A stabilized nonconforming finite element method for incompressible flow

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Abstract

In this paper we extend the recently introduced edge stabilization method to the case of nonconforming finite element approximations of the linearized Navier-Stokes equation. To get stability also in the convective dominated regime we add a term giving L^2-control of the jump in the gradient over element boundaries. An a priori error estimate that is uniform in the Reynolds number is proved and some numerical examples are presented.

Key words: stabilized methods, finite element, Crouzeix-Raviart, nonconforming, interior penalty, incompressible flow

1 Introduction

The solution of the Navier-Stokes equations for incompressible flow using finite element methods remains a challenging problem, in particular if the objective is to construct a method which remains robust and accurate for a wide range of Reynolds numbers. The discretization must assure not only satisfaction of the Babŭska-Brezzi condition but also stabilization of the convective terms and sufficient control of the incompressibility condition. Approximations using non-conforming Crouzeix-Raviart (CR) elements are attractive for the velocity approximation in combination with elementwise constant pressures, since they satisfy the Babŭska-Brezzi condition and have local conservation properties. This discretization was proposed and analyzed in [13] and a stabilized version using the streamline diffusion stabilization was analyzed in [11]. In neither of these cases the high Reynolds number limit was treated. Moreover, in a recent paper, [4], the authors showed that a stabilized nonconforming finite element method using the Crouzeix-Raviart element remains uniformly stable in the vanishing viscosity limit for the generalized Stokes' equation. However, the discretization of convection dominated problems using CR-elements are not stable without both a weak coupling on the element inflow boundary as in the discontinuous Galerkin method *and* streamline diffusion stabilization of the convective terms in the interior of the element (see [10,9,12]). In this respect, the element needs stabilization from both the streamline diffusion method *and* the discontinuous Galerkin method. Considering the fact that the method uses more degrees of freedom than the continuous Galerkin approximation this seems suboptimal. Moreover, the streamline diffusion stabilization has the drawback that it does not permit lumped mass for time stepping. In this paper we therefore propose to apply the recently introduced edge stabilization operator (see [2,3]) to the lowest order Crouzeix-Raviart element for the stabilization of the convective terms. We prove that this operator stabilizes exactly that part of the convective term which is not already included in the approximation space. In this sense this is the smallest perturbation needed to make the Crouzeix-Raviart element stable for convection-diffusion problems. This stabilization method has the advantage, as compared to other stabilized methods, that we may lump mass for efficient timestepping, we do not add any additional degrees of freedom, and we do not need any special structure of the mesh. For Oseen's equation we prove an optimal a priori error estimate in the energy norm independent of the Reynolds number. Another attractive feature of the proposed stabilization is that, unlike SUPG, here the stabilization parameter is independent of the flow regime; we illustrate this by proving an L^2 a priori error estimate for the velocities in the case of low local Reynolds number. Finally, we study the performance of the numerical scheme on some linear and nonlinear model cases.

2 A finite element method for the Oseen's equation

We consider, in $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$, the problem of solving

$$
\begin{aligned}
\sigma\, \boldsymbol{u} + \boldsymbol{\beta} \cdot \nabla \boldsymbol{u} + \nabla p - 2\mu \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) &= \boldsymbol{f}, \quad \text{in } \Omega \\
\nabla \cdot \boldsymbol{u} &= 0, \quad \text{in } \Omega \\
\boldsymbol{u} &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\tag{1}
$$

where $\boldsymbol{u}, \boldsymbol{\beta} \in [H^1_0(\Omega) \cap H_0(\text{div}; \Omega)]^d$, $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$, $p \in L^2_0(\Omega)$, $\boldsymbol{f}$ is a given source term, σ and μ are bounded positive functions. By $H_0(\text{div}; \Omega)$ we denote the functions in $[L^2(\Omega)]^d$ such that $\nabla \cdot \boldsymbol{u} = 0$, and by $L^2_0(\Omega)$ the functions in $L^2(\Omega)$ with zero mean value. The weak form of this problem is to find

$(\boldsymbol{u}, p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(p, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \\ \qquad\qquad b(q, \boldsymbol{u}) = 0 \end{cases} \tag{2}$$

$$\forall\ (\boldsymbol{v}, q) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega),$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) := (\sigma\, \boldsymbol{u},\ \boldsymbol{v}) + (\boldsymbol{\beta} \cdot \nabla \boldsymbol{u},\ \boldsymbol{v}) + 2(\mu \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))$$

$$b(p, \boldsymbol{v}) = -(p, \nabla \cdot \boldsymbol{v}) \quad \text{ and } \quad (\boldsymbol{f}, \boldsymbol{v}) := (\boldsymbol{f},\ \boldsymbol{v}).$$

We let $(\cdot, \cdot)$ denote the L_2-scalar product with the corresponding norm $\| \cdot \| = (\cdot, \cdot)^{1/2}$. The $H^s(\Omega)$ norm will be denoted by $\| \cdot \|_{s,\Omega}$. The well posedness of the above problem follows by the Lax-Milgram lemma applied in the space $[H_0^1(\Omega)]^d \cap H_0(\text{div}; \Omega)$. The finite element method consists of seeking a piecewise polynomial approximation $\boldsymbol{u}_h \in V_h$, $p_h \in Q_h$. We let V_h denote the space of the lowest order non-conforming Crouzeix-Raviart elements. Let $\mathcal{T}_h$ denote a shape regular triangulation of the domain Ω, $\mathcal{E}(K)$ the set of all faces of an element $K \in \mathcal{T}_h$, $\mathcal{E} := \cup_{K \in \mathcal{T}_h} \mathcal{E}(K)$ the set of all faces in $\mathcal{T}_h$, $\mathcal{E}_{\partial\Omega} := \{e \in \mathcal{E} : e \subset \partial\Omega\}$, and $\mathcal{E}_0 := \mathcal{E} \setminus \mathcal{E}_{\partial\Omega}$ the set of the boundary and inner faces respectively. For a given piecewise continuous function φ, the jump $[\varphi]$ and the average $\{\varphi\}$ on a face $e \in \mathcal{E}$ are defined by

$$[\varphi](\boldsymbol{x}) := \begin{cases} \lim_{s \to 0^+} (\varphi(\boldsymbol{x} + s\boldsymbol{n}) - \varphi(x - s\boldsymbol{n})) & \text{if } e \not\subset \partial\Omega \\ \lim_{s \to 0^+} = -\varphi(\boldsymbol{x} - s\boldsymbol{n}) & \text{if } e \subset \partial\Omega \end{cases}$$

$$\{\varphi\}(\boldsymbol{x}) := \begin{cases} \frac{1}{2} \lim_{s \to 0^+} (\varphi(\boldsymbol{x} + s\boldsymbol{n}) + \varphi(\boldsymbol{x} - s\boldsymbol{n})) & \text{if } e \not\subset \partial\Omega \\ \frac{1}{2} \lim_{s \to 0^+} 2\varphi(\boldsymbol{x} - s\boldsymbol{n}) & \text{if } e \subset \partial\Omega \end{cases}$$

where $\boldsymbol{n}$ is a normal unit vector on e and $\boldsymbol{x} \in e$. If $e \subset \partial\Omega$ we choose the orientation of $\boldsymbol{n}$ to be outward with respect to Ω otherwise $\boldsymbol{n}$ has an arbitrary but fixed orientation. For the nonconforming finite element functions, continuity across edges e will only be enforced with respect to

$$j_e(\boldsymbol{v}_h) := \int_e [\boldsymbol{v}_h] \,\mathrm{d}s.$$

Using this definition our finite element space may be defined as

$$V_h := \{\boldsymbol{v}_h \in [L^2(\Omega)]^d : \boldsymbol{v}_h|_K \in [P_1(K)]^d,\, \forall K \in \mathcal{T}_h, j_e(\boldsymbol{v}_h) = 0,\, \forall e \in \mathcal{E}\}.$$

Moreover we introduce the space of piecewise constants with mean value zero,

$$Q_h := \{q_h \in L_0^2(\Omega) : q_h|_K \in P_0(K)\},$$

and the subspace W_h of V_h such that

$$W_h := \{\boldsymbol{w}_h \in V_h : (\nabla \cdot \boldsymbol{w}_h, q_h)_h = 0, \forall q_h \in Q_h\}$$

where $(\nabla \cdot \boldsymbol{w}_h, q_h)_h = \sum_K (\nabla \cdot \boldsymbol{w}_h, q_h)_K$. Since the above spaces are H^1-nonconforming we introduce the broken norm equivalent of the L^2-norm

$$\|\boldsymbol{u}\|_h^2 = \sum_K \|\boldsymbol{u}\|_K^2.$$

and the broken H^1-seminorm

$$|\boldsymbol{u}|_h^2 = \sum_K |\boldsymbol{u}|_{1,K}^2.$$

The local mesh size is defined by

$$h_K := \max_K h_{\partial K},$$

and we will assume that $h_K/h_{\partial K} < C$ where C is a fixed constant. We will use C and c as generic constants taking different values every time. To indicate their provenance or main dependence, a subscript may be added, e.g., c_μ. We introduce the interpolation operator $r_h\boldsymbol{u} : [H^1(\Omega)]^d \to V^h$ defined by

$$r_h\boldsymbol{u}(\boldsymbol{x}_e) = \frac{1}{|e|}\int_e \boldsymbol{u}\,\mathrm{d}s,$$

where $\boldsymbol{x}_e$ is the midpoint of the edge e. The L^2-projections onto the spaces are also required for the analysis. Let $\pi_{0,h} : L^2(K) \to P_0(K)$, $\pi_{0,h}^d : [L^2(K)]^d \to [P_0(K)]^d$ denote the L^2-projection onto the constant functions on K and $\pi_{1,h} : [L^2(K)]^d \to V_h$ the standard L^2-projection onto the finite element space. For the above defined interpolation operator and projections we need some approximation and stability properties. These, and some inverse inequalities are collected in the following lemmas

Lemma 1 *For the interpolation operator r_h there holds, if $\boldsymbol{u} \in [H^2(\Omega)]^d$ then*

$$\|r_h\boldsymbol{u} - \boldsymbol{u}\|_h + h|r_h\boldsymbol{u} - \boldsymbol{u}|_h \leq C_r h^2. \tag{3}$$

Moreover, if $\nabla \cdot \boldsymbol{u} = 0$ then $r_h\boldsymbol{u} \in W_h$.

