Improving the Crossing Lemma
by finding more crossings in sparse graphs

János Pach[*] *Radoš Radoičić*[†] *Gábor Tardos*[‡] *Géza Tóth*[§]

Abstract

Twenty years ago, Ajtai, Chvátal, Newborn, Szemerédi, and, independently, Leighton discovered that the crossing number of any graph with v vertices and $e > 4v$ edges is at least ce^3/v^2, where $c > 0$ is an absolute constant. This result, known as the 'Crossing Lemma,' has found many important applications in discrete and computational geometry. It is tight up to a multiplicative constant. Here we improve the best known value of the constant by showing that the result holds with $c > 1024/31827 > 0.032$. The proof has two new ingredients, interesting on their own right. We show that (1) if a graph can be drawn in the plane so that every edge crosses at most 3 others, then its number of edges cannot exceed $5.5(v-2)$; and (2) the crossing number of any graph is at least $\frac{7}{3}e - \frac{25}{3}(v-2)$. Both bounds are tight up to an additive constant (the latter one in the range $4v \le e \le 5v$).

1 Introduction

Unless stated otherwise, the graphs considered in this paper have no loops or parallel edges. The number of vertices and number of edges of a graph G are denoted by $v(G)$ and $e(G)$, respectively. We say that G is *drawn* in the plane if its vertices are represented by distinct points and its edges by (possibly intersecting) Jordan arcs connecting the corresponding point pairs. If it leads to no confusion, in terminology and notation we make no distinction between the vertices of G and the corresponding points, or between the edges and the corresponding Jordan arcs. We always assume that in a drawing (a) no edge passes through a vertex different from its endpoints, (b) no three edges cross at the same point, (c) any two edges have only a finite number of interior points in common, and at these points they properly cross, i.e., one of the edges passes from one side of the other edge to the other side (see [P99], [P04]). The *crossing number* of G, denoted by $\mathrm{cr}(G)$, is the minimum number of edge crossings in a drawing of G satisfying the above conditions.

Ajtai, Chvátal, Newborn, and Szemerédi [AC82] and, independently, Leighton [L83] have proved the following result, which is usually referred to as the 'Crossing Lemma.' The crossing number of any graph with v vertices and $e > 4v$ edges satisfies

$$\mathrm{cr}(G) \ge \frac{1}{64}\frac{e^3}{v^2}.$$

This result, which is tight apart from the value of the constant, has found many applications in combinatorial geometry, convexity, number theory, and VLSI design (see [L83], [Sz95], [PS98], [ENR00], [STT02], [PTa02]). In particular, it has played a pivotal role in obtaining the best known upper bound on the number of k-sets [D98] and lower bound on the number of distinct distances determined by n points in the plane [ST01], [KT04]. According to a conjecture of Erdős and Guy [ErG73], which was verified in [PST00], as long as $e/v \to \infty$ and $e/v^2 \to 0$, the limit

$$\lim_{v\to\infty} \min_{\substack{v(G)=v\\ e(G)=e}} \frac{\mathrm{cr}(G)}{e^3/v^2}$$

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[†] Massachusetts Institute of Technology, supported by Central European University, Budapest.

[‡] Rényi Institute, Budapest.

[§] Rényi Institute, Budapest, supported by OTKA T-038397.

exists. The best known upper and lower bounds for this constant (roughly 0.09 and $1/33.75 \approx 0.029$, resp.) were obtained in [PTo97].

All known proofs of the Crossing Lemma are based on the trivial inequality $\mathrm{cr}(H) \geq e(H) - (3v(H) - 6)$, which is an immediate corollary of Euler's Polyhedral Formula ($v(H) > 2$). Applying this statement inductively to all small (and, mostly sparse) subgraphs $H \subseteq G$ or to a randomly selected one, the lemma follows. The main idea in [PTo97] was to obtain stronger inequalities for the *sparse* subgraphs H, which have led to better lower bounds on the crossing numbers of *all* graphs G. In the present paper we follow the same approach.

For $k \geq 0$, let $e_k(v)$ denote the maximum number of edges in a graph of $v \geq 2$ vertices that can be drawn in the plane so that every edge is involved in at most k crossings. By Euler's Formula, we have $e_0(v) = 3(v-2)$. Pach and Tóth [PTo97] proved that $e_k(v) \leq (k+3)(v-2)$, for $0 \leq k \leq 3$. Moreover, for $0 \leq k \leq 2$, these bounds are tight for infinitely many values of v. However, for $k = 3$, there was a gap between the lower and upper estimates. Our first theorem, whose proof is presented in Section 2, fills this gap.

Theorem 1. *Let G be a graph on $v \geq 3$ vertices that can be drawn in the plane so that each of its edges crosses at most three others. Then we have*

$$e(G) \leq 5.5(v-2).$$

Consequently, the maximum number of edges over all such graphs satisfies $e_3(v) \leq 5.5(v-2)$, and this bound is tight up to an additive constant.

