A variant of the induction theorem for Springer representations

Toshiaki Shoji*

Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya 464-8602, Japan

Abstract. Let G be a simple algebraic group over $\mathbf{C}$ with the Weyl group W. For a unipotent element $u \in G$, let $\mathcal{B}_u$ be the variety of Borel subgroups of G containing u. Let L be a Levi subgroup of a parabolic subgroup of G with the Weyl subgroup W_L of W. Assume that $u \in L$ and let $\mathcal{B}_u^L$ be a similar variety as $\mathcal{B}_u$ for L. For a certain choice of L, $u \in L$ and $e \ge 1$, we describe the W-modules $\bigoplus_{n\equiv k \mod e} H^{2n}(\mathcal{B}_u)$ for $k = 0, \dots, e-1$, in terms of the W_L-module $H^*(\mathcal{B}_u^L)$ with some additional data, which is a refinement of the induction theorem due to Lusztig. As an application, we give an explicit formula for the values of Green functions at root of unity, in the case where u is a regular unipotent element in L.

0. Introduction

Let G be a connected reductive group over an algebraically closed field k, and W the Weyl group of G. For a unipotent element $u \in G$, let $\mathcal{B}_u$ be the variety of Borel subgroups containing u. According to Springer [Sp2], Lusztig [L1], W acts naturally on the l-adic cohomology group $H^n(\mathcal{B}_u) = H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$, the so-called Springer representations of W. Assume that $k = \mathbf{C}$, or the characteristic p of k is good. Then it is known that $H^{\text{odd}}(\mathcal{B}_u) = 0$. We consider the graded W-module $H^*(\mathcal{B}_u) = \bigoplus_{n\ge 0} H^{2n}(\mathcal{B}_u)$. Let L be a Levi subgroup of a parabolic subgroup of G. Let W_L be the Weyl group of L, which is naturally a subgroup of W. If $u \in L$, the variety $\mathcal{B}_u^L$ is defined by replacing G by L, and we have a graded W_L-module $H^*(\mathcal{B}_u^L)$.

Lusztig proved in [L3] an induction theorem for Springer representations, which describes the W-module structure of $H^*(\mathcal{B}_u)$ in terms of the W_L-module structure of $H^*(\mathcal{B}_u^L)$, in the case where $u \in L$. However in this theorem, the information on the graded W-module structure is eliminated. In this paper, we try to recover partly the graded W-module structure, i.e., for a fixed positive integer e, we consider the W-modules $V_{e,k} = \bigoplus_{n\equiv k \mod e} H^{2n}(\mathcal{B}_u)$ for $k = 0, \dots, e-1$. Let G be a simple group modulo center defined over $\mathbf{C}$. We show, under a certain choice of L, u and e, that the W-module $V_{e,k}$ can be described in terms of the graded W_L-module $H^*(\mathcal{B}_u^L)$

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with some additional data. In particular, we see that $\dim V_{e,k}$ is independent of the choice of k.

In the case where $u = 1$, $H^*(\mathcal{B}_u)$ is isomorphic, as a graded W-module, to the coinvariant algebra of W. In this case $V_{e,k}$ has been studied by many authors, by Stembridge [St] for e corresponding to the regular elements in W, by Morita and Nakajima [MN1] for $W = \mathfrak{S}_n$ with e such that $1 \le e \le n$, and by Bonnafé, Lehrer and Michel [BLM] for complex reflection groups W in the most general framework. Our result partly covers the result of [BLM]. For general $u \ne 1$, Morita and Nakajima [MN2] considered certain types of unipotent elements for $G = GL_n$, which is a special case of ours.

The proof of the induction theorem in [L3] is done by passing to the finite field $\mathbf{F}_q$, and using a certain specialization argument $q \mapsto 1$ together with the properties of Deligne-Lusztig's virtual character $R_T(1)$. Our argument is a variant of that in [L3]. We use a specialization $q \mapsto \zeta$, where ζ is a primitive e-th root of unity. Thus our argument is closely related to the values of Green functions at root of unity. In the case where u is a regular unipotent element in L, we obtain an explicit formula for such values, which is regarded as a generalization of the result by Lascoux, Leclerc and Thibon [LLT] for the case of Green polynomials of GL_n.

1. The statement of the main result

1.1. Let k be an algebraic closure of a finite field with $\mathrm{ch}(k) = p > 0$ or the complex number field $\mathbf{C}$. Let G be a connected reductive group G over k. Let $\mathcal{B}$ be the variety of Borel subgroups of G, and W the Weyl group of G. For any $g \in G$, put $\mathcal{B}_g = \{B' \in \mathcal{B} \mid g \in B'\}$. We consider the Springer representations of W on $H^n(\mathcal{B}_g, \bar{\mathbf{Q}}_l)$ (or on $H^n(\mathcal{B}_g, \mathbf{C})$ in the case where $k = \mathbf{C}$).

Let L be a Levi subgroup of a parabolic subgroup P of G. The Weyl group W_L of L is naturally identified with a subgroup of W. Let $\mathcal{B}^L$ be the variety of Borel subgroups of L. For a unipotent element $u \in L$, we consider $\mathcal{B}^L_u = \{B' \in \mathcal{B}^L \mid u \in B'\}$. Thus we have a W_L-module $H^n(\mathcal{B}^L_u, \bar{\mathbf{Q}}_l)$, and a W-module $H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$. The induction theorem for Springer representations asserts that

$$\sum_{n\ge 0}(-1)^n H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l) = \mathrm{Ind}_{W_L}^W\bigl(\sum_{n\ge 0}(-1)^n H^n(\mathcal{B}^L_u, \bar{\mathbf{Q}}_l)\bigr) \tag{1.1.1}$$

as virtual W-modules.

Remark 1.2. The induction theorem was stated in [AL], with a brief indication of the proof, in the case where $k = \mathbf{C}$, and was proved in [L3] for any k. Note that if p is good, the unipotent classes in G are parametrized in the same way as the case of $k = \mathbf{C}$, independent of p. Moreover in that case, it is known that $H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l) = 0$ for odd n. Then the algorithm of computing Green functions implies that the W-module structure of $H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$ is independent of p. Thus by a general principle $H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$ is isomorphic to the W-module $H^n(\mathcal{B}_{u'}, \mathbf{C})$, where $u', \mathcal{B}_{u'}$ are the corresponding objects in the algebraic group $G_{\mathbf{C}}$ over $\mathbf{C}$. In what follows, we express $H^n(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$ or $H^n(\mathcal{B}_{u'}, \mathbf{C})$ by $H^n(\mathcal{B}_u)$ by abbreviation.

1.3. Assume that $k = \mathbf{C}$. We consider the following variant of the induction theorem. Let $\varGamma$ be a cyclic group of order e generated by a. Let ζ be a primitive e-th root of unity in $\mathbf{C}$. Let $V = \bigoplus_{n \ge 0} V_n$ be a graded W-module. Then V turns out to be a $\varGamma \times W$-module by defining the action of $\varGamma$ on V by $ax = \zeta^n x$ for $x \in V_n$. We denote by $V^{(\zeta)}$ the thus obtained $\varGamma \times W$-module V.

For $u \in L$, we consider the graded W_L-module $H^*(\mathcal{B}^L_u) = \bigoplus_{n \ge 0} H^{2n}(\mathcal{B}^L_u)$, where the degree n part is given by $H^{2n}(\mathcal{B}^L_u)$, and similarly we consider the graded W-module $H^*(\mathcal{B}_u) = \bigoplus_{n \ge 0} H^{2n}(\mathcal{B}_u)$. Let $\varGamma$ be as before. We choose $\varGamma$ such that $\varGamma \subset N_W(W_L)$, and consider the semidirect product $\widetilde{W}_L = \varGamma \ltimes W_L$. We assume that the W_L-module $H^n(\mathcal{B}^L_u)$ can be extended to a $\widetilde{W}_L$-module for each n. (In the case where $a \in Z_W(W_L)$, we have $\widetilde{W}_L = \varGamma \times W_L$. In this case, one can choose a trivial extension to $\widetilde{W}_L$, i.e., we may asssume that $\varGamma (\subset \widetilde{W}_L)$ acts trivially on $H^*(\mathcal{B}^L_u)$.) Then one can define a $\varGamma \times \widetilde{W}_L$-module $H^*(\mathcal{B}^L_u)$ as above, replacing W_L by $\widetilde{W}_L$, which we denote by $H^*(\mathcal{B}^L_u)^{(\zeta)}$. (When we need to distinguish the group $\varGamma$ as the first factor of $\varGamma \times \widetilde{W}_L$ from the subgroup of $\widetilde{W}_L$, we write the latter as $\varGamma_0$.) $\varGamma \times W$-module $H^*(\mathcal{B}_u)^{(\zeta)}$ is defined as before. Put $V^{(\zeta)} = H^*(\mathcal{B}^L_u)^{(\zeta)}$, and let $V^{(\zeta)}_n$ be the degree n-part of $V^{(\zeta)}$. Let us consider the induced W-module

$$\operatorname{Ind}_{W_L}^W V^{(\zeta)} = \bigoplus_{w \in W/W_L} w \otimes V^{(\zeta)}.$$

Then $\operatorname{Ind}_{W_L}^W V^{(\zeta)}$ turns out to be a $\varGamma \times W$-module by defining the action of $\varGamma$ by $b(w \otimes x) = \zeta^n (wb^{-1} \otimes bx)$ for $b \in \varGamma_0, x \in V^{(\zeta)}_n$, which we denote by $\varGamma\text{-}\operatorname{Ind}_{W_L}^W V^{(\zeta)}$.

