_0$ subject to $Q(u_0) < \min(\lambda_0, \lambda_1 - \lambda_0)$ and u_0 is not a ground state, satisfies for all $t \geqslant 0$ the inequalities

$$c(u(t))^2 < 1, \quad \hat{L}(u(t)) > 0 \quad \text{and} \quad c(u(t))^2 \leqslant (\hat{L}(u_0)/\lambda_0 + c(u_0)^2)C(u_0)\mathrm{e}^{-2(\nu-\eta)\lambda_0 t}. \tag{6.12}$$

Furthermore, $\int_{\mathcal{D}} A(t)\chi_0 \,\mathrm{d}\boldsymbol{x} \neq 0$ for all $t \geqslant 0$.

Proof. Lemma 6.3 states that if $Q(u_0) < \lambda_1 - \lambda_0$ then $Q(u(t)) \leqslant Q(u_0)$ for all $t \geqslant 0$. Therefore, $Q(u_0) < \min(\lambda_0, \lambda_1 - \lambda_0)$ implies that $Q(u(t)) \leqslant Q(u_0) < \min(\lambda_0, \lambda_1 - \lambda_0)$ for all $t \geqslant 0$. So with lemma 6.2 we see that $|c(u(t))| < 1$ and $\int_{\mathcal{D}} A(t)\chi_0 \,\mathrm{d}\boldsymbol{x} \neq 0$ for all $t \geqslant 0$.

From Lemma 6.1 we see that $|c(u)| < 1$ implies that $L(u) \geqslant 0$, with equality only if u is a ground state. This is impossible, because u_0 is not a ground state and the manifold of ground states is invariant. So we can conclude that $L(u(t)) > 0$; hence $\hat{L}(u(t)) > 0$ for all $t \geqslant 0$. ∎

Remark 2. Note that for any state $\bar{u} = (-c\lambda\bar{A}, \bar{A})$, $\bar{A} \in \mathcal{E}_0$ we have $Q(\bar{u}) = \lambda_0 c^2$. Hence $\bar{u}$ is a *stable* ground state if and only if $Q(\bar{u}) < \lambda_0$.

Remark 3. The condition on Q in lemma 6.4 gives a cone-like neighbourhood of the family of ground states. Define $\mu = \min(1, (\lambda_1 - \lambda_0)/\lambda_0)$. Then

$$\{u \mid Q(u) < \mu\lambda_0\} = \{u \mid c(u)^2 < \mu \text{ and } 0 \leqslant \hat{L}(u) < (\mu - c(u)^2)\lambda_0\}. \tag{6.13}$$

Indeed, if $Q(u) < \mu\lambda_0$, then by lemma 6.2 we have $|c(u)| < 1$. Therefore $\hat{L}(u) \geqslant 0$, which in turn implies

$$\lambda_0 c(u)^2 = Q(u) - \hat{L}(u) < \mu\lambda_0.$$

Thus we have proved that $c(u)^2 < \mu$. Conversely, if $c(u)^2 < \mu$ and $\hat{L}(u) < (\mu - c(u)^2)\lambda_0$, then by definition,

$$Q(u) = \hat{L}(u) + \lambda_0 c(u)^2 < \mu\lambda_0$$

and the equality (6.13) is proved.

Lemma 6.4 implies that if $u(t)$ is a solution of the RMHD equations and $Q(u(0)) < \min(\lambda_0, \lambda_1 - \lambda_0)$, then we can define the ground-state projection curve $\bar{u}(u(t))$. And $\hat{L}(u(t))$ is a measure for the distance between the solution $u(t)$ and its projection $\bar{u}(u(t))$. Without analysing the time behaviour of $\hat{L}$, lemma 6.3 gives an estimate on the distance between $u(t)$ and $\bar{u}(u(t))$, which shows that $\bar{u}(u(t))$ is a shadowing curve on the grounds states of the solution $u(t)$.

Theorem 6.5. *Assume that either the boundary conditions BC1 or BC2 hold and that $\eta < \nu < (\lambda_1/\lambda_0)\eta$. Let $u(t)$ be a solution of the dissipative RMHD equations with initial condition $u(0)$ such that $Q(u(0)) < \min(\lambda_0, \lambda_1 - \lambda_0)$. Define*

$$c(u(t)) = -\frac{I_1(u(t))}{2\lambda_0 I_2(u(t))} \quad \text{and} \quad \bar{u}(t) = \bar{u}(u(t)).$$

Define the functions $\alpha(t)$ and $\varphi(t)$ on $\mathcal{D}$, both vanishing on the boundary, to be such that

$$u(t) = \bar{u}(t) + (-\Delta(-c(u(t))\alpha(t) + \varphi(t)), \alpha(t)).$$

Then there exists some $C > 0$ such that for $t > 0$

$$[\![\varphi(t)]\!]_1^2 + [\![\Pi_0^\perp \alpha(t)]\!]_1^2 \leqslant C\mathrm{e}^{-2\lambda_0(\nu-\eta)t}\|\bar{u}(t)\|_{L^2}^2 = \frac{C}{\lambda_0}\mathrm{e}^{-2\lambda_0(\nu-\eta)t}[\![\bar{u}(t)]\!]_1^2. \tag{6.14}$$

(Recall that the norm $[\![f]\!]_1 = \int_{\mathcal{D}} |\nabla f|^2 \, d\boldsymbol{x}$.)

Proof. For $|c(u)| < 1$, $L(u)$ is equivalent to the $[\![\cdot]\!]_1^2$ norm. From lemma 6.4, it follows that $|c(u(t))| < 1$ for all $t \geqslant 0$. Hence the inequality in (6.14) follows from (6.11) by using that $L(u) = \hat{L}(u)I_2(u)$ and $I_2(u) = I_2(\bar{u}) = \frac{1}{2}\|\bar{u}\|_{L^2}^2$.

To prove the equality in (6.14) we use that $\bar{u}$ can be written as

$$\bar{u}(t) = (c_1(t)\chi_0, c_2(t)\chi_0).$$

The definition of χ_0, implies that $[\![\chi_0]\!]_1^2 = \lambda_0$ and therefore

$$[\![\bar{u}]\!]_1^2 = \lambda_0 \|\bar{u}\|_{L^2},$$

which implies the desired equality in (6.14). ∎

7. Sharper estimates for the shadowing curves

The time derivative of the functional $I_1(u(t))$ contains a boundary integral if the boundary conditions BC1 are used (see theorem 5.3). This boundary term disappears if we use the boundary conditions BC2. So the time derivative of the scaled Lyapunov functional $\hat{L}(u(t))$ will contain a boundary integral if we use the boundary conditions BC1. This will make an analysis of this time derivative very difficult.

Therefore, in this section we will use exclusively the boundary condition BC2. This enables us to improve the estimate in the theorem of the previous section by studying the time derivative of the scaled Lyapunov function $\hat{L}$. In that way we will get an improved estimate for how fast solutions are attracted to their shadowing curve.

Theorem 7.1. *Assume that $\eta < \nu < (\lambda_1/\lambda_0)\eta$. Let $u(t)$ be a solution of the dissipative RMHD equations with boundary conditions BC2 and with initial condition $u(0)$ such that*

$$Q(u(0)) \leqslant \min\left(\lambda_1 - \lambda_0, \frac{4\eta}{\nu + 3\eta}\lambda_0\right).$$

Define a shadowing curve on the family of ground states

$$\bar{A}(u) = \mathrm{sgn}\left(\int_{\mathcal{D}} A\chi_0 \, d\boldsymbol{x}\right)\sqrt{2I_2(u)}\chi_0, \quad \bar{\omega}(u) = -c(u)\lambda_0\bar{A}(u), \quad \bar{u}(u) = (\bar{\omega}(u), \bar{A}(u)),$$

where

$$c(u) = -\frac{I_1(u)}{2\lambda_0 I_2(u)}.$$

Define the functions $\alpha(t)$ and $\varphi(t)$ on $\mathcal{D}$, both vanishing on the boundary, to be such that

$$u(t) = \bar{u}(u(t)) + (-\Delta(-c(u(t))\alpha(t) + \varphi(t)), \alpha(t)).$$

Then there exist constants $C_1 > 0$ and $C_2 > 0$ such that for all $t \geqslant 0$

$$\hat{L}(u(t)) \leqslant \hat{L}(u(0))C_1 e^{-2(\lambda_1 - \lambda_0)\eta t}$$

$$[\![\varphi(t)]\!]_1^2 + [\![\Pi_0^\perp \alpha(t)]\!]_1^2 \leqslant C_2 e^{-2(\lambda_1 - \lambda_0)\eta t}\|\bar{u}(u(t))\|_{L^2} = (C_2/\lambda_0)e^{-2(\lambda_1 - \lambda_0)\eta t}[\![\bar{u}(u(t))]\!]_1.$$

(Recall that the norm $[\![f]\!]_1 = \int_{\mathcal{D}} |\nabla f|^2 \, d\boldsymbol{x}$.)

