# Self-Consistent Effective Equations Modeling Blood Flow in Medium-to-Large Compliant Arteries

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ABSTRACT. We study the flow of an incompressible viscous fluid through a long tube with compliant walls. The flow is governed by a given time dependent pressure head difference. The Navier-Stokes equations for incompressible viscous fluid are used to model the flow, and the Navier equations of a curved, linearly elastic membrane to model the wall.

Employing the asymptotic techniques typically used in thin domains, we derive a set of effective equations that hold in medium-to-large compliant vessels for laminar flow regimes. The fluid and the wall are coupled either through the undeformed interface (linear coupling) or the deformed interface (nonlinear coupling), depending on the size of the wall deformation, given via our a priori estimates. Using typical *ad hoc closure assumptions* and *linear coupling* we recover standard, effective, one-dimensional models, widely used in engineering literature. We show that *nonlinear coupling* gives rise to a new model. A bifurcation diagram showing the set of parameters for which one or the other model should be used is presented.

A major contribution of this paper is the derivation of the effective equations that do not assume any ad hoc closure. Introducing a novel approach based on homogenization techniques typically used in porous media flows, we obtain a closed system of effective equations that are of Biot type with memory. Our analysis shows, among other things, that typical ad hoc closure assumptions give rise to an error of order one in the non-stationary solution of flow with moderate Reynolds' numbers.

## 1 Introduction

In this paper we derive the effective equations that describe the flow of a viscous, incompressible Newtonian fluid in a long elastic tube. The paper is motivated by the study of blood flow in compliant arteries. Although blood is not a Newtonian fluid

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(it is a suspension of red blood cells, white blood cells and platelets in plasma), the Newtonian assumption is considered acceptable as a first approximation for the flow in medium-to-large vessels, see e.g., [18, 24, 31]. To model arterial walls we employ the Navier equations for a linearly elastic membrane, as suggested in [10, 16, 18, 31]. They model "effective" response of arterial walls, consisting of three layers (intima, media and adventitia), to the forces induced by the pulsatile blood flow.

In spite of all the simplifying assumptions made so far, the mathematical and numerical study of the fluid-structure interaction simulating blood flow in compliant arteries, is a difficult one. The primary reason lies in the relatively large wall deformations (the diameter of an artery in a healthy human varies up to 10% of the unstressed configuration). Another difficulty comes from the fact that the density of the fluid is close to that of the interface, giving rise to the "fully" coupled dynamics.

Although various numerical methods have been successfully proposed to study fluid-structure interactions arising in cardiovascular problems (see e.g. [11, 12, 22, 25, 26, 31]) they are still rather involving and time-consuming whenever larger 3-dimensional sections of the cardiovascular system are simulated. This is because the underlying phenomena are intrinsically complex and multiscale in nature. Simplified, effective models are called for.

In this vein, this paper addresses the derivation of a self-consistent, effective system of equations describing the flow of an axially symmetric, Newtonian fluid through an elastic tube with aspect ratio $\varepsilon = R/L$ ($R$=radius, $L$=length of the tube). Using rigorous mathematical approach typical for problems in thin domains (see e.g. [7]), we derive the energy and the apriori solution estimates that provide the information about the size of the vessel wall displacement and the flow regime, in terms of the parameters of the problem (Young's modulus of the vessel wall, inlet and outlet pressure data, vessel wall thickness, e.t.c.). The apriori estimates provide optimal scalings for the coupled fluid-structure interaction problem. They are used in the asymptotic expansions to obtain the effective equations. First we derive the averaged, one-dimensional equations. Assuming *ad hoc closure* and *linear coupling* we recover standard models typically used in current bioengineering literature, see e.g., [3, 4, 8, 14, 22, 23, 27, 31]. Assuming *nonlinear coupling* we derive a new model. A bifurcation diagram, showing the values of peak systolic pressure and vessel wall stiffness for which linear vs. nonlinear coupling should be used, is obtained.

To avoid using ad hoc closure, in this manuscript we introduce a novel approach that leads to a set of closed effective equations. The approach is based on the standard homogenization techniques used in porous media problems [19, 17]. Using this approach we obtain closed effective equations that are of Biot type, [1], with memory. They solve the original, 3-dimensional problem, to the $\varepsilon^2$ accuracy. As pointed to us by L. Tartar, memory terms are typical in effective equations describing wave-like phenomena in the underlying physics. In our case they describe the coupling between the waves in the fluid and the elastic structure. Our analysis shows, among other things, that standard ad hoc closure assumptions give rise to an $\mathcal{O}(1)$-error in the solutions of current one-dimensional, time-dependent models corresponding to moderate Reynolds' numbers.

This paper is organized as follows. We start by defining the problem in Section 2. Global weak formulation is presented in Section 3.1 and energy estimates are derived in Section 3.2. Based on the energy estimates we obtain apriori solution estimates first in the case when the pressure drop is zero, see Section 4.1 and then in the general case, see Section 4.2. The apriori solution estimates define the leading order behavior in asymptotic expansions, discussed in Section 5. In Section 5 we also derive a set of reduced, two-dimensional equations assuming linear and nonlinear coupling. The averaged, one-dimensional equations using standard ad hoc closure are derived in Section 6. Finally, in Section 7 we obtain the self-contained, simplified, effective one-dimensional equations without an ad hoc closure. We study two laminar flow regimes: one with zero Strouhal number and small Reynolds' number, corresponding to the creeping flow, discussed in Section 7.1, and the other with nonzero Strouhal number and moderate Reynolds' number, discussed in Section 7.2. In the second case we study two scenarios. One corresponding to large and the other to small deformations of the vessel wall. In the case of small deformations we perform asymptotic expansions with respect to the deformation. We obtain the effective, one-dimensional, closed equations corresponding to nonlinear and linear coupling presented in Sections 7.2.3 and 7.2.5, respectively.

## 2 Statement of the problem

We consider the unsteady axisymmetric flow of a Newtonian incompressible fluid in a thin elastic right cylinder whose radius is small with respect to its length. Define the aspect ratio (the ratio between the radius and the length of the cylinder) to be $\epsilon = R/L$. For each fixed $\epsilon > 0$ introduce

$$\Omega_\varepsilon(t) = \big\{x \in I\!R^3; x = (r\cos\vartheta, r\sin\vartheta, z),\ r < R + \eta^\varepsilon(z,t),\ 0 < z < L\big\}. \tag{2.1}$$

Domain $\Omega_\varepsilon(t)$ is filled with the fluid modeled by the incompressible Navier-Stokes equations. Assuming zero angular velocity, in cylindrical coordinates the Eulerian formulation of the equations in $\Omega_\varepsilon(t) \times I\!R_+$ reads

$$\rho\Big\{\frac{\partial v_r^\varepsilon}{\partial t} + v_r^\varepsilon\frac{\partial v_r^\varepsilon}{\partial r} + v_z^\varepsilon\frac{\partial v_r^\varepsilon}{\partial z}\Big\} - \mu\Big(\frac{\partial^2 v_r^\varepsilon}{\partial r^2} + \frac{\partial^2 v_r^\varepsilon}{\partial z^2} + \frac{1}{r}\frac{\partial v_r^\varepsilon}{\partial r} - \frac{v_r^\varepsilon}{r^2}\Big) + \frac{\partial p^\varepsilon}{\partial r} = 0, \tag{2.2}$$

$$\rho\Big\{\frac{\partial v_z^\varepsilon}{\partial t} + v_r^\varepsilon\frac{\partial v_z^\varepsilon}{\partial r} + v_z^\varepsilon\frac{\partial v_z^\varepsilon}{\partial z}\Big\} - \mu\Big(\frac{\partial^2 v_z^\varepsilon}{\partial r^2} + \frac{\partial^2 v_z^\varepsilon}{\partial z^2} + \frac{1}{r}\frac{\partial v_z^\varepsilon}{\partial r}\Big) + \frac{\partial p^\varepsilon}{\partial z} = 0, \tag{2.3}$$

$$\frac{\partial v_r^\varepsilon}{\partial r} + \frac{\partial v_z^\varepsilon}{\partial z} + \frac{v_r^\varepsilon}{r} = 0. \tag{2.4}$$

We assume that the lateral wall of the cylinder, $\Sigma_\varepsilon(t) = \{r = R + \eta^\varepsilon(z,t)\} \times (0, L)$, is

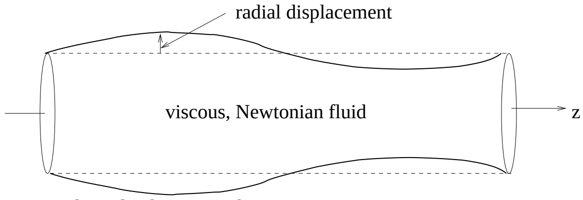


FIGURE 2.1: Domain $\Omega_\varepsilon(t)$

elastic and allows only radial displacements. Its motion, described in Lagrangian coordinates, is modeled by the Navier equations for the linearly elastic curved membrane. The radial contact force is given by

$$F_r = -\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\frac{\eta^\varepsilon}{R^2} + h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial^2\eta^\varepsilon}{\partial z^2} - \rho_w h(\varepsilon)\frac{\partial^2\eta^\varepsilon}{\partial t^2}, \tag{2.5}$$

where $F_r$ is the radial component of external forces (coming from the stresses induced by the fluid), $\eta^\varepsilon$ is the radial displacement from the reference state $\Sigma_\varepsilon^0 := \Sigma_\varepsilon(0)$, $h = h(\varepsilon)$ is the membrane thickness, $\rho_w$ the wall volumetric mass, $E = E(\varepsilon)$ is the Young's modulus, $0 < \sigma \leq 0.5$ the Poisson ratio, $G = G(\varepsilon)$ is the shear modulus and $k = k(\varepsilon)$ is the Timoshenko shear correction factor (see [16, 31]). The parameter values used in this paper, corresponding to the aortic segment including descending, thoracic and abdominal aorta, are shown in Table 2.

| *PARAMETERS* | *VALUES* |
|---|---|
| $\varepsilon$ | 0.06 |
| Characteristic radius: $R$ | 0.012 m, [31] |
| Characteristic length : $L$ | 0.2 m |
| Dynamic viscosity: $\mu$ | $3.5 \times 10^{-3}$ kg (ms)$^{-1}$ |
| Young's modulus: E | $10^5 - 8 \times 10^5$ Pa=$\frac{kg}{m\,s^2}$, [18] |
| Shear modulus: G | $E/(2(1+\sigma))$, [9, 18] |
| Wall thickness: h | $2 \times 10^{-3}$ m [31] |
| Wall density: $\rho_w$ | $1.1 kg/m^2$, [31] |
| Blood density: $\rho$ | $1050 kg/m^3$ |
| Reference pressure: $P_0$ | 13000 Pa = 97.5 mmHg |
| (Normalized) pressure drop for the aorta | 2.67 Pa= 0.02 mmHg [15] |
| $p_{inlet} - p_{ref}$ | around 1800 Pa= 13.5 mmHg |

TABLE 2.1: Parameter values

The fluid equations are coupled with the membrane equation through the lateral boundary conditions requiring continuity of velocity and continuity (balance) of forces. Depending on the size of the displacement, provided by the apriori estimates presented in Section 4.2, the coupling is evaluated at the non-deformed interface $\Sigma_\varepsilon(0)$ in case when the deformations are small (*i.e., linear coupling*), or at the deformed interface, $\Sigma_\varepsilon(t)$, in case when the deformations are large (*i.e., nonlinear coupling*). In either case, the coupling is performed in the Lagrangian framework, namely, with respect to the reference configuration $\Sigma_\varepsilon^0$. More specifically, if we assume nonlinear coupling, then we require that the fluid velocity evaluated at the deformed interface $(R+\eta^\varepsilon, z, t)$ equals the Lagrangian velocity of the membrane. Recalling that we only consider non-zero radial displacements this reads

$$v_r^\varepsilon(R+\eta^\varepsilon, z, t) = \frac{\partial\eta^\varepsilon}{\partial t}(z,t) \quad \text{on} \quad (0,L)\times I\!R_+, \tag{2.6}$$

$$v_z^\varepsilon(R+\eta^\varepsilon, z, t) = 0 \quad \text{on} \quad (0,L)\times I\!R_+. \tag{2.7}$$

Next we consider balance of forces by requiring that the radial force given by (2.5), equals the radial component of the contact force exerted by the fluid to the membrane. The fluid contact force is typically given in Eulerian coordinates and it reads

$$\big((p^\varepsilon - p_{ref})I - 2\mu D(v^\varepsilon)\big)ne_r,$$

where $D(v^\varepsilon)$ is the rate of strain tensor, *i.e.*, the symmetrized gradient of the velocity

$$D(v^\varepsilon) = \frac{1}{2}(\nabla v^\varepsilon + (\nabla v^\varepsilon)^t).$$

To perform the coupling in the Lagrangian framework we need the Jacobian of the transformation from the Eulerian to the Lagrangian coordinate system. For this purpose we consider Borel subsets $B$ of $\Sigma_\varepsilon^0$ and require that

$$\int_B \big((p^\varepsilon - p_{ref})I - 2\mu D(v^\varepsilon)\big)ne_r(R+\eta^\varepsilon(z,t))\sqrt{1+\left(\frac{\partial \eta}{\partial z}\right)^2}dz = \int_B -F_r R dz, \quad (2.8)$$

for all $B \subset \Sigma_\varepsilon^0$, where $J := \sqrt{1+\left(\frac{\partial \eta^\varepsilon}{\partial z}\right)^2}$ is the Jacobian determinant of the mapping transforming $dz$ to $\dfrac{d\Sigma_\varepsilon(t)}{2\pi R}$. Pointwise we get that on $\Sigma_\varepsilon^0 \times I\!R_+$

$$-F_r = \big((p^\varepsilon - p_{ref})I - 2\mu D(v^\varepsilon)\big)n\cdot e_r\left(1+\frac{\eta^\varepsilon}{R}\right)\sqrt{1+\left(\frac{\partial \eta^\varepsilon}{\partial z}\right)^2}. \quad (2.9)$$

Initially, the cylinder is filled with fluid and the entire structure is in an equilibrium. The equilibrium state has an initial reference pressure $P_0 = p_{ref}$ and the initial velocity zero. If we denote by $T$ the (membrane) stress tensor, then in the equilibrium (unperturbed) state only the $T_{zz}$ and $T_{\vartheta\vartheta}$ components of the stress tensor corresponding to the curved membrane $\Sigma_\varepsilon$ are not zero (see [16, 31]). Their values are $kG$ and $R\Delta P_0/h$, respectively, where $\Delta P_0$ is the difference between the reference pressure in the tube and the surrounding tissue. For simplicity we assume that $\Delta P_0 = 0$, hence $T_{\vartheta\vartheta}$ is zero in the unperturbed state. Therefore, the initial data are given by

$$\eta^\varepsilon = \frac{\partial \eta^\varepsilon}{\partial t} = 0 \quad \text{on} \quad \Sigma_\varepsilon(0)\times\{0\}. \quad (2.10)$$

A time-dependent pressure head data at the inlet and at the outlet boundary drive the problem. We also assume that the end-points of the tube are fixed, namely that the radial component of the velocity and the radial displacement are equal to zero. Therefore, we have the following inlet and outlet boundary data

$$v_r^\varepsilon = 0,\ \ p^\varepsilon + \rho(v_z^\varepsilon)^2/2 = P_1(t) + p_{ref} \quad \text{on} \quad (\partial\Omega_\varepsilon(t)\cap\{z=0\})\times I\!R_+, \quad (2.11)$$

$$v_r^\varepsilon = 0,\ p^\varepsilon + \rho(v_z^\varepsilon)^2/2 = P_2(t) + p_{ref} \quad \text{on} \quad (\partial\Omega_\varepsilon\cap\{z=L\})\times I\!R_+, \quad (2.12)$$

$$\eta^\varepsilon = 0 \ \text{ for } \ z=0, \quad \eta^\varepsilon = 0 \ \text{ for } \ z = L \ \text{ and } \forall t \in I\!R_+. \quad (2.13)$$

We will assume that the pressure drop $A(t) = P_1(t) - P_2(t) \in C_0^\infty(0,+\infty)$.

Note that physically (physiologically) one should expect non-zero displacements at the end points of the tube (vessel). Fixed outlet boundary typically gives rise to the formation of a boundary layer in the reduced set of equations, see [5, 6]. In Refs. [5, 6] we constructed the boundary layer and showed that it contaminates the flow only in a small neighborhood near the boundary (the boundary layer decays exponentially away from the fixed-end boundary). Although the boundary layer analysis in [5, 6] was performed for the Stokes problem we expect similar results to hold for the Navier-Stokes equations. It has also been our experience that periodic boundary conditions, although natural in rigid-wall geometries, do not give rise to well-posed limiting (reduced) problems when compliant walls are considered. Therefore, when studying effective, reduced equations for initial-boundary value problems in compliant vessels it is important to take two considerations into account. One is a requirement that the full 3-D axially symmetric problem is well-posed, and the other is that the reduced, effective, 1-D problem be well-posed. These were the primary reasons behind conditions (2.11), (2.12) and (2.13).

Therefore, in this paper we study the following initial-boundary-value problem for a coupled fluid-structure interaction driven by a time-dependent pressure head:

**Problem** $\mathrm{P}^{\varepsilon}$**:**

*For each fixed* $\varepsilon > 0$, *find a solution to* (2.2), (2.3) *and* (2.4) *in domain* $\Omega_\varepsilon(t)$ *defined by* (2.1), *with an elastic lateral boundary* $\Sigma_\varepsilon(t)$. *The lateral boundary conditions are given by the continuity of the velocity* (2.6) *and* (2.7), *and by the continuity of radial forces* (2.5) , *where the left hand-side of* (2.5) *is substituted by* (2.9). *The boundary conditions at the inlet and outlet boundaries are* (2.11) *and* (2.12) *and the behavior of the elastic wall there is prescribed by* (2.13). *The initial data is given by* (2.10).

We note that in the rest of the paper we will be using several different terms to describe the vessel wall: tube wall, elastic wall, membrane and structure. They should all be assumed equivalent in this manuscript.

## 3 Weak formulation and energy estimates

### 3.1 Global Weak Formulation

We consider global weak formulation of the coupled problem between the fluid and the structure. In contrast with the approach proposed by Quarteroni and Nobile in [22], where weak formulation is designed for the use of the implicit, fully coupled Arbitrary Lagrangian Eulerian (ALE) algorithms, we present here a weak formulation that is based on a fixed-point approach and apriori solution estimates, suitable for the existence proof of a solution to the nonlinear, coupled problem.

The main difficulties in defining a weak formulation stem from the following two facts:

1. *The coupling is nonlinear.* The domain geometry is time-dependent. More precisely the position of the lateral boundary (in Lagrangian coordinates) is determined by its interaction with the fluid (in Eulerian formulation), and

2. *The fluid equations are nonlinear.*

We deal with the first difficulty by deriving the apriori estimates that provide a bound on the radial displacement which determines the domain size at every time step. The apriori solution estimates are obtained in terms of the elasticity constants that describe the properties of the vessel wall, and the inlet and the outlet pressure that drive the problem. Once we have found the information about the maximum size of the domain, we introduce a *fixed*, "fictitious" domain of a larger radius, and consider the space of velocity functions defined on the entire fictitious domain, satisfying the apriori bounds that ensure the required size of the radial displacement. We define a solution set to consist of all such velocities and of the interfaces that satisfy the "continuity of velocity" condition at the interface. Among all such candidates we look for the functions that satisfy the integral form of the fluid equations with a lateral boundary condition describing continuity of forces. This is where the second difficulty arises. To deal with the nonlinearity of the equations and with the nonlinear coupling at the same time, we introduce a linearization that does not change the energy of the original problem, and then define a solution to the nonlinear problem as a fixed-point of the associated nonlinear mapping.