1.4. In the remainder of this paper, we assume that G is simple modulo center. Let $T \subset B$ be a pair of maximal torus and a Borel subgroup of G. Put $W = N_G(T)/T$. Let L be a Levi subgroup of a parabolic subgroup P of G containing B such that $L \supset T$. We have $W_L = N_L(T)/T$. Let $\varPhi \subset X(T)$ be a root system for G with respect to T, with a simple root system Π (with respect to B), where $X(T)$ is the character group of T. We denote by $\varPhi_L$ the sub system of $\varPhi$ corresponding to L with the simple root system $\Pi_L \subset \Pi$. Let Π' be the set of simple roots which are orthogonal to Π_L with respect to the standard inner product on $V = \mathbf{R} \otimes_{\mathbf{Z}} X(T)$. We denote by L' the Levi subgroup containing T corresponding to Π'. Let $W_{L'} = N_{L'}(T)/T$ be the Weyl group of L'. Then we have $W \supset W_L \times W_{L'}$, and so $W_{L'} \subset N_W(W_L)$.

We recall here the notion of regular elements of reflection groups due to Springer [Sp1]. Let W be a reflection group in $GL(V)$. A vector $v \in V$ is called regular if v is not contained in any reflecting hyperplane in V. An element $a \in W$ is called regular if a has an eigenvector v which is a regular element in V. If $av = \zeta v$, with ζ a primitive e-th root of unity, then the order of a is equal to e ([Sp1, 4.2]). In particular, if a is regular of order e, there exists an eigenvalue ζ which is a primitive e-th root of unity.

The regular elements $a \in W$ in the case of classical groups are given as follows (cf. [Sp1]).

Type A_{n-1}. In this case $W = \mathfrak{S}_n$ and there are two types of regular elements.

(a) e is a divisor of n, and a is an n/e-product of (disjoint) e-cycles in $\mathfrak{S}_n$.

(b) e is a divisor of $n-1$, and a is an $(n-1)/e$-product of e-cycles in $\mathfrak{S}_n$

Type B_n. There are two types of regular elements.

(a) e is an odd divisor of n, and a is an n/e-product of positive cycles of length e.

(b) e is an even divisor of $2n$, and a is a $2n/e$-product of negative cycles of length $e/2$.

Type D_n. In this case there are 4 types of regular elements.

(a) e is an odd divisor of n, and a is a product of positive cycles of length e.

(b) e is an odd divisor of $n-1$, and a is a product of positive cycle of length 1 and $(n-1)/e$ positive cycles of length e.

(c) n is even, and e is an even divisor of n. a is a product of negative cycles of length $e/2$.

(d) e is an even divisor of $2n-2$, and a is a product of $(n-1)/e$ negative cycles of length $e/2$ and one cycle of length 1, which is positive or negative according as $(2n-2)/e$ is even or odd.

Regular elements in the exceptional Weyl groups are listed in [Sp1].

Returning to the original setting, we consider the subgroups $W_L, W_{L'}$ of W. Let V' be the subspace of V generated by $\Pi_{L'}$. $W_{L'}$ is realized as a reflection group on V'. Assume that a is a regular element of $W_{L'}$ of order e. Let ζ be a primitive e-th root of unity, and $V(a,\zeta)$ the eigensubspace of a in V with eigenvalue ζ. Since a is regular, $V(a,\zeta)$ is not contained in any reflecting hyperplane H_α for $\alpha \in \varPhi_{L'}$. We say that a is L-regular if $V(a,\zeta)$ is not contained in any H_α for $\alpha \in \varPhi - \varPhi_L$. If L is the torus T, all the regular elements are L-regular. But if $L \ne T$, regular elements are not necessarily L-regular. For example, if L is not simple modulo center, regular elements in $W_{L'}$ are not L-regular in many cases. In the case where L is simple modulo center, L-regular elements are classified as follows.

Lemma 1.5. *Assume that L is simple modulo center.*

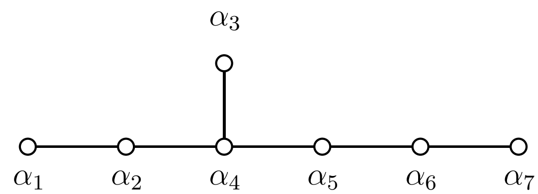
(i) *If W is of type A_n, B_n, D_n, take L such that W_L is of the same type as W of rank m, and $W_{L'}$ is of type A_{n-m-1}. Then a regular element of $W_{L'}$ of type* (a) in 1.4 is L-regular.

(ii) If W is of type G_2, F_4 or E_8, there does not exist L-regular elements for any $L \ne T$.

(iii) Assume that W is of type E_6 or E_7. Let $\Pi = \{\alpha_1, \dots, \alpha_7\}$ (resp. $\{\alpha_1, \dots, \alpha_6\}$) be the set of simple roots in E_7 (resp. in E_6) as in the figure. Take $\Pi_L = \{\alpha_k, \alpha_{k+1}, \dots, \alpha_7\}$ (resp. $\{\alpha_k, \alpha_{k+1}, \dots, \alpha_6\}$) for $k \ge 3$. Then $W_{L'}$ is of type A_j or of type $A_j + A_1$ for some j except the case where W is of type E_7 and $\Pi_L = \{\alpha_7\}$, in which case $\Pi_{L'}$ is of type D_5. In the former case, we choose a a regular element of type (a) for type A, and in the latter case, we choose a a regular element of type (a) for type D in 1.4, respectively. Then a is L-regular.

Proof. If there exists $\beta \in \varPhi - \varPhi_L$ such that β is orthogonal to $V_{L'}$, then any regular element in $W_{L'}$ cannot be L-regular. By direct inspections, one can find such β unless L is the type given in (i), (iii) of the lemma. Assume that L is as in the

$E_7 \quad (E_6)$



lemma, and let a be a regular element in $W_{L'}$. If $W_{L'}$ is of type A_j or type $A_j + A_1$, then a regular vector $v \in V'$ can be written explicitly, and one can check the L-regularity by direct inspections. If $W_{L'}$ is of type D_5 (in the case where W is of type E_7), a must be of type (a) (otherwise it is easy to see that a is not L-regular). But this element is nothing but the regular element in A_4, and the checking is reduced to the previous case. The details are omitted. □

1.6. In what follows we consider a specific cyclic group $\Gamma \in N_W(W_L)$, and $u \in L$ according to the following two cases.

Case (a): $W_{L'} \neq \{1\}$.

In this case, we assume that L is simple modulo center. We choose an L-regular element $a \in W_{L'}$, and put $\Gamma = \langle a \rangle$. Let e be the order of Γ. Thus $\Gamma \subset W_{L'}$ and we have $\Gamma \times W_L \subset W$. We take any unipotent element $u \in L$.

Case (b): $W_{L'} = \{1\}$.

In this case, we assume that L is of type $X_0 + e(A_{n_1-1} + \cdots + A_{n_r-1})$ with X_0 irreducible. We further assume that any $\beta \in \Phi - \Phi_L$ is not orthogonal to the root system $e(A_{n_1-1} + \cdots + A_{n_r-1})$. (Note: since $W_{L'} = \{1\}$, any irreducible component of the Dynkin diagram corresponding to $\Pi - \Pi_L$ consists of 1 or 2 nodes. The latter condition is satisfied for type B_n if all the irreducible components consist of one node, and for type A_n, D_n if the number of irreducible components having two nodes is at most 1.)

We choose $a \in W$ so that a permutes each component A_{n_i-1} in a cyclic way, and acts trivially on X_0. Thus $a \in \mathfrak{S}_{en_1} \times \cdots \times \mathfrak{S}_{en_r}$, and a is a product of disjoint cycles of length e. In particular, $\Gamma = \langle a \rangle \subset N_W(W_L)$, and the subgroup of W generated by Γ and W_L coincides with the semidirect product $\Gamma \ltimes W_L$. Now L is isogenic to $G_0 \times G_1 \times \cdots \times G_r$ modulo center, where G_0 is of type X_0, and $G_i \simeq GL_{n_i} \times \cdots \times GL_{n_i}$ (e-factors). We choose a unipotent element $u \in L$ so that u corresponds to $(u_0, u_1, \dots, u_r)$, where $u_0 \in G_0$ is arbitrary, and u_i is a diagonal element in G_i, i.e., $u_i = (v_i, \dots, v_i)$ with $v_i \in GL_{n_i}$ for $i = 1, \dots, r$.

We can state our main theorem, whose proof will be given in the next section.

