Finite Element Approximation to Infinite Prandtl Number Boussinesq Equations with Temperature-Dependent Coefficients - Thermal Convection Problems in a Spherical Shell

Masahisa Tabata *
Department of Mathematical Sciences, Kyushu University
Fukuoka, 812-8581 Japan

Abstract

A stabilized finite element scheme for infinite Prandtl number Boussinesq equations with temperature-dependent coefficients is analyzed. The domain is a spherical shell and the P1-element is employed for every unknown function. The finite element solution is proved to converge to the exact one in the first order of the time increment and the mesh size. The scheme is applied to Earth's mantle convection problems with viscosities strongly dependent on the temperature and some numerical results are shown.

1 Introduction

In the numerical simulation of the Earth's mantle convection phenomenon the Boussinesq equations with infinite Prandtl number are used as the fundamental mathematical model. See for instance [1],[15] and the references therein. This phenomenon has different characters in comparison with many other convection problems described by the Rayleigh-Bénard equations. As was pointed out by Ratcliff et al. [8] the rheology and the geometry are two important factors of the phenomenon. The former means that the viscosity of the mantle is strongly dependent on the temperature, and the latter means that the domain of the problem is a three-dimensional spherical shell. The corresponding mathematical model becomes a nonlinear system consisting of Stokes equations and a convection-diffusion equation in a spherical shell, coupled with the viscosity, the buoyancy and the convection. In [14] we have proved the existence of the solution of the system, and presented an efficient finite element scheme. In this paper we extend the result to the system having the temperature dependence not only in the viscosity but also in the other coefficients, and give a complete proof of the error estimate.

*Email: tabata@math.kyushu-u.ac.jp

We solve problems in a spherical shell domain, that is, a three-dimensional curved domain. While it is convenient to use the finite element method for the curved domain, it should be cheap from the computational point of view especially for the three-dimensional problem. We, therefore, employ the P1-finite element for every unknown function, which leads us to use a stabilized scheme. In the isoviscosity case we used a stabilized method of least square type [13], [11]. Here we use a stabilized method of penalty type [2] for the Stokes equations as it leads to a simpler scheme for the variable viscosity case.

The contents of this paper are as follows. In Section 2 we describe a system of infinite Prnadtl Boussinesq equations with temperature-dependent coefficients. In Section 3 we show a finite element scheme for the system, and present error estimates for the finite element solution. The proof is given in Section 4. In Section 5 we present some numerical results, which show a clear effect on the change of the ratio of temperature-dependent viscosity. We give some concluding remarks in Section 6.

Throughout this paper we denote by c a generic positive constant, which may be different at each occurrence. We denote by c_* a generic positive constant, which may be dependent on the exact solution. The symbol $(\cdot,\cdot)$ is used for the $L^2(\Omega)^3$- or $L^2(\Omega)$- inner product, and $\langle\cdot,\cdot\rangle$ is for the dual product between a Banach space and the dual space. The abbreviation $||\cdot||_m$ means the norm $||\cdot||_{H^m(\Omega)}$.

2 Infinite Prandtl number Boussinesq equations

Let $T(>0)$ be a time and Ω be a spherical domain

$$\Omega := \{x \in \mathbb{R}^3; R_1 < |x| < R_2\},$$

where $|x|$ is the Euclidian norm of $x = (x_1, x_2, x_3)$, and R_1 and R_2 are positive constants. We consider a finite element analysis of infinite Prantdl number Boussinesq equations with temperature-dependent coefficients described by the following.

$$-\nabla \cdot [2\mu(\theta)D(u)] + \nabla p + \beta(\theta)\theta = f, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla\theta - \nabla \cdot (\kappa(\theta)\nabla\theta) = g, \tag{3}$$

where the velocity u, the pressure p, and the temperature θ,

$$(u, p, \theta) : \Omega \times (0, T) \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$$

are unknown functions, f and g

$$(f, g) : \Omega \times (0, T) \to \mathbb{R}^3 \times \mathbb{R}$$

are given functions, μ, κ, and β

$$(\mu, \kappa, \beta) : \Omega \times (0, T) \times \mathbb{R} \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3$$

are continuously differentiable functions in (x, t, θ), $D(u)$ is the velocity rate tensor defined by

$$D_{ij}(u) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Let Γ_1 and Γ_2 be inner and outer boundaries and Γ be the whole boundary. The slip boundary conditions for u and Dirichlet boundary conditions for θ

$$u \cdot n = 0, \tag{4}$$

$$D(u)n \times n = 0, \tag{5}$$

$$\theta = \theta_\Gamma \tag{6}$$

are imposed on Γ, where n is the exterior unit normal and $\theta_\Gamma : \Gamma \times (0, T) \to \mathrm{I\!R}$ is a given function. Initial condition for θ at $t = 0$,

$$\theta = \theta^0 \tag{7}$$

completes our problem, where $\theta^0 : \Omega \to \mathrm{I\!R}$ is a given function.

Example 1 We consider a non-dimensional Earth's mantle convection problem with a variable viscosity dependent on the temperature. Fixing the origin at the center of the Earth, we take

$$\begin{aligned}
&R_1 = \frac{11}{9}, \quad R_2 = \frac{20}{9}, \\
&\kappa = \frac{1}{Ra}, \quad \mu(\theta) = \exp\left(-(\theta - \frac{1}{2})\log b\right), \quad \beta(x) = -\frac{x}{|x|}, \\
&f = 0, \quad g = 0, \\
&\theta_\Gamma = 1 \text{ on } \Gamma_1, \quad \theta_\Gamma = 0 \text{ on } \Gamma_2,
\end{aligned} \tag{8}$$

where Ra is the Rayleigh number, b is a positive number describing the contrast of viscosity, that is, μ is independent of x and t, normalized at $\theta = 1/2$, and the ratio of the maximum and minimum viscosity is equal to b. β is equal to the unit vector with the direction opposite to the radial vector, which describes that the gravity direction is the center of the Earth. An initial temperature θ^0 is given. Ra is defined by

$$Ra := \frac{\rho_0 g \beta_0 \Delta\theta d^3}{\kappa_0 \mu_0},$$

where g is the gravity acceleration, d is the depth of mantle, $\Delta\theta$ is the difference of temperatures on the core and the surface, κ_0 is the thermal diffusivity, β_0 is the thermal expansion coefficient, ρ_0 and μ_0 are representative density and viscosity, respectively.

