ON THE MODULI SPACE OF SINGULAR EUCLIDEAN SURFACES

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Abstract. The goal of this paper is to develop some aspects of the deformation theory of piecewise flat structures on surfaces and use this theory to construct new geometric structures on the moduli space of Riemann surfaces.

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Introduction

The Teichmüller space of a punctured surface is the space of hyperbolic metrics with cusps up to isotopy on that surface, however, it can also be seen as the space of flat metrics with conical singularities of prescribed angles at the punctures up to isotopy and rescaling. The aim of the present paper is to use this fact and show how the theory of piecewise flat surfaces and their deformations leads to new geometric structures in Teichmüller theory.

In the first section, which is rather elementary, we describe the geometry of piecewise flat surfaces. The second section describes the topology of punctured surfaces and their diffeomorphism groups. In the third section we discuss the representation space of a finitely generated group π into the group SE(2) of rigid motions in the euclidean plane. In the last two sections, we apply the previous results to construct a new geometric structure on the Riemann moduli space $\mathcal{M}_{g,n}$ of a surface Σ of genus g with n punctures. More specifically, we show that this moduli space is a good orbifold[1] which admits a family of geometric structures locally modeled on the homogeneous spaces $\Xi = \mathbb{T}^{2g} \times \mathbb{CP}^{2g+n-3}$.

We now discuss our main result. We first define a *punctured surface* $\Sigma_{g,n}$ of type (g,n) to be a fixed connected closed orientable surface S of genus g together with a distinguished set of n pairwise distinct points $p_1, p_2, ..., p_n \in \Sigma_{g,n}$. The *Teichmüller space* $\mathcal{T}_{g,n}$ of $\Sigma_{g,n}$ is the set of conformal structures on $\Sigma_{g,n}$ modulo isotopies fixing the punctures (see section 4 for a precise definition). This space is a real analytic variety in a natural way; if $2g-2+n>0$, then it is isomorphic to $\mathbb{R}^{6g-6+2n}$. The group of orientation-preserving isotopy classes of diffeomorphisms of $\Sigma_{g,n}$ fixing the punctures is called the *pure mapping class group* and denoted by $\mathrm{PMod}_{g,n}$. It acts in a natural way on the Teichmüller space $\mathcal{T}_{g,n}$.

We are now in a position to state the main result:

Theorem *Given a punctured surface $\Sigma_{g,n}$ such that $2g+n-2>0$, we can construct a group homomorphism*

$$\Phi : \mathrm{PMod}_{g,n} \to \mathcal{G} = \mathrm{Aut}(\mathbb{T}^{2g}) \times \mathrm{PGL}_{2g+n-2}(\mathbb{C})$$

and a Φ-equivariant local homeomorphism

$$\mathcal{H} : \mathcal{T}_{g,n} \to \Xi = \mathbb{T}^{2g} \times \mathbb{CP}^{2g+n-3}.$$

To say that $\mathcal{H}$ is Φ-equivariant means that $\mathcal{H}(A\mu) = \Phi(A)\cdot\mathcal{H}(\mu)$ for all $A \in \mathrm{PMod}_{g,n}$ and $\mu \in \mathcal{T}_{g,n}$.

The pair $(\mathcal{H},\Phi)$ depends on n parameters $\beta_1, \beta_2, \dots, \beta_n \in (-1,\infty)$ such that $\sum_{j=1}^{n} \beta_j = 2g-2$ and no $\beta_i \in \mathbb{Z}$.

The *moduli space* $\mathcal{M}_{g,n}$ of $\Sigma_{g,n}$ is the set of conformal structures on $\Sigma_{g,n}$ modulo diffeomorphisms fixing the punctures. It is the quotient of the Teichmüller space by the pure mapping class group of $\Sigma_{g,n}$; in other words $\mathcal{M}_{g,n}$ is a good orbifold whose universal cover is $\mathcal{T}_{g,n}$ and fundamental group is $\mathrm{PMod}_{g,n}$. In the geometric language of (G,X)-strucures on manifolds and orbifolds (see [7, 9, 14, 30]), this

[1] Recall that an *orbifold* is a space which is locally the quotient of a manifold by a finite group. A *good orbifold* is globally the quotient of a manifold by a group acting properly and discontinuously (but in general not freely).

theorem says that *we have constructed a family of geometric structures on the orbifold $\mathcal{M}_{g,n}$ which is modeled on the homogeneous space $\Xi = \mathbb{T}^{2g} \times \mathbb{CP}^{2g+n-3}$. This family is parametrized by the β_i's.*

The composition of the map $\mathcal{H}$ in the Theorem with the projection on the torus $\mathbb{T}^{2g}$ gives us a map $\rho : \mathcal{T}_{g,n} \to \mathbb{T}^{2g}$ called the *character map*. It was proved by W.A. Veech that this map is a real analytic submersion. Its fibers describe a foliation whose leaves carry a geometric structure locally modelled on the complex projective space $\mathbb{CP}^{2g+n-3}$, see [36] for proofs of these facts and a discussion of other related geometric structures on $\mathcal{T}_{g,n}$.

The proof of this theorem is based on the following strategy: we first show that the Teichmüller space can be seen as a deformation space of *flat metrics* on $\Sigma_{g,n}$ having *conical singularities* of prescribed angles at the punctures. We associate to such a metric a homomorphism, called the *holonomy* of the metric, from the fundamental group of the surface to the group SE(2) of direct isometries of the euclidean plane. We then show that such a homomorphism can be seen as a point in the variety Ξ. In brief, $\mathcal{H} : \mathcal{T}_{g,n} \to \Xi$ maps the isotopy class of a singular flat metric to its holonomy representation.

In the special case of the punctured sphere, a stronger form of this theorem has been obtained by Deligne and Mostow [10] using some techniques from algebraic geometry and by Thurston [31] using an approach closer to ours.

To conclude this introduction, let us stress that the importance of piecewise flat metrics in Teichmüller theory is illustrated by the large number of papers dedicated to this subject. In addition to the work of Veech and Thurston already quoted, let us mention the contributions of Rivin [27], Bowditch [3], Epstein and Penner [11] to name a few. Piecewise euclidean metrics also appear in quantum gravity and in topological quantum field theory, see [2, 8] and the references therein. Although the present paper starts with elementary considerations, the reader ought not to consider it as a global survey of this vast subject.

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1. Piecewise flat surfaces

1.1. **Euclidean triangulation on a surface.** A *piecewise flat surface* is a metric space obtained by properly gluing a stock of euclidean triangles in such a way that whenever two triangles meet along an edge, they are glued by an isometry along that edge. More precisely:

Definition 1.1. *A* euclidean triangulation *of a surface Σ is a set of pairs $\mathcal{T} = \{(T_\alpha, f_\alpha)\}_{\alpha \in A}$ where each T_α is a compact subset of Σ and $f_\alpha : T_\alpha \to \mathbb{R}^2$ is a homeomorphism onto a non degenerate triangle $f_\alpha(T_\alpha)$ in the euclidean plane $\mathbb{R}^2$. A subset e of T_α is an* edge *if $f_\alpha(e)$ is an edge of $f_\alpha(T_\alpha)$ and a point p of T_α is a* vertex *if its image under f_α is a vertex of $f_\alpha(T_\alpha)$.*

The eucliden triangulation $\mathcal{T}$ is subject to the following conditions:

i) The triangles cover the surface: $\Sigma = \bigcup_\alpha T_\alpha$.
ii) If $\alpha \neq \beta$, then the intersection $T_\alpha \cap T_\beta$ is either empty, or an edge or a vertex.

iii) If $T_\alpha \cap T_\beta \neq \emptyset$, then there is an element $g_{\alpha\beta} \in E(2)$ (= the group of isometries of the euclidean plane) such that $f_\alpha = g_{\alpha\beta} f_\beta$ on that intersection.

An element $(T_\alpha, f_\alpha) \in \mathcal{T}$ is called a *triangle* or a *2-simplex* of the triangulations, we often just denote it by T_α. The vertices and edges are called 0- and 1-simplices respectively.