Theorem 1.7. *Assume that G is defined over* **C**. *Let L be a Levi subgroup in G. Assume that a cyclic subgroup Γ of order e in $N_W(W_L)$ and $u \in L$ are given as in 1.4. Put $\widetilde{W}_L = \Gamma \ltimes W_L$. Then the followings hold.*

(i) *W_L-module $H^*(\mathcal{B}_u^L)$ can be extended to a $\widetilde{W}_L$-module so that $\Gamma \times \widetilde{W}_L$-module $H^*(\mathcal{B}_u^L)^{(\zeta')}$ is defined for any e-th root of unity ζ'.*

(ii) *There exists a primitive e-th root of unity ζ such that*

$$\Gamma\text{-}\mathrm{Ind}_{W_L}^{W}\big(H^*(\mathcal{B}_u^L)^{(\zeta)}\big) \simeq H^*(\mathcal{B}_u)^{(\zeta)} \tag{1.7.1}$$

as $\varGamma \times W$-modules.

Remarks 1.8. (i) The extension of W_L-module $H^*(\mathcal{B}_u^L)$ to $\widetilde{W}_L$-module is not unique. The theorem aaserts that the statement (ii) holds for some choice of extension.

(ii) The theorem asserts that (1.7.1) holds for some choice of primitive e-th root of unity ζ, but then it holds for any choice of primitive root of unity ζ'. In fact, we can write $\zeta' = \zeta^j$ for some j prime to e, and we have an automorphism τ on $\varGamma$ such that $\tau(a) = a^j$. It follows from (1.7.1) that we have an isomorphism of $\varGamma \times W$ modules, where the action of $\varGamma$ is twisted by τ. It is easy to check that the twisted $\varGamma \times W$-module $\varGamma\text{-}\mathrm{Ind}_{W_L}^W\big(H^*(\mathcal{B}_u^L)^{(\zeta)}\big)$ is isomorphic to $\varGamma\text{-}\mathrm{Ind}_{W_L}^W\big(H^*(\mathcal{B}_u^L)^{(\zeta')}\big)$, and similarly the twisted $H^*(\mathcal{B}_u)^{(\zeta)}$ is isomorphic to $H^*(\mathcal{B}_u)^{(\zeta')}$. Thus (1.7.1) holds also for ζ'.

2. Proof of Theorem 1.7

2.1. In the case where $e = 1$, Theorem 1.7 is nothing but the original induction theorem. So we assume that $e \geq 2$ in what follows. Since the structure of the W-module $H^n(\mathcal{B}_u)$ is independent of p provided that p is a good prime, it is enough to show the corresponding formula for an appropriate p. So, we assume that G is defined over $\mathbf{F}_p$, of split type, with Frobenius map F. We assume that $T \subset B$ are both F-stable, and that $L \subset P$ are F-stable. Thus F acts trivially on W and on W_L. We first note that

Lemma 2.2. *Let $a \in N_W(W_L)$ and choose $\dot a \in N_G(T) \cap N_G(L)$. Assume that $\dot a \in Z_G(u)$. Then* $\mathrm{ad}\,\dot a$ *stabilizes $\mathcal{B}_u^L$, and acts on $H^*(\mathcal{B}_u^L)$ in such a way that* $\mathrm{ad}\,\dot a(w) = awa^{-1}$ *for $w \in W_L$.*

Proof. Since $\dot a \in N_G(L)$, $\dot a$ acts on $\mathcal{B}^L$ by the adjoint action $\mathrm{ad}\,\dot a$, which stabilizes $\mathcal{B}_u^L$ since $\dot a \in Z_G(u)$. Hence $\dot a$ acts naturally on $H^*(\mathcal{B}_u^L)$. In order to compare this action with the action of W_L, we shall recall the construction of Springer representations of W_L. Let

$$\widetilde{L} = \{(x, gB) \in L \times \mathcal{B}^L \mid g^{-1}xg \in B\},$$

and $\pi : \widetilde{L} \to L$ be the first projection. Let L_r be the set of regular semisimple elements in L. Then $\pi^{-1}(L_r)$ is isomorphic to

$$\widetilde{L}_r = T_r \times L/T,$$

where $T_r = T \cap L_r$. Let $\pi_0 : \widetilde{L}_r \to L_r$ be the map defined by $\pi_0 : (t, gT) \mapsto g^{-1}tg$, which coincides with the restriction of π on $\widetilde{L}_r$ under the identification $\pi^{-1}(L_r) \simeq \widetilde{L}_r$. Then π_0 is an unramified Galois covering with group W_L, and for a constant sheaf $\bar{\mathbf{Q}}_l$ on $\widetilde{L}_r$, $\mathcal{L} = \pi_*\bar{\mathbf{Q}}_l$ is a W_L-equivariant local system on L_r. Thus $K = \mathrm{IC}(L, \mathcal{L})$ is a W_L-equivariant complex on L, and it is known by Lusztig that $K \simeq \pi_*\bar{\mathbf{Q}}_l$. Thus for each $u \in L$, the stalk $\mathcal{H}_u^i(K)$ at u of the i-th cohomology sheaf of K gives rise to a W_L-module $H^i(\mathcal{B}_u^L)$.

Now $\dot{a}$ acts on $\widetilde{L}_r$ (resp. on L_r) by $\operatorname{ad}\dot{a} : (t, gT) \mapsto (\dot{a}t\dot{a}^{-1}, \dot{a}g\dot{a}^{-1}T)$ (resp. $\operatorname{ad}\dot{a} : x \mapsto \dot{a}x\dot{a}^{-1}$), and π_0 commutes with $\operatorname{ad}\dot{a}$. Hence $\mathcal{L}$ becomes an $\dot{a}$-equivariant local system. Since $\pi_0^{-1}(t) = \{(wtw^{-1}, wT) \mid w \in W_L\}$ for $t \in T_r$, the stalk $\mathcal{L}_t$ has a natural structure of the regular W_L-module. Then the isomorphism $\mathcal{L}_{\dot{a}t\dot{a}^{-1}} \to \mathcal{L}_t$ is given by $\operatorname{ad}\dot{a}^{-1}$ under the identification $\mathcal{L}_x \simeq \bar{\mathbf{Q}}_l[W_L]$ for $x \in L_r$. It follows that $\mathcal{L}$ is $\langle\dot{a}\rangle \ltimes W_L$-equivariant, where $\langle\dot{a}\rangle$ is a cyclic group generated by $\dot{a}$, and $\dot{a}$ acts on W_L by $\operatorname{ad}\dot{a}(w) = awa^{-1}$. By the functoriality of IC functor, K turns out to be a $\dot{a}$-equivariant complex on L under the adjoint action of $\dot{a}$, which is regarded as a $\langle\dot{a}\rangle \ltimes W_L$-equivariant complex on L. Hence for $u \in L$ such that $\dot{a}u\dot{a}^{-1} = u$, $\mathcal{H}^i_u(K)$ has a structure of $\langle\dot{a}\rangle \ltimes W_L$-module.

On the other hand, $\dot{a}$ acts naturally on $\widetilde{L}$ and on L by the adjoint action, which commute with π. Thus $\pi_*\bar{\mathbf{Q}}_l$ is $\dot{a}$-equivariant, which is isomorphic to K as the complex with $\dot{a}$-action. Hence the action of $\dot{a}$ on $\mathcal{H}^i_u(K)$ coincides with the action on $H^i(\mathcal{B}^L_u)$ induced from the adjoint action of $\dot{a}$ on $\mathcal{B}^L_u$. The lemma follows from this. □

Next we show the following lemma.

Lemma 2.3. *There exists a representative* $\dot{a} \in N_G(T) \cap N_G(L) \cap Z_G(u)$ *such that* $\dot{a}$ *acts trivially on* $H^*(\mathcal{B}_u)$ *and that* $\dot{a}^e$ *acts trivially on* $H^*(\mathcal{B}^L_u)$. *In particular,* $H^*(\mathcal{B}^L_u)$ *has a structure of* $\widetilde{W}_L$*-module.*

Proof. First consider the case (a) in 1.6. Let H be the subgroup of G generated by U_α with $\alpha \in \varPhi_{L'}$, where U_α is the root subgroup corresponding to α. Then H is a connected reductive subgroup of L' whose Weyl group coincides with $W_{L'}$. Since $H \subset Z_G(u)$, we have $H \subset Z^0_G(u)$. One can choose a representative $\dot{a} \in N_H(T_1)$ of $a \in W_{L'}$, where T_1 is a maximal torus of H contained in T. Then $\dot{a} \in Z^0_G(u) \cap N_G(L)$ and $\dot{a}^e \in T_1$. Since $T_1 \subset Z_G(u)$, we see that $T_1 \subset Z^0_L(u)$. Thus, $\dot{a}^e \in Z^0_L(u)$. Hence $\dot{a}$ satisfies the condition.