Proof. In Appendix B we analyse the time behaviour of $\hat{L}(u(t))$. It is shown that

$$\frac{d}{dt}\hat{L}(u(t)) \leqslant -2(\lambda_1 - \lambda_0)\eta\hat{L}(u) + \eta\frac{2}{1 - c(u)^2}\hat{L}(u)^2$$

$$+(\lambda_1 + 3\lambda_0)(\nu - \eta)\frac{c(u)^2}{2(1 - c(u)^2)}\hat{L}(u)$$

$$+\frac{\lambda_1\eta - \lambda_0\nu}{2\lambda_0(1 - c(u)^2)F(u(0))}\max\left(\lambda_1 - \lambda_0, \frac{(3\lambda_1 - 2\lambda_0)^2}{\lambda_1 - \lambda_0}\right)(\hat{L}(u))^2, \quad (7.1)$$

where

$$F(u) = \begin{cases} (\lambda_1 - \lambda_0) - Q(u), & \text{if } \lambda_0 \geqslant \lambda_1 - \lambda_0, \\ \dfrac{\lambda_1 - \lambda_0}{\lambda_0}(\lambda_0 - Q(u)), & \text{if } \lambda_0 \leqslant \lambda_1 - \lambda_0. \end{cases}$$

Define the positive function

$$f(t) = \frac{1}{1 - c(u(t))^2}\left(2\eta + \frac{\lambda_1\eta - \lambda_0\nu}{2\lambda_0 F(u(0))}\max\left(\lambda_1 - \lambda_0, \frac{(3\lambda_1 - 2\lambda_0)^2}{\lambda_1 - \lambda_0}\right)\right)\hat{L}(u(t))$$

$$+(\lambda_1 + 3\lambda_0)(\nu - \eta)\frac{c(u(t))^2}{2(1 - c(u(t))^2)} \geqslant 0.$$

From the estimates (6.11) and (6.12) in lemmas 6.3 and 6.4, it follows that there exists a constant $C_3(u(0) > 0$ such that

$$f(t) \leqslant C_3(u(0))e^{-2(\nu - \eta)\lambda_0 t}, \tag{7.2}$$

for all $t \geqslant 0$. Setting $N(t) = \hat{L}(u(t))e^{2(\lambda_1 - \lambda_0)\eta t}$, the estimate (7.1) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) \leqslant f(t)\,N(t).$$

Applying Gronwall's lemma to this differential inequality gives

$$N(t) \leqslant N(0)\exp\left(\int_0^t f(\tau)\,\mathrm{d}\tau\right). \tag{7.3}$$

Since $\int_0^t f(\tau)\,\mathrm{d}\tau \leqslant C_3(u(0))/2\lambda_0(\nu - \eta)$ by (7.2), inequality (7.3) shows that there exist a constant $C_4(u(0)) > 0$ such that

$$\hat{L}(u(t)) \leqslant \hat{L}(u(0))C_4(u(0))\,e^{-2(\lambda_1 - \lambda_0)\eta t},$$

which proves the first estimate in the statement of the theorem. Now use this inequality, $I_2(u(t)) = I_2(\bar{u}(u(t)))$, and $[\![\bar{u}]\!]_1^2 = \lambda_0\|\bar{u}\|_{L^2}^2$ to conclude the proof of the theorem. ∎

8. Comparison to the general method

Our previous work (Derks *et al.* 1995) develops a general method of comparing solution curves of a dissipative system with symmetries to carefully chosen shadowing curves in the manifold of relative equilibria of an associated conservative system in the setting of finite dimensional systems.

Although many aspects of this general method can be recognized in the analysis of the two infinite-dimensional systems presented in this paper, there are also several important differences. First, the set-up of a classical mechanical system with symmetry on a cotangent bundle is replaced by considering systems on duals of Lie algebras. Second, the role played by the momentum map is taken over by Casimir functionals. Indeed, in the case of the Euler equations we work with the enstrophy W and in the

case of the RMHD equations with I_1 and I_2. Third, the family of stationary solutions happens to be invariant under the dynamics of the dissipative system. This remarkable extra feature considerably simplifies certain aspects of the analysis by avoiding several technical problems of adapted Lyapunov functionals which show up in the general case. Fourth, we work with a scaled Lyapunov functional $\hat{L}(u) = L(u)/F(u)$, with $F(u)$ a constant of the motion of the conservative system, instead of working with the Lyapunov functional $L(u)$ as in Derks *et al.* (1995). Since $F(u)$ is also a norm for the states of the system, this leads to sharper estimates on the attraction rate.

As stated above, in the general method for the finite dimensional case as presented in Derks *et al.* (1995), we use the unscaled Lyapunov functional $L(u)$ to analyse the rate of attraction between a solution and its shadowing curve. This analysis shows that there are two important features which determine an estimate of the rate of attraction. One determining feature is the so-called 'dissipation quotient', which measures the dissipative effects in a quasi-static approach. The other feature plays a role if the family of relative equilibria is not invariant. Then we have to take into account that all solutions are forced away from the manifold of relative equilibria.

To see which features will play a role if we use a scaled Lyapunov functional to analyse an invariant family of relative equilibria, we will give below a heuristic analysis of the time derivative of the quotient $\hat{L}(u)$. In order to do this, we have to be more specific about the dissipative system.

We write the dissipative system as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = X_H(u) + P(u), \tag{8.1}$$

with $X_H(u)$ a Hamiltonian vector field and $P(u)$ some dissipative perturbation. The conservative system generated by the Hamiltonian vector field X_H (i.e. $P = 0$) has either a momentum map $\boldsymbol{J}$, related to some symmetry group G, and/or is defined on some Poisson manifold whose Casimir functions are denoted by $I_1, \ldots, I_k$. We assume that there is a manifold of relative equilibria, where each relative equilibrium is a constrained minimum of the Hamiltonian on a level set of the conserved functionals, namely the momentum map and/or the Casimir functions $I_1, \ldots, I_k$. In other words, a relative equilibrium $\bar{u}$ is a solution of the Euler–Lagrange equation

$$DH(\bar{u}) + DJ_{\bar{\xi}}(\bar{u}) + \sum_{i=1}^{k} \lambda_i DI_i(\bar{u}) = 0,$$

with Lagrange multipliers $\bar{\xi}$ and/or $\lambda_1, \ldots, \lambda_k$ and where $J_\xi(u) = \langle \boldsymbol{J}(u), \xi \rangle$, for ξ in the Lie algebra of G. Depending on the case considered, either $\boldsymbol{J}$ or the Casimir functions $I_1, \ldots, I_k$ can be absent in this equation. In the cases considered in this paper, these manifolds of relative equilibria are the families of stable stationary solutions.

For any solution $u(t)$ of the dissipative system we define a shadowing curve $\bar{u}(t)$ on the family of relative equilibria. This curve has the characteristic property that for all $t \geqslant 0$, $\boldsymbol{J}(u(t)) = \boldsymbol{J}(\bar{u}(t))$, and/or $I_i(u(t)) = I_i(\bar{u}(t))$, $i = 1, \ldots, k$. Then $L(u) = H(u) - H(\bar{u})$ is a Lyapunov functional, measuring the distance between u and $\bar{u}$. If $F(u)$ is some constant of the motion of the conservative system, then we define the scaled Lyapunov functional

$$\hat{L}(u) = \frac{L(u)}{F(u)}.$$

Since $L(u)$ and $F(u)$ are conserved functions for the unperturbed Hamiltonian system we have $\langle DL(u), X_H(u)\rangle = 0$ and $\langle DF(u), X_H(u)\rangle = 0$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}(u(t)) = \frac{\dot{L}}{F} - \frac{L\dot{F}}{F^2} = \hat{L}\left[\frac{\langle DL(u), P(u)\rangle}{L(u)} - \frac{\langle DF(u), P(u)\rangle}{F(u)}\right]. \qquad (8.2)$$

Recall that the Lyapunov functional L was chosen using the energy-momentum method (see Simó *et al.* 1991) and/or the energy-Casimir method (see Holm *et al.* 1985). This means that for any relative equilibrium $\bar{u}$ we have $L(\bar{u}) = DL(\bar{u}) = 0$. Writing $\delta u = u - \bar{u}$ and linearizing all expressions in (8.2) gives

$$\left.\begin{aligned}
\langle DL(u), P(u)\rangle &= \langle D^2L(\bar{u})\delta u, P(\bar{u}) + DP(\bar{u})\delta u\rangle + \mathcal{O}((|\delta u| + |P(\bar{u})|)|\delta u|^2), \\
L(u) &= \tfrac{1}{2}\langle D^2L(\bar{u})\delta u, \delta u\rangle + \mathcal{O}(|\delta u|^3), \\
\langle DF(u), P(u)\rangle &= \langle DF(\bar{u}), P(\bar{u})\rangle + \mathcal{O}(|\delta u|), \\
F(u) &= F(\bar{u}) + \mathcal{O}(|\delta u|),
\end{aligned}\right\}$$
$$(8.3)$$

where we assume that $|P(\bar{u})|$ is 'small', e.g. exponentially decaying, which is reasonable if $\bar{u}$ is exponentially decaying.