As we have pointed out before, the inequality $e_0(v) \leq 3(v-2)$ immediately implies that if a graph G of v vertices has more than $3(v-2)$ edges, then every edge beyond this threshold contributes at least one to $\mathrm{cr}(G)$. Similarly, it follows from inequality $e_1(v) \leq 4(v-2)$ that, if $e(G) \geq 4(v-2)$, then every edge beyond $4(v-2)$ must contribute an additional crossing to $\mathrm{cr}(G)$ (i.e., altogether at least two crossings). Summarizing, we obtain that

$$\mathrm{cr}(G) \geq (e(G) - 3\,(v(G)-2)) + (e(G) - 4\,(v(G)-2))$$
$$\geq 2e(G) - 7\,(v(G)-2)$$

holds for every graph G. Both components of this inequality are tight, so one might expect that their combination cannot be improved either, at least in the range when $e(G)$ is not much larger that $4(v-2)$. However, this is not the case, as is shown by our next result, proved in Section 3.

Theorem 2. *The crossing number of any graph G with $v(G) \geq 3$ vertices and $e(G)$ edges satisfies*

$$\mathrm{cr}(G) \geq \frac{7}{3}e(G) - \frac{25}{3}(v(G)-2).$$

In the worst case, this bound is tight up to an additive constant whenever $4\,(v(G)-2) \leq e(G) \leq 5\,(v(G)-2)$.

As an application of the above two theorems, in Section 4 we establish the following improved version of the Crossing Lemma.

Theorem 3. *The crossing number of any graph G satisfies*

$$\mathrm{cr}(G) \geq \frac{1}{31.1}\frac{e^3(G)}{v^2(G)} - 1.06v(G).$$

If $e(G) \geq \frac{103}{16}v(G)$, we also have

$$\mathrm{cr}(G) \geq \frac{1024}{31827}\frac{e^3(G)}{v^2(G)}.$$

Note for comparison that $1024/31827 \approx 1/33.08 \approx 0.032$.

In the last section, we adapt the ideas of Székely [Sz95] to deduce some consequences of Theorem 3, including an improved version of the Szemerédi-Trotter theorem [SzT83] on the maximum number of incidences between n points and m lines. We also discuss some open problems and make a few conjectures and concluding remarks.

All drawings considered in this paper satisfy the condition that *any pair of edges have at most one point in common.* This may be either an endpoint or a proper crossing. It is well known and easy to see that every drawing of a graph G that minimizes the number of crossings meets this requirement. Thus, in the proof of Theorem 3, we can make this assumption without loss of generality. However, it is not so obvious whether the same restriction can be justified in the case of Theorem 1. Indeed, in [PTo97], the bound $e(G) \leq (k+3)(v(G)-2)$ was proved only for graphs that can be drawn with at most $k \leq 4$ crossings per edge and which satisfy this extra condition. Since for the proof of Theorem 3 we need Theorem 1 in its full generality, we have to establish the following simple statement.

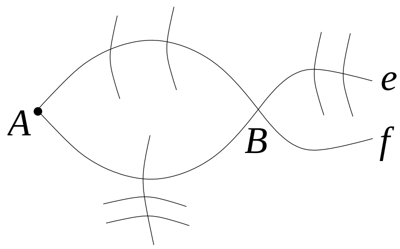

Figure 1: Two adjacent edges e and f cross, each participating in exactly 4 crossings.

Lemma 1.1. *Let $k \leq 3$, and let G be a graph of v vertices that can be drawn in the plane so that each of its edges participates in at most k crossings.*

In any drawing with this property that minimizes the total number of crossings, every pair of edges have at most one point in common.

Proof: Suppose for contradiction that some pair of edges, e and f, have at least two points in common, A and B. At least one of these points, say B, must be a proper crossing. First, try to swap the portions of e and f between A and B, and modify the new drawing in small neighborhoods of A and B so as to reduce the number of crossings between the two edges. Clearly, during this process the number of crossings along any other edge distinct from e and f remains unchanged. The only possible problem that may arise is that after the operation either e or f (say e) will participate in more than k crossings. In this case, before the operation there were at least two more crossings inside the portion of f between A and B, than inside the portion of e between A and B. Since f participated in at most three crossings (at most two, not counting B), we conclude that in the original drawing the portion of e between A and B contained no crossing. If this is the case, instead of swapping the two portions, replace the portion of f between A and B by an arc that runs very close to the portion of e between A and B, without intersecting it. $\square$

It is interesting to note that the above argument fails for $k \geq 4$, as shown in Figure 1.

2 Proof of Theorem 1

The proof goes through a series of lemmas whose proofs are omitted in this extended abstract. We use induction on v. For $v \leq 4$, the statement is trivial. Let $v \geq 4$, and suppose that the theorem has already been proved for graphs having fewer than v vertices.

