A Model Structure à la Thomason on **2-Cat**

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February 1, 2008

Abstract

We exhibit a model structure on **2-Cat**, obtained by transfer from **sSet** across the adjunction $C_2 \circ Sd^2 \dashv Ex^2 \circ N_2$.

1 Introduction

There are two well-known model category structures on the category **Cat** of small categories: the "folklore" structure, the existence of which was intuited for many years before it was finally established rigorously by Joyal and Tierney in 1991 [1], and the "topological" structure, developed by Thomason in 1980 [17] and recently corrected by Cisinski [4, 5]. In the "folklore" structure, weak equivalences are equivalences of categories, corresponding to a purely category-theoretic view of the role of categories. On the other hand, the "topological" structure is defined so that the functor $Ex^2 \circ N : \mathbf{Cat} \longrightarrow \mathbf{sSset}$ induces an equivalence of homotopy categories, where Ex is the right adjoint to the subdivision functor Sd, N is the nerve functor and **sSet** is the category of simplicial sets. In particular, a functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ between small categories is a weak equivalence if and only if $NF : N\mathbb{A} \longrightarrow N\mathbb{B}$ is a weak equivalence of simplicial sets.

Our goal in this article is to establish the existence of a Thomason-type, "topological" model category structure on **2-Cat**, the category of small 2-categories, complementing Lack's recent proof of the existence of a "folklore" structure on **Cat** [14]. More precisely, we prove that there is a model category structure on **2-Cat** such that $Ex^2 \circ N_2 : \mathbf{2\text{-}Cat} \longrightarrow \mathbf{sSet}$ induces an equivalence of homotopy categories, where N_2 denotes the 2-nerve functor. Our methods are analogous to those of Thomason and Cisinski, though the generalization to 2-categories is highly nontrivial.

We begin this article with a thorough primer on 2-category theory in section 2 . In particular we provide a careful review of the construction of limits and colimits in **2-Cat**, as well as of the definition of N_2 and its left adjoint, the 2-categorification functor C_2. We then recall the necessary elements of model category theory in section 3, including a very useful "Creation Proposition", giving conditions under which model category structure can be transfered across a pair of adjoint functors.

In section 4 we prove the existence of the Thomason-type model category structure on **2-Cat**. We first introduce the notion of right and left ideals of 2-categories, which we use then in the crucial definitions of a *distortion* between 2-functors and of a *skew immersion* of 2-categories. A distortion from a 2-functor F to a 2-functor G is a sort of left homotopy from F to G, which, in fact, induces a simplicial homotopy from N_2F to N_2G. On the other hand, a skew immersion is an inclusion of a left ideal $\mathcal{A} \hookrightarrow \mathcal{B}$ such that $\mathcal{A}$ is a sort of "strong deformation retract" (notion defined using distortions) of a right ideal $\mathcal{W}$ of $\mathcal{B}$, implying that $N_2\mathcal{A}$ truly is a strong deformation retract $N_2\mathcal{W}$ in the usual sense. The most important example of a skew immersion for our purposes is $(C_2 \circ Sd^2)(i_{k,n})$, where $i_{k,n} : \Lambda^k[n] \longrightarrow \Delta[n]$ is a horn inclusion. We establish furthermore that skew immersions are stable under pushout and that the image under N_2 of a pushout of a skew immersion along an arbitrary 2-functor is a weak pushout. Given these reults, we can finally apply the "Creation Proposition" to to show that $Ex^2 \circ N_2$ creates the desired model category structure on **2-Cat**.

In the final section of the paper, we show that Bénabou's "2-category of cylinders" gives a natural path object construction in **2-Cat**. The desire to establish this result motivated the research in this article, as it has an intriguing application in concurrency theory [13].

Given a new and interesting model category structure, it is natural to ask what properties the structure satisfies and how well we can characterize fibrations and cofibrations, as well as fibrant and cofibrant objects. It turns out that the Thomason-type structure on **2-Cat** is both cellular and proper, as we will establish in a future article. The proof of properness depends on the observation that all cofibrations in **2-Cat** are retracts of skew immersions, since all cofibrations are retracts of elements of $\mathcal{I} - cell$, where $\mathcal{I} = \{C_2 \circ Sd^2(\partial\Delta[n]) \hookrightarrow C_2 \circ Sd^2(\Delta[n]) \mid n \geq 0\}$, all elements of which are skew immersions.

2 2-Categories and 2-Nerves

2.1 2-Cat

2.1.1 2-Graphs

Definition 2.1.1 *Let $\mathbb{A}$ be a category. A preglobular object A in $\mathbb{A}$ is a $\mathbb{N}$-indexed sequence*

$$\cdots \quad A_i \underset{cod_{i-1}}{\overset{dom_{i-1}}{\rightrightarrows}} A_{i-1} \quad \cdots$$

of objects and morphisms subject to the identities

$$\begin{array}{rcl} dom_i \circ dom_{i+1} & = & dom_i \circ cod_{i+1} \\ cod_i \circ dom_{i+1} & = & cod_i \circ cod_{i+1} \end{array}$$

A is n-truncated if $i < n$. An n-graph is a n-truncated preglobular set.

Remark 2.1.1 Since an n-graph is just a presheaf, **n-Grph** is a topos for each $n \in \mathbb{N}$. In particular, **n-Grph** is complete and cocomplete. ★

Definition 2.1.2 *(i) A graph is a 1-graph with $dom \overset{def.}{=} dom_0$ and $cod \overset{def.}{=} cod_0$. Let H be a graph and $a, b \in H_0$, then*

$$H\,(a,b) \overset{def.}{=} \{u \in H_1 \;\mid\; dom\,(u) = a \;\wedge\; cod\,(u) = b\}$$

(ii) let G be a 2-graph. As in the case of graphs, the elements of G_0 are called vertices or 0-objects and those of G_1 arrows, edges or 1-morphisms. The elements of G_2 are called 2-cells or 2-morphisms. G's underlying graph $\lfloor G \rfloor$ is given by its 1-truncation $G_1 \rightrightarrows G_0$;

(iii) given $x, y \in G_0$, $G(x,y)$ *is the graph with*

$$\begin{aligned} G(x,y)_0 &\overset{def.}{=} \{f \in G_1 \mid dom_0(f) = x \wedge cod_0(f) = y\} \\ G(x,y)_1 &\overset{def.}{=} \{\alpha \in G_2 \mid dom_1(\alpha),\ cod_1(\alpha) \in G(x,y)_0\} \end{aligned}$$

and with $dom_{x,y}$, $cod_{x,y} : G(x,y)_1 \to G(x,y)_1$ *given by*

$$\begin{aligned} dom_{x,y}(\alpha) &\overset{def.}{=} dom_1(\alpha) \\ cod_{x,y}(\alpha) &\overset{def.}{=} cod_1(\alpha) \end{aligned}$$

Properties and concepts defined with respect to $G(x,y)$ (or its more structured counterparts to be introduced below) are called *local*. For instance, a morphism of graphs $h : G \to H$ is *locally injective if* $h_1 \mid_{G(x,y)}$ is an injective function for each $x, y \in G_0$.

2.1.2 Derivation Schemes and Sesquicategories

Definition 2.1.3 *A derivation scheme is a 2-graph* D *such that the underlying graph* $\lfloor D \rfloor$ *is a category. The composition in* $\lfloor D \rfloor$ *is denoted* $\circ$ *and written infix in the evaluation order. Morphisms of derivation schemes are morphisms of 2-graphs that are functors on the underlying categories.*

Proposition 2.1.1 *Derivation schemes and their morphisms form the category* **Der**. *There is an adjunction*

$$\mathbf{Der} \underset{U_{\mathbf{Der}}}{\overset{F_{\mathbf{Der}}}{\rightleftarrows}} \mathbf{2\text{-}Grph} \quad (\bot)$$

Proof. Let G be a 2-graph. The free derivation scheme $F_{\mathrm{der}}(G)$ is given by

$$\lfloor F_{\mathrm{der}}(G) \rfloor = \mathcal{F}(\lfloor G \rfloor)$$

where $\mathcal{F}(\lfloor G \rfloor)$ is the free category on $\lfloor G \rfloor$. □

Let $x, y \in G_0$. A situation involving an $\alpha \in G(x,y)_1$ such that $\mathrm{dom}(\alpha) = f$ and $\mathrm{cod}(\alpha) = g$ is customarily drawn as

$$x \underset{g}{\overset{f}{\rightrightarrows}} y \quad (\Downarrow \alpha)$$

Definition 2.1.4 *A sesquicategory* $\mathbb{S}$ *is a derivation scheme such that* $\mathbb{S}(x,y)$ *is a category for all* $x, y \in \mathbb{S}_0$. *The composition in* $\mathbb{S}(x,y)$ *is denoted* $\bullet$ *and is written infix in the evaluation order. For each* $x', x, y \in \mathbb{S}_0$ *there is an operation*

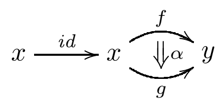
$$W_{left} : \mathbb{S}(x',x)_0 \times \mathbb{S}(x,y)_1 \to \mathbb{S}(x',y)_1$$

and for each $x, y, y' \in \mathbb{S}_0$ *there is an operation*

$$W_{right} : \mathbb{S}(x,y)_1 \times \mathbb{S}(y,y')_0 \to \mathbb{S}(x,y')_1$$

Both operations are called whiskering and are denoted $\circ$ *by abuse of notation.* W_{left} *is subject to the identities*

1. *given*

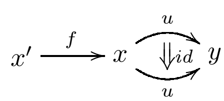


the equation

$$\alpha \circ id_x = \alpha$$

holds;

2. *given*

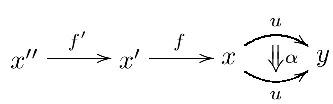


the equation

$$id_u \circ f = id_{u \circ f}$$

holds;

3. *given*

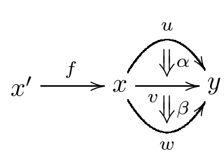


the equation

$$\alpha \circ (f \circ f') = (\alpha \circ f) \circ f'$$

holds;

4. *given*



the equation

$$(\beta \bullet \alpha) \circ f = (\beta \circ f) \bullet (\alpha \circ f)$$

holds;

5. *the rules governing* W_{right} *are defined symmetrically;*

6. given

$$x \xrightarrow{f} x \overset{f}{\underset{g}{\Downarrow \alpha}} y \xrightarrow{g} z$$

the equation $g \circ (\alpha \circ f) = (g \circ \alpha) \circ f$ holds.

Morphisms of sesquicategories, called sesqifunctors, are morphisms of the underlying derivation schemes which are locally functors and which preserve whiskering.

The equations of a sesquicategory guarantee in particular that there is no harm to write the 2-cells as strings like

$$g_m \circ \cdots \circ g_1 \circ \alpha \circ f_n \cdots f_1$$

Proposition 2.1.2 *Sesquicategories and sesquifunctors form in the category* **Sesqu** . *There is an adjunction*

$$\mathbf{Sesqu} \underset{U_{\mathbf{Sesqu}}}{\overset{F_{\mathbf{Sesqu}}}{\leftrightarrows}} \mathbf{Der} \quad (\bot)$$

A free sesquicategory $\mathcal{F}\mathbb{D}$ over a derivation scheme $\mathbb{D}$ is given by formally adding all the whiskering composites and all the vertical composites.

Definition 2.1.5 *Let $\mathbb{S}$ be a sesquicategory. A sesquicongruence on $\mathbb{S}$ is a family*

$$\left\{\sim^1_{X,Y} \subseteq \mathbb{A}\left(X,Y\right) \times \mathbb{A}\left(X,Y\right)\right\}_{X,Y \in \mathbb{A}_0}$$

of equivalence relations on morphisms and a family

$$\left\{\sim^2_{f,g} \subseteq \mathbb{A}\left(X,Y\right)\left(f,g\right) \times \mathbb{A}\left(X,Y\right)\left(f,g\right)\right\}_{\substack{X,Y \in \mathbb{S}_0 \\ f,g \in \mathbb{S}(X,Y)}}$$

of equivalence relations on 2-cells such that

(i) $\alpha \sim^2 \beta \Longrightarrow \theta \bullet \alpha \bullet \varphi \sim^2 \theta \bullet \alpha \bullet \varphi$ *and* $g \circ \alpha \circ f \sim^2 g \circ \beta \circ f$

(ii) $f \sim^1 g \Longrightarrow \phi \circ f \circ \psi \sim^2 \phi \circ g \circ \psi$

(iii) $id_f \sim^2 id_g \Longrightarrow f \sim^1 g$

Remark 2.1.2 In particular, $\sim^1$ is a congruence on $\lfloor \mathbb{S} \rfloor$. ★

Proposition 2.1.3 *An arbitrary intersection of sesquicongruences is again a sesquicongruence. The quotient $\mathbb{S}/\sim$ of a sesquicategory $\mathbb{S}$ by a sesquicongruence $\sim$ is again a sesquicategory.*

2.1.3 2-Categories

Definition 2.1.6 *Let $\mathbb{S}$ be a sesquicategory and $x, y, z \in \mathbb{S}_0$. The latter satisfy the interchange law if any diagram of the form*

$$x \underset{g}{\overset{f}{\rightrightarrows}} y \underset{g'}{\overset{f'}{\rightrightarrows}} z \qquad (\Downarrow\alpha,\ \Downarrow\alpha')$$

verifies the equation

$$(g' \circ \alpha) \bullet (\alpha' \circ f) = (f' \circ \alpha) \bullet (\alpha' \circ g) \qquad (*)$$

A 2-category is a sesquicategory in which the interchange law holds for every triple of objects. A 2-functor is a sesquifunctor between 2-categories. 2-categories and 2-functors form the category **2-Cat**.