In Section 5 we give some numerical results on Example 1. As for the derivation of the equations from the Rayleigh-Bénard equations we refer to [14]. The system of equations (1)–(3) is a generalization of Example 1. The functions μ and β are considered to be normalized, but κ may become very small as κ corresponds to $1/Ra$ and Ra becomes very large, for example, in the geophysical problem.

We now prepare three function spaces. Since (4) is an essential boundary condition, it is natural to introduce the space

$$W := \{v \in H^1(\Omega)^3;\ v \cdot n = 0 \text{ on } \Gamma\}.$$

However, as was discussed in [13], the velocity is not determined uniquely in W for the Stokes equations, to which (1) and (2) are reduced when μ is constant and $\beta = 0$. There are three freedoms of rigid body movements

$$v^{(i)} := e^{(i)} \times x \quad \text{for} \ \ i = 1, 2, 3,$$

where $e^{(i)}$ is the unit vector to the x_i-direction. Eliminating the freedoms, we seek the velocity in

$$V := \{v \in W;\ (v, v^{(i)}) = 0 \ \ (i = 1, 2, 3)\},$$

and the pressure in

$$Q := \{q \in L^2(\Omega);\ (q, 1) = 0\}.$$

We have the following result on the whole problem (1)–(7).

Proposition 1 Suppose that

$$f \in L^2(0, T; H^{-1}(\Omega)^3),\ g \in L^\infty(0, T; L^\infty(\Omega)),$$
$$\theta_\Gamma \in H^1(0, T; H^{1/2}(\Gamma)) \cap L^\infty(0, T; L^\infty(\Gamma)),\ \theta^0 \in L^\infty(\Omega).$$

Then, there exist a solution (u, p, θ) of (1)–(7),

$$u \in L^\infty(0, T; V),\ p \in L^\infty(0, T; Q),$$
$$\theta \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega)).$$

Furthermore, if

$$u \in L^2(0, T;\, W^{1,\infty}(\Omega)^3), \quad \theta \in L^2(0, T;\, W^{1,\infty}(\Omega)),$$

the solution is unique.

Proposition 1 can be proved in a similar way to Theorem 1 of [14].

3 A Finite Element Scheme

We present a finite element approximation to the problem (1)–(7). Paying attention to three-dimensional computation, we employ the cheapest element combination P1/P1/P1, that is, the velocity, the pressure, and the temperature are all approximated by the piecewise linear element. Since the combination of P1/P1 element does not work for the Stokes problem, we are required to use a stabilization method. Considering the cost of computation, we employ the stabilization of penalty type [2]. Since κ is small in our problem, (3) is convection-dominant. To solve the equation stably, we use the stream upwind Petrov/Galerkin method [6], [4].

Let Ω_h be a polyhedral approximation to Ω and $\mathcal{T}_h$ be a partition of $\bar{\Omega}_h$ by tetrahedra, where h is the maximum diameter of the tetrahedral elements. The boundary of Ω_h is denoted by Γ_h. We consider a regular family of subdivisions $\{\mathcal{T}_h\}$, $h \downarrow 0$, satisfying the inverse assumption [3]. Let S_h $(\subset H^1(\Omega_h) \cap C^0(\bar{\Omega}_h))$ be the P1 finite element space whose degrees of freedom are on the vertices of tetrahedra. We introduce finite element spaces W_h, V_h, Q_h, and Ψ_h corresponding to W, V, Q, and $\Psi := H_0^1(\Omega)$, respectively,

$$
\begin{aligned}
W_h &:= \left\{ v_h \in S_h^3 \; ; \; (v_h \cdot n_\Omega)(P) = 0 \; (\forall P) \right\}, \\
V_h &:= \left\{ v_h \in W_h \; ; \; (v_h, v^{(i)})_h = 0 \; (i = 1, 2, 3) \right\}, \\
Q_h &:= \left\{ q_h \in S_h \; ; \; (q_h, 1)_h = 0 \right\}, \\
\Psi_h &:= \left\{ \psi_h \in S_h \; ; \; \psi_h(P) = 0 \; (\forall P) \right\},
\end{aligned}
$$

where P stands for nodal point on Γ_h, n_Ω is the unit outer normal to Γ. Since we use the P1 element, every nodal point P on Γ_h is on Γ. We employ $H^1(\Omega_h)^3$-norm for W_h and V_h, $L^2(\Omega_h)$-norm for Q_h, and $H^1(\Omega_h)$-norm for Ψ_h, respectively. We define an affine space $\Psi_h(\theta_\Gamma)$ by

$$
\Psi_h(\theta_\Gamma) := \left\{ \psi_h \in S_h \; ; \; \psi_h(P) = \theta_\Gamma(P) \; (\forall P) \right\},
$$

where P stands again for the nodal point on Γ_h and θ_Γ is supposed to be continuous.