Two euclidean triangulations $\mathcal{T} = \{(T_\alpha, f_\alpha)\}_{\alpha \in A}$ and $\mathcal{T}' = \{(T_\alpha, f'_\alpha)\}_{\alpha \in A}$ of the same surface Σ are considered to be equal if they have the same simplices and, for any $\alpha \in A$, there is an isometry $g_\alpha \in E(2)$ such that $f'_\alpha = g_\alpha f_\alpha$.

Definition 1.2. *A* piecewise flat surface $(\Sigma, \mathcal{T})$ *is a surface together with a euclidean triangulation.*

A piecewise flat surface $(\Sigma, \mathcal{T})$ comes with a number of additional structures. In particular there is a well defined *area measure* which coincides with the 2-dimensional Lesbegue measure on each euclidean triangle T. We can also define the length $\ell(c)$ of an arbitrary curve $c : [0,1] \to \Sigma$ by the following axioms:

(i) if c is contained in a triangle T of $\mathcal{T}$, then $\ell(c)$ is the euclidean length.
(ii) ℓ is additive: if c is the concatenation of two curves $c_1 c_2$, then $\ell(c) = \ell(c_1) + \ell(c_2)$.

The piecewise flat surface is thus a length space (see [6] for this notion). If the surface is connected, then it is also a metric space for the distance given by the infimum of the lengths of all curves joining two given points.

There is one more structure, called the *singularity order* and which is defined as the angular excess at the vertices counted in number of turns. It tells us how singular each vertex is compared to an ordinary point; the precise definition is the following:

Definition 1.3. *The vertex $p \in \Sigma$ is said to be a* conical point of total angle θ *if*

$$\theta = \sum_{j=1}^{k} \varphi_j$$

where $\varphi_1, \dots, \varphi_k$ are the angles of all the triangles in $\mathcal{T}$ which are incident to p. The singularity order $\beta(p)$ *of a vertex p is the angular excess at p measured in number of turns:*

$$\beta(p) = \frac{\theta}{2\pi} - 1.$$

We extend the function β to all points of Σ by setting $\beta(x) = 0$ if $x \in \Sigma$ is not a vertex. The point x is then termed singular *if $\beta(x) \neq 0$ (i.e. if $\theta \neq 2\pi$) and* regular *otherwise.*

Proposition 1.4 (Gauss-Bonnet Formula)**.** *For any euclidean triangulation on a compact surface without boundary Σ, we have*

$$\chi(\Sigma) + \sum_{x \in \Sigma} \beta(x) = 0\,, \tag{1.1}$$

where $\chi(\Sigma)$ is the Euler characteristic of the surface.

The proof is a direct counting argument based on the definition of the Euler characteristic and the fact that the three internal angles of a euclidean triangle add up to π, see [32].

1.2. **The universal branched cover of a piecewise flat surface.** If $(\Sigma, \mathcal{T})$ is a piecewise flat surface, we denote by $\Sigma' = \Sigma \setminus \{p_1, ..., p_n\}$ the open surface obtained by removing the singular vertices $p_1, ..., p_n \in \Sigma$.

Definition 1.5. *A path* $c : [0,1] \to \Sigma$ *is* admissible *if it has finitely many intersections with the edges of the triangulation and if* $c(s) \in \Sigma'$ *for any* $0 < s < 1$. *A homotopy* c_t *is an* admissible homotopy *if* $s \mapsto c_t(s) \in \Sigma$ *is an admissible path for any* $0 \leq t \leq 1$.

Let us choose a fixed triangle $T_0 \in \mathcal{T}$ and call it *home* (or the *base triangle*). We also choose a base point x_0 in the interior of T_0.

Definition 1.6. *The* universal branched cover *of* $(\Sigma, \mathcal{T})$ *is the euclidean two-dimensional complex* $\widehat{\mathcal{T}}$ *obtained as follows: a* k*-simplex* $\widehat{\sigma}$ *of* $\widehat{\mathcal{T}}$, *where* $k = 0, 1$ *or* 2, *is a pair* $(\sigma, [c])$ *where* σ *is a* k*-simplex of* $\mathcal{T}$ *and* $[c]$ *is an admissible homotopy class of paths joining* T_0 *to a point in* σ.

The universal branched cover $\widehat{\mathcal{T}}$ is a simplicial complex (which is not locally finite) and there is an obvious simplicial map $\widehat{\mathcal{T}} \to \mathcal{T}$ sending $(\sigma, [c])$ to σ.

We denote by $\widehat{\Sigma}$ the geometric realization of $\widehat{\mathcal{T}}$. This is a triangulated topological space and it comes with a continuous surjective map $P : \widehat{\Sigma} \to \Sigma$ sending each simplex of $\widehat{\mathcal{T}}$ homeomorphically onto the corresponding simplex in $\mathcal{T}$. We turn $\widehat{\Sigma}$ into a metric space (in fact a length space) by requiring P to be an isometry on each simplex (concretely, we give to each simplex $\widehat{\sigma} = (\sigma, [c])$ in $\widehat{\mathcal{T}}$ the geometry of the euclidean simplex σ in $\mathcal{T}$).

Another way to understand $\widehat{\Sigma}$ is the following: let $\widetilde{\Sigma}'$ be the universal cover of Σ'. It is naturally a length space (in fact a flat Riemannian surface) and $\widehat{\Sigma}$ is its metric completion.

1.3. **The development of a piecewise flat surface.**

Definition 1.7. *An edge of the piecewise flat surface* $(\Sigma, \mathcal{T})$ *is said to be* interior *if it is not contained in the boundary of* Σ. *The* hinge *of an interior edge* e *is the unique pair of triangles* $T_1, T_2 \in \mathcal{T}$ *which are incident with* e.

Given an interior edge e with hinge (T_1, T_2) and an isometry $f_1 : T_1 \to \mathbb{R}^2$, there exists a unique isometry $f_2 : T_2 \to \mathbb{R}^2$ such that $f_1(T_1)$ and $f_2(T_2)$ have disjoint interiors and $f_1(e) = f_2(e)$. By juxtaposing these maps, we obtain a map

$$f_e = f_1 \cup f_2 : T_1 \cup T_2 \to \mathbb{R}^2$$

which is an isometry of the hinge onto a quadrilateral in the euclidean plane. The map f_e just described is called an *unfolding* of the hinge. One also says that f_2 is the *continuation* of f_1 across the edge e.

The notions of hinge, unfolding and continuation of an isometry across an edge are similarly defined on the universal branch cover $\widehat{\Sigma}$.

Proposition 1.8. *Let* $(\Sigma, \mathcal{T})$ *be a piecewise flat surface with home triangle* T_0 *and choose an isometry* f_0 *from* T_0 *onto a triangle in* $\mathbb{R}^2$. *Then there exists a unique map* $f : \widehat{\Sigma} \to \mathbb{R}^2$ *such that* f *coincides with* f_0 *on* T_0 *and* f *maps every hinge onto a quadrilateral in* $\mathbb{R}^2$.

Proof. Let $\hat{x}$ be a point in $\widehat{\Sigma}$. This point belongs to a simplex $\widehat{\sigma} = (\sigma, [c])$ in $\widehat{\mathcal{T}}$. Choose an admissible arc c connecting the base point $x_0 \in T_0$ to σ. Because c is admissible, it crosses only finitely many edges $e_1, e_2, ..., e_m$ in that order (repetitions may occur). We associate to the path c a sequence of triangles $T_1, T_2, ..., T_m \in \mathcal{T}$ by requiring that (T_0, T_1) be the hinge of e_1, (T_1, T_2) be the hinge of e_2 and so on. We then define $f_j : T_j \to \mathbb{R}^2$ to be the continuation of f_{j-1} across the edge e_j (for $1 \leq j \leq m$) and we finally set $f(\hat{x}) = f_m(P(\hat{x})$. The point $f(\hat{x}) \in \mathbb{R}^2$ only depends on the homotopy class $[c]$ and not on the representative path c.