PROOF. The proof of the interpolation estimate is given by Crouzeix and Raviart [7]. The second claim is immediate noting that

$$\int_K \nabla \cdot r_h\boldsymbol{u}\,\mathrm{d}x = \int_K \nabla \cdot \boldsymbol{u}\,\mathrm{d}x = 0$$

by the definition of the interpolant.

Lemma 2 *For the L^2 projection the following H^1 stability holds,*

$$|\pi_{1,h}\boldsymbol{u}|_h \leq C_s\|\boldsymbol{u}\|_{1,\Omega}.$$

For the error analysis, we shall use the following trace inequality

Lemma 3 *For $v \in H^1(K)$ there holds*

$$\|v\|^2_{\partial K} \leq C_t\Big(h_K^{-1}\|v\|^2_K + h_K\|v\|^2_{1,K}\Big) \quad \forall v \in H^1(K), \tag{4}$$

where C_t is a constant independent of h_K

We also need the following local inverse inequality.

Lemma 4 *Let $\boldsymbol{u}_h \in V_h$ where V_h is defined on a shape regular mesh then*

$$\|\nabla \boldsymbol{u}_h\|_K \leq h_K^{-1}C_i\|\boldsymbol{u}_h\|_K,$$

with C_i independent of K.

PROOF. For proofs of lemmas 2–4, see , respectively, Carstensen [5], Thomée [14], and Ciarlet [6].

Our finite element method reads: find $\boldsymbol{u}_h \in V_h$ such that

$$\begin{cases} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(p_h, \boldsymbol{v}_h) + j_u(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \\ \qquad\qquad\qquad\qquad\qquad\qquad b_h(q_h, \boldsymbol{u}_h) = 0 \end{cases} \tag{5}$$

$$\forall\ (\boldsymbol{v}_h, q_h) \in V_h \times Q_h,$$

where

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\sigma \boldsymbol{u}_h + \boldsymbol{\beta}\cdot\nabla\boldsymbol{u}_h, \boldsymbol{v}_h)_h - \frac{1}{2}\sum_K \langle \boldsymbol{\beta}\cdot\boldsymbol{n}[\boldsymbol{u}_h], \{\boldsymbol{v}_h\}\rangle_{\partial K} + (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_h, \tag{6}$$

$$b_h(p_h, \boldsymbol{v}_h) = -(p_h, \nabla\cdot\boldsymbol{v}_h)_h. \tag{7}$$

and the jump terms take the form

$$\begin{aligned} j_u(\boldsymbol{u}_h, \boldsymbol{v}) = &\sum_K \int_{\partial K\setminus\partial\Omega} \gamma_{\boldsymbol{\beta}} h^2_{\partial K}\boldsymbol{\beta}_h\cdot[\nabla\boldsymbol{u}_h]\boldsymbol{\beta}_h\cdot[\nabla\boldsymbol{v}]\mathrm{d}s \\ &+\sum_K \int_{\partial K} \gamma_a(\mu + |\boldsymbol{\beta}\cdot\boldsymbol{n}|h_{\partial K})h_{\partial K}[\boldsymbol{t}\cdot\nabla\boldsymbol{u}_h][\boldsymbol{t}\cdot\nabla\boldsymbol{v}]\mathrm{d}s \\ &+\sum_K \gamma_a \int_{\partial K} h_{\partial K}[(\boldsymbol{t}\cdot\nabla\boldsymbol{u}_h)\cdot\boldsymbol{n}][(\boldsymbol{t}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{n}]\mathrm{d}s. \end{aligned} \tag{8}$$

Here $\boldsymbol{\beta}_h$ is the interpolant of $\boldsymbol{\beta}$ on W_h and $\boldsymbol{t}$ is a unit vector perpendicular to $\boldsymbol{n}$. The gradient jump term serves three purposes. It stabilizes the convective terms (the first sum in (8)), it assures that Korn's inequality is satisfied (the $\gamma_a \mu$ part of the second sum in (8)), it gives additional control of the divergence inconsistency error (the third sum in (8)). The last two properties can be obtained by introducing a lower order penalizing term (see [4]), but the use of the jump of the gradient has the advantage of allowing for one point quadrature in the implementation and from the point of view of analysis it is practical. In the case of three space dimensions the tangent vector should be replaced by the tangent tensor $\nabla \boldsymbol{u}_h \times \boldsymbol{n}$. In the following we will for simplicity only consider the two dimensional case for the tangent vectors.

Remark 5 *The analysis below holds with only minor modifications if $\boldsymbol{\beta}$ is replaced by $\boldsymbol{\beta}_h$ also in the convective term. This is convenient when timestepping the Navier-Stokes equation: $\boldsymbol{\beta}$ may be taken as the solution $\boldsymbol{u}_h$ of the previous timestep.*

In the analysis we will not distinguish between the different stabilization parameters, they will all be denoted γ. We introduce the following shorthand notation

$$A[(\boldsymbol{u}_h, p_h), (\boldsymbol{w}_h, q_h)] = a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(p_h, \boldsymbol{v}_h) - b_h(q_h, \boldsymbol{u}_h).$$

To simplify the analysis we will assume that the exact solution $(\boldsymbol{u}, p)$ belongs to $[H^2(\Omega)]^d \times H^1(\Omega)$; it then follows that the formulation (5) enjoys the following consistency property.

Lemma 6 *For $\boldsymbol{u} \in H^2(\Omega)$ and $p \in H^1(\Omega)$ there holds*

$$A[(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v}_h, q_h)] + j_u(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) = R(\boldsymbol{u}, p, \boldsymbol{v}_h) \qquad (9)$$

for all $(\boldsymbol{v}_h, q_h) \in V^h \times Q_h$. Where the consistency error due to the nonconforming approximation is given by

$$R(\boldsymbol{u}, p, \boldsymbol{v}_h) = -\frac{1}{2} \sum_K \langle 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n}, [\boldsymbol{v}_h] \rangle_{\partial K} + \frac{1}{2} \sum_K \langle p, [\boldsymbol{v}_h \cdot \boldsymbol{n}] \rangle_{\partial K}$$

PROOF. This is an immediate consequence of the regularity hypothesis: if $u \in H^2(\Omega)$ then the trace of $\nabla \boldsymbol{u}$ is well defined and hence $j(\boldsymbol{u}, \boldsymbol{v}_h) = 0$. The consistency error is obtained by integration by parts,

$$\sum_K (2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)) = \sum_K (-2\mu \nabla \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{v}_h) + \frac{1}{2} \langle 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n}, [\boldsymbol{v}_h] \rangle_{\partial K},$$

and

$$-\sum_K (p, \nabla \cdot \boldsymbol{v}_h)_K = (\nabla p, \boldsymbol{v}_h) - \frac{1}{2} \sum_K \langle p, [\boldsymbol{v}_h \cdot \boldsymbol{n}] \rangle_{\partial K}.$$

In this section we will prove some preliminary results that will facilitate the analysis. The main result of this section is lemma 7 where we show that that the jump-term (8) controls the difference between the convective derivative and its quasi interpolant on the finite element space. This lemma is the key ingredient to derive error bounds that are independent of the Peclet number. Then we prove the coercivity of the bilinear form. The triple norm that we will use is given by

$$|||(\boldsymbol{w}_h, q_h)|||^2 = \|\sigma^{1/2}\boldsymbol{w}_h\|_h^2 + |\mu^{1/2}\boldsymbol{w}_h|_h^2 + \|\nabla\cdot\boldsymbol{w}_h\|_h^2 + j_u(\boldsymbol{w}_h, \boldsymbol{w}_h) + c_p\|q_h\|^2$$

where c_p is a constant depending on the problem parameters σ, μ, $\boldsymbol{\beta}$ to be specified later. Note that although for the Crouzeix-Raviart element $\nabla\cdot\boldsymbol{u}_h = 0$ on each triangle the formulation must include a stabilization of the jump of the normal velocity due to the $H(\mathrm{div}, \Omega)$ consistency error. Therefore the triple norm must be chosen as a discrete norm on $(H(\mathrm{div},\Omega) \cap \mu^{1/2}[H_0^1(\Omega)]^d) \times L_0^2(\Omega)$. The triple norm is dominated, at low Reynolds numbers, by the $H^1(\Omega)$ contribution, and at high Reynolds numbers by the part of the jump term controlling the inconsistency in the divergence. This latter term prohibits the decoupling of the velocities and the pressure and the order of the estimate can be no better that the approximation properties of the pressure space. We introduce the space of functions that are piecewise linear on each element

$$Y_h = [\{y \in L^2(\Omega) : y|_K \in P_1(K)\}]^d.$$

We now introduce a quasi interpolant based on local averages $\bar{\pi}_h : Y_h \to V_h$. Let $\boldsymbol{x}_i$ be the midpoint of the face shared by element K and element K' then

$$\bar{\pi}_h \boldsymbol{u}(\boldsymbol{x}_i) = \begin{cases} \{\boldsymbol{u}(\boldsymbol{x}_i)\} & \text{for } \boldsymbol{x}_i \text{ an interior node} \\ \boldsymbol{u}(\boldsymbol{x}_i) & \text{for } \boldsymbol{x}_i \text{ a boundary node} \end{cases}$$

In the following lemma we prove that the projection error is bounded by the jumps in the gradient.