Remark 2.1.3 The quotient of a 2-category by a sesquicongruence is again a 2-category. ★

Proposition 2.1.4 (Gray [11]) *The functor $\lfloor \_ \rfloor$:* **2-Cat** $\longrightarrow$ **Cat** *which forgets the 2-cells has a right adjoint.*

Proof. The right adjoint turns a homset into a trivial connected groupoid. □

The interchange law is often called by the name of R.Godement [10]. A 2-category $\mathcal{A}$ admits in particular a "horizontal" composition of 2-cells where $\alpha' \circ \alpha$ is given by either side of $(*)$, giving rise to a family of functors

$$\_\circ\_ :\ \mathcal{A}(y, z) \times \mathcal{A}(x, y) \to \mathcal{A}(x, z)$$

indexed by triples $x, y, z \in \mathcal{A}_0$. This is the way 2-categories are usually introduced in the literature (c.f. [3]), while the exposition above is drawn from [16].

Proposition 2.1.5 *There is an adjunction*

$$\mathbf{2\text{-}Cat} \underset{U_{\mathbf{2\text{-}Cat}}}{\overset{F_{\mathbf{2\text{-}Cat}}}{\rightleftarrows}} \mathbf{Sesqu} \qquad (\bot)$$

It is easy to see that constructing the free 2-category on a sesquicategory amounts to quotienting the latter by the sesquicongruence generated by the equations enforcing the Godement law for all triples of objects. We thus have the series of adjunctions

$$\mathbf{2\text{-}Cat} \underset{U_{\mathbf{2\text{-}Cat}}}{\overset{F_{\mathbf{2\text{-}Cat}}}{\rightleftarrows}} \mathbf{Sesqu} \underset{U_{\mathbf{Sesqu}}}{\overset{F_{\mathbf{Sesqu}}}{\rightleftarrows}} \mathbf{Der} \underset{U_{\mathbf{Der}}}{\overset{F_{\mathbf{Der}}}{\rightleftarrows}} \mathbf{2\text{-}Grph} \qquad (\bot)$$

Definition 2.1.7 *Let G be a 2-graph and*

$$\mathcal{F} \overset{def.}{=} F_{\mathbf{2\text{-}Cat}} \circ F_{\mathbf{Sesqu}} \circ F_{\mathbf{Der}}$$

The free 2-category $\mathcal{F}(G)$ on G is given by this functor.

A free 2-category on a 2-graph (or a derivation scheme) involves thus "horizontal" sequences in dimension 1 and 2 as well as "vertical" sequences in dimension 2. We write

$$< f_1; \cdots ; f_n >$$

for a 1-dimensional horizontal sequence of morphisms,

$$\ll A_1; \cdots ; A_n \gg$$

for a horizontal sequence of morphisms and/or 2-cells and

$$\ll \alpha_1 : \cdots : \alpha_m \gg$$

for a vertical sequence of 2-cells. We define the concatenation operations

$$\ll A_1; \cdots ; A_k \gg; \ll A_{k+1}; \cdots ; A_n \gg = \ll A_1; \cdots ; A_n \gg$$

and

$$\ll \alpha_1 : \cdots : \alpha_l \gg : \ll \alpha_{l+1} : \cdots : \alpha_m \gg = \ll \alpha_1 : \cdots : \alpha_m \gg$$

at any index. Those are obviously associative and can be mixed whenever it makes sense, e.g.

$$\ll \alpha : \alpha' \gg; \ll \beta : \beta' \gg = \ll \alpha; \beta \gg : \ll \alpha'; \beta' \gg$$

is an instance of the interchange law. Domains and codomains are usually clear from context. If not, we indicate them as subscripts. In case of endomorphisms or endo-2-cells we do not duplicate those subscripts, e.g

$$<>_X$$

is the empty sequence with domain and codomain X, i.e. the 1-dimensional identity at X. Similarly,

$$\ll\gg_f$$

is the 2-dimensional identity at f.

2.2 Limits and Colimits in 2-Cat

Proposition 2.2.1 **2-Cat** *is complete and cocomplete.*

Proof. Limits are obvious. Let $D : \mathbb{I} \longrightarrow$ **2-Cat** be a diagram. There is the colimiting cocone

$$\{\iota_K : (\mathcal{U} \circ D) \longrightarrow \mathrm{colim}(\mathcal{U} \circ D)\}_{K \in \mathbb{I}}$$

in **2-Grph**. Consider the 2-category

$$\mathcal{F}\left(\mathrm{colim}(\mathcal{U} \circ D)\right) / \sim$$

where $\sim$ is the sesquicongruence generated by

(i) $\ll \iota_K(\alpha) : \iota_K(\beta) \gg = \ll \iota_K(\beta \bullet \alpha) \gg$

(ii) $\ll \iota_K(\mathrm{id}_f) \gg = \ll\gg_f$

(iii) $\ll \iota_K(f); \iota_K(\alpha); \iota_K(g) \gg = \ll \iota_K(g \circ \alpha \circ f) \gg$

(iv) $< \iota_K(\mathrm{id}_X) > = <>_X$

for all $K \in \mathbb{I}$. There is the cocone

$$\{\kappa_K : D(K) \longrightarrow \mathcal{F}(\mathrm{colim}(\mathcal{U} \circ D)) / \sim\}_{K \in \mathbb{I}}$$

in **2-Cat**, given by

$$(\kappa_K)_2(\alpha) \stackrel{def.}{=} \ll \iota_K(\alpha) \gg$$

and

$$(\kappa_K)_1(f) \stackrel{def.}{=} < \iota_K(f) >$$

This cocone is colimiting. To see this, suppose there is a cocone

$$\{c_K : D(K) \longrightarrow \mathcal{C}\}_{K \in \mathbb{I}}$$

over D. Then there is the comparison morphism

$$m : \mathrm{colim}(\mathcal{U} \circ D) \longrightarrow \mathcal{U}(\mathcal{C})$$

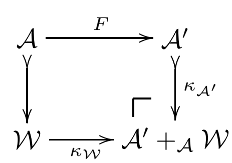
in **2-Grph**. Its transpose

$$\bar{m} : \mathcal{F}(\mathrm{colim}(\mathcal{U} \circ D)) \longrightarrow \mathcal{C}$$

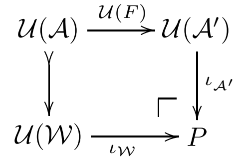
over the adjunction $\mathcal{F} \dashv \mathcal{U}$ remains defined after the passage to the quotient and is the desired comparison morphism. □

Our proof above, one of the manifold possible variants, generalizes Gabriel's and Zisman's construction of colimits in **Cat** (c.f. [9]). It is easy to see that our construction amounts to doing first the construction on the underlying category as in [9] and then to taking care of the 2-cells. It *has* to be that way because of proposition 2.1.4.

Remark 2.2.1 The calculatory recipe given in the proof of proposition 2.2.1 is quite practical indeed. Consider for instance the case of pushing inclusions out:



Then there is the pushout square



in **2-Grph**, where

$$P \cong \left(\mathcal{A}'_i + (\mathcal{W}_i \backslash \mathcal{A}_i)\right)_{0 \leqslant i \leqslant 2}$$

with structural maps given by universal property as the copairs

$$\partial^i_P = \left[\mathrm{in}_{\mathcal{A}'_i} \circ \partial^i_{\mathcal{A}'}, \left(F_i + \mathrm{id}_{\mathcal{W}_i \backslash \mathcal{A}_i}\right) \circ \partial^i_{\mathcal{W}}|_{\mathcal{W}_{i+1} \backslash \mathcal{A}_{i+1}}\right]$$

for $i \in \{0, 1\}$ and $\partial \in \{\mathrm{dom}, \mathrm{cod}\}$. On the other hand

$$\iota_{\mathcal{A}'} = \left(\mathrm{in}_{\mathcal{A}'_2}, \mathrm{in}_{\mathcal{A}'_1}, \mathrm{in}_{\mathcal{A}'_0}\right)$$

and

$$\iota_{\mathcal{W}} = \left(F_2 + \mathrm{id}_{\mathcal{W}_2 \backslash \mathcal{A}_2}, F_1 + \mathrm{id}_{\mathcal{W}_1 \backslash \mathcal{A}_1}, F_0 + \mathrm{id}_{\mathcal{W}_0 \backslash \mathcal{A}_0}\right)$$

Then

$$\begin{array}{rcl} (\kappa_{\mathcal{A}'})_0 & = & \mathrm{in}_{\mathcal{A}'_0} \\ (\kappa_{\mathcal{A}'})_1 (f) & = & < \mathrm{in}_{\mathcal{A}'_1} (f) > \\ (\kappa_{\mathcal{A}'})_2 (\alpha) & = & \ll \mathrm{in}_{\mathcal{A}'_2} (\alpha) \gg \end{array}$$

determines a morphism of 2-graphs $\kappa_{\mathcal{A}'} : \mathcal{A}' \longrightarrow \mathcal{F}(P)$ while

$$\begin{array}{rcl} (\kappa_{\mathcal{W}})_0 & = & F_0 + \mathrm{id}_{\mathcal{W}_0 \backslash \mathcal{A}_0} \\ (\kappa_{\mathcal{W}})_1 (u) & = & < \left(F_1 + \mathrm{id}_{\mathcal{W}_1 \backslash \mathcal{A}_1}\right) (u) > \\ (\kappa_{\mathcal{W}})_2 (\theta) & = & < \left(F_2 + \mathrm{id}_{\mathcal{W}_2 \backslash \mathcal{A}_2}\right) (\theta) > \end{array}$$

determines a morphism of 2-graphs

$$\kappa_{\mathcal{W}} : \mathcal{W} \longrightarrow \mathcal{F}(P)$$

so

$$\mathcal{A}' +_{\mathcal{A}} \mathcal{W} \cong \mathcal{F}(P)/\sim$$

with $\sim$ the smallest sesquicongruence making $\kappa_{\mathcal{A}'}$ and $\kappa_{\mathcal{W}}$ 2-functorial.

It follows that inclusions in **2-Cat** are stable under pushout. In particular, if an inclusion is full and locally full, then pushing it out will result in a full and locally full one. ★

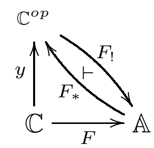
2.3 2-Nerve and 2-Categorification

2.3.1 Simplicial Sets

Lemma 2.3.1 (Kan) *Let $F : \mathbb{C} \to \mathbb{A}$ be a functor and $A \in \mathbb{A}$. The assignment*

$$A \mapsto \mathbb{A}\left(F\left(\_\right), A\right)$$

determines a functor $F_* : \mathbb{A} \to \mathbf{Set}^{\mathbb{C}^{op}}$. If $\mathbb{A}$ is cocomplete then F_* has a left adjoint $F_! = Lan_y F$ and F factors through $F_!$ by the Yoneda embedding $y : \mathbb{C} \to \mathbf{Set}^{\mathbb{C}^{op}}$:



The condition of $\mathbb{A}$ being cocomplete is stronger than the existence of the relevant Kan extension, yet it is verified in most of the cases of interest.

Definition 2.3.1 *Let* $[n] \stackrel{def.}{=} \{0 < 1 \cdots < n\}$ *be the* n*th finite ordinal and*

- $\delta^i_n : [n-1] \longrightarrow [n]$ *be the increasing injection missing* i*;*
- $\sigma^i_n : [n+1] \longrightarrow [n]$ *be the non decreasing surjection taking twice the value* i*;*

The category Δ *has finite ordinals as objects and is generated by*

$$\{\delta^i_n | n \in \mathbb{N}, 0 < n, 0 \leqslant i \leqslant n\} \cup \{\sigma^i_n | n \in \mathbb{N}, 0 \leqslant i \leqslant n\}$$

Let $\mathbb{C}$ *be a category. The category of* simplicial objects *in* $\mathbb{C}$ *is* $\mathbb{C}^{\Delta^{op}}$ *while the category of* cosimplicial objects *in* $\mathbb{C}$ *is* $\mathbb{C}^{\Delta}$*.*

As a matter of terminology, if the objects of $\mathbb{C}$ are called “gadgets” then (co)simplicial objects in $\mathbb{C}$ are called “(co)simplicial gadgets”, e.g. simplicial sets, simplicial groups, simplicial 2-categories and so on. It is customary to write **sSet** for the category of simplicial sets and, given $K \in$ **sSet**, to abbreviate $K_n \stackrel{def.}{=} K([n])$.