Let $\mathcal{G}$ denote the set of all triples $(G, G', \mathcal{D})$ where G is a graph of v vertices, $\mathcal{D}$ is a drawing of G in the plane such that every edge of G crosses at most *three* others, and G' is a *planar subgraph* of G with $V(G') = V(G)$ that satisfies the condition that no two arcs in $\mathcal{D}$ representing edges of G' cross each other. Let $\mathcal{G}' \subset \mathcal{G}$ consist of all elements $(G, G', \mathcal{D}) \in \mathcal{G}$ for which the number of edges of G is maximum. Finally, let $\mathcal{G}'' \subset \mathcal{G}'$ consist of all elements of $\mathcal{G}'$ for which the number of edges of G' is maximum. Fix a triple $(G, G', \mathcal{D}) \in \mathcal{G}''$ such that the total number crossings in $\mathcal{D}$ along all edges of G' is as small as possible. This triple remains fixed throughout the whole argument. The term *face*, unless explicitly stated otherwise, refers to a face of the planar drawing of G' induced by D. For any face Φ (of G'), let $|\Phi|$ denote its number of sides, i.e., the number of edges of G' along the boundary of Φ, where every edge whose both sides belong to the interior of Φ is counted twice. Notice that $|\Phi| \geq 3$ for every face Φ, unless G' consists of a single edge, in which case $v(G) \leq 4$, a contradiction.

It follows from the maximality of G' that every edge e of G that does not belong to G' (in short, $e \in G - G'$) crosses at least one edge of G'. The closed portion between an endpoint of e and the nearest crossing of e with an edge of G' is called a *half-edge*. We orient every half-edge from its endpoint which is a vertex of G (and G') towards its other end sitting in the interior of an edge of G'. Clearly, every edge $e \in G - G'$ has two oriented half-edges. Every half-edge lies in a face Φ and contains at most *two* crossings with edges of G in its interior. The *extension* of a half-edge is the edge of $G - G'$ it belongs to. The set of half-edges belonging to a face Φ is denoted by $H(\Phi)$.

Lemma 2.1. *Let Φ be a face of G', and let g be one of its sides. Then $H(\Phi)$ cannot contain two non-crossing half-edges, both of which end on g and cross two other edges of G (that are not necessarily the same).*

A face Φ of G' is called *simple* if its boundary is connected and it does not contain any isolated vertex of G' in its interior.

Lemma 2.2. *The number of half-edges in any simple face Φ satisfies*

$$|H(\Phi)| \le 3|\Phi| - 6.$$

A simple face Φ of G' is said to be *triangular* if $|\Phi| = 3$, otherwise it is a *big face*.

By Lemma 2.2, we have $|H(\Phi)| \le 3$, for any triangular face Φ. A triangular face Φ is called an *i-triangle* if $|H(\Phi)| = i$ $(0 \le i \le 3)$. A 3-triangle is a *$3X$-triangle* if one half-edge emanates from each of its vertices. Otherwise, it is a *$3Y$-triangle*. Observe that if Φ is a $3X$-triangle, then it has three mutually crossing half-edges, so that their extensions do not have any additional crossing and they must end in a face adjacent to Φ. Moreover, no other edges of G can enter a $3X$-triangle.

If Φ is a $3Y$-triangle, then at least two of its half-edges must end at the same side. The face adjacent to Φ along this side is called the *neighbor* of Φ.

An edge of $G - G'$ is said to be *perfect* if it starts and ends in 3-triangles and all the faces it passes through are triangular. The neighbor Ψ of a $3Y$-triangle Φ is called a *strong neighbor* if either it is a 0-triangle or it is a 1-triangle and the extension of one of the half-edges in $H(\Phi)$ ends in Ψ.

Lemma 2.3. *Let Φ be a 3-triangle. If the extensions of at least two half-edges in $H(\Phi)$ are perfect, then Φ is a $3Y$-triangle with a strong neighbor.*

Suppose that Ψ is a simple face of G' with $|\Psi| = 4$ and $|H(\Psi)| = 6$. As shown on Figure 2, there are *seven* combinatorially different possibilities for the arrangement of Ψ and the half-edges (on the sphere).

Lemma 2.4. *Let Ψ be a simple face of G' with $|\Psi| = 4$ and $|H(\Psi)| = 6$, and suppose that the arrangement of half-edges in Ψ is* not *homeomorphic with the rightmost configuration depicted in Figure 2. Then we have*

$$E(G) < 5.5\,(v(G) - 2)\,.$$

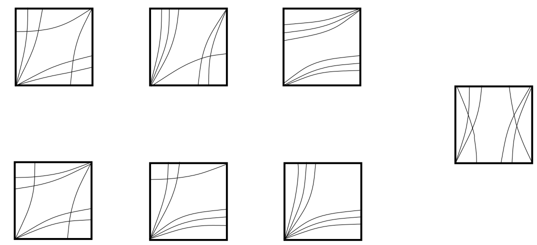

Figure 2: Seven different types of quadrilateral faces.

In view of the last lemma, from now on we may and will assume that in every simple quadrilateral face that contains 6 half-edges, these half-edges form an arrangement homeomorphic to the rightmost one depicted in Figure 2.

We define a bipartite multigraph $M = (V_1 \cup V_2, E)$ with vertex classes V_1 and V_2, where V_1 is the set of 3-triangles and V_2 is the set of all other faces of G'. For each vertex (3-triangle) $\Phi \in V_1$, separately, we add to the edge set E of M some edges incident to Φ, according to the following rules.