Lemma 7 *Let $\boldsymbol{\beta}_h \in V_h$ and $\boldsymbol{w}_h \in V_h$ then*

$$\|h^{1/2}(\boldsymbol{\beta}_h\cdot\nabla\boldsymbol{w}_h - \bar{\pi}_h(\boldsymbol{\beta}_h\cdot\nabla\boldsymbol{w}_h))\|^2 \leq j_{\boldsymbol{\beta}}(\boldsymbol{w}_h, \boldsymbol{w}_h)$$

where $j_{\boldsymbol{\beta}}(\boldsymbol{w}_h, \boldsymbol{w}_h)$ is given by

$$j_{\boldsymbol{\beta}}(\boldsymbol{w}_h, \boldsymbol{w}_h) = \sum_K \gamma_{\boldsymbol{\beta}} \int_{\partial K\setminus\partial\Omega} h_K h_{\partial K^\perp}(\boldsymbol{\beta}_h\cdot[\nabla\boldsymbol{w}_h])^2 \mathrm{d}s$$

with $h_{\partial K^\perp}$ denoting the triangle size perpendicular to the side on ∂K and $\gamma_{\boldsymbol{\beta}}$ is a parameter depending only on the number of space dimensions.

PROOF. First note that $(V_h \cdot \nabla) V_h \subset Y_h$ so that the projection $\bar{\pi}_h(\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h)$ makes sense. Now consider any triangle K and note that using the Crouzeix-Raviart basis functions $\{\varphi_i\}_{i=1}^{d}$, with φ_i associated with node $\boldsymbol{x}_i$ we may write

$$\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h|_K = \sum_{i=1}^{d+1} \boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h(\boldsymbol{x}_i) \varphi_i(\boldsymbol{x})$$

and

$$\bar{\pi}_h(\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h)|_K = \sum_{i=1}^{d+1} \{\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h(\boldsymbol{x}_i)\} \varphi_i(\boldsymbol{x}).$$

Taking the difference of the two functions in a nodal point yields

$$\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h(\boldsymbol{x}_i)|_K - \{\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h(\boldsymbol{x}_i)\} = \boldsymbol{\beta}_h(\boldsymbol{x}_i) \cdot [\nabla \boldsymbol{w}_h(\boldsymbol{x}_i)].$$

Note that if the node is on the boundary, the right hand side is zero. It follows that for any $K \in \mathcal{T}_h$ such that $\partial K \cap \partial \Omega = \emptyset$

$$\begin{aligned}
&\|h^{1/2}(\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h - \bar{\pi}_h \boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h)\|_K^2 \\
&\qquad = \int_K h_K \left(\sum_{i=1}^{d+1} (\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h(\boldsymbol{x}_i) - \{\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h\}) \varphi_i \right)^2 \mathrm{d}x \\
&\qquad \leq \int_K h_K \left(\sum_{i=1}^{d+1} \boldsymbol{\beta}_h \cdot [\nabla \boldsymbol{w}_h] \varphi_i \right)^2 \mathrm{d}x. \quad (10)
\end{aligned}$$

We now evaluate the integral using nodal point quadrature and note that since the nodes of the Crouzeix-Raviart element are on the midpoints of the element sides this is exact for second degree polynomials in two space dimensions, hence

$$\begin{aligned}
\|h_K^{1/2}(\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h - \bar{\pi}_h \boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h)\|_K^2 &= \sum_{k=1}^{3} h_K \frac{m_K}{3} \boldsymbol{\beta}_h(\boldsymbol{x}_k) \cdot [\nabla \boldsymbol{u}_h]^2 \\
&= \sum_{k=1}^{3} h_K h_{\partial K}^{\perp} \frac{m_{\partial K_k}}{6} \boldsymbol{\beta}_h(\boldsymbol{x}_k) \cdot [\nabla \boldsymbol{u}_h(\boldsymbol{x}_k)]^2 \quad (11)
\end{aligned}$$

where $m_{\partial K_k} = \int_{\partial K_k} \mathrm{d}x$ with ∂K_k the face associated with quadrature point k. In three space dimensions the midpoints of the faces has to be supplemented with the six midpoints of the edges of the tetrahedron to yield an exact quadrature formula (with weights $1/15$ for the midpoints of the faces and $3/20$ for the midpoints of the edges). One may then easily show that

$$\begin{aligned}
&\|h_K^{1/2}(\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h - \bar{\pi}_h \boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h)\|_K^2 \\
&\qquad \leq \frac{5}{6} \sum_{k=1}^{3} h_K h_{\partial K}^{\perp} \frac{m_{\partial K_k}}{6} (\boldsymbol{\beta}_h(\boldsymbol{x}_k) \cdot [\nabla \boldsymbol{u}_h(\boldsymbol{x}_k)])^2 \quad (12)
\end{aligned}$$

It follows, using the Simpson quadrature formula in two dimensions and a quadrature taking the midpoint of the face and the corner-points in three dimensions and noting that the weight for the midpoint is $\frac{d}{d+1}$, that

$$\|h^{1/2}(\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h - \bar{\pi}_h(\boldsymbol{\beta}_h \cdot \nabla \boldsymbol{w}_h))\|_K^2 \leq \gamma_d \int_{\partial K} h_K h_{\partial K^\perp} (\boldsymbol{\beta}_h \cdot [\nabla \boldsymbol{u}_h])^2 \mathrm{d}s. \quad (13)$$

Where $\gamma_d = 1/4$ in two dimensions and $\gamma_d = 10/9$ in three dimensions. We conclude by taking the sum over all triangles $K \in \mathcal{T}$ noting that all boundary contributions vanishes thanks to the definitions of the quasi interpolant.

Remark 8 *A consequence of the above proof is that the jump term edge integral may be evaluated using midpoint quadrature on the faces. In fact the first part of the jump operator given in (8) can be substituted by the discrete operator given by (11) in two space dimensions and by (12) in three to get optimal values of the stabilization constants. The integral formulation however still remains practical from a theoretical viewpoint since it is consistent for H^2-regular solutions.*

As was pointed out in the previous section we only stabilize using the jumps in the gradient. However we need to establish a result showing the equivalence between the jumps in the solution and the jumps in the tangential gradient.

Lemma 9 *For the interior penalty term* (8) *there holds*

$$\sum_K \|\mu^{1/2} h^{-1/2} [\boldsymbol{w}_h]\|_{\partial K}^2 \leq c j_u(\boldsymbol{w}_h, \boldsymbol{w}_h),$$

$$\sum_K \|h^{-1/2} [\boldsymbol{w}_h \cdot \boldsymbol{n}]\|_{\partial K}^2 \leq c j_u(\boldsymbol{w}_h, \boldsymbol{w}_h)$$

and

$$j_u(\boldsymbol{w}_h, \boldsymbol{w}_h) \leq c_\gamma |\boldsymbol{w}_h|_h$$

for all $\boldsymbol{w}_h \in V_h$.

PROOF. By the midpoint continuity of the Crouzeix-Raviart element we note that we may write, with ξ a coordinate along e with midpoint ξ_i,

$$[\boldsymbol{w}_h(\xi)]|_e = [\boldsymbol{t} \cdot \nabla \boldsymbol{w}_h]|_e (\xi - \xi_i)$$

and

$$[\boldsymbol{w}_h(\xi) \cdot \boldsymbol{n}]|_e = [(\boldsymbol{t} \cdot \nabla \boldsymbol{w}_h) \cdot \boldsymbol{n}]|_e (\xi - \xi_i).$$

Hence we have

$$\int_e [\boldsymbol{w}_h(\xi)]^2 \mathrm{d}\xi = \int_e ([\boldsymbol{t} \cdot \nabla \boldsymbol{w}_h](\xi - \xi_i))^2 \mathrm{d}\xi = \frac{1}{12} \int_e h_e^2 [\boldsymbol{t} \cdot \nabla \boldsymbol{w}_h]^2 \mathrm{d}\xi$$

which proves the first claim. The proof of the second claim is equivalent. The last claim finally is an immediate consequence of the trace inequality (4).

Lemma 10 *For the consistency error the following upper bound holds*

$$|R(\boldsymbol{u}, p, \boldsymbol{v}_h)| \leq (c_\mu h \|\boldsymbol{u}\|_{2,\Omega} + Ch\|p\|_{1,\Omega})\, j_u(\boldsymbol{v}_h, \boldsymbol{v}_h)$$

PROOF. Using the zero mean value property of the Crouzeix-Raviart space followed by the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} R(\boldsymbol{u}, p, \boldsymbol{v}_h) \leq c \left(\sum_K \|h^{1/2}(2\mu)^{1/2}(\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n} - \pi_{0,h}^d \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n})\|_{\partial K}^2 \right)^{1/2} \\ \cdot \left(\sum_K \|\mu^{1/2} h^{-1/2} [\boldsymbol{v}_h]\|_{\partial K}^2 \right)^{1/2} \\ + c \left(\sum_K h_K \|p - \pi_{0,h} p\|_{\partial K}^2 \right)^{1/2} \left(\sum_K h_K^{-1} \|[\boldsymbol{v}_h \cdot \boldsymbol{n}]\|_{\partial K}^2 \right)^{1/2} . \end{aligned} \quad (14)$$

The claim now follows using the trace inequality (4), standard interpolation and lemma 9.

Let us now investigate the coercivity properties of the discretization of the convective terms.