Next consider the case (b) in 1.6. Let L_1 be the Levi subgroup containing L of type $X_{n_0} + A_{en_1-1} + \cdots + A_{en_r-1}$. We have a natural projection $\pi : L_1 \to \bar{L}_1 = L_1/Z^0(L_1)$, and an isogeny map $\theta : \widetilde{L}_1 = G_0 \times SL_{en_1} \times \cdots \times SL_{en_r} \to \bar{L}_1$, where G_0 is the simply connected semisimple group of type X_0. Put $\bar{u} = \pi(u) \in \bar{L}_1$. Now $Z_{L_1}(u)$ acts on $H^*(\mathcal{B}_u)$. Since $Z^0(L_1)$ acts trivially on $H^*(\mathcal{B}_u)$, we have an action of $Z_{L_1}(u)/Z^0(L_1) = Z_{\bar{L}_1}(\bar{u})$ on $H^*(\mathcal{B}_u)$. Let $\widetilde{u}$ be an element in $\widetilde{L}_1$ such that $\theta(\widetilde{u}) = \bar{u}$. $\widetilde{u} = (u_0, u_1, \dots, u_r)$ can be chosen as given in 1.4. We choose $\ddot{a} \in \widetilde{L}_1$ as follows; put $\ddot{a} = (a_0, a_1, \dots, a_r)$ with $a_0 \in G_0$, and $a_i \in SL_{en_i}$ for $1 \le i \le r$. We put $a_0 = 1$ and choose $a_1, \dots, a_r$ so that $a_i \in Z^0_{SL_{en_i}}(u_i)$ and that $a_i^e \in Z(SL_{en_i})$. Such a choice is always possible for u_i of type $(n_i, \dots, n_i)$. Thus $\ddot{a} \in Z^0_{\widetilde{L}_1}(\widetilde{u})$. It follows that $\theta(\ddot{a})$ is contained in a connected subgroup of $Z_{\bar{L}_1}(\bar{u}_1)$, and by the previous remark, $\theta(\ddot{a})$ acts trivially on $H^*(\mathcal{B}_u)$. Now take $\dot{a} \in Z_{L_1}(u)$ such that $\pi(\dot{a}) = \theta(\ddot{a})$. Then $\dot{a} \in N_G(T) \cap N_G(L)$, and acts trivially on $H^*(\mathcal{B}_u)$. On the other hand, similar to π, θ, we have a map $\pi' : L \to \bar{L} = L/Z^0(L)$ and $\theta' : \widetilde{L} = G_0 \times (SL_{n_1})^e \times \cdots \times (SL_{n_r})^e \to \bar{L}$. Let $\bar{u} = \pi'(u) \in \bar{L}$, and $\widetilde{u} \in \widetilde{L}$ such that $\bar{u} = \theta'(\widetilde{u})$. Then we have an isomorphism $H^*(\mathcal{B}^L_u) \simeq H^*(\mathcal{B}^{\bar{L}}_{\bar{u}}) \simeq H^*(\mathcal{B}^{\widetilde{L}}_{\widetilde{u}})$ compatible with the actions of $Z_L(u), Z_{\bar{L}}(\bar{u})$ and $Z_{\widetilde{L}}(\widetilde{u})$ with respect to π', θ'. We have $\ddot{a}^e \in Z(SL_{n_1})^e \times Z(SL_{n_2})^e \times \cdots$. Since the action of

$Z(SL_{n_1})^e \times Z(SL_{n_2})^e \times \cdots$ can be extended to an action of $Z(GL_{n_1})^e \times Z(GL_{n_2})^e \times \cdots$ on $H^*(\mathcal{B}^{\tilde{L}}_{\tilde{u}})$, $\ddot{a}^e$ acts trivially on $H^*(\mathcal{B}^{\tilde{L}}_{\tilde{u}})$, and so $\dot{a}^e$ acts trivially on $H^*(\mathcal{B}^L_u)$. □

2.4. Let $\mathcal{Z} = Z^0_L$ be the identity component of the center of L. Put $\mathcal{B}_{\mathcal{Z}} = \{B' \in \mathcal{B} \mid \mathcal{Z} \subset B'\}$. Then $\mathcal{B}_{\mathcal{Z}}$ is decomposed into connected components

$$\mathcal{B}_{\mathcal{Z}} = \coprod_{d \in W_L \backslash W} \mathcal{B}_{\mathcal{Z},d},$$

where $\mathcal{B}_{\mathcal{Z},d} = \{{}^{xd}B \mid x \in L\}$, which is isomorphic to $\mathcal{B}^L$ under the map $B' \mapsto B' \cap L$.

Put

$$\mathcal{Z}_{\text{reg}} = \{z \in \mathcal{Z} \mid Z^0_G(z) = L\}.$$

Then for any $t \in \mathcal{Z}_{\text{reg}}$, we have $\mathcal{B}_t = \mathcal{B}_{\mathcal{Z}}$ by Lemma 2.2 (c) in [L3], and so $\mathcal{B}_{tu} = \mathcal{B}_u \cap \mathcal{B}_t = \mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}}$. It follows that

$$\mathcal{B}_{tu} = \coprod_{d \in W_L \backslash W} (\mathcal{B}_{\mathcal{Z},d} \cap \mathcal{B}_u),$$

where $\mathcal{B}_{\mathcal{Z},d} \cap \mathcal{B}_u$ is isomorphic to $\mathcal{B}^L_u$ under the map $B' \mapsto B' \cap L$. This implies that

$$H^{2n}(\mathcal{B}_{tu}) \simeq \bigoplus_{d^{-1} \in W/W_L} H^{2n}(\mathcal{B}_{\mathcal{Z},d} \cap \mathcal{B}_u). \tag{2.4.1}$$

The right hand side of (2.4.1) has a natural structure of the induced W-module $\operatorname{Ind}^W_{W_L} H^{2n}(\mathcal{B}^L_u)$. It is proved in [L3, Proposition 1.4] that (2.4.1) is actually an isomorphism of W-modules. Let $a \in W$ be as in the theorem. Since $\dot{a} \in N_G(L)$, it stabilizes $\mathcal{Z}$, and so $\dot{a}$ acts on $\mathcal{B}_{\mathcal{Z}}$ via $\operatorname{ad} \dot{a}$. It is easy to see that $\dot{a}$ induces a permutation action on the components of $\mathcal{B}_{\mathcal{Z}}$; $\dot{a} : \mathcal{B}_{\mathcal{Z},d} \mapsto \mathcal{B}_{\mathcal{Z},ad}$. It follows that $\dot{a}$ induces an automorphism on $H^{2n}(\mathcal{B}_{tu})$, which maps the factor corresponding to $d^{-1} \in W/W_L$ to $d^{-1}a^{-1} \in W/W_L$. Under the isomorphism $H^{2n}(\mathcal{B}_{\mathcal{Z},d} \cap \mathcal{B}_u) \simeq H^{2n}(\mathcal{B}^L_u)$, the factor corresponding to $d^{-1} \in W/W_L$ is written as $d^{-1} \otimes H^{2n}(\mathcal{B}^L_u)$, and $\dot{a}$ maps $d^{-1} \otimes H^{2n}(\mathcal{B}^L_u) \to d^{-1}a^{-1} \otimes H^{2n}(\mathcal{B}^L_u)$. On the other hand, by Lemma 2.3, $\dot{a}^e$ acts trivially on $H^{2n}(\mathcal{B}^L_u)$, and induces an action of $\widetilde{W}_L$ on it. Hence $\dot{a}$ induces an action of $\varGamma_0$ on $H^{2n}(\mathcal{B}_{tu}) \simeq \operatorname{Ind}^W_{W_L} H^{2n}(\mathcal{B}^L_u)$, which is given by $\dot{a} : d^{-1} \otimes x \mapsto d^{-1}a^{-1} \otimes \dot{a}x$ for each factor $d^{-1} \otimes H^{2n}(\mathcal{B}^L_u)$.

Now we define an action of $\varGamma$ on $H^*(\mathcal{B}_{tu})$ by $a : x \mapsto \zeta^n \dot{a}x$ for $x \in H^{2n}(\mathcal{B}_{tu})$, where $\dot{a}x$ is the action of $\varGamma_0$ on $H^{2n}(\mathcal{B}_{tu})$ given as above. Since the action of $\dot{a} \in G$ commutes with that of W, $H^*(\mathcal{B}_{tu})$ turns out to be a $\varGamma \times W$-module, which we denote by $H^*(\mathcal{B}_{tu})^{[\zeta]}$. The following lemma is immediate from the above discussion.

Lemma 2.5. *There exists an isomorphism of $\varGamma \times W$-modules*

$$H^*(\mathcal{B}_{tu})^{[\zeta]} \simeq \varGamma\text{-}\operatorname{Ind}^W_{W_L} H^*(\mathcal{B}^L_u)^{(\zeta)}.$$

In view of Lemma 2.5, in order to prove the theorem it is enough to show the following proposition.

Proposition 2.6. *Under an appropriate choice of (a good prime) p, there exists an isomorphism of $\varGamma \times W$-modules for any $t \in \mathcal{Z}_r$,*

$$H^*(\mathcal{B}_u)^{(\zeta)} \simeq H^*(\mathcal{B}_{tu})^{[\zeta]}.$$

2.7. The remainder of this section is devoted to the proof of the proposition. We shall prove it by modifying the arguments in [L3]. By [Sh1], [Sh2], [BS], the following fact is known; assume that G is simple modulo center. Then for each unipotent class C of G, there exists $u_1 \in C^F$, called a split unipotent element, such that F acts on $H^{2n}(\mathcal{B}_{u_1})$ as a scalar multiplication by p^n. (In the case where G is of type E_8, we assume that $p \equiv 1 \pmod 4$). Since the component group $A_G(u_1) = Z_G(u_1)/Z_G^0(u_1)$ is isomorphic to S_3, S_4, S_5 or $(\mathbf{Z}/2\mathbf{Z})^k$ for some k, there exists a positive integer s_0 (independent of p) such that F^{s_0} acts on $H^{2n}(\mathcal{B}_u)$ by a scalar multiplication by $p^{s_0 n}$ for any unipotent element u of G^F (e.g., one can take $s_0 = |S_5|$.) Similarly, F^{s_0} acts on $H^{2n}(\mathcal{B}_u^L)$ by a scalar multiplication by $p^{s_0 n}$ for any unipotent element $u \in L^F$. Note that the isomorphism in (2.4.1) is F-equivariant. Hence F^{s_0} acts also as a scalar multiplication by $p^{s_0 n}$ for $H^{2n}(\mathcal{B}_{tu})$.