It is clear from the construction that $f : \widehat{\Sigma} \to \mathbb{R}^2$ maps every hinge onto a quadrilateral in $\mathbb{R}^2$. Since f extends f_0, the proof is complete. □

Definition 1.9. *The map* $f : \widehat{\Sigma} \to \mathbb{R}^2$ *is the* development map *of the piecewise flat surface.*

If $f' : \widehat{\Sigma} \to \mathbb{R}^2$ is another development, then, clearly, $f' = g \circ f$ where $g : \mathbb{R}^2 \to \mathbb{R}^2$ is the unique isometry of the plane such that $g(f(T_0)) = f'(T_0)$.

When Σ is the boundary surface of a convex polyhedron in $\mathbb{R}^3$, the development is a very concrete operation. It is obtained by first placing the initial face (home) somewhere on the plane and then rolling without slipping the polyhedron, face after face, following an admissible path. Observe in particular that we can move our polyhedron toward any point in the plane. This is a general fact:

Proposition 1.10. *Let* Σ *be a compact piecewise flat surface without boundary. Then any development* $f : \widehat{\Sigma} \to \mathbb{R}^2$ *is surjective.*

Proof. Observe first that f is a closed map (because it is an isometry on each triangle). Suppose that $\mathbb{R}^2 \setminus f(\widehat{\Sigma}) \neq \emptyset$, then this set is open and we can find a point $y \in \mathbb{R}^2$ which lies on the boundary of $f(\widehat{\Sigma})$. Because f is closed, we can find $\hat{x} \in \widehat{\Sigma}$ with $f(\hat{x}) = y$. Let $x = P(\hat{x}) \in \Sigma$. This point cannot be in the interior of any triangle of the triangulation, thus x lies on an edge e. Moving slightly the point y if necessary, we can assume that x lies in the interior e (i.e. that x is not a vertex).

Since Σ has no boundary, e is an interior edge; the developing map f sends the hinge of e onto a quadrilateral Q in $\mathbb{R}^2$. The interior of e is sent in the interior of $Q \subset f(\widehat{\Sigma})$. This contradicts the point y lying on the boundary of $f(\widehat{\Sigma})$. □

1.4. **The holonomy of a piecewise flat surface.** The set of all admissible homotopy classes in the piecewise flat surface $(\Sigma, \mathcal{T})$ which start and end at the base point x_0 form a group π with respect to the concatenation. This group coincides with the fundamental group $\pi_1(\Sigma', x_0)$.

If $[a] \in \pi$ and $(\sigma, [b]) \in \widetilde{\mathcal{T}}$, then $(\sigma, [ba])$ is well defined, and this gives a simplicial action of π on $\widetilde{\mathcal{T}}$.

Corresponding to this simplicial action, there is an action of π on $\widehat{\Sigma}$ by isometries; the orbits space of this action coincides with the surface itself. i.e. $\Sigma = \widehat{\Sigma}/\pi$.

If $f : \widehat{\Sigma} \to \mathbb{R}^2$ is a development map of $(\Sigma, \mathcal{T})$ and $\gamma = [c] \in \pi$, then there is a unique isometry $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that the $g(f(T_0)) = f(T_0, [c])$. We denote this isometry by $g = \varphi(\gamma)$.

Proposition 1.11. *The map* $\varphi : \pi \to E(2)$ *(the group of isometries of the euclidean plane) is a group homomorphism.*

Proof. This easily follows from the construction of the development map. □

Definition 1.12. *The homomorphism $\varphi : \pi \to E(2)$ is called the* holonomy *associated to the development f.*

Proposition 1.13. *If Σ is compact without boundary, then the group $H = \varphi(\pi) \subset E(2)$ has no bounded orbit (in particular it has no fixed point).*

An obvious but important consequence is the fact that H is not conjugate to a subgroup of $O(2)$.

Proof. Suppose that there is a point $y \in \mathbb{R}^2$ such that $H \cdot y$ is bounded. Since the development map is surjective, there exists a point $\hat{x} \in \widehat{\Sigma}$ such that $f(\hat{x}) = y$. Observe that $H \cdot y = f(\pi \cdot \hat{x})$. Any point in the surface Σ can be connected to $x = P(\hat{x})$ by a path of length at most $D = \mathrm{diam}(\Sigma)$, hence any point in $\widehat{\Sigma}$ can be connected to a point in the orbit $\pi \cdot \hat{x}$ by a path of length at most D.

Since f preserves the length of all paths, it follows that any point in the image $f(\widehat{\Sigma})$ can be connected to a point in the orbit $H \cdot y$ by a path of length at most D. The last assertion contradicts the surjectivity of f. □

Recall that the development of a piecewise flat surface is not unique, it depends on the choice of an isometry of the home triangle into $\mathbb{R}^2$. However the holonomy is well defined up to conjugacy:

Proposition 1.14. *Let $f, f' : \widehat{\Sigma} \to \mathbb{R}^2$ be two development maps of the piecewise flat surface $(\Sigma, \mathcal{T})$, and let $\varphi, \varphi' : \pi \to E(2)$ be the corresponding holonomies. Then $\varphi'(\gamma) = g\varphi(\gamma)g^{-1}$ where $g \in \mathrm{E}(2)$ is the unique isometry such that $f' = g \circ f$.*

Proof. The holonomy is defined by the condition $f(T_0, \gamma) = \varphi(\gamma)(f(T_0))$, hence

$$\begin{aligned}\varphi'(\gamma)(f(T_0)) = f'(T_0, \gamma) = g \circ f(T_0, \gamma) &= g \circ \varphi(\gamma)(f(T_0)) \\ &= g \circ \varphi(\gamma) \circ g^{-1}(f'(T_0)).\end{aligned}$$

□

1.5. **The development near a singularity.** The previous notions can be clearly visualized if one restricts one's attention to a simply connected region $\Omega \subset \Sigma$ which is a union of triangles and which contains exactly one singular vertex p of order $\beta = \beta(p) \neq 0$.

Suppose that the base point x_0 sits in Ω and choose a loop c in $\Omega' = \Omega \setminus \{p\}$, based at x_0 and surrounding the point p once (so that $[c]$ is a generator of $\pi_1(\Omega', x_0) \cong \mathbb{Z}$).

Choose a connected component $\widehat{\Omega}$ of the inverse image $P^{-1}(\Omega) \subset \widehat{\Sigma}$ and still denote by P the (restriction of the) projection $P : \widehat{\Omega} \to \Omega$.

We want to describe the geometry of $\widehat{\Omega}$, of the map P as well as the development and holonomy restricted to $\widehat{\Omega}$.

It is enough to consider the case where Ω is the "star" of the vertex p, i.e. the union of all triangles incident with p (if Ω is a larger region, the other triangles will simply appear as an appendix glued to the star of p).

The space $\widehat{\Omega}$ is the geometric realization of a simplicial complex whose simplices are simplices in Ω together with an admissible homotopy class of curve connecting the base point to the given simplex.

Let us denote by $T_1, T_2, ..., T_k$ the list of all triangles (i.e. 2-simplices) incident with p and assume that $x_0 \in T_1$. Assume also that T_i has a common edge with T_{i+1} and T_k has a common edge with T_1. Then a triangle in $\widehat{\Omega}$ is given by a pair $(T_i, [a])$ where $[a]$ is the homotopy class of a curve a in Ω' from x_0 to T_i. This homotopy class is parametrized by a single integer $d \in \mathbb{Z}$ (the degree of a) which counts the number of times a turns around the point p. In other words, $\widehat{\Omega}$ is an infinite strip made out of countably many copies of each triangle $T_1, T_2, ..., T_k$ each indexed by the degree $d \in \mathbb{Z}$

$$\widehat{\Omega} = \bigcup_{d \in \mathbb{Z}} (T_{1,d} \cup T_{2,d} \cup ... \cup T_{k,d}) .$$

To develop $\widehat{\Omega}$, start with an isometry f_1 from T_1 to a triangle in the euclidean plane and continue this isometry by unfolding each hinge in $\widehat{\Omega}$. The development f then clearly satisfies

$$f(T_{i,d}) = R^d f(T_{i,0})$$

where R is a rotation of angle θ (= the sum of the angles at p of the triangles $T_1, T_2, ..., T_k$) around the point $q = f_1(p)$. The rotation $R \in E(2)$ is clearly the holonomy of the generator $[c]$ of $\pi_1(\Omega', x_0)$.