Lemma 11 *For the convective terms there holds*

$$(\boldsymbol{\beta} \cdot \nabla \boldsymbol{u}_h, \boldsymbol{u}_h)_h = \frac{1}{2} \sum_K \langle \boldsymbol{\beta} \cdot \boldsymbol{n}[\boldsymbol{u}_h], \{\boldsymbol{u}_h\} \rangle \quad (15)$$

PROOF. The result follows by integrating by parts elementwise in the left hand side and using the equality $[xy] = [x]\{y\} + \{x\}[y]$. The integration by parts yields

$$\sum_K (\boldsymbol{\beta} \cdot \nabla \boldsymbol{u}_h, \boldsymbol{u}_h)_K = \frac{1}{2} \sum_K \int_{\partial K} [\boldsymbol{\beta} \cdot \boldsymbol{n} \boldsymbol{u}_h \cdot \boldsymbol{u}_h] \, \mathrm{d}s - \sum_K (\boldsymbol{u}_h, \boldsymbol{\beta} \cdot \nabla \boldsymbol{u}_h)_K. \quad (16)$$

We rewrite the edge term in the following fashion

$$[\boldsymbol{\beta} \cdot \boldsymbol{n}\, \boldsymbol{u}_h \cdot \boldsymbol{u}_h] = 2\boldsymbol{\beta} \cdot \boldsymbol{n}[\boldsymbol{u}_h] \cdot \{\boldsymbol{u}_h\} \quad (17)$$

and the proof is completed using (17) in (16).

To prove the coercivity of our operator we also need the following discrete Korn's inequality

Lemma 12 *For $\boldsymbol{w}_h \in V_h$ there holds*

$$C_K^2 |\mu^{1/2} \boldsymbol{w}_h|_h^2 \leq \|(2\mu)^{1/2} \boldsymbol{\varepsilon}(\boldsymbol{w}_h)\|_h^2 + j_u(\boldsymbol{w}_h, \boldsymbol{w}_h).$$

PROOF. See Brenner [1].

The coercivity of our formulation is an immediate consequence of lemma 11 and lemma 12.

Lemma 13 *For all $(\boldsymbol{w}_h, q_h) \in V_h \times Q_h$ there holds*

$$C_K^2 |\mu^{1/2} \boldsymbol{w}_h|_h^2 + \|\sigma^{1/2} \boldsymbol{w}_h\|^2 \leq A[(\boldsymbol{w}_h, q_h), (\boldsymbol{w}_h, q_h)]$$

PROOF. First of all notice that the terms $b_h(\boldsymbol{u}_h, p_h)$ cancel. We may write

$$\begin{aligned} A[(\boldsymbol{w}_h, q_h), (\boldsymbol{w}_h, q_h)] = \|\sigma^{1/2} \boldsymbol{w}_h\|_h^2 + \|(2\mu)^{1/2} \boldsymbol{\varepsilon}(\boldsymbol{w}_h)\|_h^2 + (\boldsymbol{\beta} \cdot \nabla \boldsymbol{w}_h, \boldsymbol{w}_h)_h \\ - \frac{1}{2} \sum_K \langle \boldsymbol{\beta} \cdot \boldsymbol{n}[\boldsymbol{w}_h], \{\boldsymbol{w}_h\} \rangle . \end{aligned}$$

Using lemma 11 for the convective term we get

$$(\boldsymbol{\beta} \cdot \nabla \boldsymbol{w}_h, \boldsymbol{w}_h) - \frac{1}{2} \sum_K \langle \boldsymbol{\beta} \cdot \boldsymbol{n}[\boldsymbol{w}_h], \{\boldsymbol{w}_h\} \rangle = 0.$$

The proof is then completed by applying lemma 12.

3 Stability

In this section we will prove an inf-sup condition for our discretization vital for the convergence analysis.

Theorem 14 *(Stability). If $(\boldsymbol{u}_f, p_h) \in V_h \times Q_h$ then there holds*

$$c_{is} |||(\boldsymbol{u}_h, p_h)||| \leq \sup_{(\boldsymbol{w}_h, q_h) \in V_h \times Q_h} \frac{A[(\boldsymbol{u}_h, p_h), (\boldsymbol{w}_h, q_h)] + j_u(\boldsymbol{u}_h, \boldsymbol{w}_h)}{|||(\boldsymbol{w}_h, q_h)|||} \tag{18}$$

where the constant c_{is} depends only on the parameters μ,σ,$\boldsymbol{\beta}$,γ and remains bounded from below when $\mu \to 0$.

PROOF. We prove the above *inf-sup* condition in two steps. First we will prove that there exists $(\boldsymbol{w}_h, q_h) \in V_h \times Q_h$ such that

$$|||(\boldsymbol{u}_h, p_h)|||^2 \leq A[(\boldsymbol{u}_h, p_h), (\boldsymbol{w}_h, q_h)] + j_u(\boldsymbol{u}_h, \boldsymbol{w}_h) \tag{19}$$

and then we conclude by proving that

$$|||(\boldsymbol{w}_h, q_h)||| \leq C|||(\boldsymbol{u}_h, p_h)|||.$$

For the first step we note that by lemma 13 we have choosing $\boldsymbol{w}_h = \boldsymbol{u}_h$, $q_h = p_h$

$$C_K^2|\mu^{1/2}\boldsymbol{u}_h|_h^2 + \|\sigma^{1/2}\boldsymbol{u}_h\|^2 + j_u(\boldsymbol{u}_h, \boldsymbol{u}_h) \leq A[(\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)] + j_u(\boldsymbol{u}_h, \boldsymbol{u}_h) \tag{20}$$

To control the L^2-norm of the pressure we note, following [8], that by the surjectivity of the divergence operator there exists a function $\boldsymbol{v}_p \in [H_0^1(\Omega)]^d$ such that $\nabla \cdot \boldsymbol{v}_p = p_h$ and $|\boldsymbol{v}_p|_{1,\Omega} \leq c\|p_h\|$. Therefore we now choose $\boldsymbol{w}_h = r_h \boldsymbol{v}_p$ and $q_h = \nabla \cdot \boldsymbol{u}_h$. By the stability of the quasi interpolation operator r_h we have

$$\|r_h \boldsymbol{v}_p\| + |r_h \boldsymbol{v}_p|_h \leq c\|p_h\|. \tag{21}$$

Moreover, using the definition of the quasi-interpolant $r_h \boldsymbol{v}_p$ we have

$$\|p_h\|^2 = (p_h, \nabla \cdot \boldsymbol{v}_p) = \sum_e \langle [p_h], \boldsymbol{v}_p \cdot \boldsymbol{n} \rangle_e = \sum_e \langle [p_h], r_h \boldsymbol{v}_p \cdot \boldsymbol{n} \rangle_e = (p_h, \nabla \cdot r_h \boldsymbol{v}_p)_h. \tag{22}$$

As a consequence of (22) we may write

$$\begin{aligned} A[(\boldsymbol{u}_h, p_h), (r_h \boldsymbol{v}_p, \nabla \cdot \boldsymbol{u}_h)] + j_u(\boldsymbol{u}_h, r_h \boldsymbol{v}_p) &= \|p_h\|^2 + \|\nabla \cdot \boldsymbol{u}_h\|^2 \\ &+ (\sigma \boldsymbol{u}_h, r_h \boldsymbol{v}_p) + (\mu \nabla \boldsymbol{u}_h, \nabla r_h \boldsymbol{v}_p)_h + (\boldsymbol{\beta} \cdot \nabla \boldsymbol{u}_h, r_h \boldsymbol{v}_p) \\ &- \frac{1}{2} \sum_K \langle \boldsymbol{\beta} \cdot \boldsymbol{n} [\boldsymbol{u}_h], \{r_h \boldsymbol{v}_p\} \rangle_{\partial K} + j_u(\boldsymbol{u}_h, r_h \boldsymbol{v}_p). \end{aligned} \tag{23}$$

Using Cauchy-Schwarz inequality, Young's inequality, and the stability (21), we readily deduce

$$(\sigma \boldsymbol{u}_h, r_h \boldsymbol{v}_p) \geq -c_\sigma \|\sigma^{1/2} \boldsymbol{u}_h\|^2 - \frac{1}{8}\|p_h\|^2, \tag{24}$$

$$(2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(r_h \boldsymbol{v}_p))_h \geq -c_\mu |\mu^{1/2} \boldsymbol{u}_h|_h^2 - \frac{1}{8}\|p_h\|^2, \tag{25}$$

and by applying the third inequality of lemma 9

$$j_u(\boldsymbol{u}_h, r_h \boldsymbol{v}_p) \geq -j_u(\boldsymbol{u}_h, \boldsymbol{u}_h)^{1/2} j_u(r_h \boldsymbol{v}_p, r_h \boldsymbol{v}_p)^{1/2} \geq -c_\gamma j_u(\boldsymbol{u}_h, \boldsymbol{u}_h) - \frac{1}{8}\|p_h\|^2. \tag{26}$$

It now remains to bound the convective term. An integration by parts yields

$$(\boldsymbol{\beta}\cdot\nabla\boldsymbol{u}_h, r_h\boldsymbol{v}_p) - \frac{1}{2}\sum_K\langle\boldsymbol{\beta}\cdot\boldsymbol{n}[\boldsymbol{u}_h], \{r_h\boldsymbol{v}_p\}\rangle_{\partial K}$$
$$= (\boldsymbol{u}_h, \boldsymbol{\beta}\cdot\nabla r_h\boldsymbol{v}_p) - \frac{1}{2}\sum_K\langle\boldsymbol{\beta}\cdot\boldsymbol{n}\{\boldsymbol{u}_h\}, [r_h\boldsymbol{v}_p]\rangle_{\partial K} \quad (27)$$