- *Rule 0*: Connect Φ to an adjacent triangular face Ψ by two parallel edges if Ψ is a 0-triangle.
- *Rule 1*: Connect Φ to any 1-triangle Ψ by two parallel edges if there is an edge of $G - G'$ that starts in Φ and ends in Ψ.
- *Rule 2*: Connect Φ to any 2-triangle Ψ by a single edge if there is an edge of $G - G'$ that starts in Φ and ends in Ψ.
- *Rule 3*: If the extension e of a half-edge in $H(\Phi)$ passes through or ends in a big face, we may connect Φ by a single edge to the first such big face along e. However, we use this last rule only to bring the degree of Φ in M up to 2. In particular, if we have applied Rules 0 or 1, for some Φ, we do not apply Rule 3. Similarly, in no case do we apply Rule 3 for all three half-edges in $H(\Phi)$.

Notice that, besides Rules 0 and 1, the application of Rule 3 can also yield parallel edges if two half-edges in $H(\Phi)$ reach the same big face. However, we never create three parallel edges in M.

Let $d(\Phi)$ denote the degree of vertex Φ in M.

Lemma 2.5. *For any $\Phi \in V_1$, we have $d(\Phi) \geq 2$.*

To complete the proof of Theorem 1, we have to estimate from *above* the degrees of the vertices belonging to V_2 in M. If $\Psi \in V_2$ is a 1-triangle or a 2-triangle, we have $d(\Psi) \leq 2$. Every 0-triangle Ψ is adjacent to at most three 3-triangles, so its degree satisfies $d(\Psi) \leq 6$. The following lemma establishes a bound for big faces.

Lemma 2.6. *For any big face $\Psi \in V_2$, we have $d(\Psi) \leq 2|\Psi|$. Moreover, if Ψ is a simple quadrilateral face with six half-edges forming an arrangement homeomorphic to the rightmost arrangement depicted in Figure 2, we have $d(\Psi) \leq 4$.*

For any face Φ, let $t(\Phi)$ and $\bar{t}(\Phi)$ denote the number of triangles and diagonals, resp., in a triangulation of Φ. Thus, if the sum of the number of isolated vertices of G' that lie in the interior of Φ and the number of connected components of the boundary of Φ is k, we have $t = |\Phi| + 2k - 4$ and $\bar{t} = |\Phi| + 3k - 6$.

We introduce the notation $\overline{d}(\Phi) := -d(\Phi)$ for $\Phi \in V_1$, and $\overline{d}(\Psi) := d(\Psi)$ for $\Psi \in V_2$. Let $V := V_1 \cup V_2$ denote the set of all faces of G'. Then the fact that the sum of degrees of the vertices must be the same on both sides of M, can be expressed by the equation

$$\sum_{\Phi \in V} \overline{d}(\Phi) = 0.$$

Lemma 2.7. *For every face $\Phi \in V$, we have*

$$|H(\Phi)| + \frac{1}{4}\overline{d}(\Phi) \leq \frac{5}{2}t(\Phi) + 2\bar{t}(\Phi).$$

Now we can easily complete the proof of Theorem 1. Since every edge of $G - G'$ gives rise to two half-edges, we have

$$e(G) - e(G') = \frac{1}{2}\sum_{\Phi \in V} |H(\Phi)|$$

$$= \frac{1}{2}\sum_{\Phi \in V}\left(|H(\Phi)| + \frac{1}{4}\overline{d}(\Phi)\right) \leq \frac{5}{4}\sum_{\Phi \in V} t(\Phi) + \sum_{\Phi \in V} \bar{t}(\Phi),$$

where the inequality holds by Lemma 2.7. We obviously have that $\sum_{\Phi \in V} t(\Phi) = 2\,(v(G) - 2)$, which is equal to the total number of faces in any triangulation of G'. In order to obtain such a triangulation from G', one needs to add $\sum_{\Phi \in V} \bar{t}(\Phi)$ edges. Hence, we have $\sum_{\Phi \in V} \bar{t}(\Phi) = 3(v(G) - 2) - e(G')$. Notice that triangulating each face separately may create a triangulation of the plane containing some parallel edges, but this has no effect on the number of triangles or the number of edges. Now the theorem follows by simple calculation:

$$e(G) = e(G') + (e(G) - e(G'))$$

$$\leq e(G') + \frac{5}{4} \cdot 2\,(v(G) - 2) + (3\,(v(G) - 2) - e(G'))$$

$$= 5.5\,(v(G) - 2)\,.$$

This completes the proof of the inequality in Theorem 1.

Proposition 2.8. *For every $v \equiv 0 \pmod 6$, $v \geq 12$, there exists a graph G with v vertices and $5.5(v-2) - 4$ edges that can be drawn in the plane so that each of its edges crosses at most three others. That is, for these values we have $e_3(v) \geq 5.5v - 15$.*

3 Proof of Theorem 2

For any graph G drawn in the plane, let G^{free} denote the subgraph of G on the same vertex set, consisting of all crossing-free edges. Let $\triangle(G^{\text{free}})$ denote the number of triangular faces of G^{free}, containing no vertex of G in their interiors.