Since the family of relative equilibria is invariant for the dissipative equations, it follows that $P(\bar{u})$ is tangent to the family of relative equilibria. To be explicit, for every $t \geqslant 0$, fixed, we define on the family of relative equilibria the curve

$$\bar{u}(\tau, t) = (\bar{\omega}(t)\mathrm{e}^{-\nu\lambda_0(\tau-t)}, \bar{A}(t)\mathrm{e}^{-\eta\lambda_0(\tau-t)}), \quad \tau \geqslant t.$$

By theorem 5.1, this curve is a solution of

$$\frac{\mathrm{d}}{\mathrm{d}\tau}u = X_H(u) + P(u),$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\bar{u}(\tau, t) = P(\bar{u}(\tau, t)),$$

for every $\tau \geqslant t$, $t \geqslant 0$, since $\bar{u}(\tau, t)$ is a relative equilibrium; thus $X_H(\bar{u}(\tau, t)) = 0$. Using the symmetry of $D^2L(\bar{u}(t))$ and the identity $\bar{u}(t) = \bar{u}(t, t)$, we have for all variations δu and all $t \geqslant 0$

$$\langle D^2L(\bar{u}(t))\delta u, P(\bar{u}(t))\rangle = \left\langle \delta u, D^2L(\bar{u}(t, t))\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=t} u(\tau, t)\right\rangle$$
$$= \left\langle \delta u, \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=t} DL(\bar{u}(\tau, t))\right\rangle = 0,$$

since $DL(\bar{u}(\tau, t)) = 0$, because $\bar{u}$ is a relative equilibrium. This and (8.3) imply

$$\frac{\langle DL(u), P(u)\rangle}{L(u)} = \frac{\langle D^2L(\bar{u})\delta u, DP(\bar{u})\delta u\rangle}{\tfrac{1}{2}\langle D^2L(\bar{u})\delta u, \delta u\rangle} + \mathcal{O}(|P(\bar{u})| + |\delta u|),$$

$$\frac{\langle DF(u), P(u)\rangle}{F(u)} = \frac{\langle DF(\bar{u}), P(\bar{u})\rangle}{F(\bar{u})} + \mathcal{O}(|\delta u|).$$

Definition 8.1. *Define $\mathcal{S}(\bar{u})$ to be the tangent space at $\bar{u}$ to the level set*

$$\{u \mid \boldsymbol{J}(u) = \boldsymbol{J}(\bar{u}), I_i(u) = I_i(\bar{u}), i = 1, \ldots, k\};$$

in this definition, depending on the case considered, either $\boldsymbol{J}$ or the Casimir functions $I_1, \ldots, I_k$ can be absent. Then to first order, $\delta u \in \mathcal{S}(\bar{u})$. Relevant quantities for the dynamics are the dissipation coefficient

$$\beta(\bar{u}) = \max_{\delta u \in \mathcal{S}(\bar{u})} \frac{\langle D^2 L(\bar{u})\delta u, DP(\bar{u})\delta u \rangle}{\frac{1}{2}\langle D^2 L(\bar{u})\delta u, \delta u \rangle}$$

and the influence of the dissipation on the conserved functional $F(u)$

$$f(\bar{u}) = \frac{\langle DF(\bar{u}), P(\bar{u}) \rangle}{F(\bar{u})}.$$

With these notations we get the estimate

$$\frac{\langle DL(u), P(u) \rangle}{L(u)} - \frac{\langle DF(u), P(u) \rangle}{F(u)} \leqslant (\beta(\bar{u}) - f(\bar{u})) + \mathcal{O}(|P(\bar{u})| + |\delta u|)$$

and the problem becomes in estimating the last two higher-order terms. In the ideal case (something that is happening in both cases considered in this paper), one can find a constant C_1 and a positive integrable function $g_1(t)$ (meaning $\int_0^\infty g_1(\tau)\,\mathrm{d}\tau < \infty$) such that

$$\frac{\langle DL(u), P(u) \rangle}{L(u)} - \frac{\langle DF(u), P(u) \rangle}{F(u)} \leqslant (\beta(\bar{u}) - f(\bar{u})) + C_1\sqrt{\hat{L}(u)} + g_1(t).$$

Therefore the time derivative of $\hat{L}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}(u) \leqslant \hat{L}(u)(\beta(\bar{u}) - f(\bar{u})) + C_1\hat{L}(u)\sqrt{\hat{L}(u)} + g_1(t)\hat{L}(u). \tag{8.4}$$

If $\beta - f = \lim_{t \to \infty} \beta(\bar{u}) - f(\bar{u}) < 0$ and there is a positive integrable function $g_2(t)$ such that $\beta(\bar{u}) - f(\bar{u}) \leqslant (\beta - f) + g_2(t)$, then we can rewrite this estimate on the time derivative of $\hat{L}$ as an estimate on $\hat{L}$ itself. For this purpose we introduce a new function $N(t) > 0$ by

$$N(t)^2 = \exp\left(-(\beta - f)t - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\hat{L}(u(t)),$$

where $g(t) = g_1(t) + g_2(t) \geqslant 0$. Taking the time derivative of the defining relation for N and using (8.4) we get

$$\begin{aligned}
2N\frac{\mathrm{d}N}{\mathrm{d}t} &= \exp\left(-(\beta - f)t - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\left[-(\beta - f)\hat{L} - g(t)\hat{L} + \frac{\mathrm{d}}{\mathrm{d}t}\hat{L}\right] \\
&\leqslant C_1 \exp\left(-(\beta - f)t - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\hat{L}(u)\sqrt{\hat{L}(u)} \\
&= C_1 \exp\left(\tfrac{1}{2}(\beta - f)t + \frac{1}{2}\int_0^t g(\tau)\,\mathrm{d}\tau\right)N^3 \\
&\leqslant C_2 \mathrm{e}^{(\beta - f)t/2}N^3,
\end{aligned}$$

where

$$C_2 = C_1 \exp\left(\frac{1}{2}\int_0^\infty g(\tau)\,\mathrm{d}\tau\right).$$

This can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{N}\right) \geqslant -\tfrac{1}{2}C_2\mathrm{e}^{(\beta-f)t/2}$$

and so integration gives

$$\frac{1}{N(t)} - \frac{1}{N(0)} \geqslant \frac{C_2}{\beta - f}(1 - \mathrm{e}^{(\beta-f)t/2}) \geqslant \frac{C_2}{\beta - f}.$$

If $N(0) \leqslant (f - \beta)/C_2$, then this leads to

$$N(t) \leqslant N(0)\left(1 - \frac{C_2 N(0)}{f - \beta}\right)^{-1}.$$

Thus, denoting

$$C_0 = \exp\left(\int_0^\infty g(\tau)\,\mathrm{d}\tau\right),$$

this inequality gives

$$\hat{L}(u(t)) \leqslant C_0\hat{L}(u(0))\left(1 - \frac{C_2\hat{L}(u(0))}{f - \beta}\right)^{-2}\mathrm{e}^{(\beta-f)t} \tag{8.5}$$

and we can draw the following conclusion.

Conclusion 8.2. *If $L(u(t))$ is equivalent to $|u - \bar{u}|_1^2$, for some semi-norm $|\cdot|_1$ and $F(u)$ is equivalent to some norm $|u|_2^2$, then (8.5) implies*

$$|u(t) - \bar{u}|_1^2 \leqslant \tilde{C}_0\mathrm{e}^{(\beta-f)t}|u(t)|_2^2,$$

for some constant $\tilde{C}_0$, which depends on the initial condition.

If we know in advance that $\hat{L}(u(t))$ has some exponential decay (as in case of the RMHD equations), then we can substitute this decay rate into $\sqrt{\hat{L}}$ in equation (8.4). Applying Gronwall's lemma leads immediately to $\hat{L}(u(t)) \leqslant \hat{L}(u(0))C_0\mathrm{e}^{(\beta-f)t}$, for some constant C_0, independent of the initial condition.

In appendix C we will calculate $\beta(\bar{u})$ and $f(\bar{u})$ for the dissipative RMHD equation. This calculation shows that

$$\beta = -2\eta\lambda_1 \quad \text{and} \quad f = -2\eta\lambda_0.$$

Thus the optimal estimate given by (8.5) is

$$\hat{L}(u(t)) \leqslant \hat{L}(u(0))C\mathrm{e}^{-2\eta(\lambda_1-\lambda_0)t},$$

for some constant $C > 0$, which is indeed the result achieved in §6.