For the first term we clearly have

$$(\boldsymbol{u}_h, \boldsymbol{\beta}\cdot\nabla r_h\boldsymbol{v}_p) \geq -2c\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\|\boldsymbol{u}_h\|\|p_h\|. \quad (28)$$

For the second we use the H^1-regularity of $\boldsymbol{v}_p$, the trace inequality (4) and the local inverse inequality to obtain

$$\frac{1}{2}\sum_K\langle\boldsymbol{\beta}\cdot\boldsymbol{n}\{\boldsymbol{u}_h\}, [r_h\boldsymbol{v}_p - \boldsymbol{v}_p]\rangle_{\partial K} \geq -2\sum_K\|\boldsymbol{\beta}\|_{L^\infty(K)}\|\boldsymbol{u}_h\|_{\partial K}\|r_h\boldsymbol{v}_p - \boldsymbol{v}_p\|_{\partial K}$$
$$\geq -2\sum_K\|\boldsymbol{\beta}\|_{L^\infty(K)}(h_K^{-1}\|\boldsymbol{u}_h\|_K^2 + h_K\|\boldsymbol{u}_h\|_{1,K}^2)^{1/2}h_K^{1/2}\|\boldsymbol{v}_p\|_{1,K}$$
$$\geq -2C_ic\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\|\boldsymbol{u}_h\|\,\|p_h\|. \quad (29)$$

Combining (28) and (29) we obtain

$$(\boldsymbol{\beta}\cdot\nabla\boldsymbol{u}_h, r_h\boldsymbol{v}_p) - \frac{1}{2}\sum_K\langle\boldsymbol{\beta}\cdot\boldsymbol{n}[\boldsymbol{u}_h], r_h\boldsymbol{v}_p\rangle_{\partial K} \geq c_\beta\|\boldsymbol{u}_h\|_\Omega^2 - \frac{1}{8}\|p_h\|^2. \quad (30)$$

Using now (24)–(26) and (30) to find a lower bound for the expression (23) we get

$$A[(\boldsymbol{u}_h, p_h), (r_h\boldsymbol{v}_p, \nabla\cdot\boldsymbol{u}_h)] + j_u(\boldsymbol{u}_h, r_h\boldsymbol{v}_p) = \frac{1}{2}\|p_h\|^2 + \|\nabla\cdot\boldsymbol{u}_h\|^2$$
$$- C_{\mu\sigma\beta\gamma}(\|\sigma^{1/2}\boldsymbol{u}_h\|^2 + |\mu^{1/2}\boldsymbol{u}_h|_h^2 + j_u(\boldsymbol{u}_h, \boldsymbol{u}_h)) \quad (31)$$

where

$$C_{\mu\sigma\beta\gamma} = \max\left(c_\sigma, \frac{c_\beta}{\sigma}, \frac{c_\mu}{C_K^2}, c_\gamma\right).$$

Combining (20) and (31) we conclude that (19) holds for the choice $\boldsymbol{w}_h = \boldsymbol{u}_h + (2C_{\mu\sigma\beta\gamma})^{-1}r_h\boldsymbol{v}_p$, $q_h = p_h + \nabla\cdot\boldsymbol{u}_h$. More precisely we have, with $c_p = (2C_{\mu\sigma\beta\gamma})^{-1}$,

$$\frac{1}{2}|||(\boldsymbol{u}_h, p_h)|||^2 \leq A[(\boldsymbol{u}_h, p_h), (\boldsymbol{w}_h, q_h)] + j_u(\boldsymbol{u}_h, \boldsymbol{w}_h). \quad (32)$$

It remains to prove that

$$|||(\boldsymbol{w}_h, q_h)||| \leq C|||(\boldsymbol{u}_h, p_h)|||$$

this is obtained simply by noting that

$$|\mu^{1/2}\boldsymbol{w}_h|_h^2 \leq |\mu^{1/2}\boldsymbol{u}_h|_h^2 + |\mu^{1/2}c_p r_h\boldsymbol{v}_p|_h^2 \leq |\mu^{1/2}\boldsymbol{u}_h|_h^2 + \mu c_p^2 c\|p_h\|^2. \quad (33)$$

Proceeding in the same fashion for the other terms in the triple norm yields

$$
\begin{aligned}
|||(\boldsymbol{w}_h, q_h)|||^2 &= |\mu^{1/2}\boldsymbol{w}_h|_h^2 + \|\sigma^{1/2}\boldsymbol{w}_h\|^2 + j_u(\boldsymbol{w}_h, \boldsymbol{w}_h) + \|\nabla\cdot\boldsymbol{w}_h\|^2 + \|q_h\|^2 \\
&\leq |\mu^{1/2}\boldsymbol{u}_h|_h^2 + \|\sigma^{1/2}\boldsymbol{u}_h\|^2 + j(\boldsymbol{u}_h, \boldsymbol{u}_h) \\
&\quad +2\|\nabla\cdot\boldsymbol{u}_h\|^2 + C\|p_h\|^2 \\
&\leq C|||(\boldsymbol{u}_h, p_h)|||^2
\end{aligned} \tag{34}
$$

and we conclude that

$$c_{is}|||(\boldsymbol{u}_h, p_h)||| |||(\boldsymbol{w}_h, q_h)||| \leq |||(\boldsymbol{u}_h, p_h)|||^2 \leq A[(\boldsymbol{u}_h, p_h), (\boldsymbol{w}_h, q_h)] + j_u(\boldsymbol{u}_h, \boldsymbol{w}_h).$$

Clearly the constant c_{is} is independent of h and furthermore it does not vanish for vanishing μ.

4 Error estimates

We will now proceed to derive a priori error estimates in the triple norm. The estimate takes the form

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leq Ch$$

The energy norm estimate is independent of the Peclet number, indicating that the proposed method should be stable for a wide range of Reynolds numbers when applied to the full Navier-Stokes equations.

Lemma 15 *(Approximation) Consider the projection* $(\pi_{1,h}\boldsymbol{u}, \pi_{0,h}p)$ *of the exact solution*

$$(\boldsymbol{u}, p) \in [H^2(\Omega)]^d \times H^1(\Omega)$$

onto the finite element space $V_h \times Q_h$*. For the projection error there holds*

$$|||(\pi_{1,h}\boldsymbol{u} - \boldsymbol{u}, \pi_{0,h}p - p)||| \leq Ch$$

where $C \leq c(1 + \mu^{1/2} + |\boldsymbol{\beta}|^{1/2}h^{1/2} + \sigma^{1/2}h + \gamma^{1/2})\|\boldsymbol{u}\|_{2,\Omega} + c_\pi\|p\|_{1,\Omega}$.

PROOF. By the optimal approximation property of the L^2-projection we have

$$\|\sigma^{1/2}(\pi_{1,h}\boldsymbol{u} - \boldsymbol{u})\| \leq c_\sigma h^2$$

and

$$\|\pi_{0,h}p - p\| \leq Ch.$$

We now consider $r_h\boldsymbol{u} - \pi_{1,h}\boldsymbol{u}$. Noting that $r_h\boldsymbol{u} - \pi_{1,h}\boldsymbol{u} = \pi_{1,h}(r_h\boldsymbol{u} - \boldsymbol{u})$ we obtain using the H^1-stability of the L^2-projection for the Crouzeix–Raviart element on locally quasi uniform meshes, (see [5])

$$\|\nabla(r_h\boldsymbol{u} - \pi_{1,h}\boldsymbol{u})\|_h \leq C_s\|\nabla(r_h\boldsymbol{u} - \boldsymbol{u})\|_h \leq C_sC_rh \tag{35}$$

and we conclude that

$$\|\mu^{1/2}\nabla(\boldsymbol{u} - \pi_{1,h}\boldsymbol{u})\|_h \leq c_\mu h$$

and

$$\|\nabla\cdot(\boldsymbol{u} - \pi_{1,h}\boldsymbol{u})\|_h \leq Ch.$$

Finally we estimate the penalizing term $j_u(\pi_{1,h}\boldsymbol{u} - \boldsymbol{u}, \pi_{1,h}\boldsymbol{u} - \boldsymbol{u})$

$$\begin{aligned}\sum_K \int_{\partial K}[\nabla(\pi_{1,h}\boldsymbol{u} - \boldsymbol{u})]^2\mathrm{d}s &\leq 2\sum_K \int_{\partial K}(\nabla(\pi_{1,h}\boldsymbol{u} - \boldsymbol{u}))^2\mathrm{d}s \\ &\leq 2\sum_K(h_K^{-1}\|\nabla(\pi_{1,h}\boldsymbol{u} - \boldsymbol{u})\|_K^2 + h_K\|\boldsymbol{u}\|_{2,K}^2) \leq Ch\end{aligned}$$

where we have used the trace inequality (4) in the second inequality. We conclude that

$$j_u(\pi_{1,h}\boldsymbol{u} - \boldsymbol{u}, \pi_{1,h}\boldsymbol{u} - \boldsymbol{u})^{1/2} \leq c\gamma^{1/2}(1 + \mu^{1/2} + |\boldsymbol{\beta}|^{1/2}h^{1/2})h\|\boldsymbol{u}\|_{2,\Omega}$$

Lemma 16 *(Continuity). Let $\boldsymbol{\eta} = \pi_{1,h}\boldsymbol{u} - \boldsymbol{u}$ and $\zeta = \pi_{0,h}p - p$ be the projection error of the velocity and the pressure respectively, then there holds*

$$\begin{aligned}&A[(\boldsymbol{\eta}, \zeta), (\boldsymbol{w}_h, q_h)] + j_u(\boldsymbol{\eta}, \boldsymbol{w}_h) - R(\boldsymbol{u}, p, \boldsymbol{w}_h) \\ &\qquad \leq C(|||(\boldsymbol{\eta}, \zeta)||| + (c_\mu + c_\beta h^{1/2})h\|\boldsymbol{u}\|_{2,\Omega} + Ch\|p\|_{1,\Omega})|||(\boldsymbol{w}_h, q_h)|||\end{aligned}$$