Note that $\dot a$ acts trivially on $H^{2n}(\mathcal{B}_u)$ by Lemma 2.3. It follows that one can write

$$\mathrm{Tr}\,((F^s\dot a)^i w, H^*(\mathcal{B}_u)) = \sum_{n\ge 0} a_n(w)p^{isn}, \tag{2.7.1}$$

$$\mathrm{Tr}\,((w, a^i), H^*(\mathcal{B}_u)^{(\zeta)}) = \sum_{n\ge 0} a_n(w)\zeta^{in}, \tag{2.7.2}$$

for any $w \in W, 0 \le i \le e-1$ and for any positive integer s divisible by s_0, where $a_n(w) = \mathrm{Tr}\,(w, H^n(\mathcal{B}_u))$ are integers for each $n \ge 0$.

On the other hand, by the description of the action of F and of $\dot a$ on $H^n(\mathcal{B}_{tu})$ in 2.4, together with Lemma 2.5, one can write

$$\mathrm{Tr}\,((F^s\dot a)^i w, H^*(\mathcal{B}_{tu})) = \sum_{n\ge 0} b_{n,i}(w)p^{isn}, \tag{2.7.3}$$

$$\mathrm{Tr}\,((w, a^i), H^*(\mathcal{B}_{tu})^{[\zeta]}) = \sum_{n\ge 0} b_{n,i}(w)\zeta^{in}, \tag{2.7.4}$$

for w, i, s as above, where $b_{n,i}(w)$ are certain integers.

For an integer x and a prime number l, we denote by $m_l(x)$ the multiplicative order of x in $\mathbf{Z}/l\mathbf{Z}$, i.e., the smallest positive integer m such that $x^m \equiv 1 \pmod l$. The following is a key for the proof of Proposition 2.6.

Lemma 2.8. *Assume that* $p \equiv 1 \pmod 4$. *Let s_0, e be fixed positive integers coprime to p. Then there exist infinitely many prime numbers l satisfying the following properties.*

(i) *$m_l(p^s) = e$ for a certain integer s divisible by s_0.*
(ii) *$l - 1$ is divisible by e.*

Proof. By our assumption, the image of s_0e on $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ is non-zero. Hence the map $x \mapsto s_0ex+1$ induces a bijective map on $\mathbf{F}_p$. Thus there exists $c \in \mathbf{Z}$ such that the image of s_0ec+1 in $\mathbf{F}_p$ is contained in $\mathbf{F}_p^* - (\mathbf{F}_p^*)^2$. Put $\alpha = s_0ec+1$. Then α is prime to p, and so $(\alpha-1)p$ and α are coprime each other. Then by Dirichlet's theorem on arithmetic progression, there exist infinitely many prime numbers l of the form $l = n(\alpha-1)p+\alpha$ for some positive integer n. It is enough to show that these $l \ge 3$ satisfy the assertion of the lemma. For an integer a and a prime number p, let $(\frac{a}{p})$ be the Legendre symbol, i.e.,

$$(\frac{a}{p}) = \begin{cases} 1 & \text{if } x^2 \equiv a \pmod p \text{ for some } x \in \mathbf{Z}, \\ -1 & \text{otherwise.} \end{cases}$$

We show that

$$(\frac{p}{l}) = -1. \tag{2.8.1}$$

In fact, by the quadratic reciprocity law (e.g., [Se]), we have

$$(\frac{p}{l})(\frac{l}{p}) = (-1)^{\frac{p-1}{2}\cdot\frac{l-1}{2}} = 1.$$

The second equality follows from the assumption that $p \equiv 1 \pmod 4$. Hence we have $(\frac{p}{l}) = (\frac{l}{p})$. But $l \equiv \alpha \pmod p$, and so $(\frac{l}{p}) = (\frac{\alpha}{p}) = -1$ since the image of α is not contained in $\mathbf{F}_p^2$ by our choice of α. Hence (2.8.1) holds.

Now (2.8.1) is equivalent to $p^{(l-1)/2} \equiv -1 \pmod l$. It follows that $m_l(p) = l-1$. Since $l-1 = s_0ec(np+1)$, we see that $m_l(p^s) = e$ for $s = s_0c(np+1)$ and that $l-1$ is divisible by e. Thus this l satisfies the assertion of the lemma. The lemma is proved. □

2.9 For given integers $s_0 \ge 1, e \ge 2$, we choose a prime number p such that p is not a factor of e, s_0 and that $p \equiv 1 \pmod 4$, and fix it once and for all. For a multiple s of s_0, put $F' = F^s\dot{a}$ and $q = p^s$. Under the setting in 1.6, we shall describe the set $\mathcal{Z}_{\text{reg}}$ more precisely. As in [L3, Lemma 2.2], $\mathcal{Z}_{\text{reg}}$ can be written as $\mathcal{Z}_{\text{reg}} = \mathcal{Z} - \bigcup_\beta \ker(\beta|_{\mathcal{Z}})$, where β runs over all the roots in $\Phi - \Phi_L$. ($\beta|_{\mathcal{Z}}$ gives a non-trivial character of $\mathcal{Z}$ for $\beta \in \Phi - \Phi_L$).

First consider the case (a). Let L'_{der} be the derived subgroup of L', and S' be the split maximal torus of L'_{der} contained in T. Then $S' \subset \mathcal{Z}$. Put $S'_{\text{reg}} = S' \cap \mathcal{Z}_{\text{reg}}$. Now $W_{L'}$ leaves the set $\Phi - \Phi_L$ invariant. For each $\beta \in \Phi - \Phi_L$, put $H_\beta = \bigcap_{x\in\Gamma} \ker(x(\beta)|_{S'})$. Then H_β is an F'-stable subgroup of S', and we see that

$${S'}_{\text{reg}}^{F'} = {S'}^{F'} - \bigcup_{\beta\in\Phi-\Phi_L} H_\beta^{F'}. \tag{2.9.1}$$

H_β is a closed subgroup of S', and we put $e_\beta = |H_\beta/H_\beta^0|$ for each $\beta \in \Phi - \Phi_L$.

Let $\mathcal{P}'$ be the set of all prime numbers l satisfying the condition in Lemma 2.8. Thus $\mathcal{P}'$ is an infinite set. We denote by $\mathcal{P}$ the subset of $\mathcal{P}'$ consisting of l such that $l > |\varPhi - \varPhi_L|$ and that l does not divide e_β ($\beta \in \varPhi - \varPhi_L$). Thus $\mathcal{P}$ is an infinite set also.

Next we consider the case (b). We may assume that G has a connected center of dimension 1, and that the derived subgroup of G is simply connected, almost simple. Let k be an algebraic closure of $\mathbf{F}_q$. We see that there exists a subtorus S of $\mathcal{Z}$ such that $S \simeq (k^*)^c$, where c is the number of irreducible components of $\varPhi_L$. Since a permutes the factors k^* in S, we see that $S^{F'} \simeq (\mathbf{F}_{q^e}^*)^r \times (\mathbf{F}_q^*)^{r'}$, where r' is equal to 1 or 0 according to the cases where X_0 is non-empty or empty. Since $\varGamma \subset N_W(W_L)$, $\varGamma$ preserves the set $\varPhi - \varPhi_L$. For each $\beta \in \varPhi - \varPhi_L$, put $K_\beta = \bigcap_{x\in\varGamma} \ker(x(\beta)|_S)$. Then K_β is an F'-stable subgroup of S, and we have

$$S_{\text{reg}}^{F'} = S^{F'} - \bigcup_{\beta\in\varPhi-\varPhi_L} K_\beta^{F'}, \tag{2.9.2}$$

where $S_{\text{reg}} = S \cap \mathcal{Z}_{\text{reg}}$. K_β is a closed subgroup of S, and put $e_\beta = |K_\beta/K_\beta^0|$ for each $\beta \in \varPhi - \varPhi_L$. Under the identification $S^{F'} \simeq (\mathbf{F}_{q^e}^*)^r \times (\mathbf{F}_q^*)^{r'}$, we see that $K_\beta^{0F'} \simeq (\mathbf{F}_{q^e}^*)^{r-1} \times (\mathbf{F}_q^*)^{r'}$ or $K_\beta^{0F'} \simeq \mathbf{F}_{q^{e'}}^* \times (\mathbf{F}_{q^e}^*)^{r-1} \times (\mathbf{F}_q^*)^{r'}$, where e' is a proper divisor of e. (Let S_i be the subtorus of S corresponding to the factor eA_{n_i-1} for $i = 1, \dots, r$. Then the former case occurs if $\beta|_{S_i}, \beta|_{S_j}$ are non-trivial for some $i \ne j$, and the latter case occurs if $\beta|_{S_i}$ is non-trivial for only one i. Note that by our assumption in 1.4, β is non-trivial on $S_1 \times \cdots \times S_r$.)