Appendix A. Proof of lemma 6.3

The time derivative of $Q(u(t))$ for a solution $u(t)$ of the dissipative RMHD equations is

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(u(t)) = \frac{1}{I_2^2}[\dot{K}I_2 - K\dot{I}_2]$$

$$= \frac{1}{2I_2^2}\left[-\nu\int_{\mathcal{D}}\omega^2\,\mathrm{d}\boldsymbol{x}\int_{\mathcal{D}}A^2\,\mathrm{d}\boldsymbol{x} + \eta\int_{\mathcal{D}}\omega\psi\,\mathrm{d}\boldsymbol{x}\int_{\mathcal{D}}AJ\,\mathrm{d}\boldsymbol{x}\right.$$

$$\left.-\eta\int_{\mathcal{D}}J^2\,\mathrm{d}\boldsymbol{x}\int_{\mathcal{D}}A^2\,\mathrm{d}\boldsymbol{x} + \eta\left(\int_{\mathcal{D}}AJ\,\mathrm{d}\boldsymbol{x}\right)^2\right]$$

$$= -\frac{\nu}{I_2}\left[\int_{\mathcal{D}}\omega^2\mathrm{d}\boldsymbol{x} - \lambda_0\int_{\mathcal{D}}\omega\psi\,\mathrm{d}\boldsymbol{x}\right]$$

$$-\frac{1}{2I_2^2}\left[(\nu-\eta)\lambda_0\int_{\mathcal{D}}\omega\psi\,\mathrm{d}\boldsymbol{x}\int_{\mathcal{D}}A^2\,\mathrm{d}\boldsymbol{x} - \eta\int_{\mathcal{D}}\omega\psi\,\mathrm{d}\boldsymbol{x}\int_{\mathcal{D}}A(J-\lambda_0 A)\,\mathrm{d}\boldsymbol{x}\right.$$

$$\left.+\eta\int_{\mathcal{D}}(J-\lambda_0 A)^2\,\mathrm{d}\boldsymbol{x}\int_{\mathcal{D}}A^2\,\mathrm{d}\boldsymbol{x} - \eta\left(\int_{\mathcal{D}}A(J-\lambda_0 A)\,\mathrm{d}\boldsymbol{x}\right)^2\right].$$

Using the Poincaré inequality $\int_{\mathcal{D}}\omega^2 \geqslant \lambda_0\int_{\mathcal{D}}\omega\psi$ and (6.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(u(t)) \leqslant -\frac{1}{I_2}\left[(\nu-\eta)\lambda_0 Q(u)\int_{\mathcal{D}}A^2\,\mathrm{d}\boldsymbol{x} - (\nu-\eta)\lambda_0\int_{\mathcal{D}}A(J-\lambda_0 A)\,\mathrm{d}\boldsymbol{x}\right.$$

$$\left.-\eta Q(u)\int_{\mathcal{D}}A(J-\lambda_0 A)\,\mathrm{d}\boldsymbol{x} + \eta\int_{\mathcal{D}}(J-\lambda_0 A)^2\,\mathrm{d}\boldsymbol{x}\right]. \quad \text{(A 1)}$$

We write $A = \sum_{k=0}^{\infty}a_k\chi_k$, where χ_k are the L^2-normalized eigenfunctions of $-\Delta$, with eigenvalues $0 < \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(u(t)) \leqslant -\frac{1}{I_2}\sum_{k=0}^{\infty}a_k^2[(\nu-\eta)\lambda_0 Q(u) - (\nu-\eta)\lambda_0(\lambda_k-\lambda_0)$$

$$-\eta Q(u)(\lambda_k-\lambda_0) + \eta(\lambda_k-\lambda_0)^2]$$

$$= -\frac{\eta}{I_2}\sum_{k=0}^{\infty}a_k^2\left(\lambda_k - \frac{\nu}{\eta}\lambda_0\right)(\lambda_k - (\lambda_0 + Q(u))). \quad \text{(A 2)}$$

By lemma 6.2, $Q(u) \geqslant 0$ for all states u. Thus, to analyse the terms in this summation, we define for $x > 0$ and $y \geqslant 0$ the function

$$f(x,y) = \left(x - \frac{\nu}{\eta}\lambda_0\right)(x - \lambda_0 - y).$$

Since $\nu \geqslant \eta$, we see immediately that $f(\lambda_0, y) \geqslant 0$, for all $y \geqslant 0$. For all $0 \leqslant y < \lambda_1 - \lambda_0$, the inequality $\nu/\eta < \lambda_1/\lambda_0$ implies that $f(\lambda_k, y)$ is positive for all $k \in \mathbb{N}$. (Note that if $\nu/\eta > \lambda_1/\lambda_0$, then $f(\lambda_k, y) < 0$ for y sufficiently small.)

Hence $\mathrm{d}Q/\mathrm{d}t \leqslant 0$ as long as $Q(u(t)) < \lambda_1 - \lambda_0$. Thus the initial condition $Q(u_0) < \lambda_1 - \lambda_0$ implies that $Q(u(t)) \leqslant Q(u_0)$, for all time.

For fixed values of y, $f(x,y)$ is a quadratic function, with a minimum at $\hat{x}(y) = \frac{1}{2}((\nu/\eta)\lambda_0 + \lambda_0 + y)$. From the imposed conditions on ν, η, it follows that $\lambda_0 \leqslant \hat{x}(y) < \lambda_1$, if $0 \leqslant y < \lambda_1 - \lambda_0$. Hence for all $k \in \mathbb{N}$ and all $0 \leqslant Q < \lambda_1 - \lambda_0$ we have

$$f(\lambda_k, Q) \geqslant \min(f(\lambda_0, Q), f(\lambda_1, Q)) = \min\left(\lambda_0 Q\frac{\nu - \eta}{\eta}, (\lambda_1 - \lambda_0 - Q)\left(\lambda_1 - \lambda_0\frac{\nu}{\eta}\right)\right).$$

In other words, for $k \in \mathbb{N}$ and all $0 \leqslant Q \leqslant \lambda_1 - \lambda_0(\nu/\eta)$ we have

$$f(\lambda_k, Q) \geqslant \lambda_0 Q \frac{\nu - \eta}{\eta}$$

and for all $k \in \mathbb{N}$ and all $\lambda_1 - \lambda_0(\nu/\eta) < Q < \lambda_1 - \lambda_0$ we have

$$f(\lambda_k, Q) \geqslant (\lambda_1 - \lambda_0 - Q)\left(\lambda_1 - \lambda_0 \frac{\nu}{\eta}\right).$$

Thus we distinguish two situations.

(1) If $Q(u_0) \leqslant \lambda_1 - \lambda_0(\nu/\eta)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(u(t)) \leqslant -2\lambda_0(\nu - \eta)Q.$$

Hence for all $t \geqslant 0$ we have $Q(u(t)) \leqslant Q(u_0)\mathrm{e}^{-2\lambda_0(\nu-\eta)t}$.

(2) If $\lambda_1 - \lambda_0(\nu/\eta) < Q(u_0) < \lambda_1 - \lambda_0$, then

$$\frac{\mathrm{d}Q(u(t))}{\mathrm{d}t} < -2(\lambda_1\eta - \lambda_0\nu)(\lambda_1 - \lambda_0 - Q) \leqslant -2(\lambda_1\eta - \lambda_0\nu)(\lambda_1 - \lambda_0 - Q(u_0)).$$

Thus there exists some

$$0 < T < \frac{Q(u_0) - \lambda_1 + \lambda_0(\nu/\eta)}{2(\lambda_1\eta - \lambda_0\nu)(\lambda_1 - \lambda_0 - Q(u_0))}$$

such that $Q(T) = \lambda_1 - \lambda_0(\nu/\eta)$. So for $t > T$, we are in a similar situation as studied in the previous case and we can conclude that for $t > T$ we have

$$Q(u(t)) \leqslant Q(u(T))\mathrm{e}^{-2\lambda_0(\nu-\eta)(t-T)} \leqslant Q(u_0)\mathrm{e}^{-2\lambda_0(\nu-\eta)(t-T)}. \tag{A 3}$$

For $t \leqslant T$, $\mathrm{e}^{-2\lambda_0(\nu-\eta)(t-T)} \geqslant 1$. Since for all t, we have $Q(u(t)) \leqslant Q(u_0)$; this implies that the inequality in (A 3) is also valid for $0 \leqslant t \leqslant T$. Substitution of the upper bound on T into (A 3) shows

$$Q(u(t)) \leqslant Q(u_0) \exp\left(\frac{\lambda_0(\nu - \eta)(Q(u_0) - \lambda_1 + \lambda_0(\nu/\eta))}{(\lambda_1\eta - \lambda_0\nu)(\lambda_1 - \lambda_0 - Q(u_0))}\right) \mathrm{e}^{-2\lambda_0(\nu-\eta)t},$$

for $t \geqslant 0$.