PROOF. Clearly we have using Cauchy-Schwarz inequality

$$(\sigma\boldsymbol{\eta}, \boldsymbol{w}_h) \leq C\|\sigma^{1/2}\boldsymbol{\eta}\||||(\boldsymbol{w}_h, 0)||| \leq c_\sigma|||(\boldsymbol{\eta}, \zeta)||||||(\boldsymbol{w}_h, q_h)|||,$$

$$(2\mu\boldsymbol{\varepsilon}(\boldsymbol{\eta}), \boldsymbol{\varepsilon}(\boldsymbol{w}_h)) \leq C|\mu^{1/2}\boldsymbol{\eta}|_h|||(\boldsymbol{w}_h, 0)||| \leq c_\mu|||(\boldsymbol{\eta}, \zeta)||||||(\boldsymbol{w}_h, q_h)|||.$$

$$j_u(\boldsymbol{\eta}, \boldsymbol{w}_h) \leq |||(\boldsymbol{\eta}, \zeta)||||||(\boldsymbol{w}_h, q_h)|||$$

We consider now the terms expressing the pressure velocity coupling. By the orthogonality of the L^2-projection $\pi_{0,h}$ we have

$$b(\zeta, \boldsymbol{w}_h) = (\pi_{0,h}p - p, \nabla\cdot\boldsymbol{w}_h)_h = 0.$$

Using once again the Cauchy-Schwartz inequality we readily obtain

$$b(q_h, \boldsymbol{\eta}) = (q_h, \nabla\cdot\boldsymbol{\eta})_h \leq |||(\boldsymbol{w}_h, q_h)|||\ |||(\boldsymbol{\eta}, \zeta)|||.$$

It remains to treat the convective term and the nonconsistency term. Let us first consider the convective term, an integration by parts yield together with the addition and substraction of $\boldsymbol{\beta}_h$ yields

$$(\boldsymbol{\beta}\cdot\nabla\boldsymbol{\eta},\boldsymbol{w}_h)-\frac{1}{2}\sum_K\langle\boldsymbol{\beta}\cdot\boldsymbol{n}[\boldsymbol{\eta}],\{\boldsymbol{w}_h\}\rangle=-(\boldsymbol{\eta},(\boldsymbol{\beta}-\boldsymbol{\beta}_h)\cdot\nabla\boldsymbol{w}_h)-(\boldsymbol{\eta},\boldsymbol{\beta}_h\cdot\nabla\boldsymbol{w}_h)$$
$$+\frac{1}{2}\sum_K\langle\boldsymbol{\beta}\cdot\boldsymbol{n},\{\boldsymbol{\eta}\}[\boldsymbol{w}_h]\rangle=I+II+III. \quad (36)$$

The first term is controlled using a local inverse inequality and a local approximation result for $\boldsymbol{\beta}-\boldsymbol{\beta}_h$.

$$\begin{aligned}I &\leq \|\boldsymbol{\eta}\|\left(\sum_K\|\boldsymbol{\beta}\|^2_{W^{1,\infty}(K)}h_K^2\|\nabla\boldsymbol{w}_h\|^2\right)^{1/2}\\ &\leq c_{\boldsymbol{\beta}}\|\boldsymbol{\eta}\|C_i\|\boldsymbol{w}_h\|\leq c_{\boldsymbol{\beta}}h^2|||(\boldsymbol{w}_h,q_h)|||\|\,\|\boldsymbol{u}\|_{2,\Omega}\end{aligned}$$

Using lemma 7 we have for the term II

$$\begin{aligned}II &= ((\boldsymbol{\eta},(\boldsymbol{\beta}_h\cdot\nabla\boldsymbol{w}_h-\bar{\pi}_h(\boldsymbol{\beta}_h\cdot\nabla\boldsymbol{w}_h)))\\ &\leq \|h^{-1/2}\boldsymbol{\eta}\|\|h^{1/2}(\boldsymbol{\beta}_h\cdot\nabla\boldsymbol{w}_h-\bar{\pi}_h(\boldsymbol{\beta}\cdot\nabla\boldsymbol{w}_h))\|\\ &\leq \gamma^{-1/2}\|h^{-1/2}\boldsymbol{\eta}\|j_u(\boldsymbol{w}_h,\boldsymbol{w}_h)\leq c_\gamma h^{3/2}|||(\boldsymbol{w}_h,q_h)|||\,\|\boldsymbol{u}\|_{2,\Omega}. \quad (37)\end{aligned}$$

For the term III we obtain using the trace inequality (4) and the approximation properties of the L^2-projection.

$$\begin{aligned}III &\leq \left(\sum_K\left\langle|\boldsymbol{\beta}\cdot\boldsymbol{n}|^{1/2}[\boldsymbol{w}_h],[\boldsymbol{w}_h]\right\rangle_{\partial K}\right)^{1/2}\|\boldsymbol{\beta}\cdot\boldsymbol{n}\|^{1/2}_{L^\infty(K)}\|\{\boldsymbol{\eta}\}\|_{\partial K}\\ &\leq j(\boldsymbol{w}_h,\boldsymbol{w}_h)^{1/2}\sum_K\|\boldsymbol{\beta}\cdot\boldsymbol{n}\|^{1/2}_{L^\infty(K)}\|\{\boldsymbol{\eta}\}\|_{\partial K}\\ &\leq j(\boldsymbol{w}_h,\boldsymbol{w}_h)^{1/2}c_\beta h^{3/2}\,\|\boldsymbol{u}\|_{2,\Omega}.\end{aligned}$$

Only the nonconsistent residual remains to be bounded and we conclude the proof by applying lemma 10.

Theorem 17 *Let* $(\boldsymbol{u},p)\in[H^2(\Omega)]^d\times H^1(\Omega)$ *be the solution of* (1) *and let* $(\boldsymbol{u}_h,q_h)\in V_h\times Q_h$ *be the finite element solution of* (5)*. Then there holds*

$$|||(\boldsymbol{u}-\boldsymbol{u}_h,p-p_h)|||\leq(c_\sigma h+c_\mu+c_\beta h^{1/2}+c_\gamma)h\|\boldsymbol{u}\|_{2,\Omega}$$
$$+Ch\|p_h\|_{1,\Omega} \quad (38)$$

PROOF. We will consider the discrete error $\boldsymbol{e}_u^h = \pi_{1,h}\boldsymbol{u} - \boldsymbol{u}_h$ and $e_p^h = \pi_{0,h}p - p_h$ since using lemma 15 we have

$$\begin{aligned}|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \leq |||(\boldsymbol{u}-\pi_{1,h}\boldsymbol{u}, p-\pi_{0,h}p)||| \\ + |||(\pi_{1,h}\boldsymbol{u}-\boldsymbol{u}_h, \pi_{0,h}p-p_h)||| \\ \leq Ch + |||(\pi_{1,h}\boldsymbol{u}-\boldsymbol{u}_h, \pi_{0,h}p-p_h)|||\end{aligned}$$

where C is the constant given in lemma 15. By lemma 14 we have

$$c_{is}|||(\boldsymbol{e}_u^h, e_p^h)||| \leq \frac{A[(\boldsymbol{e}_u^h, e_p^h),(\boldsymbol{w}_h, q_h)] + j_u(\boldsymbol{e}_u^h, \boldsymbol{w}_h)}{|||(\boldsymbol{w}_h, q_h)|||} \tag{39}$$

Using now Galerkin orthogonality we may write

$$c_{is}|||(\boldsymbol{e}_u^h, e_p^h)||| \leq \frac{A[(\boldsymbol{\eta}, \zeta),(\boldsymbol{w}_h, q_h)] + j_u(\boldsymbol{\eta}, \boldsymbol{w}_h) - R(\boldsymbol{u}, p, \boldsymbol{w}_h)}{|||(\boldsymbol{w}_h, q_h)|||} \tag{40}$$

where $\boldsymbol{\eta} = \pi_{1,h}\boldsymbol{u} - \boldsymbol{u}$ and $\zeta = \pi_{0,h}p - p$. We conclude the proof by applying lemma 16 and lemma 15.

Remark 18 *Since the estimate is only first order, due to the low order approximation of the pressure and the inclusion of the divergence in the triple norm, the virtues of the streamline stabilization are not obvious. We could in fact proceed with an inverse inequality in term II of (36) and still have the same formal convergence order of the triple norm. It is however known that this would destroy stability of the velocities for problems with important gradients. This is illustrated in the numerical section.*

As we mentioned in the introduction the interior penalty method is independent of the Reynolds number. Of course, for low local Reynolds number the numerical scheme is stable without stabilization (except for the Korn's inequality), so that the stabilization may be eliminated. We will however show that this is unnecessary for our discretization by proving that even when keeping the stabilizing terms our discretization has optimal L^2-convergence of the velocities in the local low Reynolds number regime in spite of the fact that γ_β is independent of μ. Consider the dual continuous problem of seeking $\boldsymbol{\phi} \in [H_0^1(\Omega)]$ and $r \in L_0^2(\Omega)$ such that

$$\begin{aligned}\sigma\boldsymbol{\phi} - \boldsymbol{\beta}\cdot\nabla\boldsymbol{\phi} - 2\mu\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{\phi}) + \nabla r = \boldsymbol{e} \quad &\text{in } \Omega, \\ \nabla\cdot\boldsymbol{\phi} = 0 \quad &\text{in } \Omega\end{aligned} \tag{41}$$

where $\boldsymbol{e} := \boldsymbol{u} - \boldsymbol{u}_h$, and assume that we have the regularity estimate

$$\|\boldsymbol{\phi}\|_{H^2(\Omega)} + \|r\|_{H^1(\Omega)} \leq \|\boldsymbol{e}\|. \tag{42}$$