Let $\mathcal{P}'$ be as in the case (a). We define a subset $\mathcal{P}$ of $\mathcal{P}'$ as the set of prime numbers $l \in \mathcal{P}'$ such that $l > |\varPhi - \varPhi_L|$ and that l does not divide e_β.

The next lemma is a variant of Lemma 3.4 in [L3].

Lemma 2.10. *Assume that $l \in \mathcal{P}$, and let s be a multiple of s_0 such that $m_l(p^s) = e$ (see Lemma 2.8). Put $F' = F^s\dot{a}$. Then there exists $t \in \mathcal{Z}_{\text{reg}}$ such that $F'(t) = t$ and that $t^l = 1$.*

Proof. First consider the case (a) in 1.6. It is enough to show, for each $l \in \mathcal{P}$, that there exists $t \in {S'}_{\text{reg}}^{F'}$ such that $t^l = 1$. Note that a is a regular element of order e in $W_{L'}$. Put $V = \mathbf{R} \otimes_{\mathbf{Z}} X(S')$. Thus $W_{L'}$ acts on V as a reflection group. Let ζ be a primitive e-th root of unity, and let $a(e)$ be the dimension of the eigenspace $V(a, \zeta) \subset V$ of a with eigenvalue ζ. We show that

$$\sharp\{t \in {S'}^{F'} \mid t^l = 1\} = l^{a(e)}. \tag{2.10.1}$$

By a general formula, we have $|{S'}^{F'}| = |\det_V(qI - a)| = P_a(q)$, where $P_a(x)$ is the characteristic polynomial of $a \in W_{L'}$. Since a is regular $P_a(x)$ can be written, by [Sp1, 4.2], as

$$P_a(x) = \varPhi_e(x)^{a(e)}\varPhi'(x),$$

where $\varPhi_e(x)$ is the cyclotomic polynomial of degree e, and $\varPhi'(x)$ is a product of cyclotomic polynomials $\varPhi_{e'}(x)$ with $e' < e$. By our assumption $m_l(q) = e$, $\varPhi_e(q)$ is divisible by l, and $\varPhi'(q)$ is not divisible by l. This means that each minimal F'-stable torus M of S' corresponding to the factor $\Phi_e(x)$ contains an element of order l. Since $\{t \in M^{F'} \mid t^l = 1\} \subset \mathbf{F}_{q^e}^*$, $M^{F'}$ contains exactly l elements t such that $t^l = 1$. Thus (2.10.1) is proved.

For $\beta \in \Phi - \Phi_L$, let V_β be the subspace of V which is orthogonal to $x(\beta)$ for all $x \in \Gamma$. Then V_β can be identified with $\mathbf{R} \otimes_{\mathbf{Z}} X(H_\beta^0)$. Γ stabilizes V_β, and let $V_\beta(a, \zeta)$ be the eigenspace of a on V_β with eigenvalue ζ. Since a is L-regular, we have $\dim V_\beta(a, \zeta) < \dim V(a, \zeta) = a(e)$. It follows that the characteristic polynomial $P'_a(x)$ of a on V_β contains the factor $\Phi_e(x)$ with multiplicity less than $a(e)$. By a similar argument as above, minimal F'-stable subtori of H_β^0 corresponding to $\Phi_e(x)$ only contain elements of order l. This implies that

$$\sharp\{t \in H_\alpha^{F'} \mid t^l = 1\} = \sharp\{t \in H_\alpha^{0F'} \mid t^l = 1\} \le l^{a(e)-1}.$$

It follows, by (2.9.1), that

$$\begin{aligned}\sharp\{t \in {S'}_{\mathrm{reg}}^{F'} \mid t^l = 1\} &= \sharp\{t \in {S'}^{F'} \mid t^l = 1, t \notin \bigcup_{\beta \in \Phi - \Phi_L} H_\beta^{F'}\} \\ &\ge l^{a(e)} - N l^{a(e)-1} = l^{a(e)-1}(l - N),\end{aligned}$$

where $N = |\Phi - \Phi_L|$. Since $l > N$ by our assumption, there exists $t \in {S'}_{\mathrm{reg}}^{F'}$ such that $t^l = 1$. This proves the lemma in the case (a).

Next consider the case (b) in 1.6. It is enough to show, for each $l \in \mathcal{P}$, that there exists $t \in S_{\mathrm{reg}}^{F'}$ such that $t^l = 1$. We note that $q^{e'} - 1$ is not divisible by l for any divisor $e' < e$ of e by the assumption $m_l(q) = e$. Since $S^{F'} \simeq (\mathbf{F}_{q^e}^*)^r \times (\mathbf{F}_q^*)^{r'}$ (cf. 2.9), we have

$$\sharp\{t \in S^{F'} \mid t^l = 1\} = l^r.$$

We consider K_β given in 2.9. By the discussion in 2.9, we have

$$\sharp\{t \in K_\beta^{F'} \mid t^l = 1\} = \sharp\{t \in K_\beta^{0F'} \mid t^l = 1\} = l^{r-1}.$$

It follows, by (2.9.2), that

$$\begin{aligned}\sharp\{t \in S_{\mathrm{reg}}^{F'} \mid t^l = 1\} &= \sharp\{t \in S^{F'} \mid t^l = 1, t \notin \bigcup_{\beta \in \Phi - \Phi_L} K_\beta^{F'}\} \\ &\ge l^r - N l^{r-1} = l^{r-1}(l - N),\end{aligned}$$

where N is as before. Since $l > N$ by our assumption, the lemma holds also for the case (b). □

We need the following lemma due to Lusztig.

Lemma 2.11 ([L3, Lemma 3.2])**.** *Let H be a finite group, and ϕ a virtual character of H (over a field of characteristic 0). Assume that ϕ is integral valued. Let $x, y \in H$ be such that $xy = yx$ and $y^l = 1$ for a prime number l. Then $\phi(xy) - \phi(x) \in l\mathbf{Z}$.*

2.12. Let s_0 be as in 2.7, and $\mathcal{P}$ be as in 2.9. Let $F' = F^s\dot{a}$ be as in Lemma 2.10 for a fixed $l \in \mathcal{P}$. Let $R_{w,i} = R_{T_w}(1)$ be the Deligne-Lusztig's virtual character of $G^{F'^i}$ for $i = 1, \dots, e$, where T_w is an F'^i-stable maximal torus of G corresponding to $w \in W \simeq W(T_1)$ (here $W(T_1) = N_G(T_1)/T_1$ for an F'-stable pair $T_1 \subset B_1$). Let us choose $t \in \mathcal{Z}_{\text{reg}}$ as in Lemma 2.10. Then we have

$$\begin{aligned} \text{(2.12.1)} \qquad \operatorname{Tr}(F'^i w, H^*(\mathcal{B}_u)) &= \operatorname{Tr}(u, R_{w,i}), \\ \operatorname{Tr}(F'^i w, H^*(\mathcal{B}_{tu})) &= \operatorname{Tr}(tu, R_{w,i}). \end{aligned}$$

We remark that (2.12.1) was proved in [L2] under the assumption that p^s is large enough (which is determined only by the data of the Dynkin diagram of G). Thus if we replace s_0 in 2.7 by a suitable large number, the result in [L2] is applicable. One can also apply [Sh3, Theorem 2.2] instead of [L2], where the restriction on p^s is removed.

Since $R_{w,i}$ are integral valued, one can apply Lemma 2.11 for $H = G^{F'^i}$ and $x = u, y = t$. Hence we have

$$\operatorname{Tr}(u, R_{w,i}) = \operatorname{Tr}(tu, R_{w,i}) \mod l\mathbf{Z}.$$

It follows from (2.12.1) that

$$\text{(2.12.2)} \qquad \operatorname{Tr}(F'^i w, H^*(\mathcal{B}_u)) = \operatorname{Tr}(F'^i w, H^*(\mathcal{B}_{tu})) \mod l\mathbf{Z}.$$

Let ζ_0 be a fixed primitive e-th root of unity in $\mathbf{C}$, and R the ring of integers of the cyclotomic field $\mathbf{Q}(\zeta_0)$. Let $\mathcal{I}$ be the set of non-zero prime ideals $\mathfrak{p}$ in R such that $\mathfrak{p}$ contains one of the numbers $1 - \zeta_0^i$ for $i = 1, \dots, e-1$ and ζ_0. Let $\bar{\mathcal{I}}$ be the set of prime numbers l such that $\mathfrak{p} \cap \mathbf{Z} = l\mathbf{Z}$ for $\mathfrak{p} \in \mathcal{I}$. Since $\mathcal{I}$ is a finite set, $\bar{\mathcal{I}}$ is a finite set. So, $\mathcal{P} - \bar{\mathcal{I}}$ is an infinite set. Let $\mathcal{J}$ be the set of prime ideals $\mathfrak{p}$ of R such that $\mathfrak{p} \cap \mathbf{Z} = l\mathbf{Z}$ with $l \in \mathcal{P} - \bar{\mathcal{I}}$. Then $\mathcal{J}$ is an infinite set. Now $R/\mathfrak{p}$ is a finite extension of $\mathbf{F}_l$. Let $\bar{\zeta}_0$ be the image of ζ_0 in $R/\mathfrak{p}$. Since $l \in \mathcal{P}$, $l - 1$ is divisible by e. Hence $\bar{\zeta}_0 \in \mathbf{F}_l^*$, which has order e by our choice of $\mathfrak{p}$. Since $m_l(p^s) = e$, the image of p^s in $\mathbf{Z}/l\mathbf{Z}$ has order e. Hence there exists j such that

$$\text{(2.12.3)} \qquad p^s - \zeta_0^j \in \mathfrak{p}.$$

Note that the number j is determined by the choice of $\mathfrak{p}$, which we denote by $j(\mathfrak{p})$. For $j = 1, \dots, e-1$, let $\mathcal{J}_j$ be the set of prime ideals $\mathfrak{p}$ in $\mathcal{J}$ such that $j(\mathfrak{p}) = j$. Thus $\mathcal{J} = \bigcup_j \mathcal{J}_j$, and so there exists j_0 such that $\mathcal{J}_0 = \mathcal{J}_{j_0}$ is an infinite set. We put $\zeta = \zeta_0^{j_0}$. By (2.12.3), ζ is a primitive e-th root of unity.