Lemma 3.1. *Let G be a graph on $v(G) \geq 3$ vertices, which is drawn in the plane so that none of its edges crosses two others. Then the number of edges of G satisfies*

$$e(G) \leq 4(v(G) - 2) - \frac{1}{2}\triangle(G^{\text{free}}).$$

The proof of LEmma 3.1 is also omitted in this extended abstract. Instead of Theorem 2, we establish a slightly stronger claim.

Lemma 3.2. *Let G be a graph on $v(G) \geq 3$ vertices, which is drawn in the plane with $x(G)$ crossings. Then* *we have*

$$x(G) \geq \frac{7}{3}e(G) - \frac{25}{3}(v(G)-2) + \frac{2}{3}\triangle(G^{\text{free}}).$$

Proof: We use induction on $x(G)+v(G)$. As in the proof of Lemma 3.1, we can assume that G is 3-*connected* and that G^{free} is *maximal* in the sense that whenever the points u and v can be connected by a Jordan arc without crossing any edge of G, the edge uv belongs to G^{free}. We distinguish four cases.

Case 1. G contains an edge that crosses at least 3 other edges.

Let a be such an edge, and let G_0 be the subgraph of G obtained by removing a. Now we have, $e(G_0) = e(G)-1$, $x(G_0) \leq x(G) - 3$, and $\triangle(G_0^{\text{free}}) \geq \triangle(G^{\text{free}})$. Applying the induction hypothesis to G_0, we get

$$x(G) - 3 \geq \frac{7}{3}\left(e(G)-1\right) - \frac{25}{3}\left(v(G)-2\right) + \frac{2}{3}\triangle(G^{\text{free}}),$$

which implies the statement of the lemma.

Case 2. Every edge in G crosses at most one other edge.

Lemma 3.1 yields

$$e(G) \leq 4\left(v(G)-2\right) - \frac{1}{2}\triangle(G^{\text{free}}).$$

The statement immediately follows from this inequality, combined with the easy observation (mentioned in the Introduction) that $x(G) \geq e(G) - 3\left(v(G)-2\right)$.

Case 3. There exists an edge e of G that crosses two other edges, one of which does not cross any other edge of G.

Let zw be an edge crossing e at point x, which does not participate in any other crossing. Let u denote the endpoint of e for which the piece of e between x and u is crossing-free. Notice that u can be connected in G by crossing-free Jordan arcs to both z and w. Therefore, by the maximality of G^{free}, the edges uz and uw must belong to G^{free}. Let G_0 be the subgraph of G obtained by removing the edge e. We have $e(G_0) = e(G) - 1$ and $x(G_0) = x(G) - 2$. Clearly, G_0^{free} contains zw and all edges in G^{free}. By the 3-connectivity of G, the triangle uzw must be a triangular face of G_0^{free}, so that we have $\triangle(G_0^{\text{free}}) \geq \triangle(G^{\text{free}}) + 1$. Applying the induction hypothesis to G_0, we obtain

$$x(G) \geq \frac{7}{3}e(G) - \frac{25}{3}\left(v(G)-2\right) + \frac{2}{3}\triangle(G^{\text{free}}) + \frac{1}{3},$$

which is better than what we need.

Case 4. There exists an edge a of G that crosses precisely two other edges, b and c, and each of these edges also participates in precisely two crossings.

Subcase 4.1. b and c do not cross each other.

Let G_0 be the subgraph of G obtained by removing b. Clearly, we have $e(G_0) = e(G) - 1$, $x(G_0) = x(G) - 2$, and $\triangle(G_0^{\text{free}}) \geq \triangle(G^{\text{free}})$. Notice that c is an edge of G_0 that crosses two other edges; one of them is a, which is crossed by no other edge of G_0. Thus, we can apply to G_0 the last inequality in the analysis of Case 3 to conclude that

$$x(G) - 2 \geq \frac{7}{3}\left(e(G)-1\right) - \frac{25}{3}\left(v(G)-2\right) + \frac{2}{3}\triangle(G^{\text{free}}) + \frac{1}{3},$$

which is precisely what we need.

Subcase 4.2. b and c cross each other.

The three crossing edges, a, b, and c can be drawn on the sphere in two topologically different ways (see Figure 3). One of these possibilities is ruled out by the assumption that G is 3-connected, so the only possible configuration is the rightmost one in Figure 3. By the maximality condition, G^{free} must contain the six dashed edges in the figure. Using again the assumption that G is 3-connected, it follows that these six edges form a hexagonal face Φ in G^{free}, and the only edges of G inside this face are a, b, and c. Let G_0 be the graph obtained from G by removing the edges a, b, c, and inserting a new vertex in the interior of Φ, which is connected to every vertex of Φ by crossing-free edges. We have $v(G_0) = v(G) + 1$ and $x(G_0) = x(G) - 3$, so that we can apply the induction hypothesis to G_0. Obviously, we have $e(G_0) = e(G) + 3$ and $\triangle(G_0^{\text{free}}) = \triangle(G^{\text{free}}) + 6$. Thus, we obtain