Definition 2.3.2 *Let* $K \in$ **sSet**. *An element of* K_n *is called an* n-simplex. *The representable prefsheaf* $\Delta[n] \stackrel{def.}{=} \Delta(\_, [n]) \in$ **sSet** *is called the* standard n-simplex. *An* n*-simplex is a* face *if it is in the image of some* $\partial^n_i \stackrel{def.}{=} K(\delta^n_i)$*. It is* degenerate *if it is in the image of some* $\varepsilon^n_i \stackrel{def.}{=} K(\sigma^i_n)$*. A simplicial set is* n*-skeletal if the* m*-simplices are degenerate for* $m > n$*.*

Remark 2.3.1 The standard n-simplex $\Delta[n]$ is n-skeletal. It has precisely one non-degenerate n-simplex, namely $id_{[n]} \in \Delta([n],[n])$. The other degenerate m-simplices are all faces. ★

Definition 2.3.3 *(i) The subobject* $\partial\Delta[n] \rightarrowtail \Delta[n]$*, obtained from* $\Delta[n]$ *by removing* $id_{[n]}$*, is called boundary;*

(ii) Let $1 \leqslant k \leqslant n+1$*; the subobject* $\Lambda^k[n] \rightarrowtail \Delta[n]$*, obtained from* $\partial\Delta[n]$ *by removing* $\partial^n_k(id_{[n]}) = \delta^k_n$*, is called* k*th* horn*.*

Proposition 2.3.1 *Let*

$$\Delta_n \stackrel{def.}{=} \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n | \sum_{i=1}^{n} t_i = 1 \wedge \forall 1 \leqslant i \leqslant n. t_i \geqslant 0 \right\}$$

be the standard topological n*-simplex. The functor*

$$\begin{array}{rccc} g: & \Delta & \longrightarrow & \textbf{\textit{Top}} \\ & [n] & \longmapsto & \Delta_n \end{array}$$

determines an adjunction

$$g_! = |\_| \dashv Sing = g_*$$

The left adjoint gives the geometric realization of a simplicial set while the right adoint gives the singular complex of a topological space. In particular, singular homology is a special case of simplicial homology via this right adjoint.

2.3.2 Orientals

Definition 2.3.4 *Let $[n] \in \Delta$, $\delta_{i,j}$ be the inequality $i \leqslant j$ seen as a morphism in $[n]$ and $\bar{\Delta}_n$ be the derivation scheme given by the data*

1. $\left|\bar{\Delta}_n\right| \overset{def.}{=} \mathcal{F}([n])$;

2. $\left(\bar{\Delta}_n\right)_2 \overset{def.}{=} \{\delta_{i,j,k} \mid 0 \leq i < j < k \leq n\}$ *where*

$$dom_1(\delta_{i,j,k}) = < \delta_{i,j}; \delta_{j,k} >$$

and

$$cod_1(\delta_{i,j,k}) = < \delta_{i,k} >$$

The 2-category Δ_n is the free 2-category $\mathcal{F}\left(\bar{\Delta}_n\right)$ over $\bar{\Delta}_n$ quotiented by the relations

$$< \delta_{i,j,k}; \delta_{k,l} >: \ll \delta_{i,k,l} \gg = < \delta_{i,j}; \delta_{j,k,l} >: \ll \delta_{j,k,l} \gg$$

Following Street [15], we call the Δ_n's *2-orientals*.

Proposition 2.3.2 *The construction $\Delta_{(\_)} : \Delta \longrightarrow$ **2-Cat** is functorial and determines an adjunction*

$$C_2 \dashv N_2$$

Proof. The functoriality is immediate while $C_2 \overset{def.}{=} \Delta_{(\text{-})_!}$ and $N_2 \overset{def.}{=} \Delta_{(\text{-})_*}$. □

We call N_2 *2-nerve* and C_2 *2-categorification*.

Remark 2.3.2 Given a simplicial set K, $C_2(K)$ is the free 2-category on the derivation scheme determined by $(K_i)_{0 \leq i \leq 2}$, quotiented by the sesquicongruence generated by K_3. ★

2.4 Normal Lax Functors

Definition 2.4.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories and $F : \mathcal{A} \to \mathcal{B}$ a morphism of the underlying 2-graphs. F is a normal lax functor provided*

(i) it is locally a functor;

(ii) it preserves horizontal identites;

(iii) for any $f \in \mathcal{A}(x, y)$ and $g \in \mathcal{A}(y, z)$ there is the structural 2-cell

$$\gamma_{f,g} : F(g) \circ F(f) \Rightarrow F(g \circ f)$$

such that

(a) given any $h \in \mathcal{A}(z,a)$*, the equation*

$$\gamma_{g\circ f,h}\bullet(F(h)\circ\gamma_{f,g}) = \gamma_{f,h\circ g}\bullet(\gamma_{g,h}\circ F(f))$$

holds;

(b) given any $\alpha : f \Rightarrow f'$ *and* $\beta : g \Rightarrow g'$*, the equation*

$$\gamma_{f',g'} \bullet (F(\beta) \circ F(\alpha)) = F(\beta \circ \alpha) \bullet \gamma_{f,g}$$

holds.

Remark 2.4.1 A 2-functor is thus a special case of a normal lax functor where the structural 2-cells are all identities. ★

Remark 2.4.2 Normal lax functors compose in the obvious way and this composition is associative. The category of 2-categories and normal lax functors $\widetilde{\mathbf{2\text{-}Cat}}$ has the usual products, yet it is not finitely complete. ★

Remark 2.4.3 Let $\mathbf{NLax}([n],\mathcal{A})$be the set of normal lax functors from $[n]$ to $\mathcal{A}$. Then

$$N_2(\mathcal{A})_n = \mathbf{NLax}([n],\mathcal{A})$$

and N_2 acts on 2-functors by postcomposition. Let K be a simplicial set and let us write $S_{i_0,\dots,i_n} \in K_n$ where $i_0 < \cdots < i_n$ for an n-simplex. We use the notation

$$\partial_j(S_{i_0,\dots,i_n}) \overset{def.}{=} S_{i_0,\dots,i_{j-1},i_{j+1},\dots,i_n}$$

for the faces. The assignment

$$\delta_{p,q} \longmapsto < S_{i_p,i_q} >$$

determines a normal lax functor $S : [n] \longrightarrow C_2(K)$ with the structural 2-cells

$$\gamma_{p,q,r} = \ll S_{i_p,i_q,i_r} \gg$$

The unit $\eta_K : K \longrightarrow (N_2 \circ C_2)(K)$ of the adjunction $C_2 \dashv N_2$ is the simplicial map given in degree n by

$$S_{i_0,\dots,i_n} \longmapsto S$$

★

Remark 2.4.4 Let $\mathbb{A}$ be a category and $N_1 : \mathbf{Cat} \to \mathbf{sSet}$ be the usual categorical nerve. Let us write $[f_1,\dots,f_n]$ for a composable sequence of arrows seen as an n-simplex in the nerve. $(C_2 \circ N_1)(\mathbb{A})$ can be characterized as follows: the objects are those of $\mathbb{A}$, the arrows are generated by those of $\mathbb{A}$ (they are formal composites), while the 2-cells are generated by the collection

$$[f,g] : < f;g > \Longrightarrow < g \circ f >$$

subject to the relations

$$\ll f;[g,h] \gg : \ll [f,h\circ g] \gg = \ll [f,g];h \gg : \ll [g\circ f,h] \gg$$

In particular, $\eta_{N_1(\mathbb{A})}$ is an iso of simplicial sets for any category $\mathbb{A}$ by remark 2.4.3. ★

3 Model Category Theory

In this section, we review some classical and less classical material about model categories. Most of the section on topoi is included because of its intrinsic beauty.

3.1 Basic Facts about Model Categories

Definition 3.1.1 *Let $\mathbb{M}$ be a category. $\mathcal{L}, \mathcal{R} \subseteq \mathbb{M}_1$ form a weak factorization system $(\mathcal{L}, \mathcal{R})$ if*

1. *any morphism $f \in \mathbb{M}_1$ factors as $f = r \circ l$ with $r \in \mathcal{R}$ and $l \in \mathcal{L}$;*
2. *$\mathcal{R} = RLP(\mathcal{L})$ and $\mathcal{L} = LLP(\mathcal{R})$.*

Definition 3.1.2 *$\mathbb{M}$ is a model category if it is complete, cocomplete and has three distinguished classes of morphisms $\mathcal{C}, \mathcal{W}, \mathcal{F} \subseteq \mathbb{M}_1$ such that*

1. *$(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems;*
2. *$\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ are closed under retracts in $\mathbb{M}^{\rightarrow}$;*
3. *if two of the morphisms in a commuting triangle are in $\mathcal{W}$ so is the third one.*

It is established terminology to call morphisms in $\mathcal{F}$ *fibrations* with $\twoheadrightarrow$ as notation, those in $\mathcal{C}$ *cofibrations* with $\rightarrowtail$ as notation and those in $\mathcal{W}$ *weak equivalences* with $\xrightarrow{\sim}$as notation. It is also customary to call morphisms in $\mathcal{F} \cap \mathcal{W}$ *acyclic fibrations* and those in $\mathcal{C} \cap \mathcal{W}$ *acyclic cofibrations.*

Definition 3.1.3 *Let $\mathbb{M}$ be a cocomplete category and $I \subseteq \mathbb{M}_1$.*

1. *Let λ be an ordinal. A (λ, I)-sequence in $\mathbb{M}$ is a cocontinous functor $\lambda \rightarrow \mathbb{M}$ such that all its values on morphisms are in I.*
2. *$A \in \mathbb{M}$ is small with respect to I if there is a cardinal κ such that the covariant hom-functor $\mathbb{M}(A, \_)$ preserves colimits of all (λ, I)-sequences for all regular cardinals $\lambda \geq \kappa$.*
3. *I permits the small object argument if the domains of morphisms in I are small with respect to I.*

Definition 3.1.4 *A model category $\mathbb{M}$ is cofibrantly generated if there are sets of morphisms $I, J \subseteq \mathbb{M}_1$ permitting the small object argument and such that*

$$\mathcal{F} \cap \mathcal{W} = RLP(I)$$

and

$$\mathcal{F} = RLP(J)$$

I is called the set of the generating cofibrations while J is called the set of generating acyclic cofibrations, this since

Proposition 3.1.1 *Morphisms in I are cofibrations while those in J are acyclic cofibrations.*

Definition 3.1.5 *A continous map $f : X \longrightarrow Y$ is a* weak homotopy equivalence *if*

$$\pi_n(f, x) : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

is a homeomorphism for any choice of the basepoint $x \in X$.

Theorem 3.1.1 (Quillen) *There is a cofibrantly generated model structure on* **Top** *such that*

- *the weak equivalences are the weak homotopy equivalences;*
- $I = \{S^{n-1} \hookrightarrow D^n | n \geqslant 0\}$*;*
- $J = \{I^{n-1} \times \{0\} \hookrightarrow I^n | n \geqslant 0\}$.

The model structure of theorem 3.1.1 is called the "standard" or Serre model structure on **Top**.

3.2 Model Structures on Topoi of Presheaves

One of those topoi, namely **sSet**, is ubiquitous in homotopy theory:

Theorem 3.2.1 (Quillen) **sSet** *is a cofibrantly generated model category with*

- $\mathcal{W} = \{f \in \mathbf{sSet}_1 || f| \in \mathcal{W}_{\mathbf{Top}}\}$*;*
- $\mathcal{C} = \{Monos\}$*;*
- $I = \{\partial [n] \rightarrowtail \Delta [n] | n \in \mathbb{N}\}$*;*
- $J = \left\{\Lambda^k [n] \rightarrowtail \Delta [n] | 0 \leq k \leq n, n \in \mathbb{N} \setminus \{0\}\right\}$.

Definition 3.2.1 *Let $\mathbb{C}$ be a category with coproducts. A* cylinder $\mathcal{I} = (I, \iota_0, \iota_1, \sigma)$*on $\mathbb{C}$ is given by the following data:*

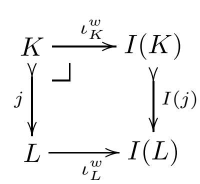
- *an endofunctor $I : \mathbb{C} \longrightarrow \mathbb{C}$;*
- *natural transformations $\iota^0, \iota^1 : id_{\mathbb{C}} \Rightarrow I$ and $\sigma : I \Rightarrow id_{\mathbb{C}}$ such that $\sigma \circ i^0 = \sigma \circ \iota^1 = id_{id_{\mathbb{C}}}$;*

A cylinder is cartesian *if*

(i) I preserves monos;

(ii) the canonical morphism $[\iota^0_C, \iota^1_C] : C + C \longrightarrow I(C)$ is mono for all $C \in \mathbb{C}$;

(iii) the naturality square



is a pullback square for all monos j and $w \in \{0,1\}$.

Definition 3.2.2 *Let $\mathbb{C}$ be a small category and $\widehat{\mathbb{C}} \stackrel{def.}{=} \mathbf{Set}^{\mathbb{C}^{op}}$ its category of presheaves. An* elementary homotopical datum *on $\mathbb{C}$ is a cartesian cylinder $\mathcal{I} = (I, \iota_0, \iota_1, \sigma)$ on $\widehat{\mathbb{C}}$ such that I preserves colimits. A* homotopical datum *on $\mathbb{C}$ is a pair $(\mathcal{I}, S)$ with $\mathcal{I}$ a homotopical datum on $\mathbb{C}$ and $S \subseteq \widehat{\mathbb{C}}_1$ a set of monos.*

As the name suggests, an elementary homotopical datum gives a notion of homotopy on morphisms of presheaves.

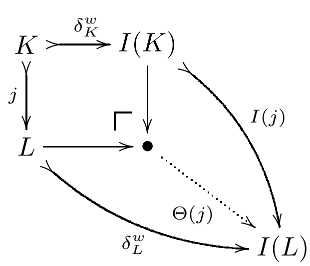
Proposition 3.2.1 *Let $\mathbb{C}$ be a small category and $\mathcal{I}$ an* elementary homotopical datum *on $\mathbb{C}$. Given morphisms of presheaves $f_0, f_1 : X \longrightarrow Y$ let*

$$f_0 \sim^1 f_1 \stackrel{def}{\Longleftrightarrow} \exists h : I(X) \longrightarrow Y. h \circ \iota_X^w = f_w$$

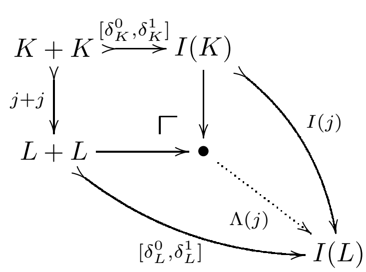
for $w \in \{0,1\}$. The equivalence relation $\sim^{\mathcal{I}}$on $\widehat{\mathbb{C}_1}$ generated by $\sim^1$is a congruence.