We start by introducing the norms that will be used to measure the size of the inlet and the outlet boundary data. Recall that the inlet and the outlet pressure head data (in fact, the deviation from the reference pressure) are denoted by $P_1(t)$ and $P_2(t)$, respectively, and that the pressure head difference $P_2(t) - P_1(t)$ is denoted by $A(t)$. Define

$$\|P_{12}(q,T)\|_{\mathcal{V}}^2 = \max\{\|P_1^2\|_\infty, \|P_2^2\|_\infty\} + 4q^2T^2\frac{1}{T}\int_0^T \max\{P'^2_1(q\tau), P'^2_2(q\tau)\}\; d\tau, \quad (3.1)$$

$$\|A(q,T)\|_{aver}^2 = \frac{1}{T}\int_0^T |A(q\tau)|^2\; d\tau, \quad (3.2)$$

$$\mathcal{P}^2 \equiv \|P_{12}(q,T)\|_{\mathcal{V}}^2 + 24\pi^2T^2\|A(q,T)\|_{aver}^2, \quad (3.3)$$

where $q$ is the frequency of oscillations. For the data presented in Table 2 a rough value of $\mathcal{P}$ is around 1800Pa (it is close to the difference between the maximum pressure and the reference pressure, since the pressure drop in the abdominal aorta is small). Motivated by the apriori estimates introduced in Section 4.2 we consider the radial displacements $\eta^\varepsilon$ and the velocities $v^\varepsilon$ such that

$$\sup_{0\le t\le T}\left\{\frac{h(\varepsilon)E(\varepsilon)}{R(1-\sigma^2)}\|\eta^\varepsilon(t)\|_{L^2(0,L)}^2 + \frac{h(\varepsilon)\rho_w R}{2}\|\frac{\partial\eta^\varepsilon}{\partial t}(t)\|_{L^2(0,L)}^2 + \frac{G(\varepsilon)h(\varepsilon)R}{2}\|\frac{\partial\eta^\varepsilon}{\partial z}(t)\|_{L^2(0,L)}^2\right\} \le 2\frac{R^3L(1-\sigma^2)}{h(\varepsilon)E(\varepsilon)}\mathcal{P}^2 \quad (3.4)$$

$$\frac{2\mu}{\pi}\int_0^T \|D(v^\varepsilon)(\tau)\|_{L^2(\Omega_\varepsilon(\tau))}^2 d\tau + \frac{\rho}{2\pi}\sup_{0\le\tau\le T}\|v^\varepsilon(\tau)\|_{L^2(\Omega_\varepsilon(\tau))}^2 \le 2\frac{R^3L(1-\sigma^2)}{h(\varepsilon)E(\varepsilon)}\mathcal{P}^2. \quad (3.5)$$

REMARK: In particular, using the estimate

$$\max_{[0,L]}|\eta^\varepsilon| \le \sqrt{2}\|\frac{\partial\eta^\varepsilon}{\partial z}\|^{1/2}\|\eta^\varepsilon\|^{1/2},$$

we calculate that if

$$16LR\mathcal{P}^2 \le h(\varepsilon)^2\sqrt{G(\varepsilon)}\Big(\frac{E(\varepsilon)}{1-\sigma^2}\Big)^{3/2} \quad (3.6)$$

then the maximum radial displacement $\eta^\varepsilon$ satisfying (3.4) is 50 percent of the non-stressed vessel radius $R$, namely,

$$\sup_{0\leq t\leq T} \|\eta^\varepsilon(t)\|_{C[0,L]} \leq \frac{R}{2}.$$

For the data presented in Table 2 the left hand side of (3.6) is approximately $10^5$, and the right hand side is of order $10^6$, so we expect that the radial displacement in a healthy aorta will always be less than 50 percent of the non-stressed radius. This is a reasonable result since it is expected that the radial displacement in healthy human arteries does not typically exceed 10 percent.

We are now ready to introduce the solution spaces $\Gamma$ and $U$ corresponding to the radial displacement and the velocity. Denote by $R_{\max}$ any number greater than or equal to the maximum radius obtained from (3.4) and let $\Omega_{R_{\max}} = (0, R_{\max}) \times (0, L)$.

Definition 1. (Solution spaces)

- *The space* $\Gamma$ *consists of all the functions* $\gamma \in L^\infty(0,T;H^1(0,L)) \cap W^{1,\infty}(0,T;L^2(0,L))$ *such that* $\gamma(t,0) = \gamma(t,L) = 0$ *and such that the bound (3.4) is satisfied.*
- *The space* $U$ *consists of all the functions* $u = (u_r, u_z) \in L^2(0,T;H^{1/2-\delta}(\Omega_{R_{\max}}) \times H^1(\Omega_{R_{\max}})) \cap L^\infty(0,T;L^2(\Omega_{R_{\max}})^2)$ *for some* $\delta > 0$*, such that* $div\, u = 0$ *in* $\Omega_{R_{\max}} \times I\!R_+$*,* $u_r = 0$ *for* $z = 0, L$ *and the bound (3.5) is satisfied.*

We look for a solution among all the functions $\gamma \in \Gamma$ and $u \in U$ that satisfy the continuity of the velocity condition at the interface $\gamma$ and are extended to the rest of the fictitious domain in a manner specified below. More precisely, let

$$\Omega_\gamma(t) = \Big\{(r,z) \mid 0 < r < R + \gamma(t,z), \quad z \in (0,L)\Big\} \tag{3.7}$$

and $\Sigma_\gamma(t) = \{r = R + \gamma(t,z)\} \times (0,L)$.

Definition 2. *The set of solution candidates* $K$ *consists of all the functions* $(\gamma, u)$*, where* $u$ *are axially symmetric, such that*

$$\begin{aligned} K = \Big\{(\gamma,u) \in \Gamma \times U \mid u_r(r,z,t) = \frac{\partial \gamma}{\partial t}(z,t) \text{ for } R + \gamma(t,z) \leq r < R_{\max} \\ u_r \in H^1(\Omega_\gamma(t)) \text{ and } u_z(r,z,t) = 0 \quad \text{for } R + \gamma(t,z) < r < R_{\max}\Big\}. \end{aligned} \tag{3.8}$$

Remark: Note that $K$ is bounded but it is not convex. Also note that the trace of $u_r$ at $r = R + \gamma(t,z)$ exists at least as an element of $H^{-1/2}$ since div $u = 0$ and $u_z(R + \gamma(t,z), z, t) = 0$.

To study the integral form of the coupled fluid-interface equations we define the space of test functions.

Definition 3. (The test space) *Let*

$$\begin{aligned} V(\Omega_\gamma(t)) \;\; = \;\; & \{\varphi = \varphi_r e_r + \varphi_z e_z \in H^1(\Omega_\gamma(t))^2 \mid \varphi_r(r,0) = \varphi_r(r,L) = 0, \\ & \varphi_z(R + \gamma(z,t), z) = 0 \text{ and } div\, \varphi = 0 \text{ in } \Omega_\gamma(t) \text{ a.e.}\} \end{aligned} \tag{3.9}$$

*The test space is the space* $H^1(0,T;V(\Omega_\gamma(t)))$.

Recall that for an axially symmetric vector valued function $\psi = \psi_r e_r + \psi_z e_z$ we have

$$D(\psi) = \begin{pmatrix} \dfrac{\partial \psi_r}{\partial r} & 0 & \dfrac{1}{2}\Big(\dfrac{\partial \psi_r}{\partial z} + \dfrac{\partial \psi_z}{\partial r}\Big) \\ 0 & \dfrac{\psi_r}{r} & 0 \\ \dfrac{1}{2}\Big(\dfrac{\partial \psi_r}{\partial z} + \dfrac{\partial \psi_z}{\partial r}\Big) & 0 & \dfrac{\partial \psi_z}{\partial z} \end{pmatrix}.$$

Define the matrix norm $|\cdot|$ through the scalar product

$$\Xi : \Psi = T_r(\Xi \cdot \Psi^t),\ \Xi, \Psi \in I\!R^9.$$

For each $\varepsilon > 0$ we study the following evolution problem.

For a given $(\gamma, u) \in K$ find $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon) \in K$ such that $\forall \varphi \in H^1(0,T; V(\Omega_\gamma(t)))$ we have

$$\begin{aligned} 2\mu \int_{\Omega_\gamma(t)} D(v^\varepsilon) : D(\varphi)\ rdrdz + \rho \int_{\Omega_\gamma(t)} \Big\{ \frac{\partial v^\varepsilon}{\partial t} + (u(t)\nabla)v^\varepsilon \Big\} \varphi\ rdrdz \\ +R \int_0^L \Big\{ h(\varepsilon)G(\varepsilon)k(\varepsilon) \frac{\partial \eta^\varepsilon}{\partial z}\frac{\partial}{\partial z}\varphi_r(R+\gamma, z, t) + \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\frac{\eta^\varepsilon}{R^2}\varphi_r(R+\gamma, z, t) \Big\}\ dz \\ +R\rho_w h \int_0^L \frac{\partial^2 \eta^\varepsilon}{\partial t^2}\varphi_r(R+\gamma(t,z), z, t)\ dz = -\int_0^R \{P_2(qt) - \frac{\rho}{2}(u_z v_z^\varepsilon)|_{z=L}\}\varphi_z|_{z=L} rdr \\ + \int_0^R \{P_1(qt) - \frac{\rho}{2}(u_z v_z^\varepsilon)|_{z=0}\}\varphi_z|_{z=0} rdr, \end{aligned} \tag{3.10}$$

and

$$\eta^\varepsilon = \frac{\partial \eta^\varepsilon}{\partial t} = 0 \quad \text{on} \quad (0,L)\times\{0\} \ \text{ and } v^\varepsilon(r,z,0) = 0. \tag{3.11}$$

Scalar products with $\frac{\partial v^\varepsilon}{\partial t}$ and $\frac{\partial^2 \eta^\varepsilon}{\partial t^2}$ should be understood as duality pairings.

The problem (3.10)-(3.11) defines a nonlinear mapping $\Phi$ defined on $K$. The apriori estimates, presented in Section 4.2, imply that $\Phi$ maps $K$ to $K$.

LEMMA 3.1. $\Phi(K) \subseteq K$.

DEFINITION 4. (WEAK SOLUTION) *The triple* $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon) \in K$ *is a weak solution for the problem* $\mathrm{P}^\varepsilon$ *if it is a fixed point for the mapping* $\Phi$.

Existence of a weak solution is studied in [20]. A related work on the existence of a solution to an incompressible fluid-elastic structure coupled problem can be found in [13]. In this paper we present the energy estimate and the apriori estimates that determine the "optimal" leading order behavior of the solution in terms of the small parameter $\varepsilon$ which we will use to derive the reduced, effective equations.

### 3.2 ENERGY ESTIMATE

The energy of this problem, obtained by using the velocity field as a test function in (3.10), consists of the elastic energy of the membrane, the kinetic and the viscous energy of the fluid, and the energy due to the outside forcing.

To get to the energy estimate we start by conveniently rewriting the elastic energy of the membrane, defined by

$$\begin{aligned} \mathcal{E}_{el} &\equiv R\int_0^L \Big\{ h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial\eta^\varepsilon}{\partial z}\frac{\partial^2\eta^\varepsilon}{\partial z\partial t} + \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\frac{\eta^\varepsilon}{R^2}\frac{\partial\eta^\varepsilon}{\partial t} \\ &+ R\rho_w h(\varepsilon)\frac{\partial^2\eta^\varepsilon}{\partial t^2}\frac{\partial\eta^\varepsilon}{\partial t}\Big\}\, dz \end{aligned}$$

in the following form.

LEMMA 3.2. *The displacement* $\eta^\varepsilon$ *satisfies*

$$\begin{aligned} \mathcal{E}_{el} &= R\frac{d}{2dt}\Big\{\rho_w h(\varepsilon)\int_0^L |\frac{\partial\eta^\varepsilon}{\partial t}|^2\, dz + h(\varepsilon)G(\varepsilon)k(\varepsilon)\int_0^L |\frac{\partial\eta^\varepsilon}{\partial z}|^2\, dz \\ &+ \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\int_0^L |\frac{\eta^\varepsilon}{R}|^2\Big\}. \end{aligned} \tag{3.12}$$

This will be used in Proposition 3.3 to obtain the variational equality from which the energy estimate will follow.

Next we introduce a time scale in the problem. We are interested in the oscillations of the membrane that are due to the coupled fluid-structure response to the time-dependent pressure (pressure head) drop $A(t)$ and the main pressure head at the inlet and at the outlet boundary, $P_1(t)$ and $P_2(t)$. These oscillations generally occur at a different time scale than the physical time $t$. The time scale should depend not only on the pressure head data but also on the parameters in the problem. For example, for a stiffer wall, the vibrations of the wall occur at a shorter time scale (high frequency) than the oscillations of a more elastic wall. To capture the waves of the coupled fluid-structure response to the outside forcing we introduce

$$\tilde{t} = \omega^\varepsilon t \tag{3.13}$$

where the characteristic frequency $\omega^\varepsilon$ will be specified later, see (4.10).

From this point on we use the rescaled time $\tilde{t}$ and drop the symbol wiggle.

We now derive the variational equality. The following identities will be useful.

$$\int_{\Omega_\varepsilon(t)} \omega^\varepsilon \frac{\partial v^\varepsilon}{\partial t} v^\varepsilon\, dx = \omega^\varepsilon \frac{d}{2dt}\int_{\Omega_\varepsilon(t)} |v^\varepsilon(t)|^2\, dx - \frac{1}{2}\int_{\partial\Omega_\varepsilon(t)} |v^\varepsilon(t)|^2 v^\varepsilon(t)\cdot n\, dS \tag{3.14}$$

$$\int_{\Omega_\varepsilon(t)} (v^\varepsilon\cdot\nabla)v^\varepsilon\cdot v^\varepsilon\, dx = \frac{1}{2}\int_{\partial\Omega_\varepsilon(t)} |v^\varepsilon(t)|^2 v^\varepsilon(t)\cdot n\, dS \tag{3.15}$$

$$\begin{aligned} \int_{\Omega_\varepsilon(t)} \big(\nabla p^\varepsilon - \mu\Delta v^\varepsilon\big)\cdot v^\varepsilon\, dx &= \int_{\Omega_\varepsilon(t)} \mathrm{Div}\, \big(p^\varepsilon I - 2\mu D(v^\varepsilon)\big)v^\varepsilon\, dx = \\ 2\mu\|D(v^\varepsilon(t))\|^2_{L^2(\Omega_\varepsilon)} &+ \int_{\partial\Omega_\varepsilon(t)} \big(p^\varepsilon I - 2\mu D(v^\varepsilon)\big)nv^\varepsilon\, dS \end{aligned} \tag{3.16}$$

Furthermore, using (2.6) in (2.9) we have

$$\int_{\Sigma_\varepsilon(t)} \big(p^\varepsilon I - 2\mu D(v^\varepsilon)\big)nv^\varepsilon\, dS = -\int_0^L F_r(t,z)\omega^\varepsilon\frac{\partial\eta^\varepsilon}{\partial t}(t,z)\, dz. \tag{3.17}$$

By keeping the rescaled time in mind, and by using the expression for the elastic energy (3.12) and the above identities we obtain

PROPOSITION 3.3. (VARIATIONAL EQUALITY) *Solution* $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon)$ *of problem (3.10)-(3.11) satisfies the following variational equality*

$$\begin{aligned}
&\omega^\varepsilon h(\varepsilon)\tfrac{d}{2dt}\Big\{\rho\textstyle\int_{\Omega_\varepsilon(t)}(\omega^\varepsilon)^2\rho_w R\|\tfrac{\partial\eta^\varepsilon(t)}{\partial t}\|^2_{L^2(0,L)} + G(\varepsilon)k(\varepsilon)R\|\tfrac{\partial\eta^\varepsilon(t)}{\partial z}\|^2_{L^2(0,L)}\\
&+\tfrac{E(\varepsilon)R}{1-\sigma^2}\|\tfrac{\eta^\varepsilon(t)}{R}\|^2_{L^2(0,L)}\Big\} + \tfrac{\rho\omega^\varepsilon}{2\pi}\tfrac{d}{2dt}\|v^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))} + \tfrac{\mu}{\pi}\|D(v^\varepsilon(t))\|^2_{L^2(\Omega_\varepsilon(t))} =\\
&-\textstyle\int_0^R P_2(qt)v_z^\varepsilon(t,r,L)\ rdr + \int_0^R P_1(qt)v_z^\varepsilon(t,r,0)\ rdr,
\end{aligned}\tag{3.18}$$

*with* $v_r^\varepsilon(t, R+\eta^\varepsilon, z) = \omega^\varepsilon \dfrac{\partial\eta^\varepsilon}{\partial t}(t,z)$ *and* $v_z^\varepsilon(t, R+\eta^\varepsilon, z) = 0$ *on* $(0,L)\times(0,T)$.

Here $q$ corresponds to the frequency of the time-oscillations of the inlet and of the outlet boundary data. Even thought nothing in the analysis presented in this paper requires time-periodic data, we have introduced the explicit frequency parameter $q$ to suggest that the blood flow application typically exhibits time-periodicity.

To get the energy estimate in terms of the data we need to estimate the right hand side of the variational equality. Notice that since the axial component of the velocity at the inlet and at the outlet boundary is not prescribed we need to estimate the right hand-side in terms of the data and the energy of the problem. Notice that on the left hand side we only have the $L^2$-norm of $D(v^\varepsilon)$ and not the $L^2$-norm of $\nabla v^\varepsilon$, and so the standard approach based on using the Gronwall estimate and the $L^2$-norm of the velocity, $\rho\int_{\Omega_\varepsilon}|v^\varepsilon(t)|^2\ rdrdz$, is insufficient to guarantee the correct order of magnitude of the velocity. To get around this difficulty we transform the right hand side term in (3.18) into a combination of a volume term and a lateral boundary term as follows

$$\begin{aligned}
&-\int_0^R P_2(qt)v_z^\varepsilon(t,r,L)\ rdr + \int_0^R P_1(qt)v_z^\varepsilon(t,r,0)\ rdr =\\
&-\int_{\Omega_\varepsilon(t)}\frac{A(qt)}{2\pi L}v_z^\varepsilon\ dx + \int_{\Sigma_\varepsilon(t)}\big(A(qt)\frac{z}{L} + P_1(qt)\big)v_r^\varepsilon(t,R,z)\frac{n_r}{2\pi}\ d\Sigma_\varepsilon(t) =\\
&-\int_{\Omega_\varepsilon(t)}\frac{A(qt)}{2\pi L}v_z^\varepsilon\ dx + R\omega^\varepsilon\int_0^L\big(A(qt)\frac{z}{L} + P_1(qt)\big)\frac{\partial\eta^\varepsilon(t)}{\partial t}(t,z)\ dz =\\
&-\int_{\Omega_\varepsilon(t)}\frac{A(qt)}{2\pi L}v_z^\varepsilon\ dx + R\omega^\varepsilon\frac{d}{dt}\int_0^L\big(A(qt)\frac{z}{L} + P_1(qt)\big)\eta^\varepsilon(t)\ dz -\\
&\qquad R\omega^\varepsilon\int_0^L\big(A'(qt)\frac{z}{L} + P_1'(qt)\big)\eta^\varepsilon(t,z)\ dz,
\end{aligned}\tag{3.19}$$

where $n_r = 1/\sqrt{1+|\frac{\partial\eta^\varepsilon(t)}{\partial z}|^2}$ and $n_rJ = 1$.