$$x(G) - 3$$
$$\geq \frac{7}{3}\left(e(G)+3\right) - \frac{25}{3}\left(v(G)-1\right) + \frac{2}{3}\left(\triangle(G^{\text{free}}) + 6\right),$$

which is much stronger than the inequality in the lemma. $\square$

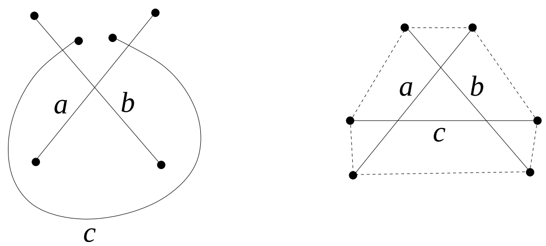

Figure 3: Proof of Lemma 3.2: Subcase 4.2.

The tightness of Theorem 2 is discussed at the end of the last section.

4 Proof of Theorem 3

Our proof is based on the following consequence of Theorem 1.

Corollary 4.1. *The crossing number of any graph G of at least 3 vertices satisfies*

$$\mathrm{cr}(G) \geq 4e(G) - \frac{103}{6}\left(v(G) - 2\right).$$

Proof: If G has at most $5\left(v(G)-2\right)$ edges, then the statement directly follows from Theorem 2. If G has more than $5\left(v(G)-2\right)$ edges, fix one of its drawings in which the number of crossings is minimum. Delete the edges of G one by one until we obtain a graph G_0 with $5\left(v(G)-2\right)$ edges. At each stage, delete one of the edges that participates in the largest number of crossings in the current drawing. Using the inequality $e_2(v) \leq 5(v-2)$ proved in [PTo97] and quoted in Section 1, at the time of its removal every edge has at least *three* crossings. Moreover, by Theorem 1, with the possible exception of the at most $\frac{1}{2}\left(v(G)-2\right)$ edges deleted last, every edge has at least *four* crossings. Thus, the total number of deleted crossings is at least

$$4\left(e(G) - 5\left(v(G)-2\right)\right) - \frac{1}{2}\left(v(G)-2\right)$$

$$= 4e(G) - \frac{41}{2}\left(v(G)-2\right).$$

On the other hand, applying Theorem 2 to G_0, we obtain that the number of crossings *not* removed during the algorithm is at least

$$\mathrm{cr}(G_0) \geq \frac{10}{3}\left(v(G)-2\right).$$

Summing up these two estimates, the result follows. □

Now we can easily complete the proof of Theorem 3. Let G be a graph drawn in the plane with $\mathrm{cr}(G)$ crossings, and suppose that $e(G) \geq \frac{103}{16}v(G)$.

Construct a *random* subgraph $G' \subseteq G$ by selecting each vertex of G independently with probability

$$p = \frac{103}{16}\frac{v(G)}{e(G)} \leq 1,$$

and letting G' be the subgraph of G induced by the selected vertices. The expected number of vertices of G' is $E[v(G')] = pv(G)$. Similarly, $E[e(G')] = p^2 e(G)$. The expected number of crossings in the drawing of G' inherited from G is $p^4\mathrm{cr}(G)$, and the expected value of the crossing number of G' is even smaller.

By Corollary 4.1, $\mathrm{cr}(G') \geq 5e(G') - \frac{103}{6}v(G')$ holds for every G'. (Note that after getting rid of the constant term in Corollary 4.1, we do not have to assume any more that $v(G') \geq 3$; the above inequality is true for every G'.) Taking expectations, we obtain

$$p^4\mathrm{cr}(G) \geq E[\mathrm{cr}(G')] \geq 4E[e(G')] - \frac{103}{6}E[v(G')]$$

$$= 4p^2e(G) - \frac{103}{6}pv(G).$$

This implies that

$$\mathrm{cr}(G) \geq \frac{1024}{31827}\frac{e^3(G)}{v^2(G)} \geq \frac{1}{31.1}\frac{e^3(G)}{v^2(G)},$$

provided that $e(G) \geq \frac{103}{6}v(G)$.

To obtain an unconditional lower bound on the crossing number of any graph G, we need different estimates when $e(G) < \frac{103}{6}v(G)$. Comparing the bounds in Theorem 2 and in Corollary 4.1 with the trivial estimates $\mathrm{cr}(G) \geq 0$ and $\mathrm{cr}(G) \geq e - 3(v(G)-2)$, a case analysis shows that

$$\frac{1024}{31827}\frac{e^3(G)}{v^2(G)} - \mathrm{cr}(G) \leq 1.06v(G).$$

The maximum is attained for a graph G with $e(G) = 4(v(G) - 2)$ and $\mathrm{cr}(G) = v(G) - 2$. In conclusion,

$$\mathrm{cr}(G) \geq \frac{1024}{31827} \frac{e^3(G)}{v^2(G)} - 1.06v(G)$$

$$\geq \frac{1}{31.1} e^3(G) v^2(G) - 1.06v(G)$$

holds for every graph G. This completes the proof of Theorem 3.