Definition 3.2.3 *Let $\mathbb{C}$ be a small category, $\mathcal{I}$ an* elementary homotopical datum *on $\mathbb{C}$ and $j : K \rightarrowtail L$ a mono in $\widehat{\mathbb{C}}$.*

(i) $\Theta(j)$ is the comparison morphism in



(ii) $\Lambda(j)$ is the comparison morphism from in



Given a set of monos $M \in \widehat{\mathbb{C}}$, let $\Theta(M) \stackrel{def.}{=} \{\Theta(j) | j \in M\}$ and $\Lambda(T) \stackrel{def.}{=} \{\Lambda(j) | j \in M\}$.

Theorem 3.2.2 (Cisinski) *Let $\mathbb{C}$ be a small category, $(\mathcal{I}, S)$ be a homotopical datum on $\mathbb{C}$ and $\mathcal{M} \in \widehat{\mathbb{C}_1}$ be a set of monos such that $LLP\left(RLP\left(\mathcal{M}\right)\right)$ is the class of all monos. Let*

– $\Lambda_0 \overset{def.}{=} S \cup \Theta(\mathcal{M})$ *and* $\Lambda_{n+1} \overset{def.}{=} \Lambda(\Lambda_n)$*;*

– $\Lambda_{\mathcal{I}}(S,\mathcal{M}) \overset{def.}{=} \bigcup_{n \geqslant 0} \Lambda_n$.

$\widehat{\mathbb{C}}$ *admits a cofibrantly generated model structure where the cofibrations are the monos and the weak equivalences are the morphisms* $f : X \longrightarrow Y$ *inducing a bijection*

$$f^* : \left(\widehat{\mathbb{C}}/\sim^{\mathcal{I}}\right)(Y,T) \cong \left(\widehat{\mathbb{C}}/\sim^{\mathcal{I}}\right)(X,T)$$

for all $T \in \mathbb{C}$ *such that* $T \xrightarrow{!_T} 1 \in RLP\left(\Lambda_{\mathcal{I}}(S,\mathcal{M})\right)$*.*

Theorem 3.2.2 works in fact for all topoi, not only those of presheaves [6].

Proof of theorem 3.2.1. Set

– $S = \varnothing$,

– $\mathcal{M} \overset{def.}{=} \{\partial\,[n] \rightarrowtail \Delta\,[n]\,|n \in \mathbb{N}\}$ and

– $\mathcal{I} \overset{def.}{=} (\_) \times \Delta[1]$

and apply theorem 3.2.2. □

However, as far as labor is concerned, there is no thing like a free lunch. What one spares with the existence is spent with the characterisations of $\mathcal{W}$ and J (I is easy). Nonetheless, 3.2.1 theorem allows to isolate the non-structural part of a task at hand.

3.3 Locally Presentable Categories for the Homotopy Theorist

Definition 3.3.1 *Suppose* $\mathbb{A}$ *has all coproducts. A family of objects* $(G_i)_{i\in I}$ *is a family of generators if the comparison morphism*

$$\gamma_C \overset{def.}{=} [f]_{i\in I,\ f\in\mathbb{A}(G_i,C)} : \left(\coprod_{i\in I,\ f\in\mathbb{A}(G_i,C)} G_i\right) \to C$$

is epi for all $C \in \mathbb{A}$*. A family of generators is*

(i) strong *if* $\gamma_C \in LLP\,(Monos)$ *for all* $C \in \mathbb{A}$*;*

(ii) dense *if, given the full subcategory* $\mathbb{G} \subseteq \mathbb{A}$ *such that* $\mathbb{G}_0 = (G_i)_{i\in I}$, $\left(C,\ (f)_{f\in\mathbb{G}/C}\right)$ *is a colimit of dom :* $\mathbb{G}/C \to \mathbb{A}$ *for all* $C \in \mathbb{A}$*.*

A one-member family of generators is called a generator (respectively a strong generator, respectively a dense generator).

A familiar example is given by the Yoneda embedding: the family of all representable presheaves $(\mathbb{B}\,(\_,B))_{B\in\mathbb{B}_0}$ over some category $\mathbb{B}$ is a dense generating family in $\mathbf{Set}^{\mathbb{B}^{op}}$.

Definition 3.3.2 *Let α be a regular cardinal. $C \in \mathbb{A}$ is α-presentable provided $\mathbb{A}(C, \_)$ preserves α -filtered colimits. It is presentable if there is an α sucht that it is α-presentable.*

An α-presentable $C \in \mathbb{A}$ is β-presentable for any regular $\beta < \alpha$. Finitely presentable groups are presentable. Presentable topological spaces are precisely the discrete ones i.e. there is no regular cardinal α for which a topological space is α-presentable. Gabriel and Ulmer observe that "...the presentable individuals are the discrete ones, an exemplary society!" [8, p.64] [1].

Definition 3.3.3 *Let α be a regular cardinal. The category $\mathbb{A}$ is locally α-presentable provided*

1. *$\mathbb{A}$ is cocomplete;*
2. *$\mathbb{A}$ has a strong family of generators $(G_i)_{i \in I}$;*
3. *each G_i is α-presentable.*

Remark 3.3.1 **2-Cat** is locally presentable. It is cocomplete by proposition 2.2.1 and it is easy to see that the 2-category $\mathbf{W}_2$ given by

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \quad (\Downarrow \alpha)$$

(a.k.a "the walking 2-cell" or "the free-living 2-cell") is a strong $\aleph_0$-presentable generator. ★

Proposition 3.3.1 *Let α be a regular cardinal and $\mathbb{A}$ be locally α-presentable. Let $\mathbb{G}$ be the full subcategory spanned by $\mathbb{A}$'s generating family $(G_i)_{i \in I}$. Then*

1. *The closure $\mathbb{P}$ of $\mathbb{G}$ under α-colimits exists and is equivalent to a small category;*
2. *$\mathbb{P}$'s α-colimits are computed as in $\mathbb{A}$;*
3. *every object in $\mathbb{P}$ is α-presentable;*
4. *$\mathbb{P}_0$ is a dense generator in $\mathbb{A}$.*

Proposition 3.3.2 *Let α be a regular cardinal and $\mathbb{A}$ be locally α-presentable. For every $C \in \mathbb{A}$ there is a regular cardinal α_C such that C is α_C-presentable.*

Corollary 3.3.1 *The small object argument applies to any set $I \subseteq \mathbb{A}_1$.*

Corollary 3.3.1 is the main reason for the interest of homotopy theorists in locally presentable categories.

[1] "Insbesondere sind die präsentierbare Individuen gereade die Diskreten, eine vorbildliche Gesellschaft!"

3.4 Creation of Model Structures by Right Adjoints

Definition 3.4.1 *Let* $\mathbb{M}$ *be a model category and*

$$\mathbb{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{M} \quad (F \dashv U)$$

be an adjunction. U *creates a model structure on* $\mathbb{C}$ *if there is a model structure on* $\mathbb{C}$ *such that* $\mathcal{F}_{\mathbb{C}} = U^{-1}(\mathcal{F}_{\mathbb{M}})$ *and* $\mathcal{W}_{\mathbb{C}} = U^{-1}(\mathcal{W}_{\mathbb{M}})$*.*

Proposition 3.4.1 *Let* $\mathbb{M}$ *be a cofibrantly generated model category with* I *and* J *the sets of generating cofibrations and acyclic cofibrations, respectively. Let* $F \dashv U$ *and* $\mathbb{C}$ *be as in definition 3.4.1. Suppose*

(i) $dom(F(i))$ *is small with respect to* $F(I)$ *for all* $i \in I$ *and* $dom(F(j))$ *is small with respect to* $F(J)$ *for all* $j \in J$*;*

(ii) the composition of any $(\lambda, \mathcal{W}_{\mathbb{M}})$*-sequence* $\lambda \longrightarrow \mathbb{M}$ *is a weak equivalence for all* $\lambda \in \mathbf{Ord}$*;*

(iii) U *preserves colimits of* λ*-sequences for all* $\lambda \in \mathbf{Ord}$*; and*

(iv) for every $A \xrightarrow{j} B \in J$ *and for every pushout*

$$\begin{array}{ccc} F(A) & \xrightarrow{f} & C \\ {\scriptstyle F(j)}\downarrow & & \downarrow{\scriptstyle g} \\ F(B) & \longrightarrow & F(B) +_{F(A)} C \end{array}$$

in $\mathbb{C}$*, the morphism* $U(g)$ *is a weak equivalence.*

Then the adjoint pair $F \dashv U$ *creates a cofibrantly generated model category structure on* $\mathbb{C}$*, where* $F(I)$ *and* $F(J)$ *are the generating cofibrations and generating acyclic cofibrations, respectively.*

Proposition 3.4.1 is an easy consequence of Kan's Theorem on creation of model category structure [12, thm. 11.3.2].

4 A Model Structure à la Thomason

This section essentially revisits and generalizes categorical techniques developed by Fritsch and Latch [7], Thomason [17] and Cisinski [4, 5]. However, it turns out that not everything carries over by tagging a "2-" in front. It is crucially the case for Cisinski's "immersions", a generalization of Thomason's "Dwyer-morphisms". We call the relevant 2-categorical notion "skew immersion".

4.1 Ideals in Categories

Definition 4.1.1 *Let $\mathbb{A} \subseteq \mathbb{B}$ be an inclusion of categories. $\mathbb{A}$ is an* L-ideal *in $\mathbb{B}$ if*

$$\forall f \in \mathbb{B}_1 . cod(f) \in \mathbb{A}_0 \ \Rightarrow f \in \mathbb{A}_1$$

and an R-ideal in $\mathbb{B}$ if

$$\forall f \in \mathbb{B}_1 . dom(f) \ \in \mathbb{A}_0 \Rightarrow f \in \mathbb{A}_1$$

In the literature, L-Ideals are called *left ideals*, *sieves* or *cribles* while R-Ideals are called *right ideals*, *cosieves* or *cocribles* [7, 17, 4, 5]

Definition 4.1.2 *Let $\mathbb{I}$ be the category generated by $L \xrightarrow{t} R$ and $\iota^L, \iota^R : 1 \longrightarrow \mathbb{I}$ be the global elements of $\mathbb{I}$ with image generated by L respectively by R. Let further $\partial^L \overset{def.}{=} cod$, $\partial^R \overset{def.}{=} dom$ and*

$$\begin{array}{rlcl} \overline{(\_)}: & \{L, R\} & \longrightarrow & \{L, R\} \\ & L & \longmapsto & R \\ & R & \longmapsto & R \end{array}$$

be the toggling map.

Proposition 4.1.1 *Let $\mathbb{A} \subseteq \mathbb{B}$ be an inclusion of categories and $\nu \in \{L, R\}$. The following are equivalent.*

(i) $\mathbb{A}$ is a ν-ideal;

(ii) there is a functor $\chi_{\mathbb{A}} : \mathbb{B} \longrightarrow I$ such that $\mathbb{A} \cong \chi_{\mathbb{A}}^*(\iota^\nu)$;

(iii) $\mathbb{A} \subseteq \mathbb{B}$ is a full inclusion and there is a commuting square

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{!} & 1 \\ \downarrow & & \downarrow \iota^\nu \\ \mathbb{B} & \xrightarrow[\chi_{\mathbb{A}}]{} & \mathbb{I} \end{array}$$

such that

$$(\chi_{\mathbb{A}})(B) = \begin{cases} \nu & B \in \mathbb{A}_0 \\ \bar{\nu} & B \in \mathbb{B}_0 \backslash \mathbb{A}_0 \end{cases}$$

Proof. $(i) \Rightarrow (ii)$ The functor given by

$$(\chi_{\mathbb{A}})_0 : \quad B \quad \longmapsto \quad \begin{cases} \nu & B \in \mathbb{A}_0 \\ \bar{\nu} & B \in \mathbb{B}_0 \backslash \mathbb{A}_0 \end{cases}$$

and

$$(\chi_{\mathbb{A}})_1 : \quad f \quad \longmapsto \quad \begin{cases} \mathrm{id}_\nu & f \in \mathbb{A}_1 \\ \theta & \partial^{\bar{\nu}}(f) \in \mathbb{A}_0 \wedge \partial^\nu(f) \in \mathbb{B}_0 \backslash \mathbb{A}_0 \\ \mathrm{id}_{\bar{\nu}} & f \in \mathbb{B}_1 \backslash \mathbb{A}_1 \end{cases}$$

is well-defined since $\mathbb{A}$ is a ν-ideal. It is immediate that $\mathbb{A} \cong \chi^*_{\mathbb{A}}(\iota^\nu)$.

(ii)⇒*(iii)* Let $f \in \mathbb{B}_1$ such that $\partial^\nu(f) \in \chi^*_{\mathbb{A}}(\iota^\nu)_0$ and $\partial^{\bar\nu}(f) \in \chi^*_{\mathbb{A}}(\iota^\nu)_0$. Then $f \in \chi^*_{\mathbb{A}}(\iota^\nu)_1$ by the underlying graph structure, so the inclusion is full.