With the estimate (6.9) on $Q(u)$, we can immediately derive some estimates on $K(u)$ and $\hat{L}(u)$. The inequality for I_2 in (5.2) implies that $I_2(u(t)) \leqslant I(u_0)\mathrm{e}^{-2\eta\lambda_0 t}$. This inequality and the remark that $K(u) = Q(u)I_2(u)$ immediately imply (6.10). Finally, $\hat{L}(u) = Q(u) - \lambda_0 c(u)^2 \leqslant Q(u)$, which implies (6.11). ∎

Appendix B. The time derivative of the scaled Lyapunov functional

Under the conditions of theorem 7.1, the time derivative of $\hat{L}(u(t))$ satisfies the following inequality.

Lemma B.1. *Assume that $\eta < \nu < (\lambda_1/\lambda_0)\eta$. Let $u(t)$ be a solution of the dissipative RMHD equations with boundary conditions BC2 and with initial condition $u(0)$ such that*

$$Q(u(0)) \leqslant \min\left(\lambda_1 - \lambda_0, \frac{4\eta}{\nu + 3\eta}\lambda_0\right).$$

Then the time derivative of $\hat{L}(u(t))$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}(u(t)) \leqslant -2(\lambda_1 - \lambda_0)\eta\hat{L}(u) + \eta\frac{2}{1 - c(u)^2}\hat{L}(u)^2$$

$$+(\lambda_1 + 3\lambda_0)(\nu - \eta)\frac{c(u)^2}{2(1 - c(u)^2)}\hat{L}(u)$$

$$+\frac{\lambda_1\eta - \lambda_0\nu}{2\lambda_0(1 - c(u)^2)F(u(0))}\max\left(\lambda_1 - \lambda_0, \frac{(3\lambda_1 - 2\lambda_0)^2}{\lambda_1 - \lambda_0}\right)(\hat{L}(u))^2, \quad (B\,1)$$

where

$$F(u) = \begin{cases} (\lambda_1 - \lambda_0) - Q(u), & \text{if } \lambda_0 \geqslant \lambda_1 - \lambda_0, \\ \dfrac{\lambda_1 - \lambda_0}{\lambda_0}(\lambda_0 - Q(u)), & \text{if } \lambda_0 \leqslant \lambda_1 - \lambda_0. \end{cases}$$

Proof. By using equation (1.1) and the boundary conditions BC2, we get (see theorem 5.3)

$$\frac{\mathrm{d}}{\mathrm{d}t}I_1(u(t)) = -(\nu + \eta)\lambda_0 I_1(u) - (\nu + \eta)\int_{\mathcal{D}}(-\Delta\varphi)(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x}$$

$$+(\nu + \eta)c(u)\int_{\mathcal{D}}j(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}I_2(u) = -2\eta\lambda_0 I_2(u(t)) - \eta\int_{\mathcal{D}}\alpha(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}K(u(t)) = -2\nu\lambda_0\,K(u) + \nu\lambda_0\int_{\mathcal{D}}(-\Delta\varphi)\varphi\,\mathrm{d}\boldsymbol{x}$$

$$-\nu\int_{\mathcal{D}}(-\Delta\varphi)^2\,\mathrm{d}\boldsymbol{x} + 2\nu c(u)\int_{\mathcal{D}}(-\Delta\varphi)(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x}$$

$$-(\eta + c(u)^2\nu)\int_{\mathcal{D}}j(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x} + \nu\lambda_0\int_{\mathcal{D}}\alpha(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x}.$$

Combining the expressions above and using the definition

$$c(u) = -\frac{I_1(u)}{2\lambda_0 I_2(u)},$$

we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}L(u(t)) = \frac{\mathrm{d}}{\mathrm{d}t}K(u(t)) + c(u)\frac{\mathrm{d}}{\mathrm{d}t}I_1(u(t)) + c(u)^2\lambda_0\frac{\mathrm{d}}{\mathrm{d}t}I_2(u(t))$$

$$+[I_1(u) + 2c(u)\lambda_0 I_2(u)]\frac{\mathrm{d}}{\mathrm{d}t}c(u(t))$$

$$= -2\nu\lambda_0 L(u) - \nu\int_{\mathcal{D}}[(-\Delta\varphi)^2 - \lambda_0|\nabla\varphi|^2]\,\mathrm{d}\boldsymbol{x}$$

$$+c(u)(\nu - \eta)\int_{\mathcal{D}}(-\Delta\varphi)(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x}$$

$$-(1 - c(u)^2)\eta\int_{\mathcal{D}}j(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x} + (\nu - c(u)^2\eta)\lambda_0\int_{\mathcal{D}}\alpha(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x}.$$

Thus the time derivative of $\hat{L} = L/I_2$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}(u(t)) = \frac{1}{I_2(u)}\left[\frac{\mathrm{d}}{\mathrm{d}t}L(u(t)) - \hat{L}(u)\frac{\mathrm{d}}{\mathrm{d}t}I_2(u(t))\right]$$

$$= -2(\nu - \eta)\lambda_0\hat{L}(u) + \eta\hat{L}(u)\frac{1}{I_2(u)}\int_{\mathcal{D}}\alpha(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x}$$

$$-\frac{1}{I_2(u)}\left[\nu\int_{\mathcal{D}}[(-\Delta\varphi)^2 - \lambda_0|\nabla\varphi|^2]\,\mathrm{d}\boldsymbol{x}\right.$$

$$-c(u)(\nu - \eta)\int_{\mathcal{D}}(-\Delta\varphi)(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x}$$

$$\left.+(1 - c(u)^2)\eta\int_{\mathcal{D}}j(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x} - (\nu - c(u)^2\eta)\lambda_0\int_{\mathcal{D}}\alpha(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x}\right].$$

Recall that we denote the L^2-orthogonal projections onto the eigenspace $\mathcal{E}_0$ by Π_0 and onto its complement by $\Pi_0^\perp$. Using eigenfunction expansions we see that $(j - \lambda_0\alpha)$, $(-\Delta\varphi - \lambda_0\varphi) \in \mathcal{E}_0^\perp$, which implies that

$$-c(u)\int_{\mathcal{D}}(-\Delta\varphi)(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x} = \int_{\mathcal{D}}(\Pi_0^\perp(-\Delta\varphi) - \tfrac{1}{2}c(u)(j - \lambda_0\alpha))^2\,\mathrm{d}\boldsymbol{x}$$

$$-\int_{\mathcal{D}}(\Pi_0^\perp(-\Delta\varphi))^2\,\mathrm{d}\boldsymbol{x} - \tfrac{1}{4}c(u)^2\int_{\mathcal{D}}(j - \lambda_0\alpha)^2\,\mathrm{d}\boldsymbol{x}$$

and also

$$\int_{\mathcal{D}}[(-\Delta\varphi)^2 - \lambda_0|\nabla\varphi|^2]\,\mathrm{d}\boldsymbol{x} = \int_{\mathcal{D}}(\Pi_0^\perp(-\Delta\varphi))^2\,\mathrm{d}\boldsymbol{x} - \lambda_0\int_{\mathcal{D}}(\Pi_0^\perp\varphi)(-\Delta\varphi)\,\mathrm{d}\boldsymbol{x}.$$

Substitution of these identities in the equation for $(\mathrm{d}/\mathrm{d}t)\hat{L}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{L}(u(t)) = -2(\nu - \eta)\lambda_0\hat{L}(u) + \eta\hat{L}(u)\frac{1}{I_2(u)}\int_{\mathcal{D}}\alpha(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x}$$

$$-\frac{1}{I_2(u)}(\nu - \eta)\int_{\mathcal{D}}(\Pi_0^\perp(-\Delta\varphi) - \tfrac{1}{2}c(u)(j - \lambda_0\alpha))^2\,\mathrm{d}\boldsymbol{x}$$

$$-\frac{1}{I_2(u)}\left[\eta\int_{\mathcal{D}}(\Pi_0^\perp(-\Delta\varphi))^2\,\mathrm{d}\boldsymbol{x} - \lambda_0\nu\int_{\mathcal{D}}(\Pi_0^\perp\varphi)(-\Delta\varphi)\,\mathrm{d}\boldsymbol{x}\right.$$

$$+(\eta - \tfrac{1}{4}c(u)^2(\nu + 3\eta))\int_{\mathcal{D}}j(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x}$$

$$\left.-(\nu - \frac{c(u)^2}{4}(\nu + 3\eta))\lambda_0\int_{\mathcal{D}}\alpha(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x}\right]. \tag{B 2}$$