We prepare the following bilinear and trilinear forms for $u,\ v \in H^1(\Omega)^3$, $q \in L^2(\Omega)$, and $\theta,\ \psi \in H^1(\Omega)$,

$$
\begin{aligned}
a(\mu, u, v) &:= 2 \int_\Omega \mu D(u) : D(v) \, dx, \\
b(v, q) &:= - \int_\Omega q \, \nabla \cdot v \, dx, \\
c_0(\kappa, \theta, \psi) &:= \int_\Omega \kappa \nabla\theta \cdot \nabla\psi \, dx, \\
c_1(u, \theta, \psi) &:= \frac{1}{2} \left\{ \int_\Omega (u \cdot \nabla\theta)\psi \, dx - \int_\Omega (u \cdot \nabla\psi)\theta \, dx \right\}.
\end{aligned}
$$

Remark 1

(i) In the finite element method every integral over Ω is replaced by that over Ω_h. In this paper we use the same notation for these two integrals, for example, a is used for the trilinear form over Ω as well as over Ω_h. Errors caused by this difference of the domains can be proved to be less than approximation errors by finite element spaces. For the details we refer to [12].

(ii) In S_h^3, the rigid body rotation $v^{(i)}$, $i = 1, 2, 3$, can be reproduced. Especially, $v^{(i)}$ belongs to W_h.

Let Δt be a time increment and set the total time step number $N_T := \lfloor T/\Delta t \rfloor$. We denote by v_h^n the value of v_h at $t = n\Delta t$ for an integer $n \in [0, N_T]$. Let X be a Banach space. We define $\ell^q(X)$-norm for a sequence $v_h \equiv \{v_h^n\}_{n=0}^{N_T} \subset X$ by

$$
||v_h||_{\ell^q(X)} := \left\{ \Delta t \sum_{n=0}^{N_T} ||v_h^n||_X^q \right\}^{1/q},
$$

where q (≥ 1) is a real number and extended naturally to ∞.

We approximate the time derivative $\partial\theta/\partial t$ at $t = (n+1)\Delta t$ by the difference $D_{\Delta t}\theta^n := (\theta^{n+1} - \theta^n)/\Delta t$. A stabilized finite element approximation to (1)–(7) is to find $(u_h^n,\, p_h^n,\, \theta_h^{n+1}) \in V_h \times Q_h \times \Psi_h(\theta_\Gamma^{n+1})$, $n = 0, \cdots, N_T$, satisfying

$$a(\mu_h(\theta_h^n), u_h^n, v_h) + b(v_h, p_h^n) = -(\beta_h(\theta_h^n)\theta_h^n, v_h) + (f_h^n, v_h), \tag{9}$$

$$b(u_h^n,\, q_h) - \delta \sum_{K\in\mathcal{T}_h} h_K^2 (\nabla p_h^n, \nabla q_h)_K = 0, \tag{10}$$

$$\begin{aligned}
&(D_{\Delta t}\theta_h^n, \psi_h) + c_0(\kappa_h(\theta_h^n), \theta_h^{n+1}, \psi_h) + c_1(u_h^n,\, \theta_h^{n+1},\, \psi_h)\\
&\qquad + \sum_{K\in\mathcal{T}_h} \tau_K^n (D_{\Delta t}\theta_h^n + u_h^n \cdot \nabla\theta_h^{n+1}, u_h^n \cdot \nabla\psi_h)_K\\
&= (g_h^{n+1},\, \psi_h) + \sum_{K\in\mathcal{T}_h} \tau_K^n (g_h^{n+1}, u_h^n \cdot \nabla\psi_h)_K
\end{aligned} \tag{11}$$

for any $(v_h, q_h, \psi_h) \in V_h \times Q_h \times \Psi_h$ and $n \in [0, N_T - 1]$ with an initial condition θ_h^0. Here $(\cdot,\cdot)_K$ represents the $(L^2)^3$- or L^2- inner product on element K, and θ_h^0 is an approximation to θ^0. A positive constant δ is a stability parameter for the Stokes equations and τ_K^n is also a stability parameter for the convection-diffusion equation defined by

$$\tau_K^n := \min\left\{ \frac{\Delta t}{2}, \frac{h_K^2}{12\kappa_K^n}, \frac{h_K}{2U_K^n} \right\}, \tag{12}$$

where h_K is the diameter of element K , $\kappa_K^n = |\kappa(\theta_h^n(G_K))|$, $U_K^n = |u_h^n(G_K)|$ and G_K is the barycenter of K. f_h^n and g_h^{n+1} are linear interpolations of $f(\cdot, n\Delta t)$ and $g(\cdot, (n+1)\Delta t)$, while κ_h, μ_h and β_h are piecewise constant functions defined by, for example, $\mu_h(\theta_h^n) = \mu(\theta_h^n(G_K))$ on K.

Remark 2 The GLS type stabilized method [7] includes the term $\nabla\cdot[\mu(\theta_h^n)\nabla u_h^n]$. The replacement of μ by μ_h leads to the penalty type stabilization (10), which reduces the computation cost. The convergence rates remain same for these two methods as we use the P1 element.

(9) and (10) are linear in u_h^n and p_h^n, and so is (11) in θ_h^{n+1}. We can show that those equations are uniquely solvable. Once θ_h^n is given, (u_h^n, p_h^n) is obtained from (9) and (10). Substituting θ_h^n and u_h^n to (11), θ_h^{n+1} is solved. Hence, starting from the initial value θ_h^0, we can obtain the finite element solution (u_h, p_h, θ_h).