(iii)⇒*(i)* Let $f \in \mathbb{B}_1$ such that $\partial^\nu(f) \in \mathbb{A}_0$. Then $\chi_{\mathbb{A}}(\partial^\nu(f)) = \nu$ by definition of $\chi_{\mathbb{A}}$and $\chi_{\mathbb{A}}(\partial^{\bar\nu}(f)) = \nu$ by the underlying graph structure, hence $\partial^{\bar\nu}(f) \in \mathbb{A}_0$. But $\mathbb{A}$ is a full subcategory so $f \in \mathbb{A}_1$. □

Definition 4.1.3 *The functor $\chi_{\mathbb{A}}$ of proposition 4.1.1 is called the ideal's* characteristic morphism.

Remark 4.1.1 An ideal is in particular always a full subcategory. A characteristic morphism in necessarily unique. ★

4.2 Ideals in 2-Categories

The notion of ideal carries over as expected to 2-categories.

Definition 4.2.1 *Let $\mathcal{A} \subseteq \mathcal{B}$ be an inclusion of 2-categories. $\mathcal{A}$ is an* L-ideal *in $\mathcal{B}$ if*

$$\forall \alpha \in \mathcal{B}_2.(cod \circ dom)(\alpha) = (cod \circ cod)(\alpha) \in \mathcal{A}_0 \Rightarrow \alpha \in \mathcal{A}_2$$

and an R-ideal in $\mathcal{B}$ if

$$\forall \alpha \in \mathcal{B}_2.(dom \circ dom)(\alpha) = (dom \circ cod)(\alpha) \in \mathcal{A}_0 \Rightarrow \alpha \in \mathcal{A}_2$$

We also call *L*-ideals *left ideals* and *R*-ideals *right ideals*.

Proposition 4.2.1 *Let $\mathcal{I}$ be the 2-category with trivial 2-cells such that $\lfloor \mathcal{I} \rfloor = \mathbb{I}$. Let $\nu \in \{L, R\}$ and $\mathcal{A} \subseteq \mathcal{B}$ be an inclusion of 2-categories The following are equivalent.*

(i) $\mathcal{A}$ is a ν-ideal in $\mathcal{B}$;

(ii) $\lfloor \mathcal{A} \rfloor$ is a ν-ideal in $\lfloor \mathcal{B} \rfloor$ and $\mathcal{A} \subseteq \mathcal{B}$ is a locally full inclusion;

(iii) there is a 2-functor $\chi_{\mathcal{A}} : \mathcal{B} \longrightarrow I$ such that $\lfloor \chi_{\mathcal{A}} \rfloor = \chi_{\lfloor \mathcal{A} \rfloor}$.

Proof. *(i)*⇒*(ii)* $\lfloor \mathcal{A} \rfloor$ is a ν-ideal by instantiating the defintion on the identity 2-cells. Suppose $\mathrm{dom}_1(\alpha) \in \mathcal{A}_1$ and $\mathrm{cod}_1(\alpha) \in \mathcal{A}_1$. Then in particular $(\partial^\nu \circ \mathrm{dom}_1)(\alpha) \in \mathcal{A}_0$ and $(\partial^\nu \circ \mathrm{cod}_1)(\alpha) \in \mathcal{A}_0$, hence $\alpha \in \mathcal{A}_2$.

(ii)⇒*(i)* Suppose $(\partial^\nu \circ \mathrm{dom}_1)(\alpha) \in \mathcal{A}_0$ and $(\partial^\nu \circ \mathrm{cod}_1)(\alpha) \in \mathcal{A}_0$. Then $\mathrm{dom}_1(\alpha) \in \mathcal{A}_1$ and $\mathrm{cod}_1(\alpha) \in \mathcal{A}_1$ since $\lfloor \mathcal{A} \rfloor$ is a ν-ideal. But $\mathcal{A}$ is a locally full sub2-category so $\alpha \in \mathcal{A}_2$.

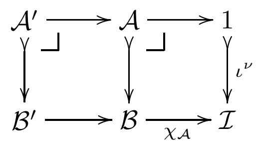
(ii)⇔*(iii)* Obvious. □

Definition 4.2.2 *The 2-functor $\chi_{\mathcal{A}} : \mathcal{B} \longrightarrow \mathcal{I}$ of proposition 4.2.1 is called the ideal's* characteristic morphism.

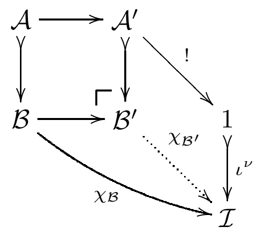
Remark 4.2.1 An ideal inclusion is in particular always full and locally full. The characteristic morphism is necessarily unique and $\mathcal{A} \cong \chi^*_{\mathcal{A}}(i^\nu)$. ★

Lemma 4.2.1 *Ideals are stable under pullback and pushout.*

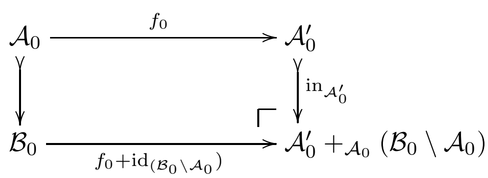
Proof. Let $\nu \in \{L, R\}$. The first assertion follows immediately from the pullback lemma:



For the second, consider the diagram



with $\chi_{\mathcal{B}'}$ given by universal property. By remark 2.2.1, $\mathcal{A}' \subseteq \mathcal{B}'$ is full and locally full and the pushout square is



on objects. We have

$$\chi_{\mathcal{B}'}|_{\mathcal{A}'_0} = \iota^\nu \circ !_{\mathcal{A}'_0}$$

and

$$(\chi_{\mathcal{B}'} \circ (f_0 + id_{(\mathcal{B}_0 \setminus \mathcal{A}_0)}))|_{\mathcal{B}_0 \setminus \mathcal{A}_0} = \chi_{\mathcal{B}'}|_{\mathcal{B}_0 \setminus \mathcal{A}_0} = \chi_{\mathcal{B}}|_{\mathcal{B}_0 \setminus \mathcal{A}_0}$$

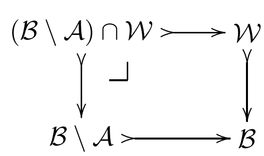
hence

$$\chi_{\mathcal{B}'}(B') = \begin{cases} \nu & B' \in \mathcal{A}'_0 \\ \bar{\nu} & B' \in (\mathcal{B}'_0 \setminus \mathcal{A}'_0) \cong (\mathcal{B}_0 \setminus \mathcal{A}_0) \end{cases}$$

so the assertion follows by proposition 4.1.1. □

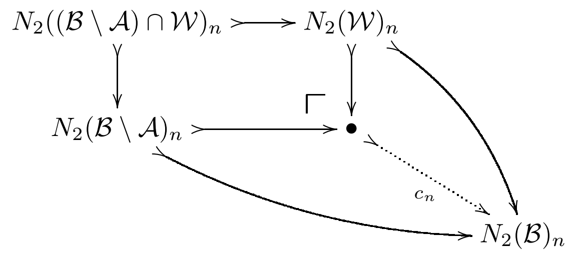
Definition 4.2.3 *Let $\mathcal{A}$ be a 2-category and $X \subseteq \mathcal{A}_0$. $\lceil X \rceil \subseteq \mathcal{A}$ is the full and locally full sub-2-category such that $\lceil X \rceil_0 = X$.*

Lemma 4.2.2 *Let $\mathcal{A} \subseteq \mathcal{W} \subseteq \mathcal{B}$ be inclusions of 2-categories with $\mathcal{A}$ a left ideal and $\mathcal{W}$ a right ideal. Let $\mathcal{B} \setminus \mathcal{A} \overset{def.}{=} \lceil \mathcal{B}_0 \setminus \mathcal{A}_0 \rceil$. The image of the pullback square*



under N_2 is a pushout square.

Proof. The comparison map c is an injection for all $n \in \mathbb{N}$:

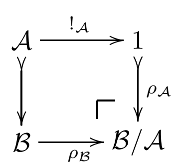


Recall that $N_2(\mathcal{A})_n = \mathbf{NLax}([n], \mathcal{A})$ (c.f. remark 2.4.3). Suppose $n = 0$. We have

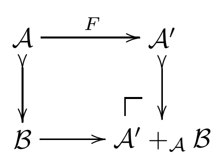
$$N_2(\mathcal{B} \setminus \mathcal{A})_0 +_{N_2((\mathcal{B}\setminus\mathcal{A})\cap\mathcal{W})} N_2(\mathcal{W})_0 \cong (\mathcal{B} \setminus \mathcal{A})_0 +_{((\mathcal{B}\setminus\mathcal{A})_0\cap\mathcal{W}_0)} \mathcal{W}_0 \cong (\mathcal{B}_0 \setminus \mathcal{A}_0) \cup \mathcal{W}_0 = \mathcal{B}_0$$

since $\mathcal{A}_0 \subseteq \mathcal{W}_0$. In particular, c_0 is a surjection. Suppose now $n > 0$. $\mathcal{B} \setminus \mathcal{A}$ is a right ideal since $\mathcal{A}$ is a left ideal and $\mathcal{W}$ is a right ideal by hypothesis, hence the image of a lax functor $[n] \longrightarrow \mathcal{B}$ is in $\mathcal{B} \setminus \mathcal{A}$ or in $\mathcal{W}$ so c_n is a surjection for all $n \in \mathbb{N}$. □

Definition 4.2.4 *Let $\mathcal{A} \rightarrowtail \mathcal{B}$ be an inclusion of 2-categories. The 2-category $\mathcal{B}/\mathcal{A}$ is given by the pushout square*



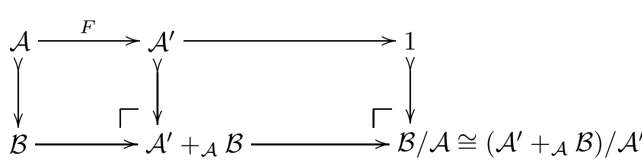
Proposition 4.2.2 *Let $I : \mathcal{A} \rightarrowtail \mathcal{B}$ be an inclusion of 2-categories, $F : \mathcal{A} \longrightarrow \mathcal{A}'$ a 2-functor and*



the corresponding pushout square. Then

$$\mathcal{B}/\mathcal{A} \cong (\mathcal{A}' +_\mathcal{A} \mathcal{B})/\mathcal{A}'$$

Proof.

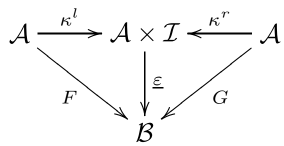


□

Corollary 4.2.1 $\mathcal{B}\setminus\mathcal{A} \cong (\mathcal{A}' +_\mathcal{A} \mathcal{B}) \setminus \mathcal{A}'$ *provided $\mathcal{A}$ is an ideal.*

4.3 Distorsions

Definition 4.3.1 *Let* $\kappa^{\nu} : \mathcal{A} \cong \mathcal{A} \times 1 \overset{id \times \iota^{\nu}}{\longrightarrow} \mathcal{A} \times \mathcal{I}$ *for* $\nu \in \{L, R\}$. *Let* $F, G : \mathcal{A} \to \mathcal{B}$ *be 2-functors. A* distortion $\varepsilon : F \rightsquigarrow G$ *is given by a normal lax functor* $\underline{\varepsilon} : \mathcal{A} \times \mathcal{I} \longrightarrow \mathcal{B}$ *such that*



commutes in $\widetilde{\textbf{2-Cat}}$.

Remark 4.3.1 N_2 extends to a product-preserving functor

$$\widetilde{N_2} : \widetilde{\textbf{2-Cat}} \to \textbf{sSet}$$

It follows that a distortion $\varepsilon : F \rightsquigarrow G$ gives rise to a simplicial homotopy $N_2(F) \simeq N_2(G)$. ★

Proposition 4.3.1 *Let* $F, G : \mathcal{A} \to \mathcal{B}$ *be 2-functors. The following are equivalent.*

(i) *There is a distortion* $\varepsilon : F \rightsquigarrow G$*;*

(ii) *there are*

- *a morphism* $\varepsilon_f : F(A) \longrightarrow G(B)$ *for all* $f \in \mathcal{A}_1$*;*
- *a 2-cell* $\varepsilon_{\alpha} : \varepsilon_f \Longrightarrow \varepsilon_{f'}$ *for all* $\alpha : f \Rightarrow f' \in \mathcal{A}_2$*;*
- *2-cells* $\varepsilon^L_{f,g} : \varepsilon_g \circ F(f) \Rightarrow \varepsilon_{g \circ f}$ *and*
- $\varepsilon^R_{f,g} : G(g) \circ \varepsilon_f \Rightarrow \varepsilon_{g \circ f}$ *for all composable* $f, g \in \mathcal{A}_1$*,*

such that

lf1 $\varepsilon_{(\beta \bullet \alpha)} = \varepsilon_{\beta} \bullet \varepsilon_{\alpha}$ *for all vertically composable* $\alpha, \beta \in \mathcal{A}_2$*;*

lf2 $\varepsilon_{id_f} = id_{\varepsilon_f}$ *for all* $f \in \mathcal{A}_1$*;*

n1 $\varepsilon^R_{cod^1(\varphi), cod^1(\theta)} \bullet (G(\theta) \circ \varepsilon_{\varphi}) = \varepsilon_{\theta \circ \varphi} \bullet \varepsilon^R_{dom^1(\varphi), dom^1(\theta)}$ *and*

n2 $\varepsilon^L_{cod^1(\varphi), cod^1(\theta)} \bullet (\varepsilon_{\theta} \circ F(\varphi)) = \varepsilon_{\theta \circ \varphi} \bullet \varepsilon^L_{dom^1(\varphi), dom^1(\theta)}$ *for all horizontally composable* $\varphi, \theta \in \mathcal{A}_2$*;*

c1 $\varepsilon^R_{g \circ f, h} \bullet \left(G(h) \circ \varepsilon^R_{f,g}\right) = \varepsilon^R_{f, g \circ h}$,

c2 $\varepsilon^L_{f, h \circ g} \bullet \left(\varepsilon^L_{g,h} \circ F(f)\right) = \varepsilon^L_{g \circ f, h}$ *and*

c3 $\varepsilon^R_{g \circ f} \bullet \left(G(h) \circ \varepsilon^L_{f,g}\right) = \varepsilon^L_{f, h \circ g} \bullet \left(\varepsilon^R_{g,h} \circ F(f)\right)$ *for all composable* $f, g, h \in \mathcal{A}_1$.