We first estimate the lateral boundary terms from (3.19). The following notation will be useful

$$\|P(t)\|^2_{\mathcal{H}} = \max\{P_1^2(qt), P_2^2(qt)\} + q^2\int_0^t \max\{P'^2_1(q\tau), P'^2_2(q\tau)\}\ d\tau.$$

LEMMA 3.4. *Let $\alpha > 0$. Radial displacement $\eta^\varepsilon$ satisfies the following estimate*

$$R\omega^\varepsilon\Big(\int_0^L \big(A(qt)\frac{z}{L} + P_1(qt)\big)\eta^\varepsilon(t,z)dz - \int_0^t\int_0^L \big(A'(q\tau)\frac{z}{L} + P'_1(q\tau)\big)\eta^\varepsilon(\tau,z)dzd\tau\Big)$$
$$\leq \frac{h(\varepsilon)E(\varepsilon)\omega^\varepsilon}{4R(1-\sigma^2)}\Big\{\|\eta^\varepsilon(t)\|^2_{L^2(0,L)} + \alpha\int_0^t \|\eta^\varepsilon(\tau)\|^2_{L^2(0,L)}d\tau\Big\} + \frac{R^3L(1-\sigma^2)\omega^\varepsilon}{2h(\varepsilon)E(\varepsilon)}\|P(t)\|^2_{\mathcal{H}}. \tag{3.20}$$

To estimate the volume forcing term in (3.19) we have two possibilities. The first one is to get an estimate in terms of the viscous energy via a variant of Korn's and Poincaré's inequalities. This approach, however, leads to an estimate in terms of the $L^4$-norm of $\eta^\epsilon$, which we do not control. The second approach is to estimate the volume term via the inertia term. This will lead to the energy estimate (3.23). More precisely, we have the following.

LEMMA 3.5. *The following estimate holds*

$$\left|\int_{\Omega_\varepsilon(t)} \frac{A(qt)}{2\pi L}{v_z}^\varepsilon r dr dz\right| \leq \frac{3R^2\pi^2|A(qt)|^2}{2L\rho\omega^\varepsilon\alpha} + \frac{\alpha\omega^\varepsilon\rho}{4\pi}\|v_z^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \frac{3\pi^2|A(qt)|^2}{2\alpha L^2\rho\omega^\varepsilon}\|\eta^\varepsilon\|^2_{L^2(0,L)}. \tag{3.21}$$

*Proof.* We have

$$|\int_{\Omega_\varepsilon(t)} \frac{A(qt)}{2\pi L}v_z^\varepsilon\ r dr dz| \leq \frac{|A(qt)|}{L}\|v_z^\varepsilon(t)\|_{L^2(\Omega_\varepsilon(t))}|\Omega_\varepsilon(t)|^{1/2} \leq$$
$$\frac{\omega^\varepsilon\rho}{4\pi}\|v_z^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))} + \frac{\pi|A(qt)|^2}{L^2\omega^\varepsilon\rho}|\Omega_\varepsilon(t)| \tag{3.22}$$

As $|\Omega_\varepsilon(t)| = \pi\int_0^L (R+\eta^\varepsilon)^2\ dz$ we get (3.21) . □

Finally, after integrating (3.18) with respect to time, and using (3.19) and the time-integrated (3.21), we get

THEOREM 3.6. (ENERGY ESTIMATE) *Radial displacement $\eta^\varepsilon$, the displacement gradient $\partial\eta^\varepsilon/\partial z$, the kinetic energy of the membrane $\rho_w\|\frac{\partial\eta^\varepsilon}{\partial t}(t)\|^2_{L^2(0,L)}$, the viscous energy $\mu\|D(v^\varepsilon)\|^2_{L^2(\Omega_\varepsilon)}$ and the kinetic energy $\rho\|v^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}$ of the fluid, satisfy the following energy estimate*

$$\omega^\varepsilon\frac{h(\varepsilon)E(\varepsilon)}{4R(1-\sigma^2)}\|\eta^\varepsilon(t)\|^2_{L^2(0,L)} + (\omega^\varepsilon)^3\frac{h(\varepsilon)\rho_w R}{2}\|\frac{\partial\eta^\varepsilon}{\partial t}(t)\|^2_{L^2(0,L)}$$
$$+\omega^\varepsilon\frac{G(\varepsilon)h(\varepsilon)R}{2}\|\frac{\partial\eta^\varepsilon}{\partial z}(t)\|^2_{L^2(0,L)} + \frac{\mu}{\pi}\int_0^t \|D(v^\varepsilon)(\tau)\|^2_{L^2(\Omega_\varepsilon(\tau))}d\tau + \frac{\omega^\varepsilon\rho}{4\pi}\|v^\varepsilon(\tau)\|^2_{L^2(\Omega_\varepsilon(\tau))} \leq$$
$$\Big\{\omega^\varepsilon\alpha\frac{h(\varepsilon)E(\varepsilon)}{4R(1-\sigma^2)} + \frac{3\pi^2\|A\|^2_{L^\infty(0,t)}}{\rho L^2\omega^\varepsilon\alpha}\Big\}\int_0^t \|\eta^\varepsilon(\tau)\|^2_{L^2(0,L)}\ d\tau$$
$$+\frac{\omega^\varepsilon\rho\alpha}{4\pi}\int_0^t \|v^\varepsilon(\tau)\|^2_{L^2(\Omega_\varepsilon(\tau))}\ d\tau + \frac{R^3L(1-\sigma^2)\omega^\varepsilon}{2h(\varepsilon)E(\varepsilon)}\|P(t)\|^2_{\mathcal{H}} + \frac{3R^2\pi^2}{\alpha\omega^\varepsilon\rho L}\int_0^t |A(q\tau)|^2\ d\tau. \tag{3.23}$$

We use this energy inequality to obtain the apriori solution estimates.

## 4 Apriori solution estimates

We first focus on the case when the pressure head difference between the inlet and the outlet boundary is zero, $A(t) = 0$.

### 4.1 Apriori solution estimates when $A(t) = 0$.

We are interested in studying the coupled response of the fluid and the structure to a time-dependent pressure head with a zero pressure (pressure head) drop. The energy stored in the membrane due to the time-dependent pressure head will impact the movement of the fluid in the tube. Our result presented below shows that the estimates for the radial displacement of the tube and for the velocity of the fluid are independent of the time scale $\omega^\varepsilon$. The amplitude of the oscillations as well as the magnitude of the fluid velocity depends on the elasticity properties of the tube walls, as well as on the radius, the length of the tube and the magnitude and frequency of the pressure head.

LEMMA 4.1. *Let $A(t) = 0$. Then the estimates for $\eta^\varepsilon$ and $v^\varepsilon$ are independent of $\omega^\varepsilon$ and they read*

$$\frac{h(\varepsilon)E(\varepsilon)}{R(1-\sigma^2)}\|\eta^\varepsilon(t)\|^2_{L^2(0,L)} + \frac{\rho}{2\pi}\|v^\varepsilon(\tau)\|^2_{L^2(\Omega_\varepsilon(\tau))} \leq$$
$$\frac{2R^3L(1-\sigma^2)}{h(\varepsilon)E(\varepsilon)}\left\{\max_{0\leq t\leq T} P_1^2(qt) + 2q^2T^2\frac{1}{T}\int_0^T |P'_1(q\tau)|^2\, d\tau\right\},\ \forall t \in [0,T] \tag{4.1}$$

*Proof.* Denote

$$y(t) = \int_0^t \left\{\frac{h(\varepsilon)E(\varepsilon)}{R(1-\sigma^2)}\|\eta^\varepsilon(t)\|^2_{L^2(0,L)} + \frac{\rho}{\pi}\|v^\varepsilon(\tau)\|^2_{L^2(\Omega_\varepsilon(\tau))}\right\} d\tau.$$

Suppose that the time oscillations in $P_1$ are of order $q$, i.e., that the period of oscillations $T = 2\pi/q$. Then for any $\alpha > 0$ we have

$$y'(t) \leq \alpha y(t) + \frac{2R^3L(1-\sigma^2)}{h(\varepsilon)E(\varepsilon)}\left\{\max_{0\leq t\leq T} P_1^2(qt) + \frac{q^2}{\alpha}\int_0^t |P'_1(q\tau)|^2\, d\tau\right\}; \quad y(0) = 0. \tag{4.2}$$

By applying the Gronwall inequality and by choosing $\alpha = \frac{1}{2T}$ we get

$$\frac{h(\varepsilon)E(\varepsilon)}{R(1-\sigma^2)}\|\eta^\varepsilon(t)\|^2_{L^2(0,L)} + \frac{\rho}{2\pi}\|v^\varepsilon(\tau)\|^2_{L^2(\Omega_\varepsilon(\tau))} \leq$$
$$\frac{4R^3L(1-\sigma^2)}{h(\varepsilon)E(\varepsilon)}\left\{\max_{0\leq t\leq T} P_1^2(qt) + 2q^2T\int_0^t |P'_1(q\tau)|^2\, d\tau\right\} \quad \forall t \in [0,T].$$

□

### 4.2 APRIORI SOLUTION ESTIMATES IN THE GENERAL CASE.

In this case we will see that the apriori solution estimates depend on frequency $\omega^\varepsilon$. Define

$$y(t) = \omega^\varepsilon \int_0^t \left\{ \frac{h(\varepsilon)E(\varepsilon)}{R(1-\sigma^2)} \|\eta^\varepsilon(t)\|^2_{L^2(0,L)} + \frac{\rho}{\pi}\|v^\varepsilon(\tau)\|^2_{L^2(\Omega_\varepsilon(\tau))} \right\} d\tau. \tag{4.3}$$

Then, energy inequality (3.23) implies

$$y'(t) \le \Big(\alpha + \frac{12\pi^2\|A\|^2_{L^\infty(0,t)}R(1-\sigma^2)}{\rho h(\varepsilon)E(\varepsilon)L^2(\omega^\varepsilon)^2\alpha}\Big)y(t)$$
$$+\frac{2R^3L(1-\sigma^2)\omega^\varepsilon}{h(\varepsilon)E(\varepsilon)}\Big\{ \max\{P_1^2(qt), P_2^2(qt)\} + \frac{q^2}{\alpha}\int_0^t \max\{P'^2_1(q\tau), P'^2_2(q\tau)\}\, d\tau\Big\}$$
$$+\frac{12R^2\pi^2}{\alpha\omega^\varepsilon\rho L}\int_0^t |A(q\tau)|^2\, d\tau; \quad y(0) = 0. \tag{4.4}$$

Without loss of generality suppose

$$\frac{12\pi^2\|A\|^2_{L^\infty(0,t)}R(1-\sigma^2)}{\rho h(\varepsilon)E(\varepsilon)L^2(\omega^\varepsilon)^2\alpha} \le \alpha \tag{4.5}$$

and then choose

$$\alpha = \frac{1}{4T}. \tag{4.6}$$

Let us note that $\omega^\varepsilon$ has dimension sec$^{-1}$ and that $\alpha$ and $T$ are dimensionless. Also $T$ is of order one.

Let $t_0 \in [0,T]$ be such that $y'(t_0) = \max_{0\le t\le T} y'(t)$. Then, instead of using the Gronwall inequality to estimate $y'(t)$ we express $y(t)$ on the right hand-side of (4.4) in terms of $y'(t_0)$, and use (4.5) to get

$$y'(t_0) \le 2\alpha T y'(t_0) + \frac{2R^3L(1-\sigma^2)\omega^\varepsilon}{h(\varepsilon)E(\varepsilon)}\Big\{ \max\{\|P_1^2\|_\infty, \|P_2^2\|_\infty\} \tag{4.7}$$
$$+4q^2T^2\frac{1}{T}\int_0^T \max\{P'^2_1(q\tau), P'^2_2(q\tau)\}\, d\tau\Big\} + \frac{48R^2\pi^2T^2}{\omega^\varepsilon\rho L}\frac{1}{T}\int_0^T |A(q\tau)|^2\, d\tau;$$
$$y(0) = 0. \tag{4.8}$$

By choosing $\alpha$ as given in (4.6) and by utilizing the notation for the norms defined in (3.1) and (3.2), we get

$$\frac{1}{2}y'(t_0) \le \frac{2R^3L(1-\sigma^2)\omega^\varepsilon}{h(\varepsilon)E(\varepsilon)}\|P_{12}(q,T)\|^2_{\mathcal{V}} + \frac{48R^2\pi^2T^2}{\omega^\varepsilon\rho L}\|A(q,T)\|^2_{aver}. \tag{4.9}$$

Now we choose the characteristic time scale, or the characteristic frequency $\omega^\varepsilon$, by requiring to see the effects of both the pressure head data, $P_1(t)$ and $P_2(t)$, as well as the pressure drop data, $A(t)$. More precisely, we choose $\omega^\varepsilon$ in (4.9) so that the coefficients on the right hand-side have the same "weight" in $\epsilon$. This leads to

$$\omega^\varepsilon = \frac{2}{L}\sqrt{\frac{2h(\varepsilon)E(\varepsilon)}{R\rho(1-\sigma^2)}} \tag{4.10}$$

Notice that $c = L\omega^\varepsilon$ is the characteristic wave speed (the local pulse wave velocity or sound speed). Expression (4.10) leads to the same characteristic wave speed as obtained in equation (16) in Fung's "Biomechanics: Circulation", [9]. For the data presented in Table 2 this leads to the pulse wave velocity at the order of 10 m/s, for the vessel wall having the Young's modulus around $6 \times 10^5$ Pa. This is in good agreement with the measured pulse wave velocity presented in [21].

With this choice of the time-scale we obtain the following apriori estimate.

LEMMA 4.2. *The radial displacement $\eta^\varepsilon$ and the fluid velocity $v^\varepsilon$ satisfy*

$$\frac{h(\varepsilon)E(\varepsilon)}{R(1-\sigma^2)}\|\eta^\varepsilon(t)\|^2_{L^2(0,L)} + \frac{\rho}{\pi}\|v^\varepsilon(\tau)\|^2_{L^2(\Omega_\varepsilon(\tau))} \le 4\frac{R^3L(1-\sigma^2)}{h(\varepsilon)E(\varepsilon)}\mathcal{P}^2,$$

*where $\mathcal{P}$ is given by* (3.3).

Notice that with this choice of $\omega^\varepsilon$ inequality (4.5) reads

$$4\pi T(1-\sigma^2)\|A\|_{L^\infty(0,T)} \le \frac{h(\varepsilon)E(\varepsilon)}{R} \tag{4.11}$$

which holds true for our data since Table 2 implies that the left hand side of (4.11) is approximately equal to $10^{-1}$, whereas the right hand side is greater than $10^4$.

After summarizing those estimates, we get an estimate which is crucial in determining the leading-order behavior in asymptotic expansions. The estimate is a basis for the *apriori* solution estimates in terms of the small parameter $\epsilon$.

PROPOSITION 4.3. *(*APRIORI ESTIMATES WHEN INERTIAL FORCES DOMINATE VISCOUS FORCES*) Solution $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon)$ of problem (3.10)-(3.11) satisfies the following apriori estimates*

$$\frac{1}{L}\|\eta^\varepsilon(t)\|^2_{L^2(0,L)} \le 4\frac{R^4(1-\sigma^2)^2}{h(\varepsilon)^2E(\varepsilon)^2}\mathcal{P}^2 \tag{4.12}$$

$$\|v^\varepsilon\|^2_{L^2(\Omega_\varepsilon(t))} \le 4\pi\frac{R^3L(1-\sigma^2)}{\rho h(\varepsilon)E(\varepsilon)}\mathcal{P}^2 \tag{4.13}$$

$$\int_0^t \left\{\|\frac{\partial v_r^\varepsilon}{\partial r}\|^2_{L^2(\Omega_\varepsilon(t))} + \|\frac{v_r^\varepsilon}{r}\|^2_{L^2(\Omega_\varepsilon(t))} + \|\frac{\partial v_z^\varepsilon}{\partial z}\|^2_{L^2(\Omega_\varepsilon(t))}\right\} d\tau \le \frac{R^2}{2\mu}\sqrt{\frac{R(1-\sigma^2)}{h(\varepsilon)E(\varepsilon)\rho}}\mathcal{P}^2 \tag{4.14}$$

$$\int_0^t \left\{\|\frac{\partial v_z^\varepsilon}{\partial r}\|^2_{L^2(\Omega_\varepsilon)} + \|\frac{\partial v_r^\varepsilon}{\partial z}\|^2_{L^2(\Omega_\varepsilon)}\right\} d\tau \le \frac{R^2}{2\mu}\sqrt{\frac{R(1-\sigma^2)}{h(\varepsilon)E(\varepsilon)\rho}}\mathcal{P}^2 \tag{4.15}$$

*Proof.* First notice that (4.12) and (4.13) are obvious consequences of (4.4). Next, (4.14) follows from

$$2\pi\int_0^t\int_0^L\int_0^{R+\eta^\varepsilon} \left\{|\frac{\partial v_r^\varepsilon}{\partial r}|^2 + \frac{1}{2}|\frac{\partial v_z^\varepsilon}{\partial r} + \frac{\partial v_r^\varepsilon}{\partial z}|^2 + |\frac{\partial v_z^\varepsilon}{\partial z}|^2 + (\frac{v_r^\varepsilon}{r})^2\right\} rdrdzd\tau =$$

$$\int_0^t \|D(v^\varepsilon(t))\|^2_{L^2(\Omega_\varepsilon(t))}\, d\tau \le \frac{\pi R^2}{\mu}\sqrt{\frac{R(1-\sigma^2)}{h(\varepsilon)E(\varepsilon)\rho}}\mathcal{P}^2. \tag{4.16}$$

It remains to prove (4.15). We start from estimate (4.16) for the shear stress term in $D(v^\varepsilon)$. It reads

$$\int_0^t \int_0^L \int_0^{R+\eta^\varepsilon} \left\{ (\frac{\partial v_r^\varepsilon}{\partial z})^2 + 2\frac{\partial v_r^\varepsilon}{\partial z}\frac{\partial v_z^\varepsilon}{\partial r} + (\frac{\partial v_z^\varepsilon}{\partial r})^2 \right\} rdrdzd\tau \leq \frac{R^2}{\mu}\sqrt{\frac{R(1-\sigma^2)}{h(\varepsilon)E(\varepsilon)\rho}}\mathcal{P}^2.$$

The difficulty comes from the term which is the product of two off-diagonal gradient terms $\frac{\partial v_r^\varepsilon}{\partial z}\frac{\partial v_z^\varepsilon}{\partial r}$. We can estimate this term by using the boundary behavior of $v^\varepsilon$, $\partial_z v_z^\varepsilon = 0$ at $z = 0, L$, and the incompressibility condition (2.4) to obtain

$$\int_{\Omega_\varepsilon(t)} \frac{\partial v_r^\varepsilon}{\partial z}\frac{\partial v_z^\varepsilon}{\partial r}\, rdrdz = -\int_{\Omega_\varepsilon(t)} v_z^\varepsilon \frac{\partial}{\partial r}(r\frac{\partial v_r^\varepsilon}{\partial z})drdz$$
$$= \int_{\Omega_\varepsilon(t)} v_z^\varepsilon \frac{\partial^2 v_z^\varepsilon}{\partial z^2}\, rdrdz = -\int_{\Omega_\varepsilon(t)} \frac{\partial}{\partial z}v_z^\varepsilon\frac{\partial v_z^\varepsilon}{\partial z}\, rdrdz = -\int_{\Omega_\varepsilon(t)} (\frac{\partial v_z^\varepsilon}{\partial z})^2 rdrdz.$$