Suppose the conditions in Proposition 1 are satisfied. From Proposition 1 we know that θ is bounded. The a priori bound

$$\begin{aligned}
&||\theta(t)||_{L^\infty(\Omega)}\\
&\leq t\,||g||_{L^\infty(0,t;L^\infty(\Omega))} + \max\{||\theta_\Gamma||_{L^\infty(0,t;L^\infty(\Gamma))} + ||\theta^0||_{L^\infty(\Omega)}\}
\end{aligned}$$

is obtained from the maximum principle to (3). Modifying μ, κ and β outside the bound, we can take positive constants μ_1, μ_2, κ_1, κ_2 and β_2 such that

$$\mu_1 \leq \mu(x,t,\xi) \leq \mu_2, \quad \kappa_1 \leq \kappa(x,t,\xi) \leq \kappa_2, \quad |\beta(x,t,\xi)| \leq \beta_2$$

for $(x,t,\xi) \in \bar{\Omega} \times [0,T] \times \mathbb{R}$. Furthermore we can take a positive constant M such that

$$|\frac{\partial \kappa}{\partial \xi}(x,t,\xi)| \leq M\kappa(x,t,\xi) \tag{13}$$

for $(x,t,\xi) \in \bar{\Omega} \times [0,T] \times \mathbb{R}$. Under such a modification and supplemental regularity assumptions on data

$$\begin{aligned} &f \in C([0,T];(H^1(\Omega) \cap C(\bar{\Omega}))^3), \quad g \in C([0,T];H^1(\Omega) \cap C(\bar{\Omega})) \\ &\theta_\Gamma \in C([0,T];C(\Gamma)) \end{aligned}$$

we can show that the finite element solution converges to the exact one.

Theorem 1 *Let (u,p,θ) be a solution of (1)–(7) such that*

$$\begin{aligned} u \in & C([0,T]\,;(H^2(\Omega) \cap W^{1,\infty}(\Omega))^3) \cap H^1(0,T\,;H^1(\Omega)^3), \\ p \in & C([0,T]\,;H^1(\Omega)), \\ \theta \in & C([0,T]\,;H^2(\Omega) \cap W^{1,\infty}(\Omega)) \cap H^1(0,T\,;H^1(\Omega)) \cap H^2(0,T\,;L^2(\Omega)). \end{aligned}$$

Suppose that the initial value θ_h^0 satisfies

$$||\theta_h^0 - \theta^0||_0 \leq c\,h||\theta^0||_1 \ . \tag{14}$$

Then, there exist positive constants $c_* = c_*(T;u,p,\theta)$, h_0 and Δt_0 such that for any $\Delta t \in (0,\Delta t_0]$ and $h \in (0,h_0]$

$$\begin{aligned} ||\theta_h - \theta||_{\ell^\infty(L^2)}\,, \quad ||\sqrt{\kappa_h}\,\nabla(\theta_h - \theta)||_{\ell^2(L^2)}\,,\ ||\sqrt{\tau_h}\,u_h \cdot \nabla(\theta_h - \theta)||_{\ell^2(L^2)}\,, \\ ||\sqrt{\tau_h}\,(D_{\Delta t} + u_h \cdot \nabla)(\theta_h - \theta)||_{\ell^2(L^2)} \leq c_*\,(\Delta t + h), \\ ||u_h - u||_{\ell^\infty(H^1)}\,, \quad ||p_h - p||_{\ell^\infty(L^2)} \leq c_*\,(\Delta t + h), \end{aligned}$$

where $\phi_h = \sqrt{\kappa_h}\,\nabla\psi_h$ means $\phi_h^{n+1} = \sqrt{\kappa_h(\theta_h^n)}\nabla\psi_h^{n+1}$, and $\phi_h = (D_{\Delta t} + u_h \cdot \nabla)\psi_h$ means $\phi_h^{n+1} = D_{\Delta t}\psi_h^n + u_h^n \cdot \nabla\psi_h^{n+1}$ and

$$||\sqrt{\tau_h}\,\phi_h||_{L^2} = \left\{ \sum_K \tau_K ||\phi_h||_{L^2(K)}^2 \right\}^{1/2} .$$

4 Proof of Theorem 1

In this section we prove Theorem 1. Throughout this section we assume the hypotheses in Theorem 1. Let (u,p,θ) be a solution of (1)–(7) stated in Theorem 1 and (u_h,p_h,θ_h) be the corresponding finite element solution. Let $(\hat{u}_h^n, \hat{p}_h^n) \in V_h \times Q_h$ be a Stokes projection of (u^n,p^n) defined by

$$\begin{aligned} &a(\mu_h(\theta_h^n),\hat{u}_h^n,v_h) + b(v_h,\hat{p}_h^n) = a(\mu_h(\theta_h^n),u^n,v_h) + b(v_h,p^n) \\ &b(\hat{u}_h^n,\,q_h) - \delta \sum_{K \in \mathcal{T}_h} h_K^2(\nabla\hat{p}_h^n,\nabla q_h)_K = 0 \end{aligned}$$

for any $(v_h, q_h) \in V_h \times Q_h$. Let $\hat{\theta}_h^n$ be a linear interpolation of θ^n. We set

$$(e_h^n, \epsilon_h^n, \zeta_h^n) := (u_h^n - \hat{u}_h^n, p_h^n - \hat{p}_h^n, \theta_h^n - \hat{\theta}_h^n) \in V_h \times Q_h \times \Psi_h.$$

Lemma 1 *There exist positive constants h_0 and $c_* = c_*(h_0, \delta, \mu, \beta, f, u, \theta)$ such that for any $h \in (0, h_0]$ and $n = 0, \cdots, N_T$*

$$||e_h^n||_1 + ||\epsilon_h^n||_0 \le c_* \left(h + ||\zeta_h^n||_0\right).$$

Proof. (e_h^n, ϵ_h^n) satisfies

$$\begin{aligned} &a(\mu_h(\theta_h^n), e_h^n, v_h) + b(v_h, \epsilon_h^n) = \langle R_h^n, v_h\rangle \\ &b(e_h^n,\, q_h) - \delta \sum_{K\in\mathcal{T}_h} h_K^2 (\nabla\epsilon_h^n, \nabla q_h)_K = 0 \end{aligned}$$

for any $(v_h, q_h) \in V_h \times Q_h$, where

$$\begin{aligned} R_h^n &:= \{\beta(\theta^n)\theta^n - \beta(\theta_h^n)\theta_h^n\} + \{a(\mu(\theta^n), u^n, \cdot) - a(\mu_h(\theta_h^n), u^n, \cdot)\} + \{f_h^n - f^n\} \\ &:= R_{h1}^n + R_{h2}^n + R_{h3}^n. \end{aligned}$$