Proof. Let $A \in \mathcal{A}_0$, $f \in \mathcal{A}_1$ and $\nu \in \{L, R\}$. The values of $\underline{\varepsilon}$ on (A, ν) respectively on (f, id_ν) are determined by F for $\nu = L$ and by G for $\nu = R$. The associated structural 2-cells are all trivial. On the other hand,

$$\varepsilon_f \overset{def.}{=} \underline{\varepsilon}(f, t)$$

and

$$\varepsilon_\alpha \overset{def.}{=} \underline{\varepsilon}(\alpha, \mathrm{id}_t)$$

are the remaining values while

$$\varepsilon^L_{f,g} \overset{def.}{=} \gamma_{(f,\mathrm{id}_L),(g,t)}$$

and

$$\varepsilon^R_{f,g} \overset{def.}{=} \gamma_{(f,t),(g,\mathrm{id}_R)}$$

are the remaining structural 2-cells. □

Remark 4.3.2 Distortions do not compose in general (neither vertically nor horizontally), yet they can be whiskered on the left as well as on the right. ★

Remark 4.3.3 Some instances of the equations governing a distortion become conveniently simpler. Let $\varepsilon : F \rightsquigarrow G$ be a distortion. Given

$$A \xrightarrow{\mathrm{id}_A} A \underset{v}{\overset{u}{\underset{\Downarrow \theta}{\rightrightarrows}}} B \xrightarrow{\mathrm{id}_B} B$$

let $\varepsilon_A \overset{def.}{=} \varepsilon_{\mathrm{id}_A}$, $\varepsilon^\nu_{A,u} \overset{def.}{=} \varepsilon^\nu_{\mathrm{id}_A,u}$ and $\varepsilon^\nu_{u,B} \overset{def.}{=} \varepsilon^\nu_{u,\mathrm{id}_B}$ for $\nu \in \{L, R\}$. We then have

n1 $\varepsilon^R_{A,v} \bullet (G(\theta) \circ \varepsilon_A) = e_\theta \bullet \varepsilon^R_{A,u}$;

n2 $\varepsilon^L_{A,v} \bullet \varepsilon_\theta = \varepsilon_\theta \bullet \varepsilon^L_{A,u}$;

c1 $\varepsilon^R_{u,B} \bullet \varepsilon^R_{A,u} = \varepsilon^R_{A,u}$;

c2 $\varepsilon^L_{A,u} \bullet \varepsilon^L_{u,B} = \varepsilon^L_{u,B}$;

c3 $\varepsilon^R_{u,B} \bullet \varepsilon^L_{A,u} = \varepsilon^L_{A,u} \bullet \varepsilon^R_{u,B}$.

★

Definition 4.3.2 *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a 2-functor. The* identity distortion $id_F : F \rightsquigarrow F$ *is given by*

(i) $(id_F)_f \overset{def.}{=} F(f)$ *for all* $f \in \mathcal{A}_1$*;*

(ii) $(id_F)_\alpha \overset{def.}{=} F(\alpha)$ *for all* $\alpha \in \mathcal{A}_2$*;*

(iii) $(id_F)^L_{f,g} = (id_F)^R_{f,g} = id$ *for all composable* $f, g \in \mathcal{A}_1$.

4.4 Skew Immersions

Definition 4.4.1 *An inclusion $J : \mathcal{A} \rightarrowtail \mathcal{B}$ of 2-categories is a skew immersion provided*

1. *$\mathcal{A}$ is a left ideal;*
2. *there is a right ideal $\mathcal{W} \subseteq \mathcal{B}$ such that the corestriction $J : \mathcal{A} \rightarrowtail \mathcal{W}$ admits a retraction $R_J : \mathcal{W} \twoheadrightarrow \mathcal{A}$ and a distortion $\varepsilon : J \circ R_J \rightsquigarrow id_{\mathcal{W}}$ with $\varepsilon J = id_J$.*

Remark 4.4.1 It follows by remark 4.3.1 that $N_2(\mathcal{A})$ is a strong deformation retract of $N_2(\mathcal{W})$ with respect to the standard model structure on **sSet**. ★

For the remaining of this section, we fix a skew immersion $J : \mathcal{A} \rightarrowtail \mathcal{B}$ and a pushout square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{U} & \mathcal{A}' \\ {\scriptstyle J}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle J'} \\ \mathcal{B} & \xrightarrow[W]{} & \mathcal{B}' \end{array}$$

along with its decomposition

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{U} & \mathcal{A}' \\ {\scriptstyle J}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle J'} \\ \mathcal{W} & \xrightarrow[V]{} & \mathcal{W}' \\ {\scriptstyle K}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle K'} \\ \mathcal{B} & \xrightarrow[W]{} & \mathcal{B}' \end{array}$$

Proposition 4.4.1 *Skew immersions are stable under pushout.*

Proof. By lemma 4.2.1, $\mathcal{A}'$ is a left ideal and $\mathcal{W}'$ is a right ideal. In particular, $\mathcal{A}$ is a left ideal in $\mathcal{W}$ while $\mathcal{A}'$ is a left ideal in $\mathcal{W}'$.

Let P be the 2-graph given by

$$\begin{array}{ccc} \mathcal{U}(\mathcal{A}) & \xrightarrow{\mathcal{U}(U)} & \mathcal{U}(\mathcal{A}') \\ \big\downarrow & \ulcorner & \big\downarrow{\scriptstyle \iota_{\mathcal{A}'}} \\ \mathcal{U}(\mathcal{W}) & \xrightarrow[\iota_{\mathcal{W}}]{} & P \end{array}$$

The colimits in a functor category being calculated pointwise, we have

$$P \cong (\mathcal{A}'_2 + (\mathcal{W}_2 \backslash \mathcal{A}_2), \mathcal{A}'_1 + (\mathcal{W}_1 \backslash \mathcal{A}_1), \mathcal{A}'_0 + (\mathcal{W}_0 \backslash \mathcal{A}_0))$$

with the copairs

$$\partial^i_P = \left[\mathrm{in}_{\mathcal{A}'_i} \circ \partial^i_{\mathcal{A}'}, \left(U_i + \mathrm{id}_{\mathcal{W}_i \backslash \mathcal{A}_i}\right) \circ \partial^i_{\mathcal{W}}|_{\mathcal{W}_{i+1} \backslash \mathcal{A}_{i+1}}\right]$$

as structural maps, for $i \in \{0,1\}$ and $\partial \in \{\text{dom}, \text{cod}\}$. The coprojections are

$$\iota_{\mathcal{A}'} = \left(\text{in}_{\mathcal{A}'_2}, \text{in}_{\mathcal{A}'_1}, \text{in}_{\mathcal{A}'_0}\right)$$

respectively

$$\iota_{\mathcal{W}} = \left(U_2 + \text{id}_{\mathcal{B}_2\backslash\mathcal{A}_2}, U_1 + \text{id}_{\mathcal{B}_1\backslash\mathcal{A}_1}, U_0 + \text{id}_{\mathcal{B}_0\backslash\mathcal{A}_0}\right)$$

Let $J' : \mathcal{A}' \rightarrowtail \mathcal{F}(P)$and $W : \mathcal{W} \to \mathcal{F}(P)$ be the morphisms of 2-graphs induced by $\iota_{\mathcal{A}'}$ respectively by $\iota_{\mathcal{W}}$. Then

$$\mathcal{W}' \cong \mathcal{F}(P)/\sim$$

with $\sim$ the smallest sesquicongruence making J' and W 2-functorial (c.f. proposition 2.2.1 and remark 2.2.1).

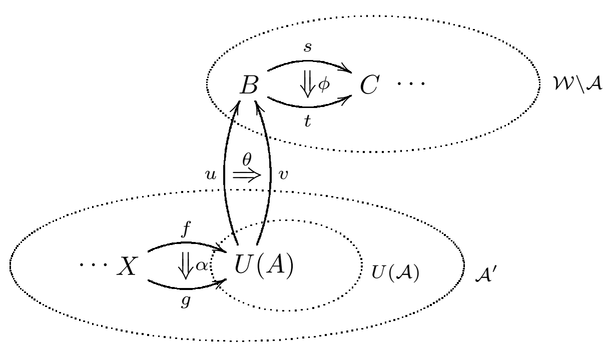
Since $\mathcal{A} \subseteq \mathcal{W}$ is a left ideal, a morphism k generating $\mathcal{W}$' has one of the following types:

1. $k \in \mathcal{A}'_1$;
2. $k \in (\mathcal{W}_1\backslash\mathcal{A}_1)\backslash(\mathcal{W}\backslash\mathcal{A})_1$;
3. $k \in (\mathcal{W}\backslash\mathcal{A})_1$ (c.f. lemma 4.2.2);

while a 2-cell ϖ generating $\mathcal{W}$' has one of the following types:

1. $\text{dom}^1(\varpi), \text{cod}^1(\varpi) \in \mathcal{A}'_1$;
2. $\text{dom}^1(\varpi), \text{cod}^1(\varpi) \in (\mathcal{W}_1\backslash\mathcal{A}_1)\backslash(\mathcal{W}\backslash\mathcal{A})_1$;
3. $\text{dom}^1(\varpi), \text{cod}^1(\varpi) \in (\mathcal{W}\backslash\mathcal{A})_1$ (c.f. lemma 4.2.2).

In particular, given a morphism k of type 2 we have $\text{dom}(k) = U(A)$ for some $A \in \mathcal{A}_0$ and $\text{cod}(k) = B$ for some $B \in \mathcal{W}_0\backslash\mathcal{A}_0$. A typical situation can be depicted as follows



General morphisms of $\mathcal{W}'$ are thus composable strings

$$< f_0; \cdots ; f_n; u; s_1; \cdots ; s_m >$$

with $f_0, \ldots, f_n$ of type 1, u of type 2 and $s_1, \ldots, s_m$ of type 3. Simlarly, general 2-cells of $\mathcal{W}'$ are horizontally composable strings

$$\ll \alpha_0; \cdots ; \alpha_n; \theta; \phi_1; \cdots ; \phi_m \gg$$

with $\alpha_0, \ldots, \alpha_n$ of type 1, θ of type 2 and $\phi_1, \ldots, \phi_m$ of type 3. On the other hand, the relations governing $\mathcal{W}'$ impose the identities

$$< f_0; \cdots ; f_n; u; s_1; \cdots ; s_m > = < f_n \circ \cdots \circ f_0 >; < s_m \circ \cdots \circ s_1 \circ u >$$

respectively

$$\ll \alpha_0; \cdots ; \alpha_n; \theta; \phi_1; \cdots ; \phi_m \gg = \ll \alpha_n \circ \cdots \circ \alpha_0 \gg; \ll \phi_m \circ \cdots \circ \phi_1 \circ \theta \gg$$

among such strings.

Finally, there is the retraction

$$R'_{J'} = \left(\left[\mathrm{id}_{\mathcal{A}'_2}, (U \circ R)_2\right], \left[\mathrm{id}_{\mathcal{A}'_1}, (U \circ R)_1\right], \left[\mathrm{id}_{\mathcal{A}'_0}, (U \circ R)_0\right]\right) : \mathcal{W}' \longrightarrow \mathcal{A}'$$

given by universal property. Let $\nu \in \{L, R\}$. There is the distortion

$$\xi : J' \circ R'_{J'} \rightsquigarrow \mathrm{id}_{\mathcal{W}'}$$

given by

- $\xi_{<f>} \overset{def.}{=} < f >$, $\xi_{\ll\alpha\gg} \overset{def.}{=} \ll \alpha \gg$ and $\xi^{\nu}_{<f>,<g>} \overset{def.}{=} \ll\gg_{<g \circ f>}$ for all f of type 1, all α of type 1 respectively all composable f and g of type 1;
- $\xi_{<p>} \overset{def.}{=} < \varepsilon_p ¿$, $\xi_{\ll\beta\gg} \overset{def.}{=} \ll \varepsilon_\beta \gg$ and $\xi^{\nu}_{<p>,<q>} \overset{def.}{=} \ll \varepsilon^{\nu}_{p,q} \gg$for all p of type 2 and 3, all β of type 2 and 3 respectively all composable p and q of type 2 or 3;
- $\xi_{<f;u>} \overset{def.}{=} < f; \varepsilon_u >$ for all composable f of type 1 and u of type 2;
- $\xi_{\ll\alpha;\theta\gg} \overset{def.}{=} \ll \alpha; \varepsilon_\theta \gg$for all horizontally composable α of type 1 and θ of type 2;
- $\xi^{\nu}_{\mathrm{dom}(f),<f,u>} \overset{def.}{=} \ll f; \xi^{\nu}_{U(\mathrm{dom}(f)),u} \gg = \ll f; \varepsilon^{\nu}_{\mathrm{dom}(f),u} \gg$ and
- $\xi^{\nu}_{<f,u>,\mathrm{cod}(u)} \overset{def.}{=} \ll f; \xi^{\nu}_{u,\mathrm{cod}(u)} \gg = \ll f; \varepsilon^{\nu}_{u,\mathrm{cod}(u)} \gg$ for all composable f of type 1 and u of type 2.