We have the following Poincaré-like inequalities:

$$\int_{\mathcal{D}}(\Pi_0^\perp(-\Delta\varphi))^2\,\mathrm{d}\boldsymbol{x} \geqslant \lambda_1\int_{\mathcal{D}}(\Pi_0^\perp\varphi)(-\Delta\varphi)\,\mathrm{d}\boldsymbol{x} \geqslant 0 \tag{B 3}$$

$$\int_{\mathcal{D}}j(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x} \geqslant \lambda_1\int_{\mathcal{D}}\alpha(j - \lambda_0\alpha)\,\mathrm{d}\boldsymbol{x} \geqslant 0. \tag{B 4}$$

The definition of L and the relations $\bar{\omega}(u) + c(u)\lambda_0\bar{A}(u) = 0$, $-\Delta\bar{A}(u) = \lambda_0\bar{A}(u)$ (valid by definition of the shadowing curve) imply that

$$2L(u) = 2L(\bar{u}) + \int_{\mathcal{D}} (-\Delta\varphi)\varphi \, \mathrm{d}\boldsymbol{x} + (1 - c(u)^2) \int_{\mathcal{D}} \alpha(j - \lambda_0\alpha) \, \mathrm{d}\boldsymbol{x}$$

$$\geqslant (1 - c(u)^2) \int_{\mathcal{D}} \alpha(j - \lambda_0\alpha) \, \mathrm{d}\boldsymbol{x},$$

which is equivalent to

$$\int_{\mathcal{D}} \alpha(j - \lambda_0\alpha) \, \mathrm{d}\boldsymbol{x} \leqslant \frac{2}{1 - c(u)^2} L(u). \tag{B 5}$$

We estimate the time derivative of $\hat{L}(u)$ neglecting the third term in the identity (B 2), and using the inequalities (B 3)–(B 5). In order to apply (B 4) we need to know that the coefficient of $\int_{\mathcal{D}} j(j - \lambda_0\alpha) \, \mathrm{d}\boldsymbol{x}$ is negative, i.e. $\eta - \frac{1}{4}c(u)^2(\nu + \eta) \geqslant 0$. This follows from lemma 6.3 and the hypothesis by using that

$$\lambda_0 c(u(t))^2 \leqslant Q(u(t)) \leqslant Q(u(0)) \leqslant \frac{4\eta}{\nu + 3\eta} \lambda_0.$$

So we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{L}(u(t)) \leqslant -2(\nu - \eta)\lambda_0 \hat{L}(u) + \eta \frac{2}{1 - c(u)^2} \hat{L}(u)^2$$

$$- \frac{\lambda_1\eta - \lambda_0\nu}{I_2(u)} \left[\int_{\mathcal{D}} (\Pi_0^\perp\varphi)(-\Delta\varphi) \, \mathrm{d}\boldsymbol{x} + (1 - c(u)^2) \int_{\mathcal{D}} \alpha(j - \lambda_0\alpha) \, \mathrm{d}\boldsymbol{x} \right]$$

$$+ \frac{c(u)^2}{4I_2(u)} \left[-4(\lambda_1\eta - \lambda_0\nu) + (\nu + 3\eta)(\lambda_1 - \lambda_0) \right] \int_{\mathcal{D}} \alpha(j - \lambda_0\alpha) \, \mathrm{d}\boldsymbol{x}.$$

Using that the third term is equal to

$$- \frac{\lambda_1\eta - \lambda_0\nu}{I_2(u)} \hat{L}(u) + \frac{\lambda_1\eta - \lambda_0\nu}{I_2(u)} \int_{\mathcal{D}} (\Pi_0\varphi)(-\Delta\varphi) \, \mathrm{d}\boldsymbol{x},$$

and employing (B 5) to bound above the integral in the last, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{L}(u(t)) \leqslant -2(\lambda_1 - \lambda_0)\eta \hat{L}(u) + \eta \frac{2}{1 - c(u)^2} \hat{L}(u)^2$$

$$+ (\lambda_1 + 3\lambda_0)(\nu - \eta) \frac{c(u)^2}{2(1 - c(u)^2)} \hat{L}(u) + \frac{\lambda_1\eta - \lambda_0\nu}{I_2(u)} \int_{\mathcal{D}} (\Pi_0\varphi)(-\Delta\varphi) \, \mathrm{d}\boldsymbol{x}.$$

$$\tag{B 6}$$

Now we are left with estimating

$$\int_{\mathcal{D}} (\Pi_0\varphi)(-\Delta\varphi) \, \mathrm{d}\boldsymbol{x} = \lambda_0 \int_{\mathcal{D}} (\Pi_0\varphi)^2 \, \mathrm{d}\boldsymbol{x}.$$

We will show that this integral is of order $(\hat{L}(u))^2 I_2(u)$.

First we observe that

$$\lambda_0^2 \int_{\mathcal{D}} (\Pi_0\varphi)^2 \, \mathrm{d}\boldsymbol{x} \int_{\mathcal{D}} (\Pi_0 A)^2 d\boldsymbol{x} = \left(\lambda_0 \int_{\mathcal{D}} \varphi(\Pi_0 A) \, \mathrm{d}\boldsymbol{x} \right)^2 = \left(\lambda_0 \int_{\mathcal{D}} \varphi(\bar{A} + \Pi_0\alpha) \, \mathrm{d}\boldsymbol{x} \right)^2.$$

$$\tag{B 7}$$

From (6.5) and using the Poincaré-like estimate

$$(\lambda_1 - \lambda_0) \int_{\mathcal{D}} |\nabla(\Pi_0^\perp\alpha)|^2 \, \mathrm{d}\boldsymbol{x} \leqslant \lambda_1 \int_{\mathcal{D}} \alpha(j - \lambda_0\alpha) \, \mathrm{d}\boldsymbol{x}$$

we get

$$\left| \lambda_0 \int_{\mathcal{D}} \varphi(\bar{A} + \varPi_0 \alpha) \, \mathrm{d}\boldsymbol{x} \right|$$

$$= \left| \lambda_0 \int_{\mathcal{D}} \varphi(\varPi_0 \alpha) \, \mathrm{d}\boldsymbol{x} + c(u) \int_{\mathcal{D}} \alpha(j - \lambda_0 \alpha) \, \mathrm{d}\boldsymbol{x} - \int_{\mathcal{D}} (-\Delta \varphi) \alpha \right|$$

$$= \left| \lambda_0 \int_{\mathcal{D}} \varphi(\varPi_0 \alpha) \, \mathrm{d}\boldsymbol{x} + c(u) \int_{\mathcal{D}} \alpha(j - \lambda_0 \alpha) \, \mathrm{d}\boldsymbol{x} \right.$$
$$\left. - \lambda_0 \int_{\mathcal{D}} \varphi(\varPi_0 \alpha) \, \mathrm{d}\boldsymbol{x} - \int_{\mathcal{D}} (-\Delta \varphi)(\varPi_0^{\perp} \alpha) \, \mathrm{d}\boldsymbol{x} \right|$$

$$\leqslant |c(u)| \int_{\mathcal{D}} \alpha(j - \lambda_0 \alpha) \, \mathrm{d}\boldsymbol{x} + \left| \int_{\mathcal{D}} \nabla \varphi \cdot \nabla(\varPi_0^{\perp} \alpha) \, \mathrm{d}\boldsymbol{x} \right|$$

$$\leqslant |c(u)| \int_{\mathcal{D}} \alpha(j - \lambda_0 \alpha) \, \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_{\mathcal{D}} |\nabla \varphi|^2 \, \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_{\mathcal{D}} |\nabla(\varPi_0^{\perp} \alpha)|^2 \, \mathrm{d}\boldsymbol{x}$$

$$\leqslant \frac{1}{2} \int_{\mathcal{D}} (-\Delta \varphi) \varphi \, \mathrm{d}\boldsymbol{x} + \frac{1}{2}(1 - c(u)^2) \int_{\mathcal{D}} \alpha(j - \lambda_0 \alpha) \, \mathrm{d}\boldsymbol{x} \left[\frac{\lambda_1 + 2|c(u)|(\lambda_1 - \lambda_0)}{(\lambda_1 - \lambda_0)(1 - c(u)^2)} \right]$$

$$\leqslant \max \left(1, \frac{\lambda_1 + 2|c(u)|(\lambda_1 - \lambda_0)}{(\lambda_1 - \lambda_0)(1 - c(u)^2)} \right) L(u). \tag{B 8}$$

Furthermore, the Poincaré-like inequality

$$(\lambda_1 - \lambda_0) \int_{\mathcal{D}} |(\varPi_0^{\perp} \alpha)|^2 \, \mathrm{d}\boldsymbol{x} \leqslant \int_{\mathcal{D}} \alpha(j - \lambda_0 \alpha) \, \mathrm{d}\boldsymbol{x}$$

implies

$$\int_{\mathcal{D}} (\varPi_0 A)^2 \, \mathrm{d}\boldsymbol{x} = 2I_2(u) - \int_{\mathcal{D}} (\varPi_0^{\perp} A)^2 \, \mathrm{d}\boldsymbol{x} = 2I_2(u) - \int_{\mathcal{D}} (\varPi_0^{\perp} \alpha)^2 \, \mathrm{d}\boldsymbol{x}$$

$$\geqslant 2I_2(u) - \frac{1}{\lambda_1 - \lambda_0} \int_{\mathcal{D}} \alpha(j - \lambda_0 \alpha) \, \mathrm{d}\boldsymbol{x}$$

$$\geqslant 2I_2(u) \left(1 - \frac{1}{(\lambda_1 - \lambda_0)(1 - c(u)^2)} \hat{L}(u) \right).$$

To analyse this last expression we use $Q(u) = \hat{L}(u) + \lambda_0 c(u)^2$ and $Q(u(t)) \leqslant Q(u(0))$, for all $t \geqslant 0$ (see lemma 6.3).