The rest of the proof is now immediate. □

COROLLARY 4.4. *We have*

$$\frac{1}{L}\|\frac{\partial \eta^\varepsilon}{\partial z}(t)\|^2_{L^2(0,L)} \leq \frac{2R^2(1-\sigma^2)}{Gkh(\varepsilon)^2E(\varepsilon)}\mathcal{P}^2 \tag{4.17}$$

$$\frac{1}{L^{1/4}}\|\eta^\varepsilon(t)\|_{L^4(0,L)} \leq \frac{2R}{h(\varepsilon)E(\varepsilon)}\sqrt{RL(1-\sigma^2)\sqrt{\frac{E(\varepsilon)(1-\sigma^2)}{Gk}}}\mathcal{P} \tag{4.18}$$

$$|\Omega_\varepsilon(t)| \leq \frac{3\pi R^2 L}{2}(1 + \frac{2(1-\sigma^2)R^2}{h^2(\varepsilon)E(\varepsilon)^2}\mathcal{P}^2) \tag{4.19}$$

$$\|\eta^\varepsilon(t)\|_{L^\infty(0,L)} \leq \frac{2R(1-\sigma^2)}{h(\varepsilon)E(\varepsilon)}\sqrt{RL(1-\sigma^2)\sqrt{\frac{E(\varepsilon)(1-\sigma^2)}{Gk}}}\mathcal{P} \tag{4.20}$$

*Proof.* Estimate (4.17) follows from the basic apriori estimate. To show (4.18) we calculate

$$|\eta^\varepsilon(t,z)|^4 = 4(\int_0^z \eta^\varepsilon(t,\xi)\frac{\partial \eta^\varepsilon}{\partial \xi}\, d\xi)^2 \leq 4(\int_0^z |\eta^\varepsilon(t,\xi)|^2\, d\xi)(\int_0^z |\frac{\partial \eta^\varepsilon}{\partial \xi}|^2\, d\xi)$$

which implies

$$\int_0^L |\eta^\varepsilon(t,z)|^4\, dz \leq 4L\|\frac{\partial \eta^\varepsilon}{\partial z}\|^2_{L^2(0,L)}\|\eta^\varepsilon(t)\|^2_{L^2(0,L)}.$$

Estimate (4.19) is immediate. Finally, (4.20) follows from

$$\max_{0\leq z\leq L}|\eta^\varepsilon(t,z)| \leq \sqrt{2}\|\frac{\partial \eta^\varepsilon}{\partial z}\|^{1/2}_{L^2(0,L)}\|\eta^\varepsilon(t)\|^{1/2}_{L^2(0,L)}.$$

□

Notice that these estimates are "reasonable". They say, among other things, that the size of the radial wall displacement is inversely proportional to the elasticity of the wall (the stiffer the wall, the smaller the amplitude of the displacement), and directly proportional to the pressure head data and to the radius of the unstressed vessel. In addition to this "general", intuitive information, our apriori estimates are "optimal" in the sense that they provide "optimal" powers describing the dependence of $\eta^\varepsilon$ and $v^\varepsilon$ on the parameters in the problem.

REMARK 1. *Obtaining the precise a priori estimate for the pressure $p^\varepsilon$ is quite technical and we don't present it here. Since the flow is incompressible pressure is a Lagrange multiplier and we can always adjust it with respect to the velocity . In the case of small Reynolds numbers such estimate was obtained in [6] . The important property of the pressure estimates is the smallness of the derivative with respect to the radial variable.*

REMARK 2. *Our goal is to obtain an effective 1D model. Clearly, the result will be local and it doesn't depend on the choice of the inlet/outlet boundary conditions. We used the given pressure head at the inlet/outlet boundaries just in order to get a simple derivation of the energy estimate. Imposing instead the pressure field at the inlet/outlet boundaries leads to complications and the energy estimate could be obtained only for the pressure drop smaller than a critical value.*

## 5 Asymptotic Expansions and the Reduced Equations

### 5.1 Leading-order asymptotic equations in non-dimensional form

To obtain the reduced equations we write the problem in non-dimensional form. Introduce the non-dimensional independent variables $\tilde{r}$ and $\tilde{z}$

$$r = R\tilde{r}, \qquad z = L\tilde{z}, \tag{5.1}$$

and recall that the time scale for the problem is determined by

$$t = \frac{1}{\omega^\varepsilon}\tilde{t}, \text{ where } \omega^\varepsilon = \frac{1}{L}\sqrt{\frac{hE}{R\rho(1-\sigma^2)}}. \tag{5.2}$$

Using the data presented in Table 2, with $E = 5 \times 10^5$ Pa for the Young's modulus, we obtain that $\omega^\varepsilon = 51$ and the time scale is around 0.02 of the physical time. For the less stiff vessels, the time-scale is closer to the physical time scale (the frequency of oscillations is smaller).

Based on the apriori estimates presented in Proposition 4.3 we introduce the following asymptotic expansions

$$v^\varepsilon = V\left\{\tilde{v}^0 + \varepsilon\tilde{v}^1 + ...\right\}, \text{ with } V = \sqrt{\frac{R(1-\sigma^2)}{\rho h E}\mathcal{P}} \tag{5.3}$$

$$\eta^\varepsilon = \Xi\left\{\tilde{\eta}^0 + \varepsilon\tilde{\eta}^1 + ...\right\}, \text{ with } \Xi = \frac{R^2(1-\sigma^2)}{hE}\mathcal{P} \tag{5.4}$$

$$p^\varepsilon = \rho V^2\left\{\tilde{p}^0 + \varepsilon\tilde{p}^1 + ...\right\}. \tag{5.5}$$

The approximate values of the scaling parameters, based on the values presented in Table 2 with $E = 6 \times 10^5$ Pa are $V = 0.17$ m/s, and $\Xi = 0.0002$ m. The scale value for the velocity is in good agreement with the results in [29] and the scale value $\Xi$ of the radial displacement is about 2 percent of the unstressed vessel radius, which is in good agreement with the observations performed in human subjects for which an average radial displacement is around 5% of the unstressed vessel radius.

We plug this into equations (2.2), (2.3) and (2.4) and collect the powers of $\varepsilon$. The incompressibility condition implies

$$\varepsilon^{-1}\frac{\partial}{\tilde{r}\partial\tilde{r}}(\tilde{r}\tilde{v}^0_{\tilde{r}}) + \frac{\partial\tilde{v}^0_{\tilde{z}}}{\partial\tilde{z}} + \frac{\partial}{\tilde{r}\partial\tilde{r}}(\tilde{r}\tilde{v}^1_{\tilde{r}}) + \varepsilon\sum_{i\geq 0}\varepsilon^i\left\{\frac{\partial\tilde{v}^{i+1}_{\tilde{z}}}{\partial\tilde{z}} + \frac{\partial}{\tilde{r}\partial\tilde{r}}(\tilde{r}\tilde{v}^{i+2}_{\tilde{r}})\right\} = 0. \tag{5.6}$$

Relation (5.6) gives

$$\tilde{v}^0_{\tilde{r}} = 0, \quad \text{and} \tag{5.7}$$

$$\frac{\partial(\tilde{v}^0_{\tilde{z}} + \varepsilon\tilde{v}^1_{\tilde{z}})}{\partial\tilde{z}} + \frac{\partial}{\tilde{r}\partial\tilde{r}}(\tilde{r}(\tilde{v}^1_{\tilde{r}} + \varepsilon\tilde{v}^2_{\tilde{r}})) = 0. \tag{5.8}$$

Because the first term in the expansion for the radial component of the velocity is zero, and since only the first two terms in the dependent variable expansions will contribute to the leading order equations, we introduce the following notation

$$\begin{aligned}
\tilde{v}_r &:= \tilde{v}^1_r + \varepsilon\tilde{v}^2_r, \text{ so that } v^\varepsilon_r = \varepsilon V\left(\tilde{v}_r + \mathcal{O}(\varepsilon^2)\right),\\
\tilde{v}_z &:= \tilde{v}^0_z + \varepsilon\tilde{v}^1_z \text{ so that } v^\varepsilon_z = V\left(\tilde{v}_z + \mathcal{O}(\varepsilon^2)\right),\\
\tilde{p} &:= \tilde{p}^0 + \varepsilon\tilde{p}^1 \text{ so that } p^\varepsilon = \rho V^2\left(\tilde{p} + \mathcal{O}(\varepsilon^2)\right),\\
\tilde{\eta} &:= \tilde{\eta}^0 + \varepsilon\tilde{\eta}^1 \text{ so that } \eta^\varepsilon = \Xi\left(\tilde{\eta} + \mathcal{O}(\varepsilon^2)\right).
\end{aligned}$$

After ignoring the terms of order $\varepsilon^2$ and smaller, the leading-order asymptotic equations describing the conservation of axial and radial momentum, and the incompressibility condition in non-dimensional variables read

$$Sh\frac{\partial\tilde{v}_z}{\partial\tilde{t}} + \tilde{v}_z\frac{\partial\tilde{v}_z}{\partial\tilde{z}} + \tilde{v}_r\frac{\partial\tilde{v}_z}{\partial\tilde{r}} + \frac{\partial\tilde{p}}{\partial\tilde{z}} - \frac{1}{Re}\left\{\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\tilde{v}_z}{\partial\tilde{r}}\right)\right\} = 0, \tag{5.9}$$

$$\frac{\partial\tilde{p}}{\partial\tilde{r}} = 0, \tag{5.10}$$

$$\frac{\partial}{\partial\tilde{r}}(\tilde{r}\tilde{v}_r) + \frac{\partial}{\partial\tilde{z}}(\tilde{r}\tilde{v}_z) = 0, \tag{5.11}$$

where $Sh := \frac{L\omega^\varepsilon}{V}$ and $Re := \frac{\rho V R^2}{\mu L}$. Using the values from Table 2 we get $Sh = 61$ and $Re = 35$ and so the viscous coefficient is of order $1/Re = 0.03 = \epsilon/2$.

One interesting consequence of our results is that the non-dimensional parameters $Sh$ and $Re$ which are typically used to determine the flow regimes, are given, as a consequence of our apriori estimates, in terms of the parameters in the problem, such as the Young's modulus, the Poisson ratio, etc. They incorporate not only the information about the fluid part of the problem (given via $V$ and $\mu$, for example) but also the information about the behavior of the membrane (given via $E$, $\omega^\varepsilon$ and $\sigma$). These parameters reflect the important information about the true nature of the coupling between the fluid and the membrane.

We continue by obtaining the leading order asymptotic equations describing the balance of forces at the vessel wall. The leading order Navier equations for the linearly elastic membrane read

$$-F_r = \mathcal{P}\left\{\tilde{\eta} - \frac{G(\varepsilon)k(1-\sigma^2)\varepsilon^2}{E(\varepsilon)}\frac{\partial^2\tilde{\eta}}{\partial\tilde{z}^2}\right\} + \mathcal{O}(\varepsilon^2). \tag{5.12}$$

Depending on the sizes of the parameters in the problem, the last term in this expression may or may not be neglected.

This force is balanced by the contact force coming from the fluid. Using (2.9) the asymptotic form of the contact force becomes

$$\big((p^\varepsilon - p_{ref})I - 2\mu D(v^\varepsilon)\big)ne_r = \rho V^2\left(\tilde{p} - \tilde{p}_{ref} + \mathcal{O}(\varepsilon^2)\right)\left(1 + \frac{\Xi}{R}\tilde{\eta}\right).$$

5.1.1 COUPLING THROUGH THE NON-DEFORMED INTERFACE. If we assume that $\frac{\rho V^2}{\mathcal{P}}\frac{\Xi}{R}$ is small (of order $\varepsilon^2$ of smaller), then the coupling takes place through the undeformed lateral boundary $\tilde{r} = 1$. In this case the leading order coupling reads

$$\frac{\rho V^2}{\mathcal{P}}\left(\tilde{p} - \tilde{p}_{ref} + \mathcal{O}(\varepsilon^2)\right) = \tilde{\eta} - \frac{G(\varepsilon)k}{E(\varepsilon)}(1-\sigma^2)\varepsilon^2\frac{\partial^2\tilde{\eta}}{\partial\tilde{z}^2} + \mathcal{O}(\varepsilon^2)$$
$$\tilde{v}_r(\tilde{z}, 1, \tilde{t}) = \frac{\partial\tilde{\eta}}{\partial\tilde{t}}, \quad \tilde{v}_z = 0.$$

If $\frac{G(\varepsilon)k}{E(\varepsilon)}(1-\sigma^2) \le 1$ we recover the Law of Laplace, which written in dimensional variables reads

$$p - p_{ref} = \left(\frac{Eh}{(1-\sigma^2)R}\right)\frac{\eta}{R}. \tag{5.13}$$

5.1.2 COUPLING THROUGH THE DEFORMED INTERFACE. Notice that in non-dimensional variables the deformed interface is defined by $\tilde{r} = 1 + \frac{\Xi}{R}\tilde{\eta}(\tilde{z}, \tilde{t})$. The leading-order coupling across the deformed lateral boundary reads

$$\frac{\rho V^2}{\mathcal{P}}\left(\tilde{p} - \tilde{p}_{ref} + \mathcal{O}(\varepsilon^2)\right)\left(1 + \frac{\Xi}{R}\tilde{\eta}\right) = \tilde{\eta} - \frac{G(\varepsilon)k}{E(\varepsilon)}(1-\sigma^2)\varepsilon^2\frac{\partial^2\tilde{\eta}}{\partial\tilde{z}^2} + \mathcal{O}(\varepsilon^2)$$
$$\tilde{v}_r(\tilde{z}, 1 + \frac{\Xi}{R}\tilde{\eta}(z,t), \tilde{t}) = \frac{\partial\tilde{\eta}}{\partial\tilde{t}}, \quad \tilde{v}_z = 0.$$

If $\frac{G(\varepsilon)k}{E(\varepsilon)}(1-\sigma^2) \le 1$, the shear modulus term is ignored and the pressure-radial displacement relationship is given by

$$\frac{\rho V^2}{\mathcal{P}}\left(\tilde{p} - \tilde{p}_{ref}\right)\left(1 + \frac{\Xi}{R}\tilde{\eta}\right) = \tilde{\eta}. \tag{5.14}$$

In dimensional variables this reads

$$p - p_{ref} = \left(\frac{Eh}{(1-\sigma^2)R}\right)\frac{\eta}{R+\eta}. \tag{5.15}$$

We call it the "nonlinear coupling version" of the Law of Laplace.

### 5.2 THE REDUCED TWO-DIMENSIONAL COUPLED PROBLEM

We summarize here the two-dimensional reduced coupled problem in non-dimensional variables. Define the scaled domain

$$\tilde{\Omega}(\tilde{t}) = \{(\tilde{z},\tilde{r}) \in I\!R^2 | \tilde{r} < 1 + \frac{\Xi}{R}\tilde{\eta}(\tilde{z},\tilde{t}), 0 < \tilde{z} < 1\},$$

and the lateral boundary $\tilde{\Sigma}(\tilde{t}) = \{\tilde{r} = 1 + \frac{\Xi}{R}\tilde{\eta}(\tilde{z},\tilde{t})\} \times (0,1)$. The problem consist of finding a $(\tilde{v}_z, \tilde{v}_r, \tilde{\eta})$ such that in $\tilde{\Omega}(\tilde{t}) \times I\!R^+$ the following is satisfied

$$Sh\frac{\partial \tilde{v}_z}{\partial \tilde{t}} + \tilde{v}_z\frac{\partial \tilde{v}_z}{\partial \tilde{z}} + \tilde{v}_r\frac{\partial \tilde{v}_z}{\partial \tilde{r}} + \frac{\partial \tilde{p}}{\partial \tilde{z}} = \frac{1}{Re}\left\{\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{v}_z}{\partial \tilde{r}}\right)\right\}, \tag{5.16}$$

$$\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{v}_r) + \frac{\partial}{\partial \tilde{z}}(\tilde{r}\tilde{v}_z) = 0, \tag{5.17}$$

$$\tilde{p} - \tilde{p}_{ref} = \frac{\mathcal{P}}{\rho V^2}\frac{1}{(1+\frac{\Xi}{R}\tilde{\eta})}\left(\tilde{\eta} - \frac{G(\varepsilon)k}{E(\varepsilon)}(1-\sigma^2)\varepsilon^2\frac{\partial^2\tilde{\eta}}{\partial \tilde{z}^2}\right), \tag{5.18}$$

$$\tilde{v}_r(\tilde{z}, 1 + \frac{\Xi}{R}\tilde{\eta}(z,t), \tilde{t}) = \frac{\partial \tilde{\eta}}{\partial \tilde{t}}, \quad \tilde{v}_z = 0, \tag{5.19}$$

with the initial and boundary conditions given by

$$\tilde{\eta} = \frac{\partial \tilde{\eta}}{\partial \tilde{t}} = 0 \quad \text{at} \quad \{\tilde{t} = 0\}, \tag{5.20}$$

$$\tilde{v}_{\tilde{r}} = 0 \;\text{ and }\; \tilde{p} = P_1(\tilde{t}) + p_{ref} \quad \text{on} \quad (\partial\tilde{\Omega}(\tilde{t}) \cap \{\tilde{z} = 0\}) \times I\!R_+, \tag{5.21}$$

$$\tilde{v}_{\tilde{r}} = 0 \;\text{ and }\; \tilde{p} = P_2(\tilde{t}) + p_{ref} \quad \text{on} \quad (\partial\tilde{\Omega}(\tilde{t}) \cap \{\tilde{z} = 1\}) \times I\!R_+, \tag{5.22}$$

$$\tilde{\eta} = 0 \;\text{ for }\; \tilde{z} = 0, \quad \tilde{\eta} = 0 \;\text{ for }\; \tilde{z} = 1 \;\text{ and }\; \forall \tilde{t} \in I\!R_+. \tag{5.23}$$

This is a closed, free-boundary problem for a two-dimensional degenerate hyperbolic system with a parabolic regularization.

REMARK 3. *We don't discuss here the asymptotic behavior close to the inlet/outlet boundaries. With our data the term $\rho(v_z^0)^2/2$ is negligible compared with the pressure and we drop it.*

In Section 7 we obtain an equivalent system in a much simpler form, which is an $\varepsilon^2$-approximation of the above problem. In fact, when radial displacement is small, we obtain a *linear system* which is an $\varepsilon^2$-approximation to this two-dimensional problem. Its existence, uniqueness and regularity is discussed in Appendix 2.