We collect in the next proposition, some of the conclusions of the previous discussion:

Proposition 1.15. *(1) The inverse image $P^{-1}(p)$ of p in $\widehat{\Omega}$ contains exactly one point $\hat{p}$;*
(2) the holonomy $\varphi(c)$ of $[c]$ is a rotation of angle $\theta = 2\pi(\beta + 1)$;
(3) if β is not an integer, then $q = f(\hat{p})$ is the unique fixed point of the rotation $\varphi(c)$.

□

1.6. **Geometric equivalence of euclidean triangulations.** Let $(\Sigma, \mathcal{T})$ be a piecewise flat surface. Choose a triangle $T_{\alpha_0} \in \mathcal{T}$ and a point q in the interior of an edge of T_{α_0}. If one connects the point q to the opposite vertex in T_{α_0} by a euclidean segment, one obtains two subtriangles $T'_{\alpha_0}, T''_{\alpha_0}$ whose union is T_{α_0}.

If one replaces the triangle T_{α_0} with T'_{α_0} and T''_{α_0} in the triangulation $\mathcal{T}_q$, one obtains a new triangulation $\mathcal{T}_q$.

Definition 1.16. *a) The triangulation $\mathcal{T}_q$ is said to be obtained from $\mathcal{T}$ by an* elementary subdivision.

b) The geometric equivalence *is the equivalence relation on the set of euclidean triangulations on a surface which is generated by elementary subdivisions.*

In other words, two euclidean triangulations $\mathcal{T}_1, \mathcal{T}_2$ on the surface Σ are geometrically equivalent if there is a common subdivison $\mathcal{T}$ which is also a euclidean triangulation.

Proposition 1.17. *The area measure dA, the length structure ℓ, the singularity order β, the development and holonomy are invariants of this equivalence relation.*

Proof. The statement is obvious for dA, ℓ and β. Observe now that if T is a triangle of $\mathcal{T}$ and T', T'' is an elementary subdivision of T, then the pair (T', T'') is the hinge of their common edge $e \subset T$. Observe also that if $f : T \to \mathbb{R}^2$ is an isometry, then f is an unfolding of that hinge.

This argument shows that the development remains unchanged when subdivising the triangulation. Since the development is invariant, so is the holonomy.

□

1.7. **Flat metrics with conical singularities.** If $(\Sigma, \mathcal{T})$ is a piecewise flat surface, then Σ' carries a well defined riemannian metric m; this metric is flat (i.e. it has no curvature) and in the neighbourhood of a conical singularity of total angle θ, we can introduce polar coordinates (r, φ), where $r \geq 0$ is the distance to p and $\varphi \in \mathbb{R}/(\theta\mathbb{Z})$ is the angular variable (it is defined modulo θ). In these coordinates, the metric reads

$$m = dr^2 + r^2 d\varphi^2.$$

A calculation shows that this metric can be written as

$$m_\beta = |z|^{2\beta}|dz|^2, \tag{1.2}$$

where $z = \frac{1}{\beta+1}(re^{i\varphi})^{\beta+1}$ (see [32]).

Definition 1.18. *A* flat surface with conical singularities (Σ, m) *is a surface* Σ *together with a singular Riemannian metric* m *which is isometric to the metric* m_β *in (1.2) in the neighbourhood of every point* $p \in \Sigma$*, where* $\beta = \beta(p) \in (-1, \infty)$.

One says that $\beta(p)$ *is the* singularity order *of* p *and* p *is a conical singularity if* $\beta(p) \neq 0$. *The singular points form a discrete set and the formal sum (with discrete support)* $\sum \beta(p)\, p$ *is called the* divisor *of the singular metric* m. *One sometimes also says that* m represents *this divisor.*

Proposition 1.19. *Any compact flat surface with conical singularities* (Σ, m) *can be geodesically triangulated. The resulting triangulation is a euclidean triangulation on* Σ *and the associated length structure coincides with the length in the metric* m.

A proof can be found in [32] and in [31]. See also [19] and [27] for further discussions on triangulations of piecewise flat surfaces.

□

Proposition 1.20. *Two euclidean triangulations* $\mathcal{T}, \mathcal{T}'$ *on a compact surface* Σ *are geometrically equivalent if and only if they give rise to the same flat surface with conical singularities* m *on* Σ.

Proof. It is clear from Proposition 1.17, that two triangulations which are geometrically equivalent give rise to the same singular flat metric. Conversely, suppose that the triangulations $\mathcal{T}$ and $\mathcal{T}'$ define the same metric, then each triangle of $\mathcal{T}$ is decomposed by $\mathcal{T}'$ in a finite number of polygonal regions. We may then further decompose these polygons in euclidean triangles, and we thus obtain a new euclidean triangulation of Σ which is a subdivision of both $\mathcal{T}$ and $\mathcal{T}'$.

□

1.8. **Relation with Riemann surfaces.** If (Σ, m) is an oriented flat surface with conical singularities, then it is covered by charts $\{(U_j, z_j)\}$ such that the metric m takes the form (1.2) in each U_j. The transition from one such coordinate z_j to another one is given by a conformal transformation. Thus Σ is a Riemann surface with a holomorphic atlas given by $\{(U_j, z_j)\}$.

Remark 1.21. *The reader should observe here that the conical singularities are invisible from the conformal viewpoint. This is a consequence of the formula (1.2) which shows that the singular metric is conformal to a smooth metric. It can also be seen as a consequence of the theorem of removability of singularities of locally bounded meromorphic functions.*

In the converse direction, we can start with a closed Riemann surface with a divisor and ask whether there is a conformal flat metric representing this divisor. The answer is positive and the following theorem classifies all compact euclidean surfaces with conical singularities.

Theorem 1.22. *Let Σ be a compact connected Riemann surface without boundary. Fix n distinct points $p_1, p_2, \dots, p_n \in \Sigma$ and n real numbers $\beta_1, \beta_2, \dots, \beta_n \in (-1, \infty)$.*

There exists a conformal flat metric m on Σ having a conical singularity of order β_j at p_j $(j = 1, \dots, n)$ if and only if the Gauss-Bonnet condition $\chi(S) + \sum_{j=1}^{n} \beta_j = 0$ holds. This metric is unique up to homothety.

See [32], a shorter proof can be found in [34, §IV].

□

Remark 1.23. *A careful examination of the proof shows that the metric m depends continuously on all the parameters: The conformal structure, the points p_j and the orders β_j.*

There is a similar theorem for the case of hyperbolic metrics with conical singularities, see [15, 23, 26, 33]. There are are also various other extensions (non constant curvature, non orientable surfaces, non compact surfaces, and surfaces with boundary, see [17, 33, 34]). The case of spherical metric is more delicate, see [12, 35] for a study of spherical metric with three conical singularities on the 2-sphere.

Theorem 1.24. *Given a compact oriented surface Σ, there are natural bijections between the following three sets:*

1) *The set of geometric equivalence classes of euclidean triangulations on Σ up to homothety;*
2) *the set of flat metrics m on Σ with conical singularities up to homothety;*
3) *the set of conformal structures on Σ together with a finite real divisor $\sum_i \beta_i p_i$ such that $\beta_i > -1$ and the Gauss-Bonnet condition (1.1) is satisfied.*

Proof. Theorem 1.22 says precisely that there is a bijection between sets (2) and (3). Proposition 1.20 shows that there is a natural injection from (1) to (2), this injection is surjective by Proposition 1.19.

□

2. Punctured surfaces

2.1. **Punctured surfaces and their fundamental groups.**

Definition 2.1. *We define a* punctured surface $\Sigma_{g,n}$ *to be an oriented, closed connected surface* Σ *of genus* g *together with a distinguished set of* n *pairwise distinct points* $p_1, p_2, ..., p_n \in \Sigma_{g,n}$.