The axioms of distortion are easily checked, e.g.

$$\begin{aligned}
\ll \xi_X; \ll \alpha; \theta \gg\gg : \xi^R_{X,<g;v>} &= \ll \alpha; \theta \gg : \xi^R_{X,<g;v>} (\xi_X = \mathrm{id}_X) \\
&= \ll \alpha; \theta \gg : \ll g; \xi^R_{U(A),v} \gg \\
&= \ll \alpha \gg; \ll \theta : \xi^R_{U(A),v} \gg \\
&= \ll \alpha; \varepsilon^R_{A,v} \bullet \theta \gg \\
&= \ll \alpha; \varepsilon^R_{A,v} \bullet (\theta \circ \varepsilon_A) \gg (\varepsilon_A = \mathrm{id}_A) \\
&= \ll \alpha; \varepsilon_\theta \bullet \varepsilon^R_{A,u} \gg \\
&= \ll \alpha \gg; \ll \xi^R_{U(A),u} : \varepsilon_\theta \gg \\
&= \ll f; \xi^R_{U(A),u} \gg : \ll \alpha; \varepsilon_\theta \gg \\
&= \xi^R_{X,<f,u>} : \xi_{\ll\alpha,\theta\gg}
\end{aligned}$$

(c.f. remark 4.3.3), while $\xi J' = \mathrm{id}_{J'}$ holds by construction. □

4.5 The 2-Thomason Model Structure

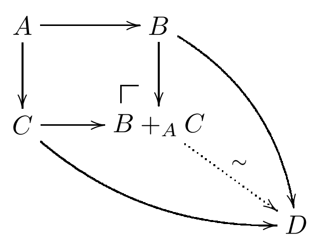
Lemma 4.5.1 $(\mathcal{B}\backslash\mathcal{A}) \cap \mathcal{W} \cong (\mathcal{B}'\backslash\mathcal{A}') \cap \mathcal{W}'$.

Proof. Let $\star_{\mathcal{A}} \in \mathcal{B}/\mathcal{A}$ be the object such that $(\rho_{\mathcal{A}} \circ !_{\mathcal{A}})(\mathcal{A}) = \lceil * _{\mathcal{A}} \rceil$ (c.f. defintion 4.2.4). Given the iso $i : \mathcal{B}/\mathcal{A} \cong \mathcal{B}'/\mathcal{A}'$ (c.f. proposition 4.2.2), it is immediate that $i(\star_{\mathcal{A}}) = \star_{\mathcal{A}'}$. On the other hand, $\mathrm{cod}(f) \in \mathcal{W}_1$ for all $f \in (\mathcal{B}_1\backslash\mathcal{A}_1) \backslash (\mathcal{B}\backslash\mathcal{A})_1$ since $\mathcal{W}$ is a right ideal and $\mathrm{cod}(f') \in \mathcal{W}'_1$ for all $f' \in (\mathcal{B}'_1\backslash\mathcal{A}'_1) \backslash (\mathcal{B}'\backslash\mathcal{A}')_1$ since $\mathcal{W}'$ is a right ideal. Hence there is a bijection

$$((\mathcal{B}\backslash\mathcal{A}) \cap \mathcal{W})_0 \cong ((\mathcal{B}'\backslash\mathcal{A}') \cap \mathcal{W}')_0$$

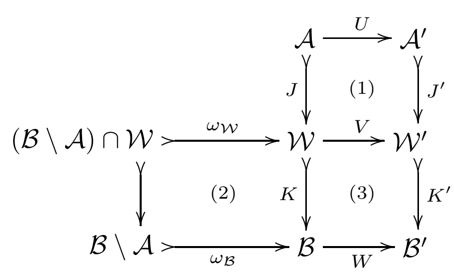
induced by i. But $(\mathcal{B}\backslash\mathcal{A}) \cap \mathcal{W}$ is a right ideal in $\mathcal{B}\backslash\mathcal{A}$ and $(\mathcal{B}'\backslash\mathcal{A}') \cap \mathcal{W}'$ is a right ideal in $\mathcal{B}'\backslash\mathcal{A}'$. In particular, both sub-2-categories are full and locally full, hence $i_{(\mathcal{B}\backslash\mathcal{A}) \cap \mathcal{W}}$ is an iso. □

Definition 4.5.1 *Let $\mathbb{M}$ be a model category. A weak pushout square in $\mathbb{M}$ is a commuting square such that the comparison map from the inscribed pushout is a weak equivalence:*



Lemma 4.5.2 *The image under N_2 of a pushout square of a skew immersion along an arbitrary 2-functor is a weak pushout square.*

Proof. Consider



By remark 4.4.1, $N_2(J)$ is part of a deformation-retraction in **sSet** and hence an acyclic cofibration. $N_2(J')$ is an acyclic cofibration for the same reason. Since the latter are preserved by pushouts in any model category, N_2 carries square (1) to a weak pushout square by 2-of-3.

On the other hand, N_2 carries square (2) to a pushout square by lemma 4.2.2. Now $\mathcal{B}\backslash\mathcal{A} \cong \mathcal{B}'\backslash\mathcal{A}'$ by corollary 4.2.1 and $(\mathcal{B}\backslash\mathcal{A}) \cap \mathcal{W} \cong (\mathcal{B}'\backslash\mathcal{A}') \cap \mathcal{W}'$ by lemma 4.5.1. Moreover,

$$V \circ \omega_{\mathcal{W}} = \omega_{\mathcal{W}'}$$

and

$$W \circ \omega_{\mathcal{B}} = \omega_{\mathcal{B}'}$$

by construction of the pushout squares (1) and (3) (c.f. remark 2.2.1). Hence the joint square (2) (3) also becomes a pushout square under N_2 by lemma 4.2.2.

But then square (3) is also transformed in a pushout square by N_2 as a consequence of the pushout lemma and the assertion follows applying the glueing lemma. □

Lemma 4.5.3 *Let $A \subseteq B$ be an inclusion of posets. $C_2N_1(A \hookrightarrow B)$ is a left ideal if A is down-closed and is a right ideal if A is upper-closed.*

Proof. Immediate. □

Lemma 4.5.4 *Let $A \subseteq B$ be an inclusion of posets with A down-closed. Let $\uparrow A$ be A's upper-closure. If $i : A \hookrightarrow \uparrow A$ admits a retraction r such that $(i \circ r)(x) \leq x$ for all $x \in \uparrow A$, then $C_2N_1(A \hookrightarrow B)$ is a skew immersion.*

Proof. By lemma 4.5.3, $C_2N_1(A)$ is a left ideal and $C_2N_1(\uparrow A)$ a right ideal. It is easy to see that the inclusion $\overline{i} : C_2N_1(A) \hookrightarrow C_2N_1(\uparrow A)$ admits a retraction $\overline{r} : C_2N_1(\uparrow A) \to C_2N_1(A)$ and that there is the family $\left(\varepsilon_x : \left(\overline{i} \circ \overline{r}\right) x \to x\right)$ given by the inequalities $(i \circ r)(x) \leq x$. Since the 2-categorification of a poset is a locally ordered 2-category, this family determines a distortion. □

Lemma 4.5.5 *Let $f :$ **Ord** $\to$ **Ord** be the functor assigning to an order the order of its non-empty totally ordered finite subsets, ordered by inclusion. Let*

$$H_{k,n} \stackrel{def.}{=} f([n]) \setminus \{(0, \ldots n), (0, \ldots, k-1, k+1, \ldots, n)\}$$

Then

$$Sd^2\left(\Lambda^k[n]\right) = N_1(f(H_{k,n}))$$

and

$$Sd^2(\Delta[n]) = N_1\left(f^2([n])\right)$$

Proof. The subdivision of a simplicial complex is the nerve of its poset of non-degenerate faces. □

Lemma 4.5.6 *Let f be as in lemma 4.5.5 and P be a finite connected poset with a greatest element $\top$. Let further $k \in P$ be a maximal element of $P \setminus \top$ and $P_k \stackrel{def.}{=} P \setminus \{k, \top\}$. Finally, let P's k-horn be given by $H_{P,k} \stackrel{def.}{=} f(P_k)$ and P's k-collar be given by $C_{P,k} \stackrel{def.}{=} f(P) \setminus \{(\top), (k), (k, \top)\}$. Then*

$$\uparrow H_{P,k} = C_{P,k}$$

and the assignment

$$\begin{array}{rlcl} r: & C_{P,k} & \to & H_{P,k} \\ & x & \mapsto & \max((\downarrow x) \cap H_{P,k}) \end{array}$$

determines a retraction such that $r(x) \subseteq x$ for all $x \in C_{P,k}$.

Proof. By the very definition, for any list $x \in C_{P,k}$ there is a list $x' \in H_{P,k}$ such that $x' \subseteq x$, hence the first assertion and also $\emptyset \neq (\downarrow x) \cap H_{P,k}$. The latter has a greatest element for

1. if $x \in H_{P,k}$ then $\max\left((\downarrow x) \cap H_{P,k}\right) = x$;
2. if $x \notin H_{P,k}$ then, by hypothesis on k, there are lists $x' \in H_{P,k}$ and $x'' \in \{(\top)(k), (k, \top)\}$ such that x is the concatenation $x = x' * x''$ hence $\max\left((\downarrow x) \cap H_{P,k}\right) = x'$.

□

Lemma 4.5.7 *Let $i_{k,n} : \Lambda^k[n] \to \Delta[n]$ be a horn inclusion. Then $C_2\left(Sd^2(i_{k,n})\right)$ is a skew immersion.*

Proof. Clearly, $f(H_{k,n}) \subseteq f^2([n])$ is down-closed. The assertion readily follows by lemma 4.5.5, lemma 4.5.6 and lemma 4.5.4. □

Theorem 4.5.1 *$Ex^2 \circ N_2$ creates a model structure on 2-Cat.*

Proof. We need to show that the conditions of proposition 3.4.1 are satisfied. Condition (i) holds since **2-Cat** is locally finitely presentable (c.f. remark 3.3.1). Condition (ii) is a well-known fact about the standard model category structure on **sSet**.

To verify condition (iii), observe that for any ordinal λ and any λ-sequence $X : \lambda \longrightarrow$ **2-Cat**

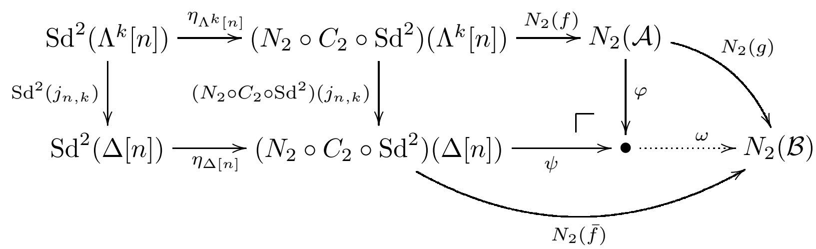
$$\begin{aligned}
(\mathrm{Ex}^2 \circ N_2)(\mathrm{colim}_\lambda X)_n &= \mathbf{sSet}\left(\Delta[n], (\mathrm{Ex}^2 \circ N_2)(\mathrm{colim}_\lambda X)\right)\\
&\cong \mathbf{sSet}\left((C_2 \circ \mathrm{Sd}^2)(\Delta[n]), \mathrm{colim}_\lambda X\right)\\
&\cong \mathrm{colim}_\lambda \mathbf{sSet}\left((C_2 \circ \mathrm{Sd}^2)(\Delta[n]), X\right)\\
&\cong \mathrm{colim}_\lambda \mathbf{sSet}\left(\Delta[n], (\mathrm{Ex}^2 \circ N_2)(X)\right)\\
&\cong \mathrm{colim}_\lambda (\mathrm{Ex}^2 \circ N_2)(X)_n
\end{aligned}$$

for all $n \geq 0$. The third equality is due to the fact that **sSet**, as any topos of presheaves, is ($\aleph_0$-) locally presentable so in particular simplicial sets are small with respect to the class of all simplicial morphisms. Since colimits are calculated dimension-wise in **sSet**, it follows that $\mathrm{Ex}^2 \circ N_2$ commutes with colimits of λ-sequences.