Remark 4.2. Pach and Tóth [PTo00] introduced two variants of the crossing number. The *pairwise crossing number* (resp. the *odd crossing number*) of G is defined as the minimum number of pairs of non-adjacent edges that cross (resp. cross an odd number of times) over all drawings of G. These parameters are at most as large as $\mathrm{cr}(G)$, but one cannot rule out the possibility that they are always equal to $\mathrm{cr}(G)$. The original proofs of the Crossing Lemma readily generalize to the new crossing numbers, and it follows that both of them are at least $\frac{1}{64}\frac{e^3(G)}{v^2(G)}$, provided that $e(G) \geq 4v(G)$. We have been unable to extend our proof of Theorem 3 to these parameters.

5 Applications, open problems, remarks

Every improvement of the Crossing Lemma automatically leads to improved bounds in all of its applications. For completeness and future reference, we include some immediate corollaries of Theorem 3 with a sketch of computations.

First, we plug Theorem 3 into Székely's method [Sz95] to improve the coefficient of the main term in the Szemerédi-Trotter theorem [SzT83], [CE90], [PTo97].

Corollary 5.1. *Given m points and n lines in the Euclidean plane, the number of incidences between them is at most* $2.5m^{2/3}n^{2/3} + m + n$.

Proof: We can assume that every line and every point is involved in at least one incidence, and that $n \geq m$, by duality. Since the statement is true for $m = 1$, we have to check it only for $m \geq 2$.

Define a graph G drawn in the plane such that the vertex set of G is the given set of m points, and join two points with an edge drawn as a straight-line segment if the two points are consecutive along one of the lines. Let I denote the total number of incidences between the given m points and n lines. Then $v(G) = m$ and $e(G) = I - n$. Since every edge belongs to one of the n lines, $\mathrm{cr}(G) \leq \binom{n}{2}$. Applying Theorem 2 to G, we obtain that $\frac{1}{31.1}\frac{(I-n)^3}{m^2} - 1.06m \leq \mathrm{cr}(G) \leq \binom{n}{2}$. Using that $n \geq m \geq 2$, easy calculation shows that

$$I - n \leq \sqrt[3]{15.55m^2n^2 + 33m^3} \leq \sqrt[3]{15.55}n^{2/3}m^{2/3} + m,$$

which implies the statement. □

It was shown in [PTo97] that Corollary 5.1 does not remain true if we replace the constant 2.5 by 0.42 .

Theorem 3 readily generalizes to multigraphs with bounded edge multiplicity, improving the constant in Székely's result [Sz95].

Corollary 5.2. *Let G be a multigraph with maximum edge multiplicity m. Then*

$$\mathrm{cr}(G) \geq \frac{1}{31.1}\frac{e^3(G)}{mv^2(G)} - 1.06m^2v(G).$$

Proof: Define a random simple subgraph G' of G as follows. For each pair of vertices v_1, v_2 of G, let $e_1, e_2, \dots e_k$ be the edges connecting them. With probability $1 - k/m$, G' will not contain any edge between v_1 and v_2. With probability k/m, G' contains precisely one such edge, and the probability that this edge is e_i is $1/m$ $(1 \leq i \leq k)$. Applying Theorem 3 to G' and taking expectations, the result follows. □

Next, we state here the improvement of another result in [PTo97].

Corollary 5.3. *Let G be a graph drawn in the plane so that every edge is crossed by at most k others, for some $k \geq 1$, and every pair of edges have at most one point in common. Then*

$$e(G) \leq 3.95\sqrt{k}v(G).$$

Proof: For $k \leq 2$, the result is weaker than the bounds given in [PTo97]. Assume that $k \geq 3$, and consider a drawing of G such that every edge crosses at most k

others. Let x denote the number of crossings in this drawing. If $e(G) < \frac{103}{16}v(G)$, then there is nothing to prove. If $e(G) \geq \frac{103}{16}v(G)$, then using Theorem 3, we obtain

$$\frac{1024}{31827}\frac{e^3(G)}{v^2(G)} \leq \text{cr}(G) \leq x \leq \frac{e(G)k}{2},$$

and the result follows. □

Recall that $e_k(v)$ was defined as the maximum number of edges that a graph of v vertices can have if it can be drawn in the plane with at most k crossings per edge. We define some other closely related functions. Let $e_k^*(v)$ denote the maximum number of edges of a graph of v vertices which has a drawing that satisfies the above requirement and, in addition, every pair of its edges meet at most once (either at an endpoint or at a proper crossing). We define $\overline{e}_k(v)$ and $\overline{e}_k^*(v)$ analogously, with the only difference that now the maximums are taken over all *triangle-free* graphs with v vertices.

It was mentioned in the Introduction (see Lemma 1.1) that $e_k(v) = e_k^*(v)$ for $0 \leq k \leq 3$, and that $e_k^*(v) \leq (k+3)(v-2)$ for $0 \leq k \leq 4$ [PTo97]. For $0 \leq k \leq 2$, the last inequality is tight for infinitely many values of v. Our Theorem 1 shows that this is not the case for $k = 3$.