## 6 THE AVERAGED EQUATIONS

To simplify the problem even further and obtain the effective equations in one space dimension we use a typical approach of averaging the two-dimensional equations across the vessel cross-section. Introduce $\tilde{A} = (1 + \frac{\Xi}{R}\tilde{\eta})^2$ and $\tilde{m} = \tilde{A}\tilde{U}$ where

$$\tilde{U} = \frac{2}{\tilde{A}^2}\int_0^{1+\frac{\Xi}{R}\tilde{\eta}} \tilde{v}_z\tilde{r}d\tilde{r} \quad \text{and} \quad \tilde{\alpha} = \frac{2}{\tilde{A}^2\tilde{U}^2}\int_0^{1+\frac{\Xi}{R}\tilde{\eta}} \tilde{v}_z^2\tilde{r}d\tilde{r}.$$

We integrate the incompressibility condition and the axial momentum equations with respect to $\tilde{r}$ from 0 to $1+\frac{\Xi}{R}\tilde{\eta}$ and obtain, after taking into account the no-slip condition at the lateral boundary,

$$\frac{\partial \tilde{A}}{\partial \tilde{t}} + \frac{\Xi}{R}\frac{\partial \tilde{m}}{\partial \tilde{z}} = 0, \tag{6.1}$$

$$Sh\frac{\partial \tilde{m}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{z}}\left(\tilde{\alpha}\frac{\tilde{m}^2}{\tilde{A}}\right) + \tilde{A}\frac{\partial \tilde{p}}{\partial \tilde{z}} = \frac{2}{Re}\sqrt{\tilde{A}}\left[\frac{\partial \tilde{v}_z}{\partial \tilde{r}}\right]_{\tilde{\Sigma}}. \tag{6.2}$$

As always when averaging of nonlinear systems is concerned, one needs a closure condition. In our case this amounts to describing the axial velocity profile. There are several ad hoc approaches in the literature. They assume the Poiseuille velocity profile

$$\tilde{v}_z = \frac{\gamma+2}{\gamma}\tilde{U}\left(1-\left(\frac{\tilde{r}}{1+\frac{\Xi}{R}\tilde{\eta}}\right)^{\gamma}\right) \tag{6.3}$$

where $\gamma = 2$, an "almost flat" velocity profile corresponding to (6.3) with $\gamma = 9$ which accounts for the non-Newtonian nature of blood [27], the flat velocity profile ("plug flow"), or the flat velocity profile with a small linear boundary layer (Bingham flow) suggested in [23]. In Section 7 we obtain a *rigorous* closure describing the velocity profile in the original three-dimensional problem approximated to order $\varepsilon^2$. Our analysis in Section 7 shows that ad hoc closures mentioned here will give rise to an error of order one in the solution of the reduced equations for moderate Reynolds numbers.

In order to compare our results thus far with those already existing in the literature, in this section we assume one of the ad hoc velocity profiles mentioned above. Namely, we consider the axial velocity profile

$$\tilde{v}_z = \frac{\gamma+2}{\gamma}\tilde{U}\left(1-\left(\frac{\tilde{r}}{1+\frac{\Xi}{R}\tilde{\eta}}\right)^{\gamma}\right). \tag{6.4}$$

Smith, Pullan and Hunter report that $\gamma = 9$ seems to be a good fit for the blood flow data [27]. This gives rise to $\alpha = 1.1$. With this assumption, the term on the right hand-side of the momentum equation becomes

$$-\frac{2}{Re}(\gamma+2)\frac{\tilde{m}}{\tilde{A}}. \tag{6.5}$$

The pressure term is specified by studying the fluid-structure coupling.

### 6.1 COUPLING THROUGH THE NON-DEFORMED INTERFACE

If the coupling is performed via a non-deformed interface then we get

$$\tilde{p} - \tilde{p}_{ref} = \frac{\mathcal{P}}{\rho V^2}\left(\tilde{\eta} - \frac{G(\varepsilon)k}{E(\varepsilon)}(1-\sigma^2)\varepsilon^2\frac{\partial^2 \tilde{\eta}}{\partial \tilde{z}^2}\right). \tag{6.6}$$

This leads to the following axial momentum equation in non-dimensional form

$$\begin{aligned} Sh\frac{\partial \tilde{m}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{z}}\left(\tilde{\alpha}\frac{\tilde{m}^2}{\tilde{A}}\right) &+ \tilde{A}\frac{\partial}{\partial \tilde{z}}\left(\frac{\mathcal{P}}{\rho V^2}\frac{R}{\Xi}\left(\sqrt{\tilde{A}}-1\right)\right) = -\frac{2}{Re}(\gamma+2)\frac{\tilde{m}}{\tilde{A}} \\ &+ \tilde{A}\frac{\partial}{\partial \tilde{z}}\left(\frac{\mathcal{P}}{\rho V^2}\frac{R}{\Xi}C_G\frac{\partial^2}{\partial \tilde{z}^2}\left(\sqrt{\tilde{A}}-1\right)\right), \end{aligned} \tag{6.7}$$

where $C_G$ is defined by

$$C_G = \frac{Gk(1-\sigma^2)}{E}\frac{R^2}{L^2}. \tag{6.8}$$

In dimensional variables the system reads

$$\frac{\partial A}{\partial t} + \frac{\partial m}{\partial z} = 0, \tag{6.9}$$

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial z}\left(\alpha\frac{m^2}{A}\right) + \frac{A}{\rho}\frac{\partial}{\partial z}\left(\frac{hE}{R(1-\sigma^2)}\left(\sqrt{\frac{A}{A_0}}-1\right)\right) = -\frac{2\mu}{\rho}(\gamma+2)\frac{m}{A}$$
$$+\frac{A}{\rho}\frac{\partial}{\partial z}\left(GkhR\frac{\partial^2}{\partial z^2}\left(\sqrt{\frac{A}{A_0}}-1\right)\right). \tag{6.10}$$

Here we used $R^2 = A_0$. If we assume $\sigma = 0.5$ and compare the advection pressure term $\frac{hE}{R(1-\sigma^2)}\left(\sqrt{\frac{A}{A_0}}-1\right)$ with the Law of Laplace $p_{\text{Lap}}(A) = \frac{4hE}{3R}\left(\sqrt{\frac{A}{A_0}}-1\right)$ then we see that they are identical. This system has been widely used by many authors (see e.g., [14, 23, 31, 9, 8, 4, 3, 18, 27]). Our analysis shows that this system is obtained by enforcing linear fluid-structure coupling.

Assuming nonlinear coupling, we get a new set of reduced, one-dimensional equations.

### 6.2 Coupling through the deformed interface

Using (5.14) we obtain $\tilde{A}\frac{\partial\tilde{p}}{\partial\tilde{z}} = \frac{\mathcal{P}}{\rho V^2}\tilde{A}\frac{\partial}{\partial\tilde{z}}\left(\frac{\tilde{\eta}}{1+\frac{\Xi}{R}\tilde{\eta}} - \frac{G(\varepsilon)k}{E(\varepsilon)}(1-\sigma^2)\varepsilon^2\frac{1}{1+\frac{\Xi}{R}\tilde{\eta}}\frac{\partial^2\tilde{\eta}}{\partial\tilde{z}^2}\right)$. Keeping in mind that $\tilde{A} = (1+\frac{\Xi}{R}\tilde{\eta})^2$ the axial momentum equation reads

$$Sh\frac{\partial\tilde{m}}{\partial\tilde{t}} + \frac{\partial}{\partial\tilde{z}}\left(\tilde{\alpha}\frac{\tilde{m}^2}{\tilde{A}}\right) + \tilde{A}\frac{\partial}{\partial\tilde{z}}\left(\frac{\mathcal{P}}{\rho V^2}\frac{R}{\Xi}\sqrt{\frac{1}{\tilde{A}}}\left(\sqrt{\tilde{A}}-1\right)\right) = -\frac{2}{Re}(\gamma+2)\frac{\tilde{m}}{\tilde{A}}$$
$$+ \tilde{A}\frac{\partial}{\partial\tilde{z}}\left(\frac{\mathcal{P}}{\rho V^2}\frac{R}{\Xi}C_G\sqrt{\frac{1}{\tilde{A}}}\frac{\partial^2}{\partial\tilde{z}^2}\left(\sqrt{\tilde{A}}-1\right)\right), \tag{6.11}$$

where $C_G$ is defined by (6.8). In dimensional variables we get

$$\frac{\partial A}{\partial t} + \frac{\partial m}{\partial z} = 0, \tag{6.12}$$

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial z}\left(\alpha\frac{m^2}{A}\right) + \frac{A}{\rho}\frac{\partial}{\partial z}\left(\frac{hE}{R(1-\sigma^2)}\sqrt{\frac{A_0}{A}}\left(\sqrt{\frac{A}{A_0}}-1\right)\right) = -\frac{2\mu}{\rho}(\gamma+2)\frac{m}{A}$$
$$+\frac{A}{\rho}\frac{\partial}{\partial z}\left(GkhR\sqrt{\frac{A_0}{A}}\frac{\partial^2}{\partial z^2}\left(\sqrt{\frac{A}{A_0}}-1\right)\right). \tag{6.13}$$

We see that this system differs from (6.9)-(6.10) by an additional factor of $\sqrt{A_0/A}$ in front of the corresponding terms arising from the leading order force balance equation.

In the bifurcation diagram shown in Figure 6.1 we show the values of the parameters $P$ (the peak systolic pressure) and $E$ (Young's modulus of the vessel wall) for which

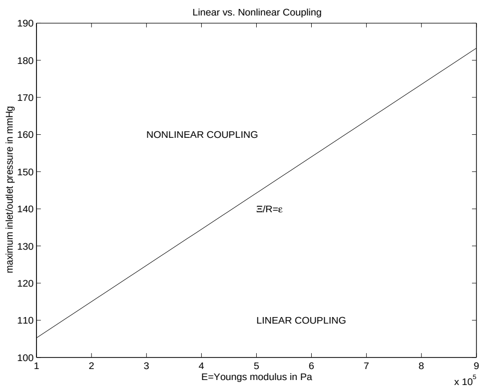


FIGURE 6.1: Bifurcation diagram showing the region of parameters $E$ and *Maximum Systolic Pressure* for which linear or nonlinear coupling should be used.

linear vs. nonlinear coupling should be used. A simple calculation shows that the condition $\frac{\rho V^2}{\mathcal{P}}\frac{\Xi}{R} < \varepsilon^2$ is equivalent to $\frac{\Xi}{R} < \varepsilon$. If we assume fixed values of $R$,$L$ and $h$ given in Table 2 and take $\sigma = 0.5$, the curve $\frac{\Xi}{R} = \varepsilon$ is the boundary of the parameter region for which the linear or the nonlinear coupling should be used. We see from Figure 6.1 that to simulate the flow of blood in the abdominal aorta in subjects with the systolic peak pressure of around 120 mmHg linear coupling can be used whenever the aortic wall is stiffer than $E = 1.4 \times 10^5$Pa. Also, the diagram shows that in hypertensive subjects nonlinear coupling should be used for a wider range of stiffness parameters of the abdominal aortic wall.

We conclude this manuscript by a derivation of a self-contained, effective model, without assuming ad hoc closure assumptions.

## 7 AN $\varepsilon^2$-APPROXIMATION WITHOUT THE AD HOC CLOSURE ASSUMPTION

In this section we obtain the one-dimensional, closed, effective equations that are an $\varepsilon^2$ approximation of the original 3-D axially symmetric fluid-structure interaction problem. The equations are simpler than those presented in (5.9)-(5.12). They can be easily solved numerically.

With our data we suppose from now on that $\frac{G(\varepsilon)k}{E(\varepsilon)}(1-\sigma^2) \ll 1$. As in the previous section, this assumption leads to the Laplace's law linking the pressure and the radial displacement. As in the linear case (see [6]) the boundary conditions for radial displacement $\eta$ are lost if and there is a boundary layer. Only the boundary condition for the pressure is kept at the inlet/outlet boundaries and the corresponding boundary conditions for the radial displacement are calculated using the Laplace's law.

We consider two flow regimes. One is the creeping flow, discussed in Section 7.1, and the other is the flow regime typical for the abdominal aorta, namely moderate Reynolds number, discussed in Section 7.2.

In the creeping flow regime it is well known that the Poiseuille profile is the unique velocity solution to the equations. We show that the displacement is described by a one-dimensional, parabolic, semi-linear equation, see (7.11), first obtained by Čanić and Mikelić in [5, 6].

In the moderate Reynolds' number regime, we consider two cases. One case corresponds to large and the other to small displacements ($\Xi/R < \varepsilon$). In the case of large radial displacements, the effective equations for the zero-th order velocity approximation form a nonlinear free-boundary problem, see (7.29)-(7.31). The $\varepsilon$-correction for the velocity is recovered by solving a linearized, fixed boundary problem. Both problems have simpler form than (5.10)-(5.12). In the case of small radial displacements, we expand the solution in terms of the small parameter $\Xi/R$ and obtain a *linear* system of equations of Biot type, see [1]. This is obtained in Section 7.2.5. These equations can be easily solved using, for example, the Laplace transform, see Appendix 1. We obtain that for a time-periodic flow regime the resulting velocity profiles are an $\varepsilon$-order correction of the Womersley profile in elastic tubes [32].

We begin our analysis with the two-dimensional system (5.9)-(5.11). Our goal is to obtain an equivalent effective problem which is closed, and for which we could show the existence of a unique solution. Furthermore, the calculation of the solution for such a system should be simple.

Motivated by the results of [17] where closed effective porous medium equations were obtained using homogenization techniques, we would like to set up a problem that would mimic a similar scenario. In this vein, we introduce

$$y = \frac{1}{\varepsilon}\tilde{z}$$

and assume periodicity in $y$ of the domain and of the velocity and pressure. Furthermore, recalling that we have a "thin" long tube with $\tilde{r} = \frac{1}{R}r = \frac{1}{\varepsilon}\frac{r}{L}$, we can assume periodicity in the radial direction thereby forming a network of a large number of strictly separated, parallel tubes. This now resembles a porous medium problem but with no flow from one horizontal tube to another. See Figure 7.1. We homogenize with respect to all directions. Since there is nothing in the physics of the problem that depends periodically on $y$ we expect to get the effective equations and the solution independent of $y$. Indeed, this is obtained in Sections 7.2.3 and 7.2.5. We note that thanks to the fact that the model contains a hydrostatic approximation of the pressure in straight tubes, the methods we use in this paper are much simpler than those used in [17]. For more details about the homogenization methods in porous media, see e.g. [19].

More precisely, we start with the following relations between the "slow" variables ($r$ and $z$ (or $\tilde{z}$)) and "fast" variables ($\tilde{r}$ and $y$)

$$z = L\tilde{z} := L\varepsilon y = Ry, \ r = R\tilde{r}. \tag{7.1}$$

We are looking for an $\varepsilon^2$-approximation of the solution to system (5.9)-(5.11) in the form of a sum of two functions: the zero and the first order approximations with respect to $\varepsilon$. We use scaling (7.1) and dependent variable expansions given in (5.3) and (5.4)

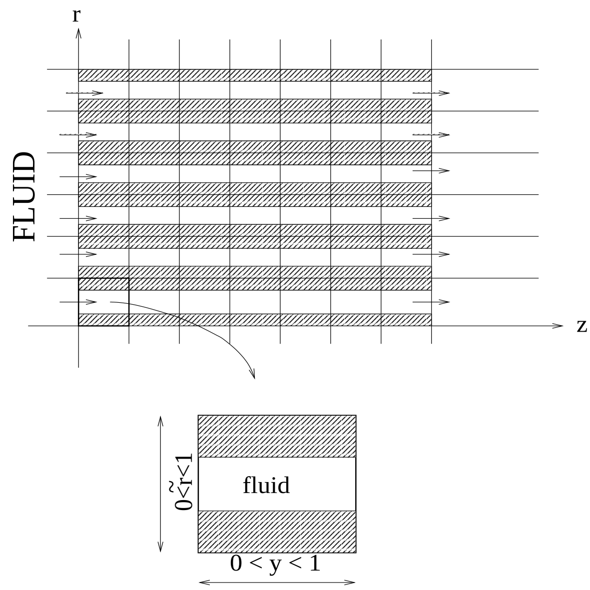


FIGURE 7.1: Homogenization domain.

to rewrite system (5.9)-(5.11). The equations at zero order read

$$Sh_0 \frac{\partial \tilde{v}_z^0}{\partial \tilde{t}} + (\tilde{v}^0 \nabla_{\tilde{r},y}) \tilde{v}_z^0 + \frac{\partial \tilde{\tilde{p}}^0}{\partial \tilde{z}} + \frac{\partial \tilde{\tilde{p}}^1}{\partial y} - \frac{1}{Re_0} \left\{ \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{v}_z^0}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{v}_z^0}{\partial y^2} \right\} = 0, \tag{7.2}$$

$$Sh_0 \frac{\partial \tilde{v}_r^0}{\partial \tilde{t}} + (\tilde{v}^0 \nabla_{\tilde{r},y}) \tilde{v}_r^0 + \frac{\partial \tilde{\tilde{p}}^0}{\partial r} + \frac{\partial \tilde{\tilde{p}}^1}{\partial \tilde{r}} - \frac{1}{Re_0} \left\{ \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{v}_r^0}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{v}_r^0}{\partial y^2} \right\} = 0, \tag{7.3}$$

$$\nabla_{\tilde{r},y} \tilde{\tilde{p}}^0 = 0, \tag{7.4}$$

$$\frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \tilde{v}_r^0 \right) + \frac{\partial}{\partial y} \left( \tilde{r} \tilde{v}_z^0 \right) = 0, \tag{7.5}$$

with

$$\tilde{v}_r^0, \tilde{v}_z^0 \text{ and } \tilde{\tilde{p}}^1 \ \ 1-\text{periodic in } y \text{ and } \tilde{v}_r^0 = \tilde{v}_z^0 = 0 \text{ at } \tilde{r} = 1 + \frac{\Xi}{R} \tilde{\eta}, \tag{7.6}$$

where $Sh_0 := \frac{\varepsilon L \omega^\varepsilon}{V}$ and $Re_0 := \frac{\rho R V}{\mu}$. Notice $Sh_0 = \varepsilon Sh$ and $Re = \varepsilon Re_0$. For the values from Table 2, $Sh_0$ is of order 1 ($Sh_0 \in (3,4)$) and $Re_0$ is around 600. We remark that equation (7.4) corresponds to the $\varepsilon^{-1}$ term. Here, a new scaling for the pressure was used to obtain equations (7.2)-(7.4). This "z-blown up" pressure scaling reads

$$p = \frac{\rho L V^2}{R} \tilde{\tilde{p}} = \rho V^2 \frac{1}{\varepsilon} \tilde{\tilde{p}} = \rho V^2 \tilde{p}, \quad \text{so} \quad \tilde{\tilde{p}} = \varepsilon \tilde{p}. \tag{7.7}$$

The leading order Navier equations for the membrane force are unchanged, see (5.12).

We now focus on the two cases corresponding to the different magnitudes of the parameters $Sh_0$ and $Re_0$.