We remark that $H^*(\mathcal{B}_{tu}) = H^*(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})$ is independent of the choice of $t \in \mathcal{Z}_{\text{reg}}$. Then in view of (2.7.1) $\sim$ (2.7.4), together with (2.12.3), we see that

$$\begin{aligned} \operatorname{Tr}((F^s\dot{a})^i w, H^*(\mathcal{B}_u)) &= \operatorname{Tr}((w, a^i), H^*(\mathcal{B}_u)^{(\zeta)}) \mod \mathfrak{p}, \\ \operatorname{Tr}((F^s\dot{a})^i w, H^*(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})) &= \operatorname{Tr}((w, a^i), H^*(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})^{[\zeta]}) \mod \mathfrak{p} \end{aligned}$$

for any $\mathfrak{p} \in \mathcal{J}_0$. Combined with (2.12.2), we have

$$\mathrm{Tr}\,((w,a^i), H^*(\mathcal{B}_u)^{(\zeta)}) = \mathrm{Tr}\,((w,a^i), H^*(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})^{[\zeta]}) \mod \mathfrak{p}$$

for $\mathfrak{p} \in \mathcal{J}_0$. Since $\mathcal{J}_0$ is an infinite set, we conclude that

$$\mathrm{Tr}\,((w,a^i), H^*(\mathcal{B}_u)^{(\zeta)}) = \mathrm{Tr}\,((w,a^i), H^*(\mathcal{B}_u \cap \mathcal{B}_{\mathcal{Z}})^{[\zeta]}).$$

Hence Proposition 2.6 is proved, and the theorem follows.

3. Applications

3.1. Let W_L be the subgroup of W, and $\varGamma$ the subgroup of W generated by $a \in N_W(W_L)$ such that $\varGamma$ and W_L generate the semidirect product group $\widetilde{W}_L = \varGamma \ltimes W_L$. Let $V = V^{(\zeta)}$ be the $\varGamma \times \widetilde{W}_L$-module as in 1.3. (We write $\varGamma$ as $\varGamma_0$ if it is regarded as a subgroup of $\widetilde{W}_L$, cf. 1.3.) Then V can be decomposed as $V = \bigoplus_{i \in \mathbf{Z}/e\mathbf{Z}} V^{(i)}$, where $V^{(i)}$ is the eigenspace of $a \in \varGamma$ with eigenvalue ζ^i, which is a $\widetilde{W}_L$-submodule of V. Then we have

$$\begin{aligned}\mathrm{Ind}_{W_L}^W V &= \bigoplus_i \bigoplus_{w \in W/\widetilde{W}_L} \bigoplus_j wa^j \otimes V^{(i)} \\ &= \bigoplus_i \bigoplus_{w \in W/\widetilde{W}_L} \bigoplus_k wb_k \otimes V^{(i)},\end{aligned}$$

where $b_k = \sum_j \zeta^{jk} a^j \in \mathbf{C}[\varGamma]$ (the group ring of $\varGamma$). For each $i \in \mathbf{Z}$, let $\psi^{(i)}$ the linear character of $\varGamma$ defined by $\psi^{(i)}(a) = \zeta^i$. Then $\varGamma$-module $\mathbf{C}b_k$ is afforded by $\psi^{(-k)}$. Let $V_n^{(i)}$ be the eigenspace of $a \in \varGamma_0$ on the $\widetilde{W}_L$-module $V^{(i)}$ with eienvalue ζ^n. Let $(\varGamma\text{-}\mathrm{Ind}_{W_L}^W V)^{(k)}$ be the eigensapce of $a \in \varGamma$ with eigenvalue ζ^k. Then we have the following lemma.

Lemma 3.2. (i) *Let the notations be as above. We have*

$$(\varGamma\text{-}\mathrm{Ind}_{W_L}^W V)^{(k)} \simeq \bigoplus_{w \in W/\widetilde{W}_L} \bigoplus_{j \in \mathbf{Z}/e\mathbf{Z}} \bigoplus_{0 \le n < e} wb_{k-n-j} \otimes V_n^{(j)} \tag{3.2.1}$$

as vector spaces. In particular, $\dim(\varGamma\text{-}\mathrm{Ind}_{W_L}^W V^{(\zeta)})^{(k)}$ *is independent of the choice of* $k \in \mathbf{Z}/e\mathbf{Z}$*, which is given by*

$$\dim(\varGamma\text{-}\mathrm{Ind}_{W_L}^W V)^{(k)} = [W : \widetilde{W}_L] \dim V. \tag{3.2.2}$$

(ii) *Assume that* $\varGamma$ *commutes with* W_L*. Then we have*

$$(\varGamma\text{-}\mathrm{Ind}_{W_L}^W V)^{(k)} \simeq \bigoplus_{j \in \mathbf{Z}/e\mathbf{Z}} \mathrm{Ind}_{\varGamma \times W_L}^W (\psi^{(-k+j)} \otimes V^{(j)}) \tag{3.2.3}$$

as W-modules.

Proof. Under the action of $\varGamma$ on $\varGamma\text{-}\mathrm{Ind}_{W_L}^W V$, $wb_k \otimes V_n^{(i)}$ is contained in an eigenspace of a with eigenvalue ζ^{k+n+j}. Then (i) follows easily from the discussion in 3.1. Now assume that $\varGamma$ commutes with W_L. Then $b_k \otimes V^{(i)}$ has a structure of $\varGamma \times W_L$-module given by $\psi^{(-k)} \otimes V^{(i)}$. (ii) follows from the formula (3.2.1) by noticing that $V^{(i)} = V_0^{(i)}$. The lemma is proved. $\square$

We consider a Levi subgroup $L \subset G$ and a unipotent element $u \in L$, and take $\varGamma = \langle a \rangle \subset N_W(W_L)$ satisfying the condition in 1.6. We apply the preceding argument to the situation $V^{(\zeta)} = H^*(\mathcal{B}_u^L)^{(\zeta)}$. Then as a corollary to Theorem 1.7, we have

Proposition 3.3. *Under the setting in Theorem 1.7, we have, for $0 \le k \le e-1$,*

$$\bigoplus_{n \equiv k \mod e} H^{2n}(\mathcal{B}_u) \simeq \bigoplus_{w \in W/\widetilde{W}_L} \bigoplus_{j \in \mathbf{Z}/e\mathbf{Z}} \bigoplus_{0 \le n < e} wb_{k-n-j} \otimes H^{2j}(\mathcal{B}_u^L)_n \tag{3.3.1}$$

as vector spaces, where $b_i \in \mathbf{C}[\varGamma]$ and $H^{2n}(\mathcal{B}_u^L)_n$ is the eigenspace of $a \in \varGamma_0$ with eigenvalue ζ^n. In particular, $\dim\big(\bigoplus_{n \equiv k \mod e} H^{2n}(\mathcal{B}_u)\big)$ *is independent of the choice of k. In the case* (a) *in* 1.6, (3.3.1) *can be made more precise as follows;*

$$\bigoplus_{n \equiv k \mod e} H^{2n}(\mathcal{B}_u) \simeq \mathrm{Ind}_{\varGamma \times W_L}^W \Big(\bigoplus_{j \in \mathbf{Z}/e\mathbf{Z}} \psi^{(-k+j)} \otimes H^{2j}(\mathcal{B}_u^L) \Big) \tag{3.3.2}$$

as W-modules.

Proof. By Theorem 1.7 $\varGamma\text{-}\mathrm{Ind}_{W_L}^W H^*(\mathcal{B}_u^L)^{(\zeta)}$ is isomorphic to $H^*(\mathcal{B}_u)^{(\zeta)}$ as $\varGamma \times W$-modules. Since $(H^*(\mathcal{B}_u)^{(\zeta)})^{(k)} = \bigoplus_{n \equiv k \mod e} H^{2n}(\mathcal{B}_u)$, the corollary follows from Lemma 3.2. $\square$

Remarks 3.4. (i) In the case where $u = 1$, the cohomology ring $H^*(\mathcal{B}_u) = H^*(\mathcal{B})$ coincides with the coinvariant algebra R of W. In the special case where G is of type A_{n-1}, i.e., $W \simeq \mathfrak{S}_n$, we consider $W_L \simeq \mathfrak{S}_{n-re}$ for $1 \le e \le n$. Then $W_{L'} \simeq \mathfrak{S}_{re}$, and if we choose a regular element $a \in W_{L'}$ as a product of disjoint cycles of length e, Proposition 3.3 can be applied. This recovers the formula obtained by Morita and Nakajima [MN1].