Since a is coercive on V_h and b satisfies the weak inf-sup condition on $V_h \times Q_h$ (Lemma 8 [13]), we have from the theory of the stability finite element method [5]

$$||e_h^n||_1 + ||\epsilon_h^n||_0 \le c(h_0, \delta, \mu)||R_h^n||_{V_h'}.$$

Each term of R_h can be estimated as

$$\begin{aligned} ||R_{h1}^n||_0 &\le c(\beta, ||\theta^n||_1)\left(h + ||\zeta_h^n||_0\right) \\ ||R_{h2}^n||_{V_h'} &\le c(\mu, ||u^n||_{W^{1,\infty}}, ||\theta^n||_1)\left(h + ||\zeta_h^n||_0\right) \\ ||R_{h3}^n||_0 &\le c||f^n||_1 h, \end{aligned}$$

which complete the proof.

We now prove Theorem 1. $\{\zeta_h^n\}$ satisfies the following variational form for $n = 0, \cdots, N_T - 1$ and $\psi_h \in \Psi_h$,

$$\begin{aligned} &(D_{\Delta t}\zeta_h^n, \psi_h) + c_0(\kappa_h(\theta_h^n), \zeta_h^{n+1}, \psi_h) + c_1(u_h^n,\, \zeta_h^{n+1},\, \psi_h) \\ &+ \sum_{K\in\mathcal{T}_h} \tau_K^n (D_{\Delta t}\zeta_h^n + u_h^n \cdot \nabla\zeta_h^{n+1}, u_h^n \cdot \nabla\psi_h)_K = \sum_{i=1}^{5} \langle R_i^{n+1}, \psi_h\rangle, \end{aligned} \tag{15}$$

where

$$
\begin{aligned}
R_1^{n+1} &:= \left(\frac{\partial \theta}{\partial t}\right)^{n+1} - D_{\Delta t}\hat{\theta}_h^n, \\
R_2^{n+1} &:= c_1(u^{n+1}, \theta^{n+1}, \cdot) - c_1(u_h^n, \hat{\theta}_h^{n+1}, \cdot), \\
R_3^{n+1} &:= c_0(\kappa(\theta^{n+1}), \theta^{n+1}, \cdot) - c_0(\kappa_h(\theta_h^n), \hat{\theta}_h^{n+1}, \cdot), \\
R_4^{n+1} &:= \sum_{j=1}^{3} R_{4j}^{n+1}, \\
\langle R_{41}^{n+1}, \psi_h \rangle &:= \sum_{K\in\mathcal{T}_h} \tau_K^n \left(\left(\frac{\partial \theta}{\partial t}\right)^{n+1} - D_{\Delta t}\hat{\theta}_h^n,\ u_h^n \cdot \nabla\psi_h \right)_K, \\
\langle R_{42}^{n+1}, \psi_h \rangle &:= \sum_{K\in\mathcal{T}_h} \tau_K^n \left(u^{n+1}\cdot\nabla\theta^{n+1} - u_h^n \cdot \nabla\hat{\theta}_h^{n+1},\ u_h^n \cdot \nabla\psi_h \right)_K, \\
\langle R_{43}^{n+1}, \psi_h \rangle &:= -\sum_{K\in\mathcal{T}_h} \tau_K^n \left(\nabla\cdot\left(\kappa(\theta^{n+1})\nabla\theta^{n+1}\right),\ u_h^n \cdot \nabla\psi_h \right)_K, \\
\langle R_5^{n+1}, \psi_h \rangle &:= (g_h^{n+1} - g^{n+1}, \psi_h) + \sum_{K\in\mathcal{T}_h} \tau_K^n \left(g_h^{n+1} - g^{n+1},\ u_h^n \cdot \nabla\psi_h \right)_K.
\end{aligned}
$$

We substitute ζ_h^{n+1} into ψ_h in (15). Using the inequalities

$$
(b \pm a)b = \frac{1}{2}b^2 - \frac{1}{2}a^2 + \frac{1}{2}(b \pm a)^2
$$

and (12), we estimate the left-hand side as

$$
\begin{aligned}
& D_{\Delta t}\left(\frac{1}{2}||\zeta_h^n||_0^2\right) + \frac{\Delta t}{2}||D_{\Delta t}\zeta_h^n||_0^2 + c_0(\kappa_h(\theta_h^n), \zeta_h^{n+1}, \zeta_h^{n+1}) \\
&\quad + c_1(u_h^n,\, \zeta_h^{n+1},\, \zeta_h^{n+1}) + \sum_{K\in\mathcal{T}_h} \tau_K^n (D_{\Delta t}\zeta_h^n + u_h^n\cdot\nabla\zeta_h^{n+1},\ u_h^n\cdot\nabla\zeta_h^{n+1})_K \\
\geq\ & D_{\Delta t}\left(\frac{1}{2}||\zeta_h^n||_0^2\right) + \frac{\Delta t}{4}||D_{\Delta t}\zeta_h^n||_0^2 + ||\sqrt{\kappa_h^n}\nabla\zeta_h^{n+1}||_0^2 \\
&\quad + \frac{1}{2}\sum_{K\in\mathcal{T}_h} \tau_K^n \left(||u_h^n\cdot\nabla\zeta_h^{n+1}||_{L^2(K)}^2 + ||D_{\Delta t}\zeta_h^n + u_h^n\cdot\nabla\zeta_h^{n+1}||_{L^2(K)}^2 \right). \qquad (16)
\end{aligned}
$$