The points $p_1, ..., p_n$ are considered to be special places (with some geometric significance) on the surface. We call them the *punctures* and we denote by $\Sigma'_{g,n}$ the surface obtained by removing them:

$$\Sigma'_{g,n} = \Sigma_{g,n} \setminus \{p_1, p_2, ..., p_n\}.$$

The connected sum of two punctured surfaces is defined by removing a disk containing no puncture in each surface and then gluing them along their boundary. The resulting surface is again a punctured surface. In fact we have

$$\Sigma_{g_1,n_1} \# \Sigma_{g_2,n_2} = \Sigma_{g_1+g_2,n_1+n_2},$$

where the symbol # means the connected sum. In particular

$$\Sigma_{g,n} = \Sigma_{g,0} \# \Sigma_{0,n}. \tag{2.3}$$

We easily deduce from this observation that the Euler characteristic of $\Sigma'_{g,n}$ is given by

$$\chi(\Sigma'_{g,n}) = 2 - 2g - n. \tag{2.4}$$

If $n > 0$, then $\Sigma'_{g,n}$ can be homotopically retracted onto a bouquet of $2g + n - 1$ circles and the fundamental group $\pi_{g,n}$ of $\Sigma'_{g,n}$ is thus a free group on $2g + n - 1$ generators.

Note that $\pi_{g,n}$ also admits the following presentation with $2g+n$ generators and one relation:

$$\pi_{g,n} = \langle a_1, ..., a_g, b_1, ..., b_g, c_1, ..., c_n \,\big|\, \Pi\, [a_i, b_i] = \Pi c_j \rangle, \tag{2.5}$$

this presentation is a consequence of the identity (2.3) and Van Kampen's Theorem.

2.2. **Uniformization of a punctured Riemann surface.** Let us fix a conformal structure $[m]$ on $\Sigma_{g,n}$. Assuming that $2 - 2g - n < 0$, the uniformization Theorem states that (Σ', m) is conformally equivalent to $\mathbb{U}/\Gamma$ where $\mathbb{U} = \{z \in \mathbb{C} \,\big|\, \mathrm{Re}(z) > 0\}$ is the upper-half plane, and $\Gamma \subset PSL_2(\mathbb{R})$ is a Fuchsian group of the first kind[2] isomorphic to $\pi_{g,n}$.

The isomorphism $\pi_{g,n} \to \Gamma$ is compatible with the punctures in the sense that the generator c_i is sent to a parabolic element of Γ and the generators a_i, b_i are sent to hyperbolic elements (here, the letters a_i, b_i, c_j refer to the presentation (2.5)).

Let us denote by $\Upsilon \subset \partial\mathbb{U} = \mathbb{R} \cup \{\infty\}$ the set *cusp points* of Γ, i.e. the set of fixed points of all parabolic elements in Γ. Following [29, page 10], we define a topology on $\widehat{\mathbb{U}} = \mathbb{U} \cup \Upsilon$ as follows: for a point $z \in \mathbb{U}$, the family of hyperbolic disks $D(z, \rho)$ is a fundamental system of neighborhoods of z. For a point $y \in \Upsilon$ the family of horodisks centered at y is a fundamental system of neighborhoods of y. With this topology, $\widehat{\mathbb{U}}$ is a Hausdorff space and Γ acts by homeomorphisms. The space is not locally compact and Υ is topologically a discrete space. Standard

[2]Recall that a *Fuchsian group* is a discrete subgroup of $PSL_2(\mathbb{R})$, it is of the f*irst kind* if there is a fundamental domain $D \subset \mathbb{U}$ of finite hyperbolic area.

arguments from hyperbolic geometry (see e.g. [29]) show that the projection map $P : \mathbb{U} \to \mathbb{U}/\Gamma = \Sigma'$ extends to a surjective continuous map

$$\widehat{P} : \widehat{\mathbb{U}} \to \Sigma \tag{2.6}$$

where $\widehat{\mathbb{U}} = \mathbb{U} \cup \Upsilon$. This extension maps Υ to the punctures $\{p_i\} \subset \Sigma$.

Remarks 1) The previous considerations show that there exists a unique metric m_{-1} on Σ' of constant curvature -1 which is complete, has finite volume and belongs to the conformal structure $[m]$. This metric has a cusp at each puncture p_i, it is the unique metric such that P^*m_{-1} is the Poincaré metric on $\mathbb{U}$; its existence can also be proved by directly solving the prescribed curvature equation (see [16, 17]).

2) We know from Theorem 1.22 that the conformal class $[m]$ also contains a metric m_0 on Σ', unique up to homothety, which is flat and has a conical singularity of order β_j at p_j $(j = 1, \dots, n)$ provided (1.1) holds.

This flat metric lifts as a flat conformal metric $\widetilde{m}_0 = P^*(m_0)$ on $\mathbb{U}$. For this metric, $\mathbb{U}$ is not complete and its completion is given by $\widehat{\mathbb{U}} = \mathbb{U}$. The map $\widehat{P} : \widehat{\mathbb{U}} \to \Sigma$ is thus a concrete model of the universal branched covering introduced earlier.

We identify the set Υ as a subset of Γ as follows: we first fix a base point $\tilde{z}_0 \in \mathbb{U}$ and let $z_0 = P(\tilde{z}_0) \in \Sigma$. For $y \in \Upsilon$, let us denote by $\tilde{\gamma}_y$ the hyperbolic ray in $\mathbb{U}$ starting at $\tilde{z}_0$ and asymptotic to the point y, and let $\gamma_y = P(\tilde{\gamma}_y)$, this is a path joining z_0 to a puncture $p_i = P(y)$. Now let $D_i \subset \Sigma$ be a small disk around p_i containing no other puncture, and let $\gamma'_y = \gamma_y \setminus D_i$.

We now define $c_y \in \pi_1(\Sigma', z_0)$ to be the homotopy class of the path obtained by following γ'_y, then ∂D_i (in the positive direction) and then $(\gamma'_y)^{-1}$.

Recall that we have a canonical isomorphism, $\Gamma \cong \pi_1(\Sigma', z_0) = \pi_{g,n}$, we have thus constructed a map

$$\begin{array}{ccc} \Upsilon & \to & \Gamma \\ y & \mapsto & c_y \end{array} \tag{2.7}$$

It is clear that $c_y \in \Gamma$ is a parabolic element fixing y, in particular, the map $\Upsilon \to \Gamma$ is injective.

2.3. **Some Groups of Diffeomorphisms of a Punctured surface.** Given a punctured surface $\Sigma = \Sigma_{g,n}$, we define $\mathrm{Diff}_{g,n}$ to be the group of diffeomomorphisms $h : \Sigma \to \Sigma$ which leaves the set $\{p_1, ..., p_n\}$ of punctures invariant. We also introduce the following subgroups: $\mathrm{Diff}^+_{g,n} \subset \mathrm{Diff}_{g,n}$ is the subgroup of orientation preserving diffeomomorphisms, $\mathrm{PDiff}_{g,n}$ is the subgroup of pure diffeomomorphisms, i.e. diffeomomorphisms fixing each puncture p_i individually and $\mathrm{PDiff}^+_{g,n} = \mathrm{PDiff}_{g,n} \cap \mathrm{Diff}^+_{g,n}$.

Every element $h \in \mathrm{Diff}^+_{g,n}$ permutes the punctures and we have an exact sequence

$$1 \to \mathrm{PDiff}^+_{g,n} \to \mathrm{Diff}^+_{g,n} \to \mathrm{Sym}(n) \to 1.$$

where $\mathrm{Sym}(n)$ is the permutation group of $\{p_1, ..., p_n\}$.

We also define $\mathrm{Diff}^0_{g,n} \subset \mathrm{PDiff}^+_{g,n}$ to be the group of diffeomorphisms which are isotopic to the identity through an isotopy fixing the punctures. The quotient

$$\mathrm{Mod}_{g,n} = \pi_0(\mathrm{Diff}^+_{g,n}) = \mathrm{Diff}^+_{g,n} / \mathrm{Diff}^0_{g,n},$$

is called the *mapping class group* or the *modular group* of the punctured surface $\Sigma_{g,n}$, and

$$\mathrm{PMod}_{g,n} = \pi_0(\mathrm{PDiff}^+_{g,n}) = \mathrm{PDiff}^+_{g,n} / \mathrm{Diff}^0_{g,n},$$

is the *pure mapping class group*.