Conjecture 5.4. *We have $e_k(v) = e_k^*(v)$ for every k and v.*

Using the proof technique of Theorem 1, it is not hard to improve the bound $e_4^*(v) \leq 7(v-2)$. In particular, in this case Lemma 2.2 holds with $3(|\Phi|-2)$ replaced by $4(|\Phi|-2)$. Moreover, an easy case analysis shows that every triangular face Φ with four half-edges satisfies at least one of the following two conditions:

1. The extension of at least one of the half-edges in Φ either ends in a triangular face with fewer than four half-edges, or enters a big face.

2. Φ is adjacent to an empty triangle.

Based on this observation, one can modify the arguments in Section 2 to obtain the upper bound $e_4^*(v) \leq (7 - \frac{1}{9})v - O(1)$.

Conjecture 5.5. $e_4^*(v) \leq 6v - O(1)$.

As for the other two functions, we have $\overline{e}_k(v) = \overline{e}_k^*(v)$ for $0 \leq k \leq 3$, and $\overline{e}_k^*(v) \leq (k+2)(v-2)$ for $0 \leq k \leq 2$. If $0 \leq k \leq 1$, these bounds are attained for infinitely many values of v. These estimates were applied by Czabarka et al. [CS03] to obtain some lower bounds on the so-called *biplanar crossing number* of complete graphs.

Given a triangle-free graph drawn in the plane so that every edge crosses at most 2 others, an easy case analysis shows that each quadrilateral face that contains four half-edges is adjacent to a face which is either non-quadrilateral or does not have four half-edges[1]. As in the proof of Theorem 1 (before Lemma 2.5), we can use a properly defined bipartite multigraph M to establish the bound

$$\overline{e}_2(v) \leq \left(4 - \frac{1}{10}\right)v - O(1).$$

Conjecture 5.6. $\overline{e}_2(v) \leq 3.5v - O(1)$.

The coefficient 3.5 in the above conjecture cannot be improved as shown by the triangle-free (actually bipartite!) graph in Figure 4, whose vertex set is the set of vertices of a $4 \times v/4$ grid.

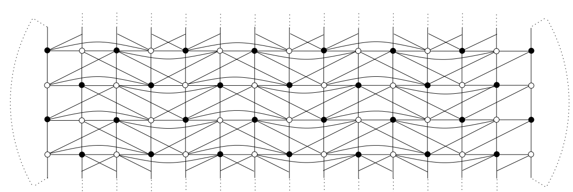

Figure 4: $\overline{e}_2(v) \geq 3.5v - 16$.

Let $\text{cr}(v, e)$ denote the minimal crossing number of a graph with $v \geq 3$ vertices and e edges. Clearly, we have $\text{cr}(v, e) = 0$, whenever $e \leq 3(v-2)$, and $\text{cr}(v, e) = e - 3(v-2)$ for $3(v-2) \leq e \leq 4(v-2)$. To see that these values are indeed attained by the function, consider the graph constructed in [PTo97], which (if v is a multiple of 4) can be obtained from a planar graph with v vertices,

[1]This statement actually holds under the assumption that G and G' are maximal, in the sense described at the beginning of Section 2.

$2(v-2)$ edges, and $v-2$ quadrilateral faces, by adding the diagonals of the faces. If $e < 4(v-2)$, delete as many crossing-free edges as necessary.

In the next interval, i.e., when $4(v-2) \le e \le 5(v-2)$, Theorem 2 gives tight bound on $\mathrm{cr}(v,e)$ up to an additive constant. To see this, consider a planar graph with only pentagonal and quadrilateral faces and add all diagonals in every face. If no two faces of the original planar graph shared more than a vertex or an edge, for the resulting graph the (first) inequality of Theorem 2 holds with equality. For certain values of v and e, no such construction exists, but we only lose a constant.

If $5(v-2) \le e \le 5.5(v-2)$, the best known bound, $\mathrm{cr}(v,e) \ge 3e - \frac{35}{3}(v-2)$, follows from Theorem 2, while for $e \ge 5.5(v-2)$ the best known bound is either the one in Corollary 4.1 or the one in Theorem 3. We do not believe that any of these bounds are optimal.

Conjecture 5.7 $\mathrm{cr}(v,e) \ge \frac{25}{6}e - \frac{35}{2}(v-2)$.

Note that, if true, this bound is tight up to an additive constant for $5(v-2) \le e \le 6(v-2)$. To see this, consider a planar graph with only pentagonal and hexagonal faces and add all diagonals of all faces. If no two faces of the planar graph shared more than a vertex or an edge, the resulting graph shows that Conjecture 5.7 cannot be improved. As a first step toward settling this conjecture, we can show the following statement, similar to Lemma 3.1.

Lemma 5.8 *Let G be a graph on $v(G) \ge 3$ vertices drawn in the plane so that every edge is involved in at most two crossings. Then*

$$e(G) \le 5(v(G)-2) - \triangle(G^{\text{free}}).$$

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