MR2322810 (Review) 53A17

Yüce, Salim; Kuruoğlu, Nuri

The Holditch sickles for the open homothetic motions. (English summary)

Appl. Math. E-Notes **7** (2007), 175–178 (*electronic*).

Back in 1858, H. Holditch stated and proved an interesting theorem. An oval path is given together with a rigid rod of length l. Each end of the rod (say A and B) touches the oval. Whenever the rod completes a full turn while moving along the oval path, a given fixed point in the rod (say at the distance a from A and b from B so that $l = a + b$) also describes a closed curve, but one which is not necessarily convex. The surface part between the center of the oval and the curve traced out by the fixed point on the rod at distance a from A is then removed from the surface bounded by the oval curve. This yields Holditch's ring, the surface of which obeys a simple and elegant formula

$$F = ab\pi.$$

Such a result has been extended to various other curves that are not necessarily convex (as in the case for the oval) or that are not closed (for instance for infinite "rails" on which the rod slides). Corresponding formulas for what can be defined as "the remaining ring" which is obviously not a ring anymore in case the curve is given by a pair of infinite rails. For instance, H. Pottmann [Arch. Math. (Basel) **44** (1985), no. 4, 373–378; MR0788954 (86i:53006)] has shown that a rotational angle corresponding to the two extreme excursions of parallel tangents δ can be defined. Then he also showed how to define the ring, much in the same way as for the classical Holditch ring based on the surface separating the guiding curve (one of the rails) and the curve traced by the point fixed on the rod. One difficulty is that the curve can loop back on itself and therefore parts of the surface need to be added with a negative sign. Nevertheless, the surface still keeps a simple expression which is

$$F = ab\delta/2.$$

In the paper under review, a further generalization is obtained for movements that not only follow a given curve which is not closed, but for which the linked frame, attached to the selected moving point in-between A and B, distorts according to a homothetic scale $h(t)$ (i.e. a sort of continuous "zooming in and out" motion). Therefore, two moving frames in the Euclidean planes are defined, one for the fixed reference frame $E' = \{O'; \mathbf{e}'_1, \mathbf{e}'_2\}$ and one for the moving referential $E = \{O; \mathbf{e}_1, \mathbf{e}_2\}$. For a closed homothetic motion, the moving frame is related to the fixed one through

$$\mathbf{e}_1 = \cos(\varphi(t))\mathbf{e}'_1 + \sin(\varphi(t))\mathbf{e}'_2,$$
$$\mathbf{e}_2 = -\sin(\varphi(t))\mathbf{e}'_1 + \cos(\varphi(t))\mathbf{e}'_2,$$

and the coordinates $\mathbf{x}' = (x'_1\ x'_2)^T$ are related to $\mathbf{x} = (x_1\ x_2^T)$ through

$$\mathbf{x}' = h(t)\mathbf{x} - \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix},$$

where $h(t)$ is the homothetic periodic scale, $\varphi(t)$ is a periodic rotation and $u_1(t)$ and $u_2(t)$ are periodic translation functions, that is, there exists $T > 0$ such that $h(t+T) = h(t)$, $\varphi(t+T) = \varphi(t) + 2\varphi\nu$, $u_1(t+T) = u_1(t)$ and $u_2(t+T) = u_2(t)$. The integer $\nu \in \mathbb{Z}$ is called the rotation number of the closed planar homothetic motion. An open homothetic motion is one for which one of the above periodicity does not hold anymore and therefore leads to a possible unbounded motion. For the open homothetic case, the paper gives the area of the associated ring as

$$F = h^2(t_0)ab\delta/2$$

for a certain $t_0 \in [0, T]$. It is shown that t_0 exists but it is not constructed explicitly (it is a consequence of the mean-value theorem).

The paper also gives a formula for a spatial generalization of the above result leading to the volume cut-out by a straight line guided by convex bodies. The region defined as the spatial Holditch sickle $S \subset \mathbb{R}^3$ is the point set bounded by the ruled surface and the cylinder parts between two horizontal planes $z = 0$ and $z = k$. The volume traced out is shown to be equal to

$$V = h^2(t_0)\frac{\delta}{2}k^2 \cot^2 \beta$$

for a certain $t_0 \in [0, T]$ and where β is the constant ruling angle $\arcsin(k/hl)$, l being, as above, the length of the line.

Reviewed by *Philippe A. Müllhaupt*