The estimates of terms R_1^{n+1} and R_2^{n+1},

$$
\begin{aligned}
& |\langle R_1^{n+1},\, \zeta_h^{n+1}\rangle| \\
\leq\ & c\left\{\sqrt{\Delta t}||\frac{\partial^2\theta}{\partial t^2}||_{L^2(t_n,t_{n+1};L^2)} + \frac{h}{\sqrt{\Delta t}}||\frac{\partial\theta}{\partial t}||_{L^2(t_n,t_{n+1};H^1)}\right\} ||\zeta_h^{n+1}||_0, \\
& |\langle R_2^{n+1}, \zeta_h^{n+1}\rangle| \leq c\left\{||u||_{C(H^2)} + ||p||_{C(H^1)} + ||\theta||_{C(H^2)}\right\} \\
&\qquad \times \left\{\sqrt{\Delta t}||\frac{\partial u}{\partial t}||_{L^2(t_n,t_{n+1};H^1)} + h||\theta||_{C(H^2)} + ||e_h^n||_1\right\} ||\zeta_h^{n+1}||_0 \\
\leq\ & c_*\left\{\sqrt{\Delta t}||\frac{\partial u}{\partial t}||_{L^2(t_n,t_{n+1};H^1)} + h + ||e_h^n||_1\right\} ||\zeta_h^{n+1}||_0
\end{aligned}
$$

are same as Theorem 11 in [13]. We estimate the term R_3^{n+1} as

$$
\begin{aligned}
&\langle R_3^{n+1}, \zeta_h^{n+1}\rangle \\
&= c_0\left(\kappa(\theta^{n+1}) - \kappa(\theta^n), \theta^{n+1}, \zeta_h^{n+1}\right) + c_0\left(\kappa(\theta^n), \theta^{n+1} - \hat{\theta}_h^{n+1}, \zeta_h^{n+1}\right) \\
&\quad + c_0\left(\kappa(\theta^n) - \kappa_h(\hat{\theta}_h^n), \hat{\theta}_h^{n+1}, \zeta_h^{n+1}\right) + c_0\left(\kappa_h(\hat{\theta}_h^n) - \kappa_h(\theta_h^n), \hat{\theta}_h^{n+1}, \zeta_h^{n+1}\right) \\
&\le \frac{c\kappa_2}{\sqrt{\kappa_1}}\Big\{(\Delta t||\frac{\partial\theta}{\partial t}||_{L^\infty(L^2)} + h||\theta^n||_1 + ||\zeta_h^n||_0)M||\theta^{n+1}||_{W^{1,\infty}} + h||\theta^{n+1}||_2\Big\} \\
&\qquad \times ||\sqrt{\kappa_h^n}\nabla\zeta_h^{n+1}||_0 \\
&\le c_*\{\Delta t + h + ||\zeta_h^n||_0\}||\sqrt{\kappa_h^n}\nabla\zeta_h^{n+1}||_0.
\end{aligned}
$$

The term R_4^{n+1} is evaluated as

$$
\begin{aligned}
&|\langle R_4^{n+1}, \zeta_h^{n+1}\rangle| \\
&\le c\left\{\sqrt{\tau_{\max}}\left(\sqrt{\Delta t}||\frac{\partial^2\theta}{\partial t^2}||_{L^2(t_n,t_{n+1};L^2)} + \frac{h}{\sqrt{\Delta t}}||\frac{\partial\theta}{\partial t}||_{L^2(t_n,t_{n+1};H^1)}\right.\right. \\
&+ \left.\left(||e_h^n||_0 + h||u^n||_1 + \sqrt{\Delta t}||\frac{\partial u}{\partial t}||_{L^2(t_n,t_{n+1};L^2)}\right)||\theta^{n+1}||_{W^{1,\infty}} + h||\theta^{n+1}||_2||u^n||_{L^\infty}\right) \\
&\quad \left. + h\sqrt{\kappa_2}(M||\theta^{n+1}||^2_{W^{1,\infty}} + ||\theta^{n+1}||_2)\right\}||\sqrt{\tau_h^n}u_h^n\cdot\nabla\zeta_h^{n+1}||_0. \\
&\le c_*\left\{\sqrt{\Delta t}||\frac{\partial^2\theta}{\partial t^2}||_{L^2(t_n,t_{n+1};L^2)} + \frac{h}{\sqrt{\Delta t}}||\frac{\partial\theta}{\partial t}||_{L^2(t_n,t_{n+1};H^1)}\right. \\
&\qquad \left. +\sqrt{\Delta t}||\frac{\partial u}{\partial t}||_{L^2(t_n,t_{n+1};L^2)} + h + ||e_h^n||_0\right\}||\sqrt{\tau_h^n}u_h^n\cdot\nabla\zeta_h^{n+1}||_0,
\end{aligned}
$$

where we have used (12). Finally the term R_5^{n+1} is estimated as

$$
\begin{aligned}
|\langle R_5^{n+1}, \zeta_h^{n+1}\rangle| &\le ch||g||_{C(H^1)}\left(||\zeta_h^{n+1}||_0 + \sqrt{\tau_{\max}}||\sqrt{\tau_h^n}u_h^n\cdot\nabla\zeta_h^{n+1}||_0\right) \\
&\le c_*h\left(||\zeta_h^{n+1}||_0 + ||\sqrt{\tau_h^n}u_h^n\cdot\nabla\zeta_h^{n+1}||_0\right).
\end{aligned}
$$