### 7.1 CASE I: $Sh_0 = 0$ AND $Re_0$ SUFFICIENTLY SMALL

In this case the Poiseuille profile

$$\tilde{v}_z^0 = -Re\frac{\partial\tilde{\tilde{p}}^0}{\partial\tilde{z}}(\tilde{z},\tilde{t})\frac{(1+\Xi\tilde{\eta}^0/R)^2-\tilde{r}^2}{4},\ \tilde{v}_r^0 = 0 \tag{7.8}$$

is the unique velocity which solves (7.2)-(7.6). To complete the solution we need to calculate the pressure and the radial displacement. They are related through the coupling and the lateral boundary

$$\tilde{\tilde{p}} - \tilde{\tilde{p}}_{\mathrm{ref}} = \frac{\mathcal{P}R}{\rho V^2 L}\tilde{\eta} = \frac{hE}{(1-\sigma^2)L\mathcal{P}}\tilde{\eta}. \tag{7.9}$$

Here, for simplicity, we are assuming that the shear modulus term is negligible. We average the continuity equation

$$\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\tilde{v}_r^0\right) + \varepsilon\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\tilde{v}_r^1\right) + \varepsilon\frac{\partial}{\partial\tilde{z}}\left(\tilde{r}\tilde{v}_z^0\right) + \varepsilon^2\frac{\partial}{\partial\tilde{z}}\left(\tilde{r}\tilde{v}_r^1\right) = 0,$$

keeping in mind that $\tilde{v}_r^0 = 0$, $\tilde{v}_r^1 = \partial\tilde{\eta}^0/\partial\tilde{t}$, and ignoring the term at $\varepsilon^2$. As before, we get the following

$$\frac{\partial\tilde{A}}{\partial\tilde{t}} + \frac{\Xi}{R}\frac{\partial\tilde{m}}{\partial\tilde{z}} = 0, \tag{7.10}$$

where $\tilde{A} = (1+\Xi\tilde{\eta}^0/R)^2$. Express $\tilde{m}$ explicitly using the Poiseuille velocity profile (7.8), namely

$$\tilde{m} = \tilde{A}\tilde{U} = 2\int_0^{1+\Xi\tilde{\eta}^0/R}\tilde{r}\tilde{v}_z^0 d\tilde{r},$$

to obtain the following semilinear parabolic equation for the cross-sectional area

$$\frac{\partial\tilde{A}}{\partial\tilde{t}} = \frac{Re}{8}\frac{R}{\rho VL}\sqrt{\frac{hE\rho}{R(1-\sigma^2)}}\frac{\partial}{\partial\tilde{z}}\left(\tilde{A}^2\frac{\partial\sqrt{\tilde{A}}}{\partial\tilde{z}}\right) = \frac{Re}{8}\frac{R}{L}\frac{\partial}{\partial\tilde{z}}\left(\tilde{A}^2\frac{\partial\sqrt{\tilde{A}}}{\partial\tilde{z}}\right). \tag{7.11}$$

In dimensional variables this reads

$$8\mu\frac{\partial A}{\partial t} = \frac{hE}{R^2(1-\sigma^2)}\frac{\partial}{\partial z}\left(A^2\frac{\partial\sqrt{A}}{\partial z}\right). \tag{7.12}$$

This is a semi-linear variant of the equations obtained by Čanić and Mikelić in [6], where a parabolic equation for the pressure was obtained. The effective equation holds in axi-symmetric domains, and it is an $\varepsilon^2$-approximation of the 3-D axially symmetric flow away from the boundary. There was no ad hoc closure assumption made on the form of the velocity profile.

We proceed in the same spirit, but for a more complicated scenario.

### 7.2 CASE II: $Sh_0 > 0$ AND MODERATE $Re_0$

In this case, for a given pressure gradient $\frac{\partial \tilde{\tilde{p}}^0}{\partial \tilde{z}}$, the non-stationary, axially symmetric system (7.2)-(7.6) admits a unique unidirectional, but strongly non-stationary solution. The unidirectional solution refers to a solution independent of $y$. We will write the solution of system (7.2)-(7.6) as a sum of this unidirectional solution and a small perturbation of it. This perturbation satisfies a linearized system, see (7.2)-(7.6), where the linearization is calculated around the unidirectional solution. This system is *closed.*

7.2.1 THE ZERO-TH ORDER APPROXIMATION: THE UNIDIRECTIONAL FLOW. For every given smooth $\tilde{\tilde{p}}^0$, system (7.2)-(7.6) has a unique strong solution (see e.g., [30])

$$\tilde{v}_z^0 = w(\tilde{r}, \tilde{z}, t), \ \tilde{v}_r^0 = 0, \tag{7.13}$$

where $w$ satisfies

$$Sh_0 \frac{\partial w}{\partial \tilde{t}} - \frac{1}{Re_0} \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial w}{\partial \tilde{r}} \right) = -\frac{\partial \tilde{\tilde{p}}^0}{\partial \tilde{z}} (\tilde{z}, \tilde{t}) = -\left( \frac{Eh}{\mathcal{P}L(1-\sigma^2)} \right) \frac{\partial \tilde{\eta}^0}{\partial \tilde{z}} \tag{7.14}$$

$$w(0, \tilde{z}, \tilde{t}) \text{ bounded, } w(1 + \Xi \tilde{\eta}^0(\tilde{z}, \tilde{t})/R, \tilde{z}, \tilde{t}) = 0 \text{ and } w(\tilde{r}, \tilde{z}, 0) = 0. \tag{7.15}$$

Furthermore, solution $\tilde{\tilde{p}}^1$ is a linear function of $y$, independent of $\tilde{r}$. Due to 1-periodicity with respect to $y$ we get $\tilde{\tilde{p}}^1 = \tilde{\tilde{p}}^1(\tilde{z}, \tilde{t})$.

This is a free-boundary problem because the condition at the lateral boundary depends on the solution. For a known pressure or the radial displacement (or the cross-sectional area) this is well-posed. However, to close the system for the unknown functions $w$, $\tilde{\tilde{p}}^0$ and $\tilde{\eta}^0$ we need to specify one more condition. The averaged continuity equation provides the necessary closure. Therefore, the following closed system provides the unidirectional solution

$$\begin{aligned} \frac{\partial \tilde{A}}{\partial \tilde{t}} + \frac{\Xi}{R} \frac{\partial \tilde{m}}{\partial \tilde{z}} &= 0, \\ Sh_0 \frac{\partial w}{\partial \tilde{t}} + \frac{\partial \tilde{\tilde{p}}^0}{\partial \tilde{z}} (\tilde{z}, \tilde{t}) &= \frac{1}{Re_0} \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial w}{\partial \tilde{r}} \right), \quad \tilde{\tilde{p}}^0 = (\frac{Eh}{\mathcal{P}L(1-\sigma^2)}) \tilde{\eta}^0 \end{aligned}$$

with $w(0, \tilde{z}, \tilde{t})$ bounded, $w(1 + \Xi \tilde{\eta}^0(\tilde{z}, \tilde{t})/R, \tilde{z}, \tilde{t}) = 0$ and $w(\tilde{r}, \tilde{z}, 0) = 0$. We can eliminate $\tilde{\tilde{p}}^0$, and use the definitions of $\tilde{A}$ and $\tilde{m}$ to write this in terms of $w$ and $\tilde{A}$ as

$$\frac{\partial \tilde{A}}{\partial \tilde{t}} + \frac{\Xi}{R} \frac{\partial}{\partial \tilde{z}} \int_0^{\sqrt{\tilde{A}}} 2\tilde{r} w d\tilde{r} = 0, \tag{7.16}$$

$$Sh_0 \frac{\partial w}{\partial \tilde{t}} + \left( \frac{R}{\Xi} \right)^2 \frac{R}{L} \frac{\partial \sqrt{\tilde{A}}}{\partial \tilde{z}} = \frac{1}{Re_0} \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial w}{\partial \tilde{r}} \right), \tag{7.17}$$

with

$$w(0, \tilde{z}, \tilde{t}) \text{ bounded, } w(\sqrt{\tilde{A}}, \tilde{z}, \tilde{t}) = 0 \tag{7.18}$$

$$\tilde{A}(\tilde{z}, 0) = 0, \quad w(\tilde{r}, \tilde{z}, 0) = 0 \tag{7.19}$$

$$\tilde{A}(0, \tilde{t}) = A_1(\tilde{t}), \quad \tilde{A}(L, \tilde{t}) = A_2(\tilde{t}). \tag{7.20}$$

This is a two-dimensional, free-boundary problem of hyperbolic-parabolic type. It has a simpler form than system (5.9)-(5.11).

7.2.2 THE FIRST-ORDER CORRECTION: PERTURBATION OF THE UNIDIRECTIONAL FLOW. We will be using the zero-th order approximation to the solution consisting of the velocity $(w, 0)$ and displacement $\tilde{\eta}^0$ (or, equivalently, the pressure $\tilde{\tilde{p}}^0$) to find an $\varepsilon$-correction by solving (5.9)-(5.11), linearized around the zero-th order approximation:

$$Sh_0 \frac{\partial \tilde{v}_z^1}{\partial \tilde{t}} + \tilde{v}_z^0 \left\{ \frac{\partial \tilde{v}_z^1}{\partial y} + \frac{\partial \tilde{v}_z^0}{\partial \tilde{z}} \right\} + \tilde{v}_r^1 \frac{\partial \tilde{v}_z^0}{\partial \tilde{r}} + \frac{\partial \tilde{\tilde{p}}^1}{\partial \tilde{z}} + \frac{\partial \tilde{\tilde{p}}^2}{\partial y} = \frac{1}{Re_0} \left\{ \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{v}_z^1}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{v}_z^1}{\partial y^2} \right\} \tag{7.21}$$

$$Sh_0 \frac{\partial \tilde{v}_r^1}{\partial \tilde{t}} + \tilde{v}_z^0 \frac{\partial \tilde{v}_r^1}{\partial y} + \frac{\partial \tilde{\tilde{p}}^2}{\partial \tilde{r}} = \frac{1}{Re_0} \left\{ \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{v}_r^1}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{v}_r^1}{\partial y^2} \right\} \tag{7.22}$$

$$\frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \tilde{v}_r^1 \right) + \frac{\partial}{\partial y} \left( \tilde{r} \tilde{v}_z^1 \right) + \frac{\partial \tilde{v}_z^0}{\partial \tilde{z}} = 0, \tag{7.23}$$

$$\tilde{v}_r^1, \tilde{v}_z^1, \tilde{\tilde{p}}^2 \ 1 - \text{periodic in } \ y; \ \tilde{v}_r^1 = \frac{\partial \tilde{\eta}^0}{\partial \tilde{t}}, \ \tilde{v}_z^0 = 0 \ \text{ at } \ \tilde{r} = 1 + \frac{\Xi}{R} \tilde{\eta}^0. \tag{7.24}$$

This is a linear system which is known as a non-stationary Oseen's system. Since $\tilde{\eta}^0$ is known from the previous calculation, the problem is posed on a fixed cylindrical domain of radius $\tilde{r} = 1 + \frac{\Xi}{R}\tilde{\eta}^0$. Notice, however, that the system does not appear to be closed since $\tilde{\tilde{p}}^1 = \tilde{\tilde{p}}^1(\tilde{z}, \tilde{t})$ and $\tilde{\tilde{p}}^2 = \tilde{\tilde{p}}^2(\tilde{r}, y, \tilde{z}, \tilde{t})$ are unknown as well. Nevertheless, since $\tilde{\tilde{p}}^1 = \tilde{\tilde{p}}^1(\tilde{z}, \tilde{t})$ is zero at the boundary $\tilde{r} = 1 + \Xi\tilde{\eta}^0/R$ and it depends only on $(\tilde{z}, \tilde{t})$, $\tilde{\tilde{p}}^1$ must be zero. We will show below that $\tilde{\tilde{p}}_2 = 0$ which will lead to a closed system. To show that this is, indeed, the case we first suppose that $\tilde{v}_z^1 = \tilde{v}_z^1(\tilde{r}, \tilde{z}, \tilde{t})$ and calculate $\tilde{v}_r^1$ using (7.23). We get an explicit formula for $\tilde{v}_r^1(\tilde{r}, \tilde{z}, \tilde{t})$ in terms of the unidirectional solution

$$\tilde{r}\tilde{v}_r^1(\tilde{r}, \tilde{z}, \tilde{t}) = (1 + \Xi\tilde{\eta}^0/R) \frac{\partial \tilde{\eta}^0}{\partial \tilde{t}} + \int_{\tilde{r}}^{1+\Xi\tilde{\eta}^0/R} \frac{\partial \tilde{v}_z^0}{\partial \tilde{z}}(\xi, \tilde{z}, \tilde{t}) \ \xi \ d\xi. \tag{7.25}$$

Next using (7.22) we find $\tilde{\tilde{p}}^2$ in the form $\tilde{\tilde{p}}^2 = \alpha(\tilde{r}, \tilde{z}, \tilde{t}) + \varphi(y, \tilde{z}, \tilde{t})$, where $\varphi$ is an arbitrary function, 1-periodic in $y$. If we plug this into the axial momentum equation for $\tilde{v}_z^1$

$$Sh_0 \frac{\partial \tilde{v}_z^1}{\partial \tilde{t}} - \frac{1}{Re_0} \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} \frac{\partial \tilde{v}_z^1}{\partial \tilde{r}} \right) + \frac{\partial \varphi}{\partial y}(y, \tilde{z}, \tilde{t}) = -\tilde{v}_r^1 \frac{\partial \tilde{v}_z^0}{\partial \tilde{r}} - \frac{\partial}{\partial \tilde{z}} \left( \frac{(\tilde{v}_z^0)^2}{2} + \tilde{\tilde{p}}^1 \right) \tag{7.26}$$

$$\tilde{v}_z^1(0, \tilde{z}, \tilde{t}) \text{ bounded}, \ \tilde{v}_z^1(1 + \Xi\tilde{\eta}^0(\tilde{z}, \tilde{t})/R, \tilde{z}, \tilde{t}) = 0 \tag{7.27}$$

$$\tilde{v}_z^1(\tilde{r}, \tilde{z}, 0) = 0 \tag{7.28}$$

we see that the axial momentum equation implies, together with the periodicity in $y$, that $\varphi = 0$. Since $\alpha(\tilde{r}, \tilde{z}, \tilde{t})$ is zero at $\tilde{r} = 1 + \Xi\tilde{\eta}^0/R$ we conclude that $\tilde{\tilde{p}}_2 = 0$. Therefore, correction $(\tilde{v}_z^1, \tilde{v}_r^1)$, $(\tilde{\tilde{p}}^1, \tilde{\tilde{p}}^2) = (0, 0)$ is obtained by solving (7.26)-(7.28) with $\varphi = \tilde{\tilde{p}}^1 = 0$. This way we have obtained a *closed* problem.

Functions $(\tilde{v}_z^0 + \varepsilon\tilde{v}_z^1, \varepsilon\tilde{v}_r^1)$ and $\tilde{\eta}^0$ (namely, $\tilde{\tilde{p}}^0$) also satisfy problem (5.16)-(5.23) to $\varepsilon^2$-order. More precisely, since $\tilde{\tilde{p}}^0 = \varepsilon\tilde{p}$ and due to the boundary conditions for the pressure, we have that $\tilde{\tilde{p}}^0$ is of order $\varepsilon$. Consequently, both $\tilde{v}_z^0$ and $\tilde{v}_r^1$ are of order $\varepsilon$. We have

PROPOSITION 7.1. *The velocity field* $(\tilde{v}_z^0 + \varepsilon \tilde{v}_z^1)$ *and the pressure field* $\frac{1}{\varepsilon}\tilde{p}^0$ *satisfy equations* (5.16)-(5.23) *to* $\mathcal{O}(\varepsilon^2)$.

*Proof.* The functions $(\tilde{v}_z^0 + \varepsilon \tilde{v}_z^1)$ and $\frac{1}{\varepsilon}\tilde{p}^0$ satisfy conservation of momentum (5.16) and conservation of mass (5.17) to order $\mathcal{O}(\varepsilon^3)$ and $\mathcal{O}(\varepsilon^2)$ respectively:

$$Sh\frac{\partial}{\partial \tilde{t}}(\tilde{v}_z^0 + \varepsilon \tilde{v}_z^1) + (\tilde{v}_z^0 + \varepsilon \tilde{v}_z^1)\frac{\partial}{\partial \tilde{z}}(\tilde{v}_z^0 + \varepsilon \tilde{v}_z^1) + \tilde{v}_r^0\frac{\partial}{\partial \tilde{r}}(\tilde{v}_z^0 + \varepsilon \tilde{v}_z^1) + \frac{1}{\varepsilon}\frac{\partial \tilde{p}}{\partial \tilde{z}}$$
$$-\frac{1}{Re}\Delta_r(\tilde{v}_z^0 + \varepsilon \tilde{v}_z^1) = \varepsilon\left(\tilde{v}_z^0\frac{\partial \tilde{v}_z^1}{\partial \tilde{z}} + \tilde{v}_z^1\frac{\partial \tilde{v}_z^0}{\partial \tilde{z}} + \tilde{v}_r^1\frac{\partial \tilde{v}_z^1}{\partial \tilde{r}}\right) + \varepsilon^2\tilde{v}_z^1\frac{\partial \tilde{v}_z^1}{\partial \tilde{z}} = \mathcal{O}(\varepsilon^3)$$
$$\text{and} \quad \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}(\tilde{r}\tilde{v}_r^1) + \frac{\partial}{\partial \tilde{z}}(\tilde{v}_z^0 + \varepsilon \tilde{v}_z^1) = \varepsilon\frac{\partial \tilde{v}_z^1}{\partial \tilde{z}} = \mathcal{O}(\varepsilon^2).$$

In the pressure-radius relationship, ignoring the shear modulus term, and recalling that the relation between the pressure and the radial displacement was used up to order $\varepsilon^2$, we see that the functions $(\tilde{v}_z^0 + \varepsilon \tilde{v}_z^1)$ and $\frac{1}{\varepsilon}\tilde{p}^0$ satisfy (5.16)-(5.23) to $\mathcal{O}(\varepsilon^2)$. □

We summarize the main steps in the derivation of the model and the final equations in dimensional form. The summary is provided as a self-contained algorithm ready for the development of a numerical solver.

7.2.3 SUMMARY: THE PROBLEM WITH NONLINEAR COUPLING IN DIMENSIONAL FORM. The following is an $\varepsilon^2$-approximation of the 3-D axially symmetric flow of an incompressible, Newtonian fluid in an elastic tube described in Section 2 as Problem $P^\varepsilon$. The unknown functions are the velocity $(v_z^0 + \varepsilon v_z^1, \varepsilon v_r^1)$ and the radial displacement $\eta^0$. The pressure $p = p_{ref} + p^0$ is then recovered via the pressure-radius relationship (7.34). The radius of the deformed vessel at every time step is given by $r_{\text{vessel}} = R + \eta^0(z,t)$.