We recall that since $\boldsymbol{u}_h \in W_h$ and $r_h\boldsymbol{\phi}_h \in W_h$ we have $\nabla \cdot \boldsymbol{e} = 0$ and $\nabla \cdot (\boldsymbol{\phi} - r_h\boldsymbol{\phi}) = 0$ elementwise. Multiply the first line of (41) by $\boldsymbol{e}$ and integrate by parts to obtain

$$\|\boldsymbol{e}\|^2 = a_h(\boldsymbol{e},\boldsymbol{\phi}) + j_u(\boldsymbol{e},\boldsymbol{\phi}) - \sum_K \langle 2\,\mu \boldsymbol{n}\cdot\boldsymbol{\varepsilon}(\boldsymbol{\phi}), [\boldsymbol{e}]\rangle_{\partial K} + \sum_K \langle r, [\boldsymbol{n}\cdot\boldsymbol{e}]\rangle_{\partial K}.$$

Where we have used the partial integration

$$\begin{aligned} -(\boldsymbol{e},\boldsymbol{\beta}\cdot\nabla\boldsymbol{\phi}) &= \textstyle\sum_K(\boldsymbol{\beta}\cdot\nabla\boldsymbol{e},\boldsymbol{\phi})_K - \sum_K \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\boldsymbol{e},\boldsymbol{\phi}\rangle_{\partial K} \\ &= \textstyle\sum_K(\boldsymbol{\beta}\cdot\nabla\boldsymbol{e},\boldsymbol{\phi})_K - \frac{1}{2}\sum_K \langle \boldsymbol{\beta}\cdot\boldsymbol{n}[\boldsymbol{e}],\{\boldsymbol{\phi}\}\rangle_{\partial K}\end{aligned}$$

and the divergence free property of the error and of $\boldsymbol{\phi}$. Using Galerkin orthogonality, the divergence free property of the interpolant and the zero mean value property of the Crouzeix-Raviart element, we obtain

$$\begin{aligned}\|\boldsymbol{e}\|^2 &= a_h(\boldsymbol{e},\boldsymbol{\phi}-r_h\boldsymbol{\phi}) + j_u(\boldsymbol{e},\boldsymbol{\phi}-r_h\boldsymbol{\phi}) \\ &\quad - \textstyle\sum_K\langle 2\,\mu(\boldsymbol{n}\cdot\boldsymbol{\varepsilon}(\boldsymbol{\phi}) - \pi^d_{0,h}\boldsymbol{n}\cdot\boldsymbol{\varepsilon}(\boldsymbol{\phi})), [\boldsymbol{e}]\rangle_{\partial K} \\ &\quad - \textstyle\sum_K\langle 2\,\mu(\boldsymbol{n}\cdot\boldsymbol{\varepsilon}(\boldsymbol{\phi}) - \pi^d_{0,h}\boldsymbol{n}\cdot\boldsymbol{\varepsilon}(\boldsymbol{\phi})), [\boldsymbol{\phi}-r_h\boldsymbol{\phi}]\rangle_{\partial K} \\ &\quad + \textstyle\sum_K\langle r - \pi_{0,h}r, [\boldsymbol{n}\cdot\boldsymbol{e}]\rangle_{\partial K} \\ &\quad + \textstyle\sum_K\langle p - \pi_{0,h}p, [\boldsymbol{n}\cdot(\boldsymbol{\phi}-r_h\boldsymbol{\phi})]\rangle_{\partial K} \\ &\leq |||(\boldsymbol{e},0)|||\,|||(\boldsymbol{\phi}-r_h\boldsymbol{\phi},0)||| + c_\beta j_u(\boldsymbol{e},\boldsymbol{e})^{1/2}\left(\textstyle\sum_K \|\boldsymbol{\phi}-r_h\boldsymbol{\phi}\|^2_{\partial K}\right)^{1/2} \\ &\quad + \left(\textstyle\sum_K h_K\|2\mu^{1/2}(\boldsymbol{n}\cdot\boldsymbol{\varepsilon}(\boldsymbol{\phi}) - \pi^d_{0,h}\boldsymbol{n}\cdot\boldsymbol{\varepsilon}(\boldsymbol{\phi}))\|^2_{L_2(\partial K)}\right)^{1/2} |||(\boldsymbol{e},0)||| \\ &\quad + \left(\textstyle\sum_K h_K\|r-\pi_{0,h}r\|_{L_2(\partial K)}\right)^{1/2} |||(\boldsymbol{e},0)||| \\ &\quad + \left(\textstyle\sum_K h_K\|2\mu^{1/2}(\boldsymbol{n}\cdot\boldsymbol{\varepsilon}(\boldsymbol{\phi}) - \pi^d_{0,h}\boldsymbol{n}\cdot\boldsymbol{\varepsilon}(\boldsymbol{\phi}))\|^2_{L_2(\partial K)}\right)^{1/2} |||(\boldsymbol{\phi}-r_h\boldsymbol{\phi},0)||| \\ &\quad + \left(\textstyle\sum_K h_K\|p-\pi_{0,h}p\|_{L_2(\partial K)}\right)^{1/2} |||(\boldsymbol{\phi}-r_h\boldsymbol{\phi},0)|||.\end{aligned}$$

Using lemma 15, the trace inequality (4) and error estimates for r_h and for piecewise constant interpolation, we arrive at

$$\|\boldsymbol{e}\|^2 \leq Ch\left(|||(\boldsymbol{e},0)||| + h\|\boldsymbol{u}\|_{H^2(\Omega)} + h\|p\|_{H^1(\Omega)}\right)\left(\|\boldsymbol{\phi}\|_{H^2(\Omega)} + \|r\|_{H^1(\Omega)}\right),$$

and thus we have

Theorem 19 *Under the regularity assumption (42), the L_2–error in the ve-*

locities can be estimated as

$$\|\boldsymbol{e}\| \leq Ch^2 \left(\|\boldsymbol{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \right). \tag{43}$$

5 Numerical results

5.1 Convergence study in the case of small viscosity

Let $\lambda = (\mu^{-1} - (\mu^{-2} + 16\pi 2)^{1/2})/2$. Then the exact solution to (1) is given by

$$u_1(x_1, x_2) = 1 - e^{\lambda x_1} \cos 2\pi\, x_2,$$
$$u_2(x_1, x_2) = \tfrac{\lambda x_1}{2\pi} e^{\lambda} \sin 2\pi\, x_2,$$
$$p = \tfrac{1}{2} e^{2\lambda x} + C,$$

with $\boldsymbol{\beta} = \boldsymbol{u}$, $\sigma = 0$ and a right hand side matching the exact solution. In our examples, we also chose C to give zero mean pressure. We solved this problem approximatively on $\Omega = (-1/2, 3/2) \times (0, 2)$, using stability parameters $\gamma_u = \gamma = 1/100$ and $\gamma_\beta = 1/4$.

In Figure 1 we show the convergence for $\mu = 10^{-3}$, and in Figure 2 for $\mu = 10^{-5}$. Note that the absolute value of the pressure decreases linearly with the inverse of μ in L_2, which is why the absolute error in pressure is smaller in Fig. 2.

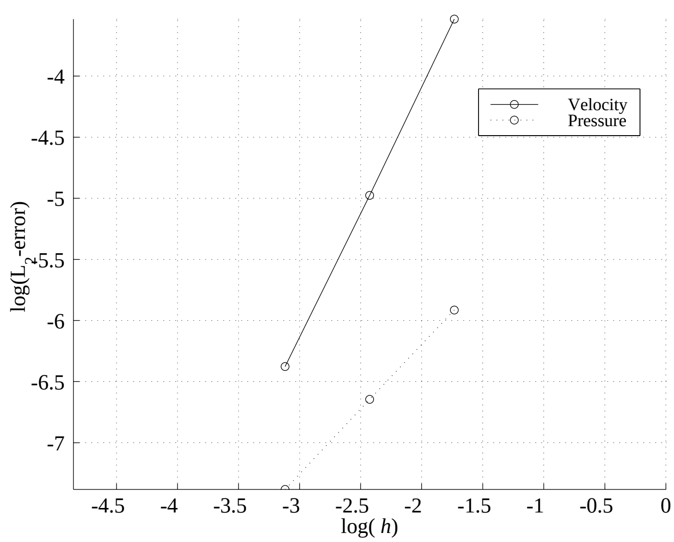

Fig. 1. Convergence for $\mu = 10^{-3}$

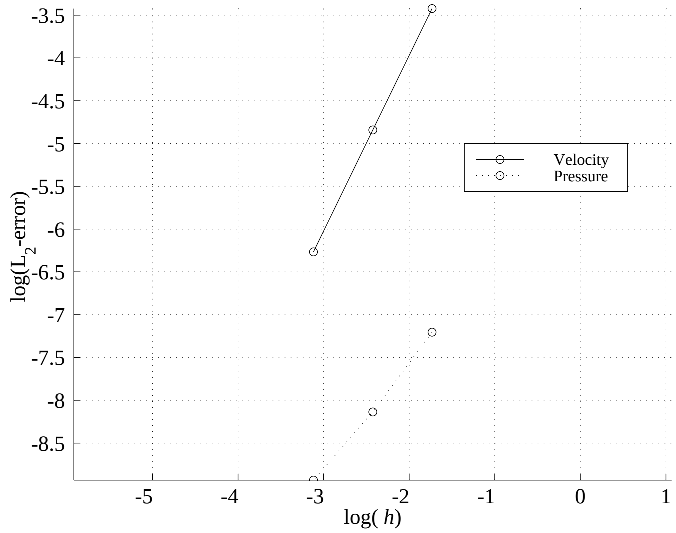

Fig. 2. Convergence for $\mu = 10^{-5}$

5.2 *Stability in the Navier-Stokes case*

We show the influence of the different stabilizing terms for the lid-driven cavity flow in $\Omega = (0,1) \times (0,1)$ and with $\mu = 10^{-3}$. In Figure 3 we show the numerical solution of Navier-Stokes equations (with $\boldsymbol{\beta} = \boldsymbol{u}$) after 15 fixed point iterations, we present the solution using only upwind fluxes, as well as the fully stabilized solution. Note that we do not get a wiggle free solution without the jump in the convective derivative. On the other hand, in Figure 4 we show the solution obtained using only the jump in convective derivative as stabilization. This solution is markedly less diffusive, yet still completely stable.