More generally, consider the Weyl group W acting on the real vector space V as the reflection module. For $v \in V$, let W_v be the stabilizer of v in W, and N_v the stabilizer of the line $\mathbf{R}v$ in W. Note that W_v is normal in N_v, and W_v coincides with W_L for a certain Levi subgroup in G. Then for any $\varGamma = \langle a \rangle$ such that $\varGamma \subset N_v$, Bonnafé, Lehrer and Michel [BLM] have proved a similar formula as in Proposition 3.3. So our formula (3.3.2) can be regarded as a special case of theirs. (Note that they treat a more general case, where W is a complex reflection group and $\varGamma$ is not necessarily cyclic, in a framework of coinvariant algebras.)

(ii) We consider a unipotent element $u \in L$ in the case where $G = GL_n$. $u \in G$ can be written as $u = u_\mu$ by a partition $\mu = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})$ of n. Take a positive integer $e \ge 2$, and let I be a subset of $\{1, \dots, n\}$ such that $e \le m_i$ for $i \in I$.

We consider a Levi subgroup L of type $X_0 + e\sum_{i\in I} A_{i-1}$, where $X_0 = A_k$ with $k = \sum_{i\notin I} im_i + \sum_{i\in I} i(m_i - e) - 1$. Then we have $W_{L'} = \{1\}$, and one can choose $u \in L$ so that it satisfies the assumption of the case (b) in 1.6. Thus Proposition 3.3 can be applied. This covers the results on the stability of dimensions obtained in [MN2], [MN3], where they considered the case $|I| = 1$ or the case all the m_i are divisible by e.

Returning to the general setup, we consider the case where u is a regular unipotent element in L. Then $H^*(\mathcal{B}_u^L) = H^0(\mathcal{B}_u^L) \simeq \mathbf{C}$ is a trivial W_L-module. Thus Proposition 3.3 implies the following.

Corollary 3.5. *Let G be a simple algebraic group modulo center, and L a Levi subgroup in G. Let u be a regular unipotent element in L. Let $\varGamma = \langle a\rangle$ be a subgroup of $N_W(W_L)$ of order e satisfying the conditions in 1.6. Then for $k = 0, \dots, e-1$, we have*

$$\bigoplus_{n\equiv k \mod e} H^{2n}(\mathcal{B}_u) \simeq \operatorname{Ind}_{\varGamma\ltimes W_L}^{W} \widetilde{\psi}^{(-k)}$$

as W-modules, where $\widetilde{\psi}^{(-k)}$ is the character of $\varGamma \ltimes W_L$ obtained as the pull back of $\psi^{(-k)}$ under the projection $\varGamma \ltimes W_L \to \varGamma$.

Proof. In the case (a), the assertion follows from (3.3.2). So we consider the case (b). In the setup of 3.1, $V^{(i)}$ is a trivial W_L-module $\mathbf{C}$ for $i = 0$ and zero otherwise. Then we see that $V^{(0)} = V_0^{(0)}$, and $b_k \otimes V^{(0)}$ has a structure of $\widetilde{W}_L$-module $\widetilde{\psi}^{(-k)}$. The assertion follows from the formula in 3.1. □

3.6. Let G be a simple algebraic group defined over $\mathbf{F}_q$ with Frobenius map F. We assume that G^F is of split type. The Green function Q_{T_w} is defined as the restriction of the Deligne-Lusztig's virtual character $R_{T_w}(1)$ to the set of unipotent elements in G^F. We assume that $p = \operatorname{ch}\mathbf{F}_q$ is good, and in the case where G is of type E_8, we further assume that $q \equiv 1 \pmod 4$. Then as explained in 2.7, for each unipotent class C of G, there exists a split element $u \in C^F$. As in 2.12, we have

$$Q_{T_w}(u) = \sum_{n\ge 0} \operatorname{Tr}(w, H^{2n}(\mathcal{B}_u))q^n. \tag{3.6.1}$$

Hence there exists a polynomial $\mathbf{Q}_{w,C}(x) \in \mathbf{Z}[x]$ such that $Q_{T_w}(u) = \mathbf{Q}_{w,C}(q)$. Concerning the values of Green functions at root of unity, we have the following.

Proposition 3.7. *Suppose that G, L and $u \in L$ are as in Corollary 3.5. Then we have*

$$\mathbf{Q}_{w,C}(\zeta^j) = |W_L|^{-1}\sharp\{x \in W \mid x^{-1}wx \in a^jW_L\} \tag{3.7.1}$$

for $j = 0, \dots, e-1$. In particular, the value $\mathbf{Q}_{w,C}(\zeta')$ is independent of the choice of a primitive e-th root of unity ζ'.

Proof. Put $c_i(w) = \sharp\{x \in W \mid x^{-1}wx \in a^iW_L\}$ for $i = 0, \dots, e-1$. Then

$$\begin{aligned}\big(\operatorname{Ind}_{\varGamma\ltimes W_L}^{W} \widetilde{\psi}^{(-k)}\big)(w) &= |\varGamma \ltimes W_L|^{-1}\sum_{i=0}^{e-1}\sum_{\substack{x\in W\\ x^{-1}wx\in a^iW_L}} \widetilde{\psi}^{(-k)}(x^{-1}wx) \\ &= |\varGamma \ltimes W_L|^{-1}\sum_{i=0}^{e-1} c_i(w)\zeta^{-ki}.\end{aligned}$$

It follows, by (3.6.1) together with Corollary 3.5, that

$$\begin{aligned}\mathbf{Q}_{w,C}(\zeta^j) &= \sum_{k=0}^{e-1}\zeta^{kj}\sum_{n\equiv k \mod e}\operatorname{Tr}(w, H^{2n}(\mathcal{B}_u)) \\ &= |\varGamma \ltimes W_L|^{-1}\sum_{i=0}^{e-1}c_i(w)\sum_{k=0}^{e-1}\zeta^{(j-i)k} \\ &= |W_L|^{-1}c_j(w).\end{aligned}$$

Hence we obtain the formula (3.7.1). Let ζ^j be a primitive e-th root of unity. There exists an element $\tau \in \operatorname{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ such that $\tau(\zeta) = \zeta^j$. By (3.6.1), we see that $\mathbf{Q}_{w,C}(\zeta) \in \mathbf{Q}(\zeta)$ and that $\tau(\mathbf{Q}_{w,C}(\zeta)) = \mathbf{Q}_{w,C}(\zeta^j)$. But since $\mathbf{Q}_{w,C}(\zeta) \in \mathbf{Z}$ by (3.7.1), we conclude that $\mathbf{Q}_{w,C}(\zeta) = \mathbf{Q}_{w,C}(\zeta^j)$. This proves the proposition. □

Remark 3.8. In the case where $G = GL_n$ and L is of type $A_{m-1} + \cdots + A_{m-1}$ (e-times) with $n = em$, take a regular unipotent element u in L. Then $u = u_\mu \in G$ with $\mu = (m^e)$. For $w \in W = \mathfrak{S}_n$, let $\lambda(w) = (1^{l_1}, 2^{l_2}, \dots)$ be the partition of n corresponding to the cycle decomposition of w. Then one can show by a direct computation (cf. [M, (6.2)]) that

$$|W_L|^{-1}\sharp\{x \in W \mid x^{-1}wx \in aW_L\} = \begin{cases} e^{l(\lambda(w))} & \text{if } e \mid l_i \text{ for all } i, \\ 0 & \text{otherwise,}\end{cases}$$

where $l(\lambda)$ is the number of parts for a partition λ. Thus we recover the formula in [LLT, Theorem 3.2, Theorem 3.4] concerning the values of Green polynomials of GL_n at roots of unity.

3.9. We give some more examples where Proposition 3.3 can be applied.

(i) Assume that G is of type B_n and L is a Levi subgroup of type B_m with $m < n$. Then L' is of type A_{n-m-1}. For any $u \in L$ and a divisor e of $n-m$, the proposition can be applied. Similar results hold also for C_n or D_n.

(ii) Assume that $G = Sp_{2n}$. Then a unipotent element $u \in G$ can be written as $u = u_\mu$ as an element of GL_{2n}, where $\mu = (1^{m_1}, 2^{m_2}, \dots)$ is a partition of $2n$ such that m_i is even for odd i. Take an even integer $e \geq 2$, and let I be a subset of odd integers $\{1, 3, \dots, 2n-1\}$ such that $e \leq m_i$ for $i \in I$. We consider a Levi subgroup L of type $X_0 + e\sum_{i\in I}A_{i-1}$, where X_0 is of type C_k with $2k = \sum_{i\notin I} im_i + \sum_{i\in I} i(m_i - e)$.

Then $W_{L'} = \{1\}$, and as in Remarks 3.4 (ii), one can find $u \in L$ so that the case (b) in 1.6 can be applied. Similar results hold also for type B_n and D_n.

(iii) Assume that G is of type E_7, and choose L of type A_2 so that L' is of type A_4. Take any unipotent element $u \in L$. Then the proposition can be applied with $e = 5$.

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