**STEP 1.**(THE ZERO-TH ORDER APPROXIMATION)
Look for $v_z^0 = v_z^0(r,z,t)$ and $\eta^0 = \eta^0(z,t)$ and then recover $p^0 = p^0(z,t)$ by solving the following free-boundary problem defined on the domain $0 \le z \le L$, $0 \le r \le R + \eta^0(z,t)$

$$\frac{\partial (R+\eta^0)^2}{\partial t} + \frac{\partial}{\partial z}\int_0^{R+\eta^0} r v_z^0 dr = 0, \tag{7.29}$$

$$\rho\frac{\partial v_z^0}{\partial t} + \frac{\partial}{\partial z}\left(\frac{hE}{R(1-\sigma^2)}\left(\frac{(R+\eta^0)^2}{R^2} - 1\right)\right) = \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z^0}{\partial r}\right), \tag{7.30}$$

$$v_z^0(0,z,t) \text{ bounded, } v_z^0(R+\eta^0(z,t),z,t) = 0 \text{ and } v_z^0(r,z,0) = 0, \tag{7.31}$$

with the following inlet and outlet boundary conditions

$$p = P_1(t) + p_{ref} \quad \text{for} \quad z = 0,\ 0 \le r \le R \quad \text{and} \quad \forall t \in I\!R_+, \tag{7.32}$$

$$p = P_2(t) + p_{ref} \quad \text{for} \quad z = L,\ 0 \le r \le R \quad \text{and} \quad \forall t \in I\!R_+. \tag{7.33}$$

The pressure $p$ is linked to $\eta^0$ by

$$p(z,t) = p_{ref} + \frac{hE}{R(1-\sigma^2)}\left(\frac{(R+\eta^0(z,t))^2}{R^2} - 1\right). \tag{7.34}$$

**STEP 2.**(THE $\varepsilon$-CORRECTION FOR THE VELOCITY)
Solve for $v_z^1 = v_z^1(r,z,t)$ and $v_r^1 = v_r^1(r,z,t)$ by first recovering $v_r^1$ via

$$rv_r^1(r,z,t) = (R+\eta^0)\frac{\partial \eta^0}{\partial t} + \int_r^{R+\eta^0} \frac{\partial v_z^0}{\partial z}(\xi,z,t)\ \xi\ d\xi$$

and then solve the following linear *fixed* boundary problem for $v_z^1$, defined on the domain $0 \le z \le L$, $0 \le r \le R+\eta^0(z,t)$

$$\frac{\partial v_z^1}{\partial t} - \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z^1}{\partial r}\right) = -S_{v_z^1}(r,z,t)$$
$$v_z^1(0,z,t) \text{ bounded, } v_z^1(R+\eta^0(z,t),z,t) = 0$$
$$v_z^1(r,0,t) = v_z^1(r,L,t) = 0 \quad \text{and} \quad v_z^1(r,z,0) = 0,$$

where $S_{v_z^1}(r,z,t)$ contains the already calculated functions and is defined by

$$S_{v_z^1}(r,z,t) = v_r^1\frac{\partial v_z^0}{\partial r} + v_z^0\frac{\partial v_z^0}{\partial z}.$$

Here $\nu = \mu/\rho$ is the kinematic viscosity coefficient. Notice that the boundary condition is evaluated at the deformed boundary whose $\varepsilon^2$-approximation is obtained in the previous step.

In the next section we obtain an $\varepsilon^2$-approximation of Problem $\mathrm{P}^\varepsilon$ in the case when the radial displacement is small and so linear coupling between the flow and the elastic structure is justified. In this case we expand the equations with respect to the radial displacement and obtain a set of linear equations for the velocity and the displacement that approximate the 3-D axially symmetric flow to the order $\varepsilon^2$.

7.2.4 EXPANSION WITH RESPECT TO THE RADIAL DISPLACEMENT: THE LINEAR MODEL . Assume that $\frac{\Xi}{R} \le \varepsilon$. In this section we will show how the expansion with respect to the radial displacement $\frac{\Xi}{R}\eta$ leads to a linear problem of Biot type, [1], with memory. As pointed out by Tartar, see [28], effective equations with memory typically occur in problems with wave-like phenomena. Biot type equations have been used in modeling seismic waves and in general, describing waves in deformable, porous media.

Introduce the following expansions of the dependent variables $(w+\varepsilon\tilde{v}_z^1, \varepsilon\tilde{v}_r^1)$, $\tilde{\eta}^0$ and $\tilde{\tilde{p}}^0$ :

$$\tilde{\eta}^0 = \tilde{\eta}^{0,0} + \frac{\Xi}{R}\tilde{\eta}^{0,1} + \ldots, \qquad \tilde{\tilde{p}}^0 = \tilde{\tilde{p}}^{0,0} + \frac{\Xi}{R}\tilde{\tilde{p}}^{0,1} + \ldots$$
$$w = w^0 + \frac{\Xi}{R}w^1 + \ldots, \quad \tilde{v}_z^1 = \tilde{v}_z^{1,0} + \ldots, \quad \tilde{v}_r^1 = \tilde{v}_r^{1,0} + \ldots.$$

Plug these expansions into (7.14)-(7.15) and obtain the following equations of order zero and one, respectively:

$$Sh_0\frac{\partial w^0}{\partial \tilde{t}} - \frac{1}{Re_0}\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial w^0}{\partial \tilde{r}}\right) = -\frac{\partial \tilde{\tilde{p}}^{0,0}}{\partial \tilde{z}}(\tilde{z},\tilde{t}) = -\left(\frac{Eh}{\mathcal{P}L(1-\sigma^2)}\right)\frac{\partial \tilde{\eta}^{0,0}}{\partial \tilde{z}} \tag{7.35}$$

$$w^0(0,\tilde{z},\tilde{t}) \text{ bounded,} \quad w^0(1,\tilde{z},\tilde{t}) = 0 \quad \text{and} \quad w^0(\tilde{r},\tilde{z},0) = 0, \tag{7.36}$$

and

$$Sh_0 \frac{\partial w^1}{\partial \tilde{t}} - \frac{1}{Re_0}\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial w^1}{\partial \tilde{r}}\right) = -\frac{\partial \tilde{p}^{0,1}}{\partial \tilde{z}}(\tilde{z},\tilde{t}) = -\left(\frac{Eh}{\mathcal{P}L(1-\sigma^2)}\right)\frac{\partial \tilde{\eta}^{0,1}}{\partial \tilde{z}}, \tag{7.37}$$

$$w^1(0,\tilde{z},\tilde{t}) \text{ bounded}, \quad w^1(1,\tilde{z},\tilde{t}) = -\tilde{\eta}^{0,0}\frac{\partial w^0}{\partial \tilde{r}}(1,\tilde{z},\tilde{t}) \text{ and } w^1(\tilde{r},\tilde{z},0) = 0, \tag{7.38}$$

where we have linearized the lateral boundary condition. Note that $\tilde{\eta}^{0,0}\frac{\partial w^0}{\partial \tilde{r}}(1,\tilde{z},\tilde{t}) = 0$ at $\tilde{t} = 0$. Both of these problems can be solved efficiently by using the auxiliary (homogeneous) problem

$$\frac{\partial \tilde{\zeta}}{\partial \tilde{t}} - \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \tilde{\zeta}}{\partial \tilde{r}}\right) = 0 \quad \text{in } (0,1)\times(0,\infty) \tag{7.39}$$

$$\tilde{\zeta}(0,\tilde{t}) \text{ is bounded }, \quad \tilde{\zeta}(1,\tilde{t}) = 0 \text{ and } \tilde{\zeta}(\tilde{r},0) = 1. \tag{7.40}$$

Then, by linear parabolic theory, $\tilde{\zeta}$ decays in time exponentially, with the rate equal to the first zero of the Bessel function $J_0$. $\tilde{\zeta}$ is non-negative by the maximum principle. We set

$$\tilde{\mathcal{K}}(t) = 2\int_0^1 \tilde{\zeta}(\tilde{r},\tilde{t})\ \tilde{r}d\tilde{r}. \tag{7.41}$$

We now solve equations (7.35)-(7.38). The following operators will be useful. Let $f = f(\tilde{z},\tilde{t})$. Define $\left(\tilde{\zeta}\star f\right)(\tilde{r},\tilde{z},\tilde{t})$ and $\left(\tilde{\mathcal{K}}\star f\right)(\tilde{z},\tilde{t})$ to be the following integral operators with the kernels $\tilde{\zeta}$ and $\tilde{\mathcal{K}}$ respectively

$$\left(\tilde{\zeta}\star f\right)(\tilde{r},\tilde{z},\tilde{t}) := \int_0^{\tilde{t}} \tilde{\zeta}(\tilde{r},\frac{\tilde{t}-\tau}{Sh_0 Re_0})f(\tilde{z},\tau)d\tau,$$

$$\left(\tilde{\mathcal{K}}\star f\right)(\tilde{z},\tilde{t}) := \int_0^{\tilde{t}} \tilde{\mathcal{K}}(\frac{\tilde{t}-\tau}{Sh_0 Re_0})f(\tilde{z},\tau)d\tau.$$

Then, the solution of (7.35) in terms of $\frac{\partial \tilde{p}^{0,0}}{\partial \tilde{z}}$ can be written as

$$w^0(\tilde{r},\tilde{z},\tilde{t}) = -\frac{1}{Sh_0}\left(\tilde{\zeta}\star\frac{\partial \tilde{p}^{0,0}}{\partial \tilde{z}}\right)(\tilde{r},\tilde{z},\tilde{t}). \tag{7.42}$$

Using

$$\frac{\partial \tilde{p}^{0,0}}{\partial \tilde{z}} = \left(\frac{Eh}{\mathcal{P}L(1-\sigma^2)}\right)\frac{\partial \tilde{\eta}^{0,0}}{\partial \tilde{z}} \tag{7.43}$$

and (7.42) in (7.10) we get the following equation for the first term of the radial displacement, $\tilde{\eta}^{0,0}$, at the zero-th order, holding in $(0,L)\times(0,\infty)$

$$2Sh_0\frac{\partial \tilde{\eta}^{0,0}}{\partial \tilde{t}}(\tilde{z},\tilde{t}) - \frac{hE}{L\mathcal{P}(1-\sigma^2)}\frac{\partial}{\partial \tilde{z}}\left(\tilde{\mathcal{K}}\star\frac{\partial \tilde{\eta}^{0,0}}{\partial \tilde{z}}\right)(\tilde{z},\tilde{t}) = 0. \tag{7.44}$$

For the explicit formula for the Laplace transform of $\tilde{\eta}^{0,0}$ see Appendix 1.

Next we calculate $\tilde{\eta}^{0,1}$ by first expressing $w^1$. Integrate (7.37) and take the linearized boundary condition in (7.38) into account to obtain

$$w^1(\tilde{r},\tilde{z},\tilde{t}) = -\tilde{\eta}^{0,0}\frac{\partial w^0}{\partial \tilde{r}}(1,\tilde{z},\tilde{t}) - \frac{1}{Sh_0}\left(\tilde{\zeta}\star\frac{\partial \tilde{p}^{0,1}}{\partial \tilde{z}}\right)(\tilde{r},\tilde{z},\tilde{t}) + \left(\tilde{\zeta}\star\frac{\partial}{\partial \tilde{t}}\left[\tilde{\eta}^{0,0}\left.\frac{\partial w^0}{\partial \tilde{r}}\right|_{\tilde{r}=1}\right]\right)(\tilde{r},\tilde{z},\tilde{t}). \tag{7.45}$$

Here, the last expression means

$$\left(\tilde{\zeta}\star\frac{\partial}{\partial \tilde{t}}\left[\tilde{\eta}^{0,0}\left.\frac{\partial w^0}{\partial \tilde{r}}\right|_{\tilde{r}=1}\right]\right)(\tilde{r},\tilde{z},\tilde{t}) = \int_0^{\tilde{t}}\tilde{\zeta}(\tilde{r},\frac{\tilde{t}-\tau}{Sh_0Re_0})\frac{\partial}{\partial\tau}\left(\tilde{\eta}^{0,0}(\tilde{z},\tau)\frac{\partial w^0}{\partial \tilde{r}}(1,\tilde{z},\tau)\right)\,d\tau.$$

From here we get the integral of $w^1$ in terms of the kernel $\tilde{\mathcal{K}}$

$$2\int_0^1 w^1(\tilde{r},\tilde{z},\tilde{t})\;\tilde{r}d\tilde{r} = -\tilde{\eta}^{0,0}\frac{\partial w^0}{\partial \tilde{r}}(1,\tilde{z},\tilde{t}) - \frac{1}{Sh_0}\left(\tilde{\mathcal{K}}\star\frac{\partial \tilde{p}^{0,1}}{\partial \tilde{z}}\right)(\tilde{z},\tilde{t}) + \left(\tilde{\mathcal{K}}\star\frac{\partial}{\partial \tilde{t}}\left[\tilde{\eta}^{0,0}\left.\frac{\partial w^0}{\partial \tilde{r}}\right|_{\tilde{r}=1}\right]\right)(\tilde{z},\tilde{t}). \tag{7.46}$$

Plugging (7.46) into (7.10) we obtain the equation for $\tilde{\eta}^{0,1}$ in $(0,L)\times(0,\infty)$

$$2Sh_0\frac{\partial\tilde{\eta}^{0,1}}{\partial\tilde{t}} - \frac{hE}{L\mathcal{P}(1-\sigma^2)}\frac{\partial}{\partial\tilde{z}}\left(\tilde{\mathcal{K}}\star\frac{\partial\tilde{\eta}^{0,1}}{\partial\tilde{z}}\right)(\tilde{z},\tilde{t}) = -S_{\tilde{\eta}^{0,1}}(\tilde{z},\tilde{t}), \tag{7.47}$$

where

$$S_{\tilde{\eta}^{0,1}}(\tilde{z},\tilde{t}) := 2Sh_0\eta^{0,0}\frac{\partial\eta^{0,0}}{\partial\tilde{t}} - Sh_0\frac{\partial}{\partial\tilde{z}}\left(\tilde{\eta}^{0,0}\frac{\partial w^0}{\partial\tilde{r}}(1,\tilde{z},\tilde{t})\right) + Sh_0\frac{\partial}{\partial\tilde{z}}\left(\tilde{\mathcal{K}}\star\frac{\partial}{\partial \tilde{t}}\left[\tilde{\eta}^{0,0}\left.\frac{\partial w^0}{\partial \tilde{r}}\right|_{\tilde{r}=1}\right]\right)(\tilde{z},\tilde{t}).$$

We perform the same expansions for the correction of order $\varepsilon$. Equation (7.25) implies the following zero-order approximation of $\tilde{v}_r^{1,0}$ with respect to $\frac{\Xi}{R}$:

$$\tilde{r}w^0(\tilde{r},\tilde{z},\tilde{t}) = \frac{\partial\tilde{\eta}^{0,0}}{\partial\tilde{t}} + \int_{\tilde{r}}^1\frac{\partial w^0}{\partial\tilde{z}}(\xi,\tilde{z},\tilde{t})\;\xi\;d\xi. \tag{7.48}$$

Equations (7.26)-(7.28) imply, after taking into account $\varphi = \tilde{\tilde{p}}^1 = 0$, the following zero-order equation for $\tilde{v}_z^{1,0}$:

$$Sh_0\frac{\partial\tilde{v}_z^{1,0}}{\partial\tilde{t}} - \frac{1}{Re_0}\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\tilde{v}_z^{1,0}}{\partial\tilde{r}}\right) = -S_{\tilde{v}_z^{1,0}}(\tilde{r},\tilde{z},\tilde{t})$$

$$\tilde{v}_z^{1,0}(0,\tilde{z},\tilde{t})\text{ bounded, }\tilde{v}_z^{1,0}(1,\tilde{z},\tilde{t}) = 0\quad\text{and}\quad\tilde{v}_z^{1,0}(\tilde{r},\tilde{z},0) = 0$$

where

$$S_{\tilde{v}_z^{1,0}}(\tilde{r},\tilde{z},\tilde{t}) = \tilde{v}_r^{1,0}\frac{\partial w^0}{\partial\tilde{r}} + w^0\frac{\partial w^0}{\partial\tilde{z}}.$$

Therefore, the solution is given by

$$\tilde{v}_z^{1,0}(\tilde{r},\tilde{z},\tilde{t}) = -\frac{1}{Sh_0}\int_0^{\tilde{t}} \tilde{\zeta}(r, \frac{\tilde{t}-\tau}{Sh_0 Re_0}) S(\tilde{r},\tilde{z},\tau)\, d\tau. \tag{7.49}$$

With the calculations presented in this section we have derived a set of linear equations that are an $\varepsilon^2$-approximation of Problem $P^\varepsilon$ when $\Xi/R \leq \varepsilon$. First one recovers $\tilde{\eta}^{0,0}$ by solving (7.44) with the appropriate initial and boundary conditions. Next, the zero-th approximation of the unidirectional velocity $w^0$ is recovered by calculating (7.42). Pressure $\tilde{\tilde{p}}^{0,0}$ follows from (7.43). The 2nd-order correction $\tilde{\eta}^{0,1}$ for the displacement is obtained by solving (7.47) with appropriate initial and boundary conditions. This correction is necessary to recover the next term in the approximation of the unidirectional flow $w^1$ via (7.2.4). Finally, the $\varepsilon$-correction for the unidirectional velocity is obtained by recovering $\tilde{v}_z^{1,0}$ and $\tilde{v}_r^{1,0}$ via (7.42) and (7.48) respectively.

Notice that since the radius of the vessel in non-dimensional variables reads

$$1+\frac{\Xi}{R}\tilde{\eta} = 1+\frac{\Xi}{R}\left(\tilde{\eta}^0 + \varepsilon\tilde{\eta}^1 + ...\right) = 1+\frac{\Xi}{R}\tilde{\eta} = 1+\frac{\Xi}{R}\left(\left(\tilde{\eta}^{0,0} + \frac{\Xi}{R}\tilde{\eta}^{0,1} + ...\right) + \varepsilon\tilde{\eta}^1 + ...\right),$$

assumption $\frac{\Xi}{R} \leq \varepsilon$ implies that the $\varepsilon^2$ approximation of the solution is achieved already with the $\tilde{\eta}^{0,0}$ term. However, we need to calculate $\tilde{\eta}^{0,1}$ in order to recover the $\varepsilon^2$ approximation of the velocity. A similar argument holds for the calculation of the scaled pressure $\tilde{\tilde{p}}$.

We summarize the main steps written in dimensional form, in the following section.

### 7.2.5 SUMMARY: THE PROBLEM WITH LINEAR COUPLING IN DIMENSIONAL FORM.

Assuming that $\Xi/R \leq \varepsilon$ we are looking for an $\varepsilon^2$-approximation of the solution consisting of: (1) the velocity field $((v_z^{0,0} + \Xi/R\, v_z^{0,1}) + \varepsilon v_z^{1,0}, \varepsilon v_r^{1,0})$ where $(v_z^{0,0} + \Xi/R\, v_z^{0,1}, 0)$ is the unidirectional velocity and $(\varepsilon v_z^{1,0}, \varepsilon v_r^{1,0})$ is its $\varepsilon$-correction, (2) the radial displacement $\eta^{0,0}$ and (3) the leading-order pressure $p = p_{\text{ref}} + p^{0,0}$.