To complete the proof we must show that for any pushout diagram in **2-Cat**

$$\begin{array}{ccc}
(C_2 \circ \mathrm{Sd}^2)(\Lambda^k[n]) & \xrightarrow{f} & \mathcal{A} \\
{\scriptstyle (C_2 \circ \mathrm{Sd}^2)(j_{n,k})}\downarrow & \ulcorner & \downarrow{\scriptstyle g} \\
(C_2 \circ \mathrm{Sd}^2)(\Delta[n]) & \xrightarrow[\bar{f}]{} & \mathcal{B}
\end{array}$$

for any $n > 0$ and $0 \leq k \leq n$, $(\mathrm{Ex}^2 \circ N_2)(g)$ is a weak equivalence of simplicial sets. Consider



with ω the comparison morphism. Since $\mathrm{Sd}^2(K)$ is the 1-nerve of a poset for any simplicial set K, the unit maps η are isos by remark 2.4.4, so in particular weak equivalences. Furthermore, there is the obvious commutative diagram

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \mathrm{Sd}^2\Lambda^k[n] \\ {\scriptstyle j_{n,k}}\downarrow & & \downarrow{\scriptstyle \mathrm{Sd}^2(j_{n,k})} \\ \Delta[n] & \longrightarrow & \mathrm{Sd}^2\Delta[n] \end{array}$$

in which the horizontal maps induce homeomorphisms after geometric realization and are therefore weak equivalences. Hence, by 2-of-3, $\mathrm{Sd}^2(j_{n,k})$ is also a weak equivalence. Thus, applying 2-of-3 to the lefthand square of diagram (*), we obtain that $(N_2 \circ C_2 \circ \mathrm{Sd}^2)(j_{n,k})$ is a weak equivalence as well, which implies that φ is a weak equivalence, since acyclic cofibrations are preserved under pushout in any model category.

On the other hand, ω is also a weak equivalence, as $(C_2 \circ \mathrm{Sd}^2)(j_{n,k})$ is a skew immersion by lemma 4.5.7. Thus, $N_2(g) = \omega \circ \varphi$ is a weak equivalence, which implies that $(\mathrm{Ex}^2 \circ N_2)(g)$ is a weak equivalence since Ex preserves the latter, which completes the proof. □

We call the model structure of theorem 4.5.1 the *2-Thomason model structure* since it is conceptually similar to the model structure on **Cat** due to R.W.Thomason [17].

5 Homotopy

Definition 5.0.2 *Let $\mathcal{A}$ be a 2-category and $f, g \in \mathcal{A}_1$.*

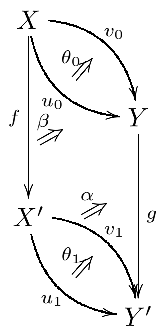
1. *A lax square $(u_0, u_1, \alpha) : f \longrightarrow g$ is given by the diagram*

$$\begin{array}{ccc} X & \xrightarrow{u_0} & Y \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[u_1]{} & Y' \end{array}$$

Let $(v_0, v_1, \beta) : g \longrightarrow h$ be a further lax square. Their pasting composite *is the lax square*

$$(v_0, v_1, \beta) \circledast (u_0, u_1, \alpha) \overset{def.}{=} (v_0 \circ u_0, v_1 \circ u_1, (\beta \circ u_0) \bullet (v_1 \circ \alpha)) : f \longrightarrow h$$

2. *A cylinder* $(\theta_0, \theta_1) : (u_0, u_1, \alpha) \longrightarrow (v_0, v_1, \beta)$ *is given by the diagram*



where $(g \circ \theta_0) \bullet \alpha = \beta \bullet (\theta_1 \circ f)$

Proposition 5.0.1 *Let $\mathcal{A}$ be a 2-category.*

1. *There is a 2-category Cyl($\mathcal{A}$) given by the data*

   – *Objects: morphisms of $\mathcal{A}$;*

   – *Morphisms: lax squares;*

   – *2-cells: cylinders.*

   Composition of morphisms is given by pasting while the operations on 2-cells are those of $\mathcal{A}$ taken componentwise.

2. *The assignments*

$$\begin{array}{rlcl} dom_{\mathcal{A}}: & Cyl(\mathcal{A}) & \longrightarrow & \mathcal{A} \\ & f & \longmapsto & dom(f) \\ & (u_0, u_1, \alpha) & \longmapsto & u_0 \\ & (\theta_0, \theta_1) & \longmapsto & \theta_0 \end{array}$$

$$\begin{array}{rlcl} cod_{\mathcal{A}}: & Cyl(\mathcal{A}) & \longrightarrow & \mathcal{A} \\ & f & \longmapsto & cod(f) \\ & (u_0, u_1, \alpha) & \longmapsto & u_1 \\ & (\theta_0, \theta_1) & \longmapsto & \theta_1 \end{array}$$

   and

$$\begin{array}{rlcl} I_{\mathcal{A}}: & \mathcal{A} & \longrightarrow & Cyl(\mathcal{A}) \\ & X & \longmapsto & id_X \\ & f & \longmapsto & (f, f, id_f) \\ & \alpha & \longmapsto & (\alpha, \alpha) \end{array}$$

   are 2-functorial.

Following Bénabou, we call Cyl$(\mathcal{A})$ the *2-category of cylinders over* $\mathcal{A}$ [2]. The name stems from the "geometry" of 2-cells. Notice that Cyl$(\mathcal{A})$ is a generalization of the familiar category of arrows. The construction is 2-functorial, yet this fact is not relevant for the present development.

Remark 5.0.1 Let $\mathcal{A}$ be a 2-category and $q \in \mathbb{N}$. A normal lax functor $F : [q] \longrightarrow \mathrm{Cyl}(\mathcal{A})$ is determined by

- a morphism $F(k) : F(k)^- \longrightarrow F(k)^+$ for all $0 \leqslant k \leqslant q$;
- a lax square $F(k < l) \overset{def.}{=} (F(k,l)^-, F(k,l)^+, F(k,l)) : F(k) \longrightarrow F(l)$ as in

$$\begin{array}{ccc} F(k)^- & \xrightarrow{F(k,l)^-} & F(l)^- \\ {\scriptstyle F(k)}\downarrow & \overset{F(k,l)}{\nearrow} & \downarrow{\scriptstyle F(l)} \\ F(k)^+ & \xrightarrow[F(k,l)^+]{} & F(l)^+ \end{array}$$

  for all $0 \leqslant k < l \leqslant q$;
- a cylinder

$$F(k < l < m) \overset{def.}{=} \big((F(k,l,m)^-, F(k,l,m)^+\big) : F(l < m) \circledast F(k < l) \longrightarrow F(k < m)$$

  for all $0 \leqslant k < l < m \leqslant q$

such that

$$F(k,m,n)^s \bullet (F(m,n)^s \circ F(k,l,m)^s) = F(k,l,n)^s \bullet (F(l,m,n)^s \circ F(k,l)^s)$$

for all $s \in \{-,+\}$ and $0 \leqslant k < l < m < n \leqslant q$. ★

Definition 5.0.3 *Let $F, G :\ \mathcal{A} \to \mathcal{B}$ be 2-functors. A lax transformation $\alpha :\ F \Rightarrow G$ is given by*

- *a morphism $\alpha_X :\ F(X) \to G(X)$ for each $X \in \mathcal{A}$ and*
- *a 2-cell*

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & \swarrow{\scriptstyle \alpha_f} & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\alpha_Y]{} & G(Y) \end{array}$$

  for each morphism $f :\ X \to Y$

such that

(i) $\alpha_{f'} \bullet (G(\theta) \circ \alpha_X) = (\alpha_Y \circ F(\theta)) \bullet \alpha_f$ *for each 2-cell $\theta :\ f \Rightarrow f' : X \to Y$;*

(ii) $(\alpha_g \circ F(f)) \bullet (G(g) \circ \alpha_f) = \alpha_{g \circ f}$ *for each $f :\ X \to Y$ and $g :\ Y \to Z$.*

Proposition 5.0.2 *The following are equivalent* *(i) There is a lax transformation* $\alpha : F \Rightarrow G$;

(ii) There is a 2-functor $\overline{\alpha} : \mathcal{A} \to Cyl(\mathcal{B})$ *such that*

$$\begin{array}{ccc} & & Cyl(\mathcal{B}) \\ & \overset{\bar{\alpha}}{\nearrow} & \downarrow{\scriptstyle \langle dom_{\mathcal{B}}, cod_{\mathcal{B}} \rangle} \\ \mathcal{A} & \xrightarrow[\langle F,G \rangle]{} & \mathcal{B} \times \mathcal{B} \end{array}$$

commutes.

Our 2-category of cylinders is in fact the strict case of Bénabou's *bicategory of cylinders*. He defined lax transformations for lax functors among bicategories in terms of this classifing device [2].

Definition 5.0.4 *Let* $\mathbb{M}$ *be a model category and* $P, B \in \mathbb{M}$. P *is a path object on* B *if there is a a morphism* $p_B : P \longrightarrow B \times B$ *and commuting diagram*

$$\begin{array}{ccc} & & P \\ & \overset{\sim}{\nearrow} & \downarrow{\scriptstyle p_B} \\ B & \xrightarrow[\Delta]{} & B \times B \end{array}$$

Proposition 5.0.3 $Cyl(\mathcal{A})$ *is a path object on* $\mathcal{A}$ *in the 2-Thomason model structure.*

Proof. It is immediate that

$$\begin{array}{ccc} & & \mathrm{Cyl}(\mathcal{A}) \\ & \overset{I_{\mathcal{A}}}{\nearrow} & \downarrow{\scriptstyle \langle \mathrm{dom}_{\mathcal{A}}, \mathrm{cod}_{\mathcal{A}} \rangle} \\ \mathcal{A} & \xrightarrow[\Delta]{} & \mathcal{A} \times \mathcal{A} \end{array}$$

commutes. On the other hand, there is a simplicial homotopy

$$\begin{array}{ccccc} N_2\left(\mathrm{Cyl}(\mathcal{A})\right) & \xrightarrow{i_0} & N_2\left(\mathrm{Cyl}(\mathcal{A})\right) \times [1] & \xleftarrow{i_1} & N_2\left(\mathrm{Cyl}(\mathcal{A})\right) \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow{\scriptstyle H} & \swarrow {\scriptstyle N_2(I_{\mathcal{A}} \circ \mathrm{dom}_{\mathcal{A}})} & \\ & & N_2\left(\mathrm{Cyl}(\mathcal{A})\right) & & \end{array}$$

It can be constructed as a family

$$H_i^n : N_2(\mathrm{Cyl}(\mathcal{A}))_n \longrightarrow N_2(\mathrm{Cyl}(\mathcal{A}))_{n+1}, 0 \leqslant i \leqslant n, n \geqslant 0$$

enjoying the well-known properties. Let

$$\mathrm{cart}_F(s,t) \overset{def.}{=} \left(F(s,t)^-, F(t) \circ F(s,t)^-\right) : \mathrm{id}_{F(s)^-} \longrightarrow F(t)$$

and

$$\mathrm{dom}_F(s,t) \overset{def.}{=} \left(F(s,t)^-, F(s,t)^-\right) : \mathrm{id}_{F(s)^-} \longrightarrow \mathrm{id}_{F(t)^-}$$

be lax squares for $0 \leqslant s < t \leqslant n$. Let $F \in N_2(\mathrm{Cyl}(\mathcal{A}))_n \cong \mathbf{NLax}\left([n], \mathrm{Cyl}(\mathcal{A})\right)$. The normal lax functor

$$H_i^n(F) : [n+1] \longrightarrow \mathrm{Cyl}(\mathcal{A})$$

is given by the following data:

– $H_i^n(F)(p) \overset{def.}{=} \begin{cases} \mathrm{id}_{F(p)^-} & p \leqslant i \\ F(p-1) & p > i \end{cases}$

– $H_i^n(F)(p<q) \overset{def.}{=} \begin{cases} \mathrm{dom}_F(p,q) & p,q \leqslant i \\ \mathrm{cart}_F(p,q-1) & p \leqslant i, q > i \\ F(p-1 < q-1) & p,q > i \end{cases}$

– $H_i^n(F)(p<q<r) \overset{def.}{=}$

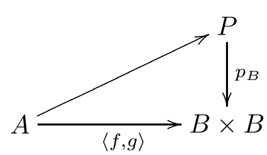
$$\begin{cases} (F(p,q,r)^-, F(p,q,r)^-) & p,q,r \leqslant i \\ (F(p,q,r-1)^-, F(r-1) \circ F(p,q,r-1)^-) & p,q \leqslant i \wedge r > i \\ (F(p,q-1,r-1)^- \ , & p \leqslant i \wedge q,r > i \\ (F(r-1) \circ F(p,q-1,r-1)^-) \bullet (F(q-1,r-1) \circ F(p,q-1)^-)) & \\ F(p-1 < q-1 < r-1) & p,q,r \geqslant i \end{cases}$$

(c.f. remark 5.0.1). A laborious yet straightforward calculation shows that the coherence conditions hold and that the H_i^n's commute with faces and degeneracies as required. It thus follows (by functoriality) that there is a homotopy

$$|N_2\left(I_{\mathcal{A}} \circ \mathrm{dom}_{\mathcal{A}}\right)| \sim \mathrm{id}_{|N_2(\mathrm{Cyl}(\mathcal{A}))|}$$

hence $I_{\mathcal{A}}$ is a homotopy equivalence so in particular a weak equivalence. □

Definition 5.0.5 *Let $\mathbb{M}$ be a model category. Given $f, g : A \to B$, there is a right homotopy $f \simeq g$ if there is a path object over B such that $\langle f, g \rangle$ factors through p_B as in*



Theorem 5.0.2 *Lax transformations are right homotopies in the 2-Thomason model structure.*

Proof. Direct consequence of proposition 5.0.3. □

Remark 5.0.2 Reversing the direction of the 2-cell in the definition of a lax square yields the dual notion of *oplax square* and those of *opcylinder* and of *oplax transformation* respectively. It is easy to see that oplax cylinders are path objects and, consequently, oplax transformations are right homotopies in the 2-Thomason model structure. ★

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