These groups have been intensely studied since the pioneer work of Dehn and Nielsen. We refer to [5, 20, 24, 37] among many other papers, for more information.

2.4. **Outer automorphisms.** The mapping class group is related to the group of outer automorphisms of the fundamental group of Σ'. Let us recall this purely algebraic notion: If π is an arbitrary group, we denote by $\mathrm{Aut}(\pi)$ the group of all its automorphisms and by $\mathrm{Inn}(\pi) \subset \mathrm{Aut}(\pi)$ the subgroup of inner automorphisms (i.e. conjugations $\gamma \to \alpha\gamma\alpha^{-1}$). This is a normal subgroup.

Definition 2.2. *The group of* outer automorphisms *of* π *is the quotient*

$$\mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\,\mathrm{Inn}(\pi).$$

An outer automorphism is thus an automorphism of π *defined up to conjugacy.*

Lemma 2.3. *There is a naturally defined group homomorphism*

$$\mathrm{Mod}_{g,n} \to \mathrm{Out}(\pi_{g,n}).$$

Proof. This homomorphism is defined as follows. Let $h \in \mathrm{Diff}(\Sigma')$ be an arbitrary diffeomorphism and fix a base point $*$ and a path δ in Σ' connecting $*$ to $h(*)$. If γ is a loop in Σ' based at $*$, then we set

$$h_\delta(\gamma) = \delta^{-1}(h \circ \gamma)\delta.$$

This defines an automorphism $h_{\delta,\#} \in \mathrm{Aut}(\pi_{g,n})$.

If δ' is another path connecting $*$ to $h(*)$, then $h_{\delta,\#}$ and $h_{\delta',\#}$ are conjugate by $\delta^{-1}\delta'$. The outer automorphism $h_\# \in \mathrm{Out}(\pi_{g,n})$ is thus well defined independently of the choice of the path δ and it is clear that if h is homotopic to the identity, then it acts trivially on $\pi_{g,n}$, i.e. we have defined a map $\mathrm{Mod}_{g,n} \to \mathrm{Out}(\pi_{g,n})$. It is routine to check that it is a group homomorphism. □

Introducing the group $\mathrm{POut}(\pi_{g,n}) \subset \mathrm{Out}(\pi_{g,n})$ of all outer automorphisms preserving the conjugacy class of each c_i $(i = 1, ..., n)$ in the presentation (2.5), we have the following deep result:

Theorem 2.4. *If* $g > 0$ *and* $n > 0$*, then the homomorphism defined in the previous lemma induces an isomorphism*

$$\Phi : \mathrm{PMod}_{g,n} \xrightarrow{\sim} \mathrm{POut}(\pi_{g,n}). \tag{2.8}$$

This is the so called *Dehn-Nielsen-Baer Theorem*, see [20, 37] for a proof.

□

2.5. **Lifting the group** $\mathrm{Diff}^0(\Sigma')$ **on** $\mathbb{U}$**.** Using the notations of section 2.2, one writes the universal branched covering of $\Sigma_{g,n}$ as $\widehat{P} : \widehat{\mathbb{U}} \to \Sigma$, where $\widehat{\mathbb{U}} = \mathbb{U} \cup \Upsilon$ (we still assume $2 - 2g - n < 0$).

We denote by $\mathrm{Diff}^+(\mathbb{U})$ the group of orientation preserving diffeomorphisms of $\mathbb{U} = \widetilde{\Sigma'}$ and we define the *normalizer* $N(\Gamma)$ and the *centralizer* $C(\Gamma)$ of Γ in $\mathrm{Diff}^+(\mathbb{U})$ by

$$N(\Gamma) = \{h \in \mathrm{Diff}^+(\mathbb{U}) \,\big|\, h\Gamma = \Gamma h\}.$$

and

$$C(\Gamma) = \{h \in \mathrm{Diff}^+(\mathbb{U}) \,\big|\, h \circ \gamma = \gamma \circ h \text{ for all } \gamma \in \Gamma\}.$$

Observe that $C(\Gamma) = \ker(\psi)$, where $\psi : N(\Gamma) \to \mathrm{Aut}(\Gamma)$ is defined by $\psi(h) : \gamma \to h\gamma h^{-1}$.

The *center* of Γ is the intersection $Z(\Gamma) = \Gamma \cap C(\Gamma)$; it is the largest abelian subgroup of Γ.

Lemma 2.5. *Let Γ be an arbitrary Fuchsian group, then $Z(\Gamma)$ is trivial unless Γ is cyclic.*

Proof. This follows from classical Fuchsian group theory. Indeed, it is well known that if γ_1, γ_2 are non-trivial elements in $PSL_2(\mathbb{R})$, then they commute if and only if they have the same fixed points (see e.g. [21, theorem 2.3.2]). So if $Z(\Gamma)$ contains a non-trivial element γ_0, then any $\gamma \in \Gamma \setminus \{id\}$ must have the same fixed points as γ_0 and it follows from [21, theorem 2.3.5]) that Γ is cyclic.

□

Recall the projection $P : \mathbb{U} \to \Sigma' = \mathbb{U}/\Gamma$. For any element $h \in N(\Gamma)$, we define $P_*h : \Sigma' \to \Sigma'$ by $P_*h(x) = P(h(\tilde{x}))$ where $\tilde{x} \in \mathbb{U}$ is an arbitrary point in $P^{-1}(x)$. This map is well-defined, because the condition $h\Gamma = \Gamma h$ means precisely that h maps Γ-orbits in U to Γ-orbits, and it is clearly a diffeomorphism. We thus have defined a map

$$P_* : N(\Gamma) \to \mathrm{Diff}^+(\Sigma'),$$

and it is obviously a group homomorphism.

Proposition 2.6. *Γ is a normal subgroup in $N(\Gamma)$ and P_* defines an isomorphism from $N(\Gamma)/\Gamma$ to $\mathrm{Diff}^+(\Sigma')$.*

Proof. It is obvious that $\Gamma \subset N(\Gamma)$ is normal and that $P_*(\Gamma) = \{id\}$. In particular P_* factors through a well defined homomorphism $N(\Gamma)/\Gamma \to \mathrm{Diff}^+(\Sigma')$. This homomorphism is surjective since every diffeomorphism of Σ' lifts to the universal cover $\mathbb{U}$ of Σ'.

Suppose now that $P_*h = id$. Then $h(x) \in \Gamma \cdot x$ for all $x \in U$. This means that there exists a map $U \to \Gamma$, $x \to \gamma_x$ such that $h(x) = \gamma_x\, x$ for all $x \in U$. Since h is continuous, so is this map, but this implies that $x \mapsto \gamma_x$ is constant because Γ is a discrete group. It follows that $h \in \Gamma$ and we have shown that $P_* : N(\Gamma)/\Gamma \to \mathrm{Diff}^+(\Sigma')$ is also injective.

□

Lemma 2.7. *P_* maps $C(\Gamma)$ isomorphically onto $\mathrm{Diff}^0(\Sigma')$.*

Proof. Suppose that $P_*h \in \mathrm{Diff}^0(\Sigma')$. Then there exists an isotopy $h_t \in N(\Gamma)$ such that $h_0 = id$ and $h_1 = h$. Hence $\psi(h_t) \in \mathrm{Aut}(\Gamma)$ is constant by continuity. Because $\psi(h_0) = \psi(id) \in \mathrm{Aut}(\Gamma)$ is the trivial element, we have $h \in \ker\psi = C(\Gamma)$.

In the reverse direction, we use an argument going back to Nielsen: Suppose that $h \in \ker\psi = C(\Gamma)$ and define $h_t(x) \in U$ to be the point on the hyperbolic segment $[x, h(x)]$ such that $d(x, h_t(x)) = td(x, h(x))$ (where d is the hyperbolic distance in $\mathbb{U}$). Since $h \in \ker\psi$ and Γ preserves the hyperbolic distance in $\mathbb{U}$, the segment $[\gamma x, h(\gamma x)]$ coincides with $[\gamma x, \gamma h(x)]$ for any $x \in \mathbb{U}$ and any $\gamma \in \Gamma$. Therefore we have $h_t(\gamma x) = \gamma h_t(x)$, i.e. $h_t \in C(\Gamma) \subset N(\Gamma)$. The path $P_*h_t \in \mathrm{Diff}(\Sigma')$ is an isotopy from P_*h to the identity and we conclude that $P_*h \in \mathrm{Diff}^0(\Sigma')$.