(1) If $\lambda_0 \geqslant \lambda_1 - \lambda_0$, we have

$$(\lambda_1 - \lambda_0)(1 - c(u)^2) - \hat{L}(u) = (\lambda_1 - \lambda_0) - Q(u) + c(u)^2(2\lambda_0 - \lambda_1)$$
$$\geqslant (\lambda_1 - \lambda_0) - Q(u(0)).$$

(2) If $\lambda_0 \leqslant \lambda_1 - \lambda_0$, we have

$$(\lambda_1 - \lambda_0)(1 - c(u)^2) - \hat{L}(u) = \frac{\lambda_1 - \lambda_0}{\lambda_0}(\lambda_0 - Q(u)) + \frac{\lambda_1 - 2\lambda_0}{\lambda_0} \hat{L}(u)$$
$$\geqslant \frac{\lambda_1 - \lambda_0}{\lambda_0}(\lambda_0 - Q(u(0))).$$

To capture both situations, we define

$$F(u) = \begin{cases} (\lambda_1 - \lambda_0) - Q(u), & \text{if } \lambda_0 \geqslant \lambda_1 - \lambda_0, \\ \dfrac{\lambda_1 - \lambda_0}{\lambda_0}(\lambda_0 - Q(u)), & \text{if } \lambda_0 \leqslant \lambda_1 - \lambda_0. \end{cases}$$

then $F(u(0)) > 0$ by the hypothesis of the theorem and we can write

$$\int_{\mathcal{D}} (\Pi_0 A)^2 \, \mathrm{d}\boldsymbol{x} \geqslant 2I_2(u) \frac{F(u(0))}{(\lambda_1 - \lambda_0)(1 - c(u)^2)}. \tag{B 9}$$

Substitution of the estimates (B 9) and (B 8) into (B 7) gives

$$\frac{2\lambda_0^2 F(u(0))}{(\lambda_1 - \lambda_0)(1 - c(u)^2)} I_2(u) \int_{\mathcal{D}} (\Pi_0 \varphi)^2 \, \mathrm{d}\boldsymbol{x}$$
$$\leqslant \max\left(1, \frac{(\lambda_1 + 2|c(u)|(\lambda_1 - \lambda_0))^2}{(\lambda_1 - \lambda_0)^2(1 - c(u)^2)^2}\right)(L(u))^2.$$

Using that $0 \leqslant |c(u)| < 1$, we get

$$\int_{\mathcal{D}} (\Pi_0 \varphi)^2 \, \mathrm{d}\boldsymbol{x} \leqslant \frac{(\hat{L}(u))^2 I_2(u)}{2\lambda_0^2(1 - c(u)^2)F(u(0))} \max\left(\lambda_1 - \lambda_0, \frac{(3\lambda_1 - 2\lambda_0)^2}{\lambda_1 - \lambda_0}\right).$$

Substitution of this estimate in (B 6) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{L}(u(t)) \leqslant -2(\lambda_1 - \lambda_0)\eta \hat{L}(u) + \eta \frac{2}{1 - c(u)^2} \hat{L}(u)^2$$
$$+ (\lambda_1 + 3\lambda_0)(\nu - \eta) \frac{c(u)^2}{2(1 - c(u)^2)} \hat{L}(u)$$
$$+ \frac{\lambda_1 \eta - \lambda_0 \nu}{2\lambda_0(1 - c(u)^2)F(u(0))} \max\left(\lambda_1 - \lambda_0, \frac{(3\lambda_1 - 2\lambda_0)^2}{\lambda_1 - \lambda_0}\right)(\hat{L}(u))^2. \tag{B 10}$$

∎

Appendix C. The dissipation coefficient of the RMHD equations

In this appendix we will calculate $\beta(\bar{u})$ and $f(\bar{u})$ for the dissipative RMHD equation. In the case of the dissipative RMHD equation, we have $\mathcal{S}(\bar{u}) = \mathcal{E}_0^\perp \times \mathcal{E}_0^\perp$. By definition $L(u) = H(u) + c(u)I_1(u) - (1 - c(u)^2)\lambda_0 I_2(u)$, with

$$c(u) = -\frac{I_1(u)}{2\lambda_0 I_2(u)},$$

so we get

$$DL(u) = DH(u) + c(u)DI_1(u) - (1 - c(u)^2)\lambda_0 DI_2(u)$$

and

$$Dc(u) = -\frac{1}{2\lambda_0 I_2(u)}[DI_1(u) + 2\lambda_0 c(u)DI_2(u)].$$

Both $DI_1(\bar{u})$ and $DI_2(\bar{u})$ are elements of $\mathcal{E}_0 \times \mathcal{E}_0$, thus $\langle Dc(\bar{u}), \delta u \rangle = 0$, if $\delta u \in \mathcal{S}(\bar{u})$. Thus for any $\delta u, \Delta u$, we have

$$\langle D^2 L(\bar{u})\delta u, \Delta u \rangle = \langle (D^2 H(\bar{u}) + c(\bar{u})D^2 I_1(\bar{u}) - (1 - c(\bar{u})^2)\lambda_0 D^2 I_2((\bar{u}))\delta u, \Delta u \rangle$$
$$+ (\langle DI_1(\bar{u}), \delta u \rangle + 2c(\bar{u})\lambda_0 \langle DI_2(\bar{u}), \delta u \rangle)\langle Dc(\bar{u}), \Delta u \rangle$$
$$= \langle (D^2 H(\bar{u}) + c(\bar{u})D^2 I_1(\bar{u}) - (1 - c(\bar{u})^2)\lambda_0 D^2 I_2(\bar{u}))\delta u, \Delta u \rangle$$
$$+ 2\lambda_0 I_2(\bar{u})\langle Dc(\bar{u}), \delta u \rangle \langle Dc(\bar{u}), \Delta u \rangle.$$

Hence if either $\delta u \in \mathcal{S}$ or $\Delta u \in \mathcal{S}$, we have

$$\langle D^2 L(\bar{u})\delta u, \Delta u \rangle = \langle (D^2 H(\bar{u}) + c(\bar{u})D^2 I_1(\bar{u}) - (1 - c(\bar{u})^2)\lambda_0 D^2 I_2(u)))\delta u, \Delta u \rangle. \tag{C 1}$$

Using (C 1) and writing $\delta u = (\delta\omega, \delta A) = (-\Delta(\delta\psi), \delta A) \in \mathcal{S}(\bar{u})$, we get

$$\langle D^2 L(\bar{u})\delta u, \delta u\rangle$$

$$= \int_{\mathcal{D}} (\delta\omega\delta\psi + \delta A\delta J) \, d\boldsymbol{x} + 2c(\bar{u}) \int_{\mathcal{D}} \delta\omega\delta A \, d\boldsymbol{x} - (1 - c(\bar{u})^2)\lambda_0 \int_{\mathcal{D}} (\delta A)^2 \, d\boldsymbol{x}$$

$$= \int_{\mathcal{D}} (\delta\omega + c(\bar{u})\delta J)(\delta\psi + c(\bar{u})\delta A) \, d\boldsymbol{x} + (1 - c(\bar{u})^2) \int_{\mathcal{D}} \delta A(\delta J - \lambda_0\delta A) \, d\boldsymbol{x}.$$