We can replace $||e_h^n||_i$, $i = 0, 1$, by $h + ||\zeta_h^n||_0$ in the above estimates in virtue of Lemma 1. These estimates and (16) leads to

$$
\begin{aligned}
&D_{\Delta t}\left(\frac{1}{2}||\zeta_h^n||_0^2\right) + \frac{\Delta t}{4}||D_{\Delta t}\zeta_h^n||_0^2 + \frac{1}{2}||\sqrt{\kappa_h^n}\nabla\zeta_h^{n+1}||_0^2 \\
&\quad + \frac{1}{4}||\sqrt{\tau_h^n}u_h^n\cdot\nabla\zeta_h^{n+1}||_0^2 + \frac{1}{2}||\sqrt{\tau_h^n}(D_{\Delta t}\zeta_h^n + u_h^n\cdot\nabla\zeta_h^{n+1})||_0^2 \\
&\le c_*\left(\Delta t||\frac{\partial^2\theta}{\partial t^2}||^2_{L^2(t_n,t_{n+1};L^2)} + \frac{h^2}{\Delta t}||\frac{\partial\theta}{\partial t}||^2_{L^2(t_n,t_{n+1};H^1)} + \Delta t||\frac{\partial u}{\partial t}||^2_{L^2(t_n,t_{n+1};H^1)}\right. \\
&\qquad \left. + h^2 + ||\zeta_h^n||_0^2\right) + ||\zeta_h^{n+1}||_0^2.
\end{aligned}
$$

Applying Gronwall's inequality with (14), we obtain

$$
\begin{aligned}
||\zeta_h||_{\ell^\infty(L^2)}, \quad ||\sqrt{\kappa_h}\nabla\zeta_h||_{\ell^2(L^2)}, \quad ||\sqrt{\tau_h}\, u_h\cdot\nabla\zeta_h||_{\ell^2(L^2)},\\
||\sqrt{\tau_h}(D_{\Delta t}+u_h\cdot\nabla)\zeta_h||_{\ell^2(L^2)} \le c_*\,(\Delta t + h),
\end{aligned}
$$

which implies from Lemma 1

$$
||e_h||_{\ell^\infty(H^1)}, \quad ||\epsilon_h||_{\ell^\infty(L^2)} \le c_*\,(\Delta t + h).
$$

Triangle inequalities

$$
||\theta-\theta_h|| \le ||\theta-\hat{\theta}_h||+||\zeta_h||,\ ||u-u_h|| \le ||u-\hat{u}_h||+||e_h||,\ ||p-p_h|| \le ||p-\hat{p}_h||+||\epsilon_h||
$$

lead to the desired result.

5 Numerical Results

We present numerical results of (1)–(7) in the case of Example 1 described in Section 2. The boundary temperature θ_Γ is normalized. The choice of the viscosity function in the temperature

$$
\mu(\theta) = \exp[-(\theta - \frac{1}{2})\log b]
$$

is based on a linearized Arrhenius law [8]. We take the initial temperature θ^0

$$
\theta^0(x) = \theta^*(r) + \epsilon \sin\pi(\frac{R_2 - r}{R_2 - R_1})Y_3^2(\varphi,\psi)\,, \tag{17}
$$

where (r,φ,ψ) is the spherical coordinate of x, $\theta^*(r)$ is the conductive solution defined by

$$
\theta^*(r) = \frac{R_1}{R_2 - R_1}(\frac{R_2}{r} - 1),
$$

Y_n^m is the normalized spherical harmonic function of degree n and order m, and $\epsilon = 0.1$. This initial condition was used in [8]. We set $Ra = 7,000$.

We performed a numerical simulation for this problem by the stabilized finite element scheme (9)–(11) with $\delta = 0.005$. Figure 1 shows the mesh and Table 1 shows the data for the computation. We consider five cases of $b = 1$, 10, 10^2, 10^3, and 10^4. Starting from the initial temperature (17), we got a numerically stationary solution (u_h, p_h, θ_h) for each case. In Figs.2–4 we show the isothermal surfaces of θ_h=0.2, 0.5, and 0.8. In the left of Figure 2 the isothermal surfaces of the initial temperature (17) are shown. As the viscosity ratio b increases from 1 to 10^4, the viscosity near the surface of the Earth, where the temperature is low, increases. The plume heads, therefore, flatten much more as the viscosity ratio increases as observed clearly in Figs. 2–4. In the case of $b = 10^4$ the number of plumes increases to 12, while the numbers of plumes remain four for $b = 1$, 10, 10^2, and 10^3.

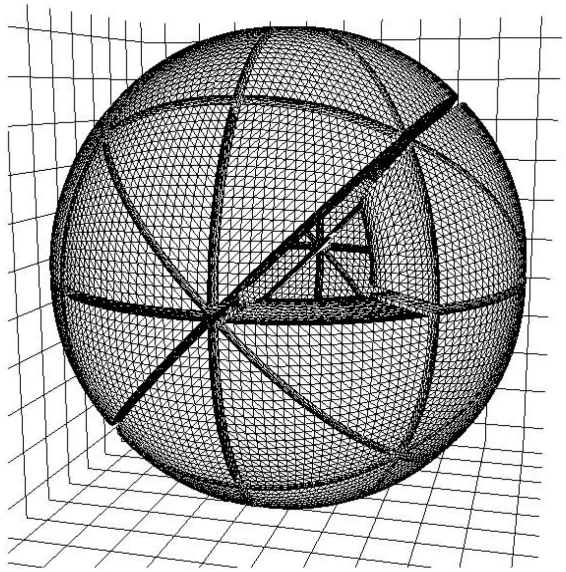

Figure 1: Mesh

Table 1: Discretization parameters

# of nodes	# of elements	h	Δt
117,540	664,320	0.2	3.0

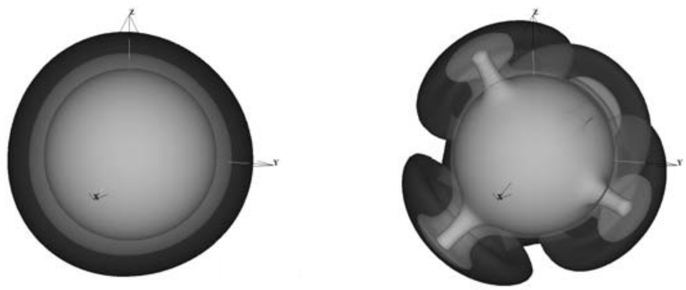

Figure 2: $Ra = 7,000, t = 0$(left), $b = 1$(right) [9]