We have proved that $P_*^{-1}(\mathrm{Diff}^0(\Sigma')) = C(\Gamma)$. It is now clear that $P_* : C(\Gamma) \to \mathrm{Diff}^0(\Sigma')$ is an isomorphism since its kernel is $C(\Gamma) \cap \Gamma = Z(\Gamma) = \{id\}$.

□

Corollary 2.8. *P_* induces an isomorphism from $N(\Gamma)/(\Gamma \times C(\Gamma))$ to the modular group* $\mathrm{Mod}_{g,n}$.

3. The representation variety of a finitely generated group in $SE(2)$

Given a finitely generated group π and an algebraic Lie group G, it is easy to see that the set $\mathrm{Hom}(\pi, G)$ is an algebraic set. The group G itself acts on $\mathrm{Hom}(\pi, G)$ by conjugation: $g \cdot \varphi(\gamma) = g^{-1}\varphi(\gamma)g$. The quotient space is called the *representation variety* of π in G and denoted by

$$\mathcal{R}(\pi, G) = \mathrm{Hom}(\pi, G)/G.$$

This variety plays an important role in the study of geometric structures on manifolds, see e.g. [14].

The discussion in section 1.4 shows that an element of the representation variety $\mathcal{R}(\pi, \mathrm{E}(2))$ is associated to any piecewise flat surface $(\Sigma, \mathcal{T})$ (where $\pi = \pi_1(\Sigma', x_0)$). In the present section, we investigate the structure of $\mathcal{R}(\pi, \mathrm{E}(2))$, in fact, for convenience, we shall restrict ourself to the subgroup $\mathrm{SE}(2) \subset \mathrm{E}(2)$ of orientation preserving isometries of the euclidean plane (this is a subgroup of index 2).

3.1. On the cohomology of groups.

We will need some elementary results from group cohomology; here we recall a few basic definitions and facts.

Let π be an arbitrary group and A be a π-module, i.e. an abelian group with a representation $\rho : \pi \to \mathrm{Aut}(A)$.

Definition 3.1. *(1) A* 1-cocycle *in A is a map $\sigma : \pi \to A$ such that*

$$\sigma(\gamma_1\gamma_2) = \sigma(\gamma_1) + \rho(\gamma_1) \cdot \sigma(\gamma_2)$$

for any $\gamma_1, \gamma_2 \in \pi$. The set of 1-cocycles is an abelian group denoted by $Z^1(\pi, A)$.

(2) The 1-cocycle $\sigma \in Z^1(\pi, A)$ is a 1- coboundary *if it can be written as*

$$\sigma = \delta_a(\gamma) = \rho(\gamma) \cdot a - a.$$

for some element $a \in A$. The set of 1-coboundaries is a subgroup of $Z^1(\pi, A)$ denoted by $B^1(\pi, A)$.

(3) The quotient

$$H^1(\pi, A) = Z^1(\pi, A)/B^1(\pi, A)$$

is the first cohomology group of π with values in A.

Example. Let us compute the first cohomology group when $A = k$ is a field and π is a finitely generated group. We denote by k_ρ the π-module k with the representation $\rho : \pi \to \mathrm{Aut}(k)$.

Assume first that the representation $\rho : \pi \to \mathrm{Aut}(k)$ is a scalar representation, i.e. $\rho : \pi \to k^* \subset \mathrm{Aut}(k)$ and that $\pi = F_s = \langle a_1, a_2, ..., a_s \rangle$ is a free group on s generators.

Since π is free, the homomorphism $\rho : \pi \to k^*$ is completely determined by the vector $r = (\rho(a_1), \rho(a_2), ..., \rho(a_s)) \in (k^*)^s$. Likewise, a cocycle is given by the vector

$$t = (\tau(a_1), \tau(a_2), ..., \tau(a_s)) \in k^s.$$

There is no restriction on the vector $t \in k^s$ (again because π is free) and thus

$$Z^1(\pi, k_\rho) \cong k^s. \tag{3.9}$$

An element $\sigma \in Z^1(\pi, k_\rho)$ is a coboundary if $\sigma = u(id - \rho)$ for some $u \in k$, thus

$$B^1(\pi, k_\rho) \cong k \cdot (1 - \rho(a_1), 1 - \rho(a_2), ..., 1 - \rho(a_s)) \in k^s.$$

Let us choose a linear form $\mu : k^s \to k$ such that $\mu \equiv 0$ if ρ is trivial and

$$\mu(1 - \rho(a_1), 1 - \rho(a_2), ..., 1 - \rho(a_s)) \neq 0$$

else. It is easy to check that

$$B^1(\pi, k_\rho) \oplus \ker \mu = k^s = Z^1(\pi, k_\rho)$$

in k^s and we thus obtain the following

Proposition 3.2. *For any free group on s generators, we have*

$$H^1(\pi, k_\rho) = Z^1(\pi, k_\rho)/B^1(\pi, k_\rho) = \ker \mu \cong \begin{cases} k^s & \text{if } \rho \text{ is trivial,} \\ k^{s-1} & \text{otherwise.} \end{cases}$$

□

General case. Let us compute the first cohomology group when $A = k$ is a field and π is a finitely presented group with presentation

$$\pi = \langle S \mid R \rangle.$$

Here $S = \{a_1, a_2, ..., a_s\} \subset \pi$ is a finite set generating the group and $R = \{r_1, r_2, ..., r_m\} \subset F(S)$ (= the free group on S) is a finite set of words in S defining all the relations among the elements of S. We denote by k_ρ the π-module k with the representation $\rho : \pi \to \mathrm{Aut}(k)$.

For any relation $r = a_{i_1} a_{i_2} \cdots a_{i_p} \in R$, we introduce the linear form $\lambda_r : k^s \to k$ defined by

$$\lambda_r(\sigma) = \sum_{\mu=1}^{p} \left(\prod_{\nu < \mu} \rho(a_{i_\nu}) \right) \sigma(a_{i_\mu}) \tag{3.10}$$

and we define $\Lambda : k^s \to k^m$, by

$$\Lambda(\sigma) = (\lambda_{r_1}(\sigma), ..., \lambda_{r_m}(\sigma)). \tag{3.11}$$

Lemma 3.3. *Th space of 1-cocycles in k_ρ is given by*

$$Z^1(\pi, k_\rho) = \ker \Lambda = \bigcap_{r \in R} \ker \lambda_r \subset k^s.$$

Proof. If $\sigma \in Z^1(\pi, k_\rho)$ and $r = a_{i_1}a_{i_2}\cdots a_{i_p} \in R$, then we deduce from the cocycle relation that

$$\begin{aligned} 0 = \sigma(r) &= \sigma(a_{i_1}a_{i_2}\cdots a_{i_p}) = \sigma(a_{i_1}) + \rho(a_{i_1})\sigma(a_{i_2}\cdots a_{i_p}) \\ &= \sigma(a_{i_1}) + \rho(a_{i_1})\sigma(a_{i_2}) + \rho(a_{i_{i_1}})\rho(a_{i_2})\sigma(a_{i_3}\cdots a_{i_p}) \\ &= \sum_{\mu=1}^{p}\left(\prod_{\nu<\mu}\rho(a_{i_\nu})\right)\sigma(a_{i_\mu}). \end{aligned}$$

□

On the other hand, since any 1-coboundary in k_ρ is a multiple of $\rho - 1$, we have

$$B^1(\pi, k_\rho) = k\cdot(\rho-1) \subset k^s.$$

We have proved the following

Proposition 3.4. *The first cohomology group of the finitely presented group* $\pi = \langle S \,|\, R\rangle$ *with value in* k_ρ *is given by*

$$H^1(\pi, k_\rho) = \ker\Lambda/(k(\rho-1)).$$

□

In particular, if π has exactly one non trivial relation, then

$$H^1(\pi, k) \cong \begin{cases} k^{s-1} & \text{if } \rho \text{ is trivial,} \\ k^{s-2} & \text{otherwise.} \end{cases}$$

where $s = \mathrm{Card}(S)$ is the number of generators.