Since $P(u) = (\nu\Delta\omega, \eta\Delta A)$, we have $DP(\bar{u}) \cdot \delta u = (\nu\Delta\delta\omega, \eta\Delta\delta A)$ so that, using the boundary conditions BC2 in the integration by parts

$$\nu \int_{\mathcal{D}} \Delta\delta\omega\delta A = \nu \int_{\mathcal{D}} \delta\omega\delta J,$$

the identity (C 1) gives

$$\langle D^2 L(\bar{u})\delta u, DP(\bar{u})\delta u\rangle$$

$$= -\nu \int_{\mathcal{D}} (\delta\omega)^2 \, d\boldsymbol{x} - \eta \int_{\mathcal{D}} (\delta J)^2 \, d\boldsymbol{x} - c(\bar{u})(\eta + \nu) \int_{\mathcal{D}} \delta\omega\delta J \, d\boldsymbol{x}$$

$$+ (1 - c(\bar{u})^2)\lambda_0\eta \int_{\mathcal{D}} \delta A\delta J \, d\boldsymbol{x}$$

$$= -\frac{1}{2}(\nu - \eta) \int_{\mathcal{D}} (\delta\omega)^2 \, d\boldsymbol{x} - \frac{1}{2}(\nu + \eta) \int_{\mathcal{D}} (\delta\omega + c(\bar{u})\delta J)^2 \, d\boldsymbol{x}$$

$$+ \frac{1}{2}c(\bar{u})^2(\nu - \eta) \int_{\mathcal{D}} \delta J^2 \, d\boldsymbol{x} - (1 - c(\bar{u})^2)\eta \int_{\mathcal{D}} \delta J(\delta J - \lambda_0\delta A) \, d\boldsymbol{x}.$$

From the estimate

$$\int_{\mathcal{D}} (\delta\omega + c(\bar{u})\delta J)^2 \, d\boldsymbol{x} \geqslant \lambda_1 \int_{\mathcal{D}} (\delta\omega + c(\bar{u})\delta J)(\delta\psi + c\delta A) \, d\boldsymbol{x}$$

for $\delta\omega + c(\bar{u})\delta J \in \mathcal{E}_0^\perp$ and omitting the first negative term on the right-hand side of the previous inequality we get

$$\langle D^2 L(\bar{u})\delta u, DP(\bar{u})\delta u\rangle \leqslant -\frac{1}{2}\lambda_1(\eta + \nu) \int_{\mathcal{D}} (\delta\omega + c\delta J)(\delta\psi + c\delta A) \, d\boldsymbol{x}$$

$$- [(1 - c(\bar{u})^2)\eta - \tfrac{1}{2}c(\bar{u})^2(\nu - \eta)] \int_{\mathcal{D}} \delta J(\delta J - \lambda_0\delta A) \, d\boldsymbol{x}$$

$$+ \frac{1}{2}c(\bar{u})^2(\nu - \eta)\lambda_0 \int_{\mathcal{D}} \delta J\delta A \, d\boldsymbol{x}. \tag{C 2}$$

Since $\delta A, \delta J \in \mathcal{E}_0^\perp$ we have the estimates

$$\int_{\mathcal{D}} \delta J(\delta J - \lambda_0\delta A) \, d\boldsymbol{x} \geqslant \lambda_1 \int_{\mathcal{D}} \delta A(\delta J - \lambda_0\delta A) \, d\boldsymbol{x},$$

and

$$\int_{\mathcal{D}} \delta A\delta J \, d\boldsymbol{x} \geqslant \lambda_1 \int_{\mathcal{D}} \delta A^2 \, d\boldsymbol{x},$$

the latter being equivalent to

$$\int_{\mathcal{D}} \delta A\delta J \, d\boldsymbol{x} \leqslant \frac{\lambda_1}{\lambda_1 - \lambda_0} \int_{\mathcal{D}} \delta A(\delta J - \lambda_0\delta A) \, d\boldsymbol{x}.$$

Requiring

$$(1 - c(\bar{u})^2)\eta - \tfrac{1}{2}c(\bar{u})^2(\nu - \eta) > 0, \tag{C 3}$$

we get from (C 2)

$$\langle D^2 L(\bar{u})\delta u, DP(\bar{u})\delta u\rangle \leqslant -\tfrac{1}{2}\lambda_1(\eta + \nu)\int_{\mathcal{D}}(\delta\omega + c\delta J)(\delta\psi + c\delta A)\,\mathrm{d}\boldsymbol{x}$$

$$-[(1 - c(\bar{u})^2)\eta - \tfrac{1}{2}c(\bar{u})^2(\nu - \eta)]\lambda_1\int_{\mathcal{D}}\delta A(\delta J - \lambda_0\delta A)\,\mathrm{d}\boldsymbol{x}$$

$$+\tfrac{1}{2}c(\bar{u})^2(\nu - \eta)\frac{\lambda_0\lambda_1}{\lambda_1 - \lambda_0}\int_{\mathcal{D}}\delta A(\delta J - \lambda_0\delta A)\,\mathrm{d}\boldsymbol{x}$$

$$= -\tfrac{1}{2}\lambda_1(\eta + \nu)\int_{\mathcal{D}}(\delta\omega + c(\bar{u})\delta J)(\delta\psi + c(\bar{u})\delta A)\,\mathrm{d}\boldsymbol{x}$$

$$-\frac{\lambda_1}{2(1 - c(\bar{u})^2)}\left[2\eta - \frac{c(\bar{u})^2}{\lambda_1 - \lambda_0}((\lambda_1\nu - \lambda_0\eta) + (\lambda_1 - \lambda_0)\eta)\right]$$

$$\times(1 - c(\bar{u})^2)\int_{\mathcal{D}}\delta A(\delta J - \lambda_0\delta A)\,\mathrm{d}\boldsymbol{x}. \tag{C 4}$$

If, in addition,

$$2\eta - \frac{c(\bar{u})^2}{\lambda_1 - \lambda_0}((\lambda_1\nu - \lambda_0\eta) + (\lambda_1 - \lambda_0)\eta) > 0, \tag{C 5}$$

an inequality which implies (C 3), we get

$$\langle D^2 L(\bar{u})\delta u, DP(\bar{u})\delta u\rangle \leqslant -\tfrac{1}{2}\lambda_1\langle D^2 L(\bar{u})\delta u, \delta u\rangle$$

$$\times \min\left(\nu + \eta, \frac{1}{1 - c(\bar{u})^2}\left[2\eta - \frac{c(\bar{u})^2}{\lambda_1 - \lambda_0}((\lambda_1\nu - \lambda_0\eta) + (\lambda_1 - \lambda_0)\eta)\right]\right). \tag{C 6}$$

Next we analyse under what conditions inequality (C 5) is valid. Recall from §5, lemma 6.4, the estimate (6.12):

$$c(u(t))^2 \leqslant (\hat{L}(u(0))/\lambda_0 + c(u(0))^2)C(u(0))\mathrm{e}^{-2(\nu - \eta)\lambda_0 t},$$

valid for either boundary conditions BC1 or BC2, provided that $u(0)$ satisfies $Q(u(0)) < \min(\lambda_0, \lambda_1 - \lambda_0)$ and $\nu < \eta < \lambda_1\eta/\lambda_0$. The constant $C(u(0))$ is large, only if $Q(u(0))$ is near $(\lambda_1 - \lambda_0)$, and not if $u(0)$ is small. We now place ourselves in these conditions and conclude hence that $c(\bar{u}) \to 0$ as $t \to \infty$. Thus, there is some $T_1 > 0$ such that for all $t > T_1$ inequality (C 5) holds which then means that (C 4) is valid. Moreover, since $\nu > \eta$ it follows that there is a $T_2 > 0$ such that for all $t > T_2$ we have

$$\nu > \frac{1 + c(\bar{u})^2}{1 - c(\bar{u})^2}\eta,$$

which in turn implies that

$$\nu + \eta > \frac{1}{1 - c(\bar{u})^2}\left[2\eta - \frac{c(\bar{u})^2}{\lambda_1 - \lambda_0}((\lambda_1\nu - \lambda_0\eta) + (\lambda_1 - \lambda_0)\eta)\right]$$

and so for $t > \max(T_1, T_2)$, the inequality (C 6) implies

$$\langle D^2 L(\bar{u})\delta u, DP(\bar{u})\delta u\rangle \leqslant \beta(\bar{u})\tfrac{1}{2}\langle D^2 L(\bar{u})\delta u, \delta u\rangle, \tag{C 7}$$

where

$$\beta(\bar{u}) = -\frac{\lambda_1}{1 - c(\bar{u})^2}\left[2\eta - \frac{c(\bar{u})^2}{\lambda_1 - \lambda_0}((\lambda_1\nu - \lambda_0\eta) + (\lambda_1 - \lambda_0)\eta)\right]. \tag{C 8}$$

From (C 8) we see that

$$\beta = \lim_{t\to\infty}\beta(\bar{u}) = -2\eta\lambda_1. \tag{C 9}$$

Since $F(u) = I_2(u)$, we conclude that

$$f(\bar{u}) = \frac{\langle DI_2(\bar{u}), P(\bar{u}) \rangle}{I_2(\bar{u})} = -2\eta\lambda_0 = f.$$

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