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Standard bases for local rings of branches and their modules of differentials. (English summary)
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A topological and analytical classification of planar branches introduces some invariant numbers associated with a given planar branch that distinguishes the type of singularity of the branch. For example, a planar branch in the ordinary plane is cut out by two formal power series in x and y, and one attempt to classify the branch is by performing successive coordinate transformations to write the resulting expression in a normal form in which the type and singularity can be readily read off. Abstracting a little further, as is now widely done in algebraic geometry, a branch over an algebraically closed field K is a prime ideal $C = \langle f_1, f_2, \dots, f_r \rangle$ of the ring of formal power series $K[[X_1, X_2, \dots, X_n]]$ such that its local ring has Krull dimension one. Then the question is how to provide the algebraic classification and the singularity type of this local ring.

The paper proposes a computational technique that is based on the extension of classical polynomial ring theory over a field to more general rings, i.e., the generalization of algorithms such as the Buchberger division process and the Gröbner basis construction. The main difference here is that the local ring is not necessarily generated over a field and this accounts for some difficulties in the stopping condition for the construction of the basis. These bases are called standard bases.

The singularity type of the branch is closely linked to a specific valuation on the implicitization of the branch. For instance, $x^2 = y^3$ admits $x = t^3$ and $y = t^2$ as an implicit parametrization, and the valuation is then given by the degree of t. The standard basis algorithm proposed gives a set of formal power series representations associated with the branch. The monomials of highest order of each member of the standard basis reveal the type of singularity. The conductor—an important topological invariant of the branch—is the lowest number c of the semigroup of valuation Γ. This semigroup is characterized by the lowest degree of t in the components of the implicitization of the branch. Whenever an element of the local ideal of the branch has valuation $c_1 \geq c$, then necessarily there exists another element in the ideal with valuation $c_1 + 1$. The main proposition is the following: $G = \{h_0, \dots, h_g\} \subset \mathcal{O}$ is a minimal standard basis for $\mathcal{O}$ if, and only if, $\{v(h_0), \dots, v(h_g)\}$ is a minimal set of generators of the semigroup of values Γ of $\mathcal{O}$. (Here, $v(\cdot)$ stands for the valuation in t.) Two irreducible algebroid curves are said to be equisingular if they have the same semigroup of values Γ.

As an example for computing Γ, consider the branch given in implicit form $x = t^8$, $y = t^{10} + t^{13}$ and $z = t^{12} + at^{15}$. Since $8, 10, 12 \in \Gamma$ and $27 = v(x^3 - z^2) \in \Gamma$ if $a \neq 0$ or $23 = v(y^2 - xz) \in \Gamma$ if $a = 0$, it is simple to verify that the consecutive integers $54, 55, \dots, 61$ belong to Γ. Hence the conductor $c \leq 54$. The minimal S-processes (i.e., analogs of S-polynomials in classical Gröbner bases) obtained by the proposed algorithm are $y^4 - x^5$, $y^2 - xz$, $z^2 - x^3$, $y^2z - x^4$, $z^3 - x^2y^2$ and $z^4 - x^6$. Then, reducing modulo x, y, z using the implicitization, it is shown that $F_1 = \{t^8, t^{10} + t^{13}, t^{12}, 2t^{23} + t^{26}\}$ so that the set of degrees of the leading monomials becomes the generators of Γ, i.e., $\Gamma = \langle 8, 10, 12, 23\rangle$. Now, adding elements of Γ together gives rise to the sequence $8, 10, 12, 16, 18, 20, 22, 23, 24, 26, 28, 30, 31, 32, 34, 35, \ldots$, that is, all consecutive integers beyond 34 are generated through such combinations, so that the conductor is $c = 34$, which is smaller than 54.

Similar constructions are obtained for the module of Kähler differentials, giving a powerful procedure for computing Milnor's μ and Tyurina's τ numbers (analytical invariants). Let $\Lambda = v(\mathcal{O}d\mathcal{O}/\mathcal{T})$, where $\mathcal{T}$ is the torsion submodule of the module of differentials $\mathcal{O}d\mathcal{O}$.

Using the relation

$$\tau = l(\mathcal{T}) = \mu - \#(\Lambda \setminus \Gamma), \tag{1}$$

where $\#$ indicates the number of elements in the set, together with its relation with the conductor, $\mu = c$, the computational procedures proposed can be used to compute τ. For example, consider C as the curve parameterized by $x = t^8$, $y = t^{12} + t^{13}$ whose associated semigroup of values is $\Gamma = \langle 8, 12, 25\rangle$ with conductor $c = 80$. Computing the differentials dx, dy, and dz where $z = y^2 - x^3$, the proposed standard basis algorithm provides: $\omega_1 = 3ydx - 2xdy$, $\omega_2 = 8xdz - 25zdx$, $\omega_3 = 12ydz - 25zdy$, and ω_4 as the reduction of $25x^2zdx - 8y^2dz$ by $202/25zdz$ and ω_5 as the reduction of $yzdx + 8x^3\omega_1$ by $6z\omega_1$. (The reduction resorts to a dedicated algorithm that goes along with the one used for the basis construction.) Therefore, after implicitization, $\psi(\omega_1) = -2t^{20}$, $\psi(\omega_2) = 8t^33$, $\psi(\omega_3) = -38t^{37} - 13t^{38}$, $\psi(\omega_4) = \frac{204}{25}t^{50} + \frac{52}{25}t^{51}$ and $\psi(\omega_5) = -4t^{46}$, from which

$$\begin{aligned}\tau = c - \#(\Lambda \setminus \Gamma) = \\ 80 - \#\{21, 29, 34, 38, 42, 46, 47, 51, 54, 55, 59, 63, 67, 71, 79\} = 65.\end{aligned}$$

The paper gives all the details on the above ideas with complete descriptions of algorithms and proofs. It shows how existing identities such as (1) can be put to good use so as to guide the algorithm in obtaining a standard basis. Such algorithms are useful in various situations for establishing key topological and analytical invariants.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.