**STEP 1.** (THE ZEROTH ORDER APPROXIMATION) Find $v_z^{0,0}(r,z,t)$, $\eta^{0,0}(z,t)$ and $p^{0,0}(z,t)$ such that

$$\frac{\partial(\eta^{0,0})}{\partial t} + \frac{1}{R}\frac{\partial}{\partial z}\int_0^R r v_z^{0,0} dr = 0$$

$$\rho\frac{\partial v_z^{0,0}}{\partial t} - \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z^{0,0}}{\partial r}\right) = -\frac{\partial p^{0,0}}{\partial z}(z,t), \quad \frac{\partial p^{0,0}}{\partial z}(z,t) = \left(\frac{Eh}{R^2(1-\sigma^2)}\right)\frac{\partial \eta^{0,0}}{\partial z}$$

$$\begin{aligned}
&\text{with} \quad v_z^{0,0}(0,z,t) \text{ bounded}, \quad v_z^{0,0}(R,z,t) = 0,\\
&p^{0,0}(z,0) = p_{\text{ref}}, \quad \eta^{0,0}(z,0) = v_z^{0,0}(r,z,0) = 0,\\
&\eta^{0,0}(0,t) = \frac{R^2(1-\sigma^2)}{Eh}P_1(t), \quad \eta^{0,0}(L,t) = \frac{R^2(1-\sigma^2)}{Eh}P_2(t).
\end{aligned}$$

Then recover the $\Xi/R$-correction $v_z^{0,1}(r,z,t)$, $\eta^{0,1}(z,t)$ and $p^{0,1}(z,t)$ by solving

$$\frac{\partial(\eta^{0,1})}{\partial t} + \frac{1}{R}\frac{\partial}{\partial z}\int_0^R r v_z^{0,1} dr = 0$$

$$\rho\frac{\partial v_z^{0,1}}{\partial t} - \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z^{0,1}}{\partial r}\right) = -\frac{\partial p^{0,1}}{\partial z}(z,t), \quad \frac{\partial p^{0,1}}{\partial z}(z,t) = \left(\frac{Eh}{R^2(1-\sigma^2)}\right)\frac{\partial \eta^{0,1}}{\partial z}$$

with

$$v_z^{0,1}(0,z,t) \text{ bounded}, \quad v_z^{0,1}(R,z,t) = -\eta^{0,1}\frac{\partial v_z^{0,0}}{\partial r}(R,z,t),$$
$$p^{0,1}(z,0) = 0, \quad \eta^{0,1}(z,0) = v_z^{0,1}(r,z,0) = 0$$
$$\eta^{0,1}(0,t) = \eta^{0,1}(L,t) = 0.$$

We can solve these problems efficiently by considering the auxiliary problem

$$\frac{\partial \zeta}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \zeta}{\partial r}\right) = 0 \quad \text{in } (0,R)\times(0,\infty) \tag{7.50}$$

$$\zeta(0,t) \text{ is bounded}, \quad \zeta(R,t) = 0 \text{ and } \zeta(r,0) = 1, \tag{7.51}$$

and the mean of $\zeta$ in the radial direction

$$\mathcal{K}(t) = 2\int_0^R \zeta(r,t)\, r dr, \tag{7.52}$$

which can both be evaluated in terms of the Bessel's functions. One can show that $\tilde{\zeta} = R^2\omega^\varepsilon\zeta$ and $\tilde{\mathcal{K}} = \omega^\varepsilon\mathcal{K}$. The unidirectional solution can then be written in terms of the following operators

$$(\zeta \star f)(r,z,t) := \int_0^t \zeta(r, \frac{\mu(t-\tau)}{\rho}) f(z,\tau) d\tau,$$
$$(\mathcal{K} \star f)(z,t) := \int_0^t \mathcal{K}(\frac{\mu(t-\tau)}{\rho}) f(z,\tau) d\tau.$$

We write the explicit solution strategy below. We need to solve linear initial-value problems of Biot type with memory. Denote $C = Eh/(R^2(1-\sigma^2))$.

**STEP 1. REVISITED.** (EXPLICIT SOLUTION METHOD)

- **STEP 1a.** Find $\eta^{0,0}, p^{0,0}, v_z^{0,0}$ by solving the following initial-boundary value problem with memory:

$$2\frac{\rho}{R}\frac{\partial \eta^{0,0}}{\partial t}(z,t) = C\,\frac{\partial^2(\mathcal{K}\star\eta^{0,0})}{\partial z^2}(z,t) \text{ on } (0,L)\times(0,+\infty)$$
$$\eta^{0,0}(0,t) = P_1(t)/C, \quad \eta^{0,0}(L,t) = P_2(t)/C \quad \text{and} \quad \tilde{\eta}^{0,0}(z,0) = 0.$$

Recover $\frac{\partial p^{0,0}}{\partial z}(z,t) = C\frac{\partial \eta^{0,0}}{\partial z}(z,t).$

Calculate $v_z^{0,0}(r,z,t) = -\frac{1}{\rho}\left(\zeta \star \frac{\partial p^{0,0}}{\partial z}\right)(r,z,t).$

- **STEP 1b.** Find $\eta^{0,1}, p^{0,1}, v_z^{0,1}$ by solving the following initial-boundary value problem with memory:

$$2\frac{\rho}{R}\frac{\partial \eta^{0,1}}{\partial t}(z,t) = C\,\frac{\partial^2(\mathcal{K}\star\eta^{0,1})}{\partial z^2}(z,t) - S_{\eta^{0,1}}(z,t), \text{ on } (0,L)\times(0,+\infty)$$
$$\eta^{0,1}(0,t) = \eta^{0,1}(L,t) = 0 \text{ and } \eta^{0,1}(z,0) = 0,$$

where

$$S_{\eta^{0,1}}(z,t) := 2\frac{\rho}{\mathcal{P}}C\eta^{0,0}\frac{\partial \eta^{0,0}}{\partial t} - \rho\frac{\partial}{\partial z}(\eta^{0,0}\frac{\partial v_z^{0,0}}{\partial r}|_{r=R}) + \rho\frac{\partial}{\partial z}\left(\mathcal{K}\star\frac{\partial}{\partial t}\left(\eta^{0,0}\frac{\partial v_z^{0,0}}{\partial r}|_{r=R}\right)\right).$$

Recover $\quad \dfrac{\partial p^{0,1}}{\partial z}(z,t) = C\dfrac{\partial \eta^{0,1}}{\partial z}(z,t).$

Calculate

$$v_z^{0,1}(r,z,t) = -\eta^{0,0}\frac{\partial v_z^{0,0}}{\partial r}(R,z,t) - \frac{R^2}{\rho}\left(\zeta\star\frac{\partial p^{0,1}}{\partial z}\right)(r,z,t)$$
$$+R^2\left(\zeta\star\frac{\partial}{\partial t}\left[\eta^{0,0}\left.\frac{\partial v_z^{0,0}}{\partial r}\right|_{r=R}\right]\right)(r,z,t).$$

This way we have recovered the unidirectional velocity $(v_z^{0,0} + \frac{\Xi}{R}v_z^{0,1}, 0)$, the $\varepsilon^2$-approximation of the radial displacement $\eta^{0,0}$ and the $\varepsilon^2$-approximation of the pressure

$$p(z,t) = p_{ref} + \frac{hE}{R(1-\sigma^2)}\left(\frac{(R+\eta^{0,0}(z,t))^2}{R^2} - 1\right). \tag{7.53}$$

**STEP 2.**(THE $\varepsilon$-CORRECTION FOR THE VELOCITY)
Solve for $v_z^{1,0} = v_z^{1,0}(r,z,t)$ and $v_r^{1,0} = v_r^{1,0}(r,z,t)$ by first recovering $v_r^{1,0}$ via

$$rv_r^{1,0}(r,z,t) = R\frac{\partial \eta^{0,0}}{\partial t} + \int_r^R \frac{\partial v_z^{0,0}}{\partial z}(\xi,z,t)\;\xi\; d\xi$$

and then solve the following linear problem for $v_z^{1,0}$

$$\frac{\partial v_z^{1,0}}{\partial t} - \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z^{1,0}}{\partial r}\right) = -S_{v_z^{1,0}}(r,z,t) \quad \text{on} \quad (0,R)\times(0,L)\times(0,\infty)$$
$$v_z^{1,0}(0,z,t) \text{ bounded, } v_z^{1,0}(R,z,t) = 0$$
$$v_z^{1,0}(r,0,t) = v_z^{1,0}(r,L,t) = 0 \quad \text{and} \quad v_z^{1,0}(r,z,0) = 0,$$

where $S_{v_z^{1,0}}(r,z,t)$ contains the already calculated functions and is defined by

$$S_{v_z^{1,0}}(r,z,t) = v_r^{1,0}\frac{\partial v_z^{0,0}}{\partial r} + v_z^{0,0}\frac{\partial v_z^{0,0}}{\partial z}.$$

The solution is given by $v_z^{1,0}(r,z,t) = -\dfrac{1}{\rho}\left(\zeta\star S_{v_z^{1,0}}\right)(r,z,t).$

We have completed an algorithm for the calculation of an $\varepsilon^2$-approximation of the solution to Problem $\mathrm{P}^\varepsilon$ in the case when $\Xi/R \le \varepsilon$. The solution consists of the velocity $(v_z^{0,0} + \frac{\Xi}{R}v_z^{0,1} + \varepsilon v_z^{1,0}, \varepsilon v_r^{1,0})$, the radial displacement $\eta^{0,0}$ and the pressure $p$ determined from $\eta^{0,0}$ via (7.53).

## 8 Final Remarks

We conclude by a couple of remarks related to the validity of the models. Our approximations are expected to work not only at moderate but also at high but laminar Reynolds numbers. In the turbulent flow regime, however, our approach in not likely to work and it should be modified. Also, taking the limit as $Re \to \infty$ in our analysis, even if formally possible, would very likely lead to wrong models. The reader interested in such flow regimes can look at [2] where the "Euler" variant of system (5.16)-(5.19) was studied, corresponding to $1/Re = 0$, with no-slip boundary conditions and with a convex velocity profile.

## 9 Appendix 1. (Explicit Laplace transform solution)

In this Appendix we calculate the Laplace transform of the zero-th order approximation of the displacement when linear coupling is considered. We apply the Laplace transform to the auxiliary problem (7.39), (7.40). The Laplace transform $\hat{\tilde{\zeta}}$ of $\tilde{\zeta}$ satisfies

$$p\hat{\tilde{\zeta}}(p,\tilde{r}) - \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \hat{\tilde{\zeta}}}{\partial \tilde{r}}(p,\tilde{r})\right) = 1 \quad \text{on} \quad (0,1) \tag{9.1}$$

$$\hat{\tilde{\zeta}}(p,0) \quad \text{is bounded,} \quad \hat{\tilde{\zeta}}(p,1) = 0. \tag{9.2}$$

This problem has a unique solution for all $p > 0$ and the solution is given by

$$\hat{\tilde{\zeta}}(p,\tilde{r}) = \frac{1}{p}\left\{1 - \frac{J_0(i\sqrt{p}\tilde{r})}{J_0(i\sqrt{p})}\right\} = \frac{1}{p}\left\{1 - \frac{I_0(\sqrt{p}\tilde{r})}{I_0(\sqrt{p})}\right\} \tag{9.3}$$

where $J_0$ is the Bessel function of order zero and $I_0$ is the modified Bessel function of order zero. The Laplace transform $\hat{\tilde{\mathcal{K}}}$ of the convolution kernel $\tilde{\mathcal{K}}$ is

$$\begin{aligned}\hat{\tilde{\mathcal{K}}}(p) = 2\int_0^1 \hat{\tilde{\zeta}}(0,\tilde{r})\tilde{r}\; d\tilde{r} &= \frac{2}{p}\left\{\frac{1}{2} - \frac{1}{I_0(\sqrt{p})}\int_0^1 \tilde{r} I_0(\sqrt{p}\tilde{r})d\tilde{r}\right\} \\ &= \frac{2}{p}\left\{\frac{1}{2} - \frac{1}{\sqrt{p}}\frac{I_1(\sqrt{p})}{I_0(\sqrt{p}}\right\},\end{aligned}$$

where $I_1(x) = -iJ_1(ix)$ is the modified Bessel function of order 1.

We use this to explicitly calculate the Laplace transform of the solutions to the evolution problems (7.44) and (7.47) for $\tilde{\eta}^{0,0}$ and $\tilde{\eta}^{0,1}$. The homogeneous problems for both $\tilde{\eta}^{0,0}$ and $\tilde{\eta}^{0,1}$ have the form

$$Sh_0\frac{\partial \tilde{\eta}}{\partial \tilde{t}} - \beta_0\frac{\partial^2}{\partial \tilde{z}^2}\left\{\tilde{\mathcal{K}}\left(\frac{\cdot}{Re_0 Sh_0}\right) \star \tilde{\eta}\right\} = 0, \quad \text{in} \quad (0,1)\times I\!R^+, \tag{9.4}$$

$$\tilde{\eta}(0,\tilde{z}) = 0, \quad \tilde{\eta}(\tilde{t},0) = \tilde{\eta}_0(\tilde{t}) \quad \text{and} \quad \tilde{\eta}(\tilde{t},L) = \tilde{\eta}_L(\tilde{t}), \tag{9.5}$$

where $\beta_0 > 0$ is a given constant. We apply the Laplace transform to (9.4), (9.5) and obtain

$$Sh_0 p\; \hat{\tilde{\eta}}(p,\tilde{z}) - \beta_0 Re_0 Sh_0\; \hat{\tilde{\mathcal{K}}}(Re_0 Sh_0 p)\frac{\partial^2}{\partial \tilde{z}^2}\hat{\tilde{\eta}}(p,\tilde{z}) = 0,$$

$$\hat{\tilde{\eta}}(p,0) = \hat{\tilde{\eta}}_0(p) \quad \text{and} \quad \hat{\tilde{\eta}}(p,L) = \hat{\tilde{\eta}}_L(p).$$

Let

$$\beta(p) := Sh_0 \frac{p^2}{\beta_0} \left\{ 1 - \frac{2}{\sqrt{Re_0 Sh_0 p}} \frac{I_1(\sqrt{Re_0 Sh_0 p})}{I_0(\sqrt{Re_0 Sh_0 p})} \right\}^{-1}.$$

Then the solution of (9.4), (9.5) is given by

$$\hat{\tilde{\eta}}(p,\tilde{z}) = \frac{\hat{\tilde{\eta}}_L(p) - \hat{\tilde{\eta}}_0(p)\cosh(\sqrt{\beta(p)}L)}{\sinh(\sqrt{\beta(p)}L)} \sinh(\sqrt{\beta(p)}\tilde{z}) + \hat{\tilde{\eta}}_0(p)\cosh(\sqrt{\beta(p)}\tilde{z}). \tag{9.6}$$

## 10 APPENDIX 2. (EXISTENCE, REGULARITY AND UNIQUENESS FOR THE BIOT SYSTEM)

We prove here that the system studied in Section 7.2.4, has a unique solution.

Consider

$$\frac{\partial \eta}{\partial t} + \gamma_1 \frac{\partial}{\partial z} \int_0^1 r v_z \, dr = -\frac{\partial \eta_L}{\partial t} \quad \text{in} \quad (0,L)\times(0,T), \tag{10.1}$$

$$\frac{\partial v_z}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left( r \frac{\partial v_z}{\partial r}\right) = -\gamma_2 \frac{\partial \eta}{\partial z} - \gamma_2 \frac{\partial \eta_L}{\partial z} \quad \text{in} \quad (0,1)\times(0,L)\times(0,T), \tag{10.2}$$

$$v_z(1,z,t) = 0, \quad v_z(0,z,t) \ \text{ bounded}, \quad \eta(0,t) = \eta(L,t) = 0 \tag{10.3}$$

$$\eta(z,0) = v_z(r,z,0) = 0 \quad \text{on} \quad (0,1)\times(0,L), \tag{10.4}$$

where

$$\eta_L(z,t) = \frac{\eta_L(t) - \eta_0(t)}{L} z + \eta_0(t),$$

and $\eta_0, \eta_L \in C_0^\infty(0,\infty)$, $\gamma_1$ and $\gamma_2$ are positive constants. System (10.1)-(10.4) implies the following energy equalities

$$\frac{1}{2}\frac{d}{dt}\left\{ \int_0^L |\eta(z,t)|^2 \, dz + \frac{\gamma_1}{\gamma_2}\int_0^L \int_0^1 |v_z(r,z,t)|^2 \, r \, dr \, dz \right\}$$
$$+ \frac{\gamma_1}{\gamma_2}\int_0^L \int_0^1 |\frac{\partial v_z}{\partial r}|^2 r \, dr \, dz = \int_0^L \frac{\partial \eta_L}{\partial t}\eta dz - \gamma_1 \int_0^L \int_0^1 r v_z \frac{\partial \eta_L}{\partial z} dr \, dz \tag{10.5}$$

$$\frac{1}{2}\frac{d}{dt}\left\{ \int_0^L |\frac{\partial \eta}{\partial t}|^2 dz + \frac{\gamma_1}{\gamma_2}\int_0^L \int_0^1 |\frac{\partial v_z}{\partial t}|^2 \, r \, dr \, dz \right\}$$
$$+ \frac{\gamma_1}{\gamma_2}\int_0^L \int_0^1 |\frac{\partial^2 v_z}{\partial t \partial r}|^2 \, r \, dr \, dz = \int_0^L \frac{\partial^2 \eta_L}{\partial t^2}\frac{\partial \eta}{\partial t} \, dz - \gamma_1 \int_0^L \int_0^1 r \frac{\partial v_z}{\partial t}\frac{\partial \eta_L}{\partial z} \, dr \, dz. \tag{10.6}$$

Since (10.6) guarantees that $\frac{\partial v_z}{\partial t} \in L^2$, we write (10.2) in the form

$$-\Delta_r v_z := -\frac{1}{r}\frac{\partial}{\partial r}\left( r\frac{\partial v_z}{\partial r}\right) = -\gamma_2 \frac{\partial \eta}{\partial z}(z,t) - \gamma_2 \frac{\partial \eta_L}{\partial z} - \frac{\partial v_z}{\partial t}$$

and, consequently

$$\int_0^1 r v_z dr = -\frac{\gamma_2}{16}\frac{\partial \eta}{\partial z}(z,t) - \frac{\gamma_2}{16}\frac{\partial \eta_L}{\partial z} - \int_0^1 r(-\Delta_r)^{-1}\frac{\partial v_z}{\partial t} \, dr.$$

Using this expression in equation (10.1) we get

$$\frac{\partial \eta}{\partial t} - \frac{\gamma_1\gamma_2}{16}\frac{\partial^2 \eta}{\partial z^2} = -\frac{\partial \eta_L}{\partial t} + \frac{\gamma_1\gamma_2}{16}\frac{\partial^2 \eta_L}{\partial z^2} + \gamma_1 \frac{\partial}{\partial z}\int_0^1 r(-\Delta_r)^{-1}\frac{\partial v_z}{\partial t} dr.$$

We multiply this equation by $\eta$ and integrate. After taking into account that $\frac{\partial^2 \eta_L}{\partial z^2} = 0$ we obtain a standard energy estimate for the heat equation

$$\frac{1}{2}\int_0^L |\eta(z,t)|^2 dz + \frac{\gamma_1\gamma_2}{16}\int_0^t \int_0^L |\frac{\partial \eta}{\partial z}|^2 dz\ d\tau = -\int_0^t \int_0^L \frac{\partial \eta_L}{\partial t}\ \eta\ dz\ d\tau$$
$$-\gamma_1 \int_0^t \int_0^L \left(\int_0^1 r(-\Delta_r)^{-1}\frac{\partial v_z}{\partial t} dr\right)\frac{\partial \eta}{\partial z}\ dz\ d\tau. \tag{10.7}$$

The apriori estimates (10.5), (10.6) and (10.7) imply existence of a unique solution $\{\eta, v_z\} \in H^1((0,L)\times(0,T)) \times H^1(0,t;L^2((0,1)\times(0,L)))$, $\sqrt{r}\frac{\partial v_z}{\partial r} \in H^1(0,T;L^2((0,1)\times(0,L))$ for the Biot system (10.1)-(10.4). This regularity guarantees uniqueness of a solution to system (7.2)-(7.6). We have proved the following

Theorem 10.1. *The Biot system* (10.1)-(10.4) *has a unique solution* $\{\eta, v_z\} \in H^1((0,L)\times(0,T)) \times H^1(0,t;L^2((0,1)\times(0,L)))$ *with* $\sqrt{r}\frac{\partial v_z}{\partial r} \in H^1(0,T;L^2((0,1)\times(0,L))$.

Corollary 10.2. *A solution of system* (7.2)-(7.6) *is unique.*

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