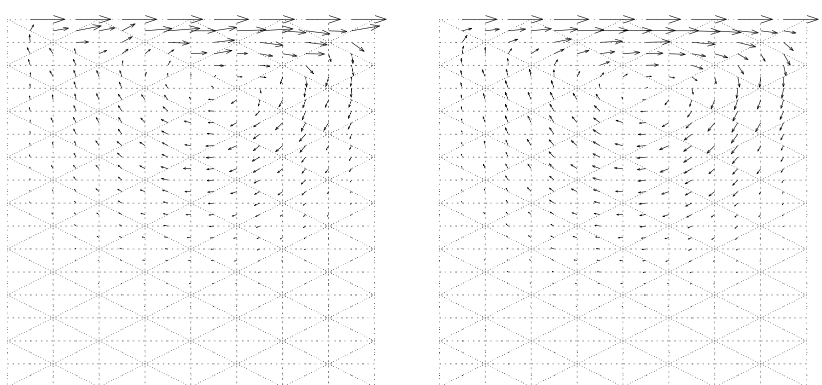

Fig. 3. Approximate solution of the velocities, left:with fluxes only and right: with fluxes plus jump of the convective derivative.

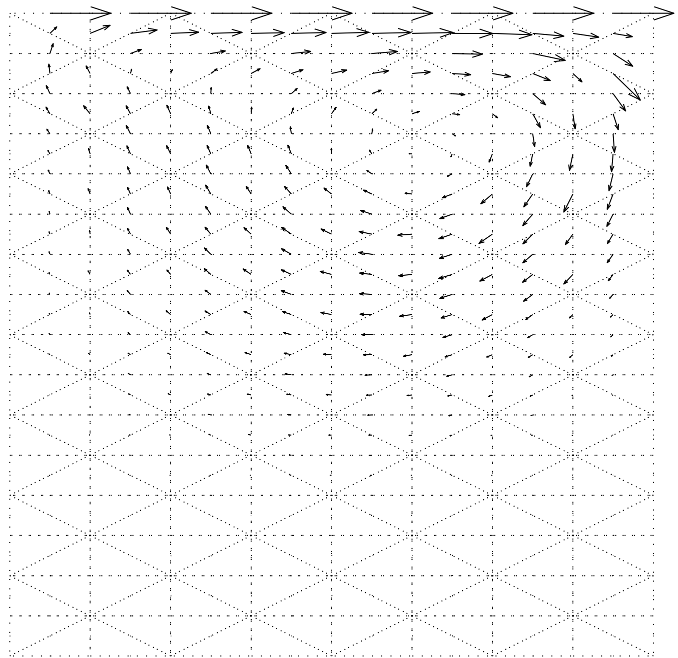

Fig. 4. Approximate solution of the velocities with jump of the convective derivative only.

5.3 Navier-Stokes flow over a step

Finally, we give an example of Navier–Stokes flow over a step using increasing Reynolds numbers. The computational domain is given by $\Omega = (0,4) \times (0,1) \setminus (1.2,1.6) \times (0,0.4)$, and the boundary conditions are: $\boldsymbol{u} = (0,0)$ at the upper and lower parts of $\partial\Omega$; $\boldsymbol{u} = (4\,x_2\,(1-x_2),0)$ at the inflow; natural boundary condition at outflow (not traction free: the viscous operator was written as $-\mu\Delta\boldsymbol{u}$ for this example).

We give the velocities and pressures (shown L^2–projected onto the continuous space $\{v \in C^0(\Omega) : v|_K \in P_1(K), \forall K \in \mathcal{T}_h\}$ for ease of presentation) computed without the edge fluxes, and compare in Figure 8 with a computation with fluxes. Clearly, the fluxes introduce too much artificial viscosity into the method (at least when combined with the gradient jumps). This could be improved by tuning the gradient jump parameter, but the conclusion is that the flux terms are indeed not necessary.

6 Conclusion

We have studied a nonconforming stabilized finite element method for incompressible flow. The velocities were approximated using the Crouzeix-Raviart

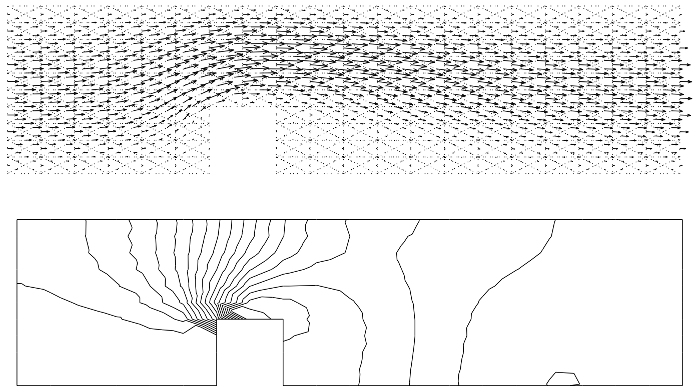

Fig. 5. Approximate solution of the velocities and pressures at Reynolds number 100.

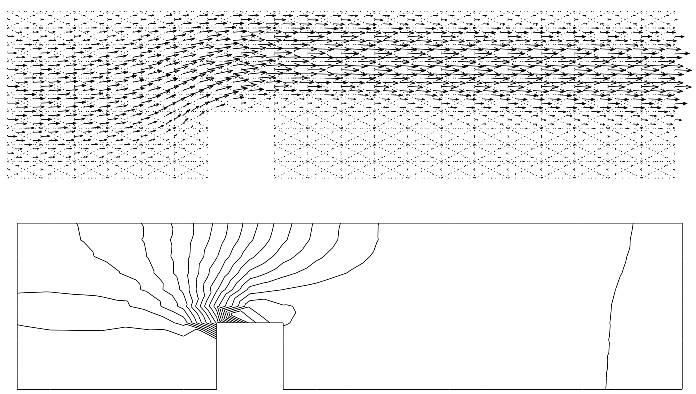

Fig. 6. Approximate solution of the velocities and pressures at Reynolds number 1000.

element and the pressures were chosen as piecewise constants. The numerical scheme proposed is of interior penalty type and remains stable in all flow regimes without streamline-diffusion type stabilization. Instead we stabilize the jump in the streamline derivative between adjacent elements. We prove that this stabilization controls the part of the streamline derivative which is not already in the approximating space, allowing an optimal order a priori error estimate in the energy norm which is uniform in the Peclet (Reynolds) number. Moreover the stabilizing term has the right asymptotic behaviour

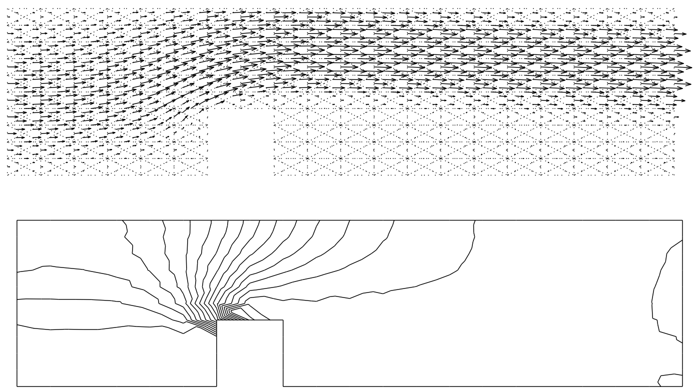

Fig. 7. Approximate solution of the velocities and pressures at Reynolds number 10000.

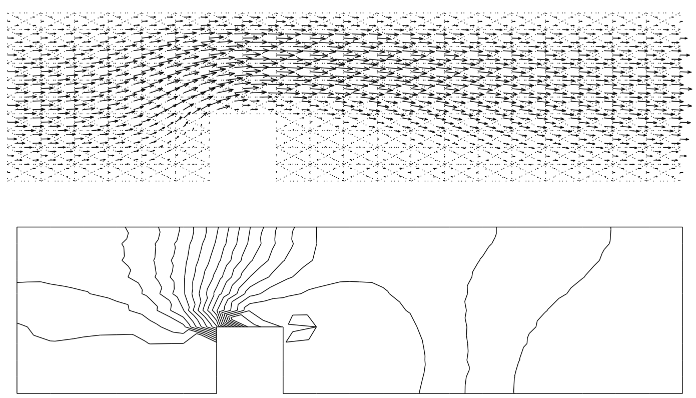

Fig. 8. Approximate solution of the velocities and pressures at Reynolds number 10000, with fluxes.

in the low Peclet regime and optimal order L^2 estimates for the velcities are proved in this case. We present numerical results for different Reynolds numbers showing the robustness of the method and indicating optimal convergence of the error in the L^2-norm. We also test the stability of the scheme on the lid-driven cavity flow and observe that the jumps in the streamline gradient is the most important stabilizing term.

We believe that this scheme offers an attractive alternative to the ones proposed in [13] and in [11]. We have stability for all Reynolds numbers and may still lump mass for efficient timestepping.

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