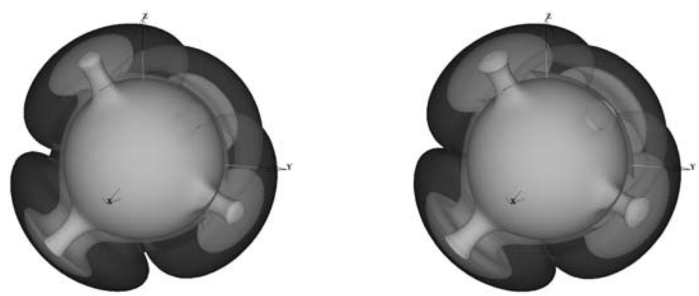

Figure 3: $Ra = 7,000, b = 10$(left), $b = 10^2$(right) [9]

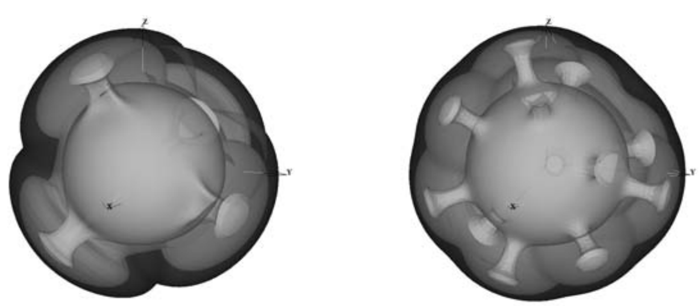

Figure 4: $Ra = 7,000, b = 10^3$(left), $b = 10^4$(right) [9]

6 Concluding Remarks

We have analyzed a 3D stabilized finite element scheme for infinite Prandtl number Boussinesq equations with temperature dependent coefficients. Our finite elements are P1-elements for all unknown functions. The convergence rate has been proved to be of order $\Delta t + h$ for the unsteady problems. We have performed numerical experiments and found that the plume number increases from 4 to 12 when the viscosity ratio increases from 10^3 to 10^4 under some condition. The detail of the numerical results will be presented in a forthcoming paper [9].

Acknowledgments

This paper was finished while the author was visiting the Centre Bernoulli of École Polytechnique Fédérale de Lausanne. Thanks are due to Professor A. Quarteroni for the many courtesies extended to the author.

References

[1] J. R. Baumgardner. Three-dimensional treatment of convective flow in the Earth's mantle. *Journal of Statistical Physics*, 39(5/6):501–511, 1985.

[2] F. Brezzi and J. Pitkäranta. On the stabilization of finite element approximations of the Stokes equations. In W. Hachbush, editor, *Efficient Solutions of Elliptic Systems*, volume 10, *Notes on Numerical Fluid Mechanics*, pages 11–19. Braunschweig, Wiesbaden, 1984.

[3] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems*. SIAM, New York, 2002.

[4] L. P. Franca, S. L. Frey, and T. J. R. Hughes. Stabilized finite element methods: I. Application to the advective-diffusive model. *Computer Methods in Applied Mechanics and Engineering*, 95:253–276, 1992.

[5] L. P. Franca and R. Stenberg. Error analysis of some Galerkin least squares methods for the elasticity equations. *SIAM J. Numer. Anal.*, 28:1680–1697, 1991.

[6] T. J. R. Hughes and A. Brooks. A theoretical framework for Petrov-Galerkin methods with discontinuous weighting functions: Application to the streamline-upwind procedure. In R. H. Gallagher et al., editors, *Finite Elements in Fluids*, volume 4, pages 47–65. John Wiley & Sons, 1982.

[7] T. J. R. Hughes and L. P. Franca. A new finite element formulation for computational fluid dynamics: VII. the Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity/pressure

spaces. *Computer Methods in Applied Mechanics and Engineering*, 65:85–96, 1987.

[8] J. T. Ratcliff, G. Schubert, and A. Zebib. Three-dimensional variable viscosity convection of an infinite Prandtl number Boussinesq fluid in a spherical shell. *Geophysical Research Letters*, 22:2227–2230, 1995.

[9] A. Suzuki and M. Tabata. Numerical computation of an Earth's mantle convection problem with temperature dependent viscosity, to appear.

[10] A. Suzuki and M. Tabata. Finite element matrices in congruent subdomains and their effective use for large scale computatuions, to appear.

[11] A. Suzuki, M. Tabata, and S. Honda. Numerical solution of an unsteady Earth's mantle convection problem by a stabilized finite element method. *Theoretical and Applied Mechanics*, 48:371–378, 1999.

[12] M. Tabata. Uniform solvability of finite element solutions in approximate domains. *Japan Journal of Industrial and Applied Mathematics*, 18:567–585, 2001.

[13] M. Tabata and A. Suzuki. A stabilized finite element method for the Rayleigh-Bénard equations with infinite Prandtl number in a spherical shell. *Computer Methods in Applied Mechanics and Engineering*, 190:387–402, 2000.

[14] M. Tabata and A. Suzuki. Mathematical modeling and numerical simulation of Earth's mantle convection. In P. G. Ciarlet, I. Babuska and T. Miyoshi, editors, *Mathematical Modeling and Numerical Simulation in Continuum Mechanics*, volume 19, *Lecture Notes in Computational Science and Engineering*, pages 219–231. Springer, Berlin, 2002.

[15] P. J. Tackley. Self-consistent generation of tectonic plates in time-dependent, three-dimensional mantle convection simulations 1. Pseudoplastic yielding. *Geochemistry Geophysics Geosystems, An Electronic Journal of the Earth Sciences*, 1(2000GC000036):1–45, 2000.