3.2. **Abelian Representations.** Representations of a finitely presented group π in an abelian Lie group are easy to describe:

Lemma 3.5. *If* G *is an abelian group, then* $\mathcal{R}(\pi, G) = \mathrm{Hom}(\pi, G)$. *This set is itself an abelian topological group.*

Proof. Since there are no non trivial inner automorphisms in an abelian group, it is clear that $\mathcal{R}(\pi, G) = \mathrm{Hom}(\pi, G)$.

We endow $\mathrm{Hom}(\pi, G)$ with the compact open topology and we define a product on this space by

$$(\varphi_1\varphi_2)(\gamma) = \varphi_1(\gamma)\varphi_2(\gamma)$$

for $\varphi_1, \varphi_2 \in \mathrm{Hom}(\pi, G)$ and $\gamma \in \pi$. The following calculation shows that $\mathrm{Hom}(\pi, G)$ is a group for this multiplication:

$$\begin{aligned} (\varphi_1\varphi_2)(\gamma_1\gamma_2) &= \varphi_1(\gamma_1\gamma_2)\varphi_2(\gamma_1\gamma_2) \\ &= \varphi_1(\gamma_1)\varphi_1(\gamma_2)\varphi_2(\gamma_1)\varphi_2(\gamma_2) \\ &= \varphi_1(\gamma_1)\varphi_2(\gamma_1)\varphi_1(\gamma_2)\varphi_2(\gamma_2) \\ &= (\varphi_1\varphi_2)(\gamma_1)(\varphi_1\varphi_2)(\gamma_2). \end{aligned}$$

The identity e in $\mathrm{Hom}(\pi, G)$ is the trivial representation. Observe finally that this group is abelian since $\varphi_1(\gamma)\varphi_2(\gamma) = \varphi_2(\gamma)\varphi_1(\gamma)$.

□

Recall that the abelianized group of π is the abelian group

$$Ab(\pi) = \pi/[\pi, \pi].$$

Another useful remark is that if G is abelian, then

$$\mathrm{Hom}(\pi' \times \pi'', G) = \mathrm{Hom}(\pi', G) \times \mathrm{Hom}(\pi'', G)$$

for any groups π', π''.

Assume now that π is a finitely generated group. $Ab(\pi)$ is then an abelian group of finite type, hence

$$Ab(\pi) = \pi/[\pi, \pi] = \mathbb{Z}^r \oplus F$$

where F is a finite abelian group (the torsion) and $r \in \mathbb{N}$ is the *abelian rank* of π.

We obviously have $\mathrm{Hom}(\pi, G) = \mathrm{Hom}(Ab(\pi), G)$ and it is clear that all representation varieties of a finitely generated group π in an abelian Lie group G can be deduced from the following special cases:

1) $\mathrm{Hom}(\mathbb{Z}, \mathbb{R}) = \mathbb{R}$;
2) $\mathrm{Hom}(\mathbb{Z}, U(1)) = U(1)$;
3) $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{R}) = 0$;
4) $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, U(1)) = \{z \in \mathbb{C} \mid z^m = 1\}$.

For instance, if π is the free group on s generators, then $Ab(\pi) = \mathbb{Z}^s$. Thus $\mathrm{Hom}(\pi, \mathbb{R}) = \mathbb{R}^s$ and $\mathrm{Hom}(\pi, U(1)) = \mathbb{T}^s$.

Another simple example, with torsion, is the group $\pi' = \langle a, b, c \mid [a, b] = c^m \rangle$. We have $Ab(\pi') = \mathbb{Z}^2 \oplus \mathbb{Z}/m\mathbb{Z}$, therefore $\mathrm{Hom}(\pi', \mathbb{R}) = \mathbb{R}^2$ and

$$\mathrm{Hom}(\pi', U(1)) = \mathbb{T}^2 \oplus \{e^{2ki\pi/m} \mid m = 0, 1, \dots, m-1\}.$$

3.3. **Representations in** SE(2)**.** We denote by $SE(2) = \mathrm{Iso}^+(\mathbb{R}^2)$ the group of orientation preserving isometries of the euclidean plane.

We may identify the euclidean plane with the complex line $\mathbb{C}$: any $g \in SE(2)$ can then be written as $g(z) = u \cdot z + v$ where $u \in U(1) \subset \mathbb{C}^*$ and $v \in \mathbb{C}$. We thus identify SE(2) with the subgroup of $GL_2(\mathbb{C})$ consisting of matrices of the form

$$\mathrm{SE}(2) = \left\{ \left(\begin{array}{cc} u & v \\ 0 & 1 \end{array} \right) \middle| \, u, v \in \mathbb{C}, \ |u| = 1 \right\}.$$

In particular SE(2) is a semidirect product $U(1) \rtimes \mathbb{C}$ and any representation $\varphi \in \mathrm{Hom}(\pi, SE(2))$ can be written as

$$\varphi = \left(\begin{array}{cc} \rho_\varphi & \tau_\varphi \\ 0 & 1 \end{array} \right) \tag{3.12}$$

where $\rho_\varphi : \pi \to U(1)$ and $\tau_\varphi : \pi \to \mathbb{C}$. Observe the following:

Lemma 3.6. *The map $\rho_\varphi : \pi \to U(1)$ is a group homomorphism. It only depends on the conjugacy class of φ.*

The proof is elementary.

Definition 3.7. *The homomorphism $\rho_\varphi : \pi \to U(1)$ is the* character *of the representation class $\varphi \in \mathrm{Hom}(\pi, SE(2))$.*

Remark: In the literature on group representations, the character $\chi_\varphi : \pi \to K$ of a representation $\varphi \in \mathrm{GL}_n(K)$ is classically defined to be the trace of the representation. The two notions of characters are equivalent as shown by the formula

$$\chi_\varphi = \mathrm{Tr}\,\varphi = 1 + \rho_\varphi.$$

Any homomorphism $\rho \in \mathrm{Hom}(\pi, U(1))$ defines a structure of $\pi-$module on $\mathbb{C}$. We will denote by $\mathbb{C}_\rho$ this $\pi-$module, and we have:

Proposition 3.8. *Given any pair of maps $\rho : \pi \to U(1)$ and $\tau : \pi \to \mathbb{C}$, the map $\varphi : \pi \to SE(2)$ given by (3.12) is a group homomorphism if and only if $\rho \in \mathrm{Hom}(\pi, U(1))$ and $\tau \in Z^1(\pi, \mathbb{C}_\rho)$.*

Proof. Suppose that $\varphi : \pi \to SE(2)$ is given by (3.12). Then we have

$$\varphi(\gamma_1\gamma_2) = \begin{pmatrix} \rho(\gamma_1\gamma_2) & \tau(\gamma_1\gamma_2) \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} \varphi(\gamma_1)\varphi(\gamma_2) &= \begin{pmatrix} \rho(\gamma_1) & \tau(\gamma_1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho(\gamma_2) & \tau(\gamma_2) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \rho(\gamma_1)\rho(\gamma_2) & \tau(\gamma_1) + \rho(\gamma_1)\tau(\gamma_2) \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

It follows that φ is a group homomorphism (i.e. $\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1)\varphi(\gamma_2)$) if and only if

$$\rho(\gamma_1\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)$$

and

$$\tau(\gamma_1\gamma_2) = \tau(\gamma_1) + \rho(\gamma_1)\tau(\gamma_2).$$

In other words φ is a homomorphism if and only if $\rho : \pi \to U(1)$ is a homomorphism and τ is a 1-cocycle in the corresponding π-module $\mathbb{C}_\rho$.

□

This Proposition says that the map from $\mathrm{Hom}(\pi, \mathrm{SE}(2))$ to the set

$$\{(\rho, \tau) \,|\, \rho \in \mathrm{Hom}(\pi, U(1)) \text{ and } \tau \in Z^1(\pi, \mathbb{C}_\rho)\}$$

given by $\varphi \to (\rho_\varphi, \tau_\varphi)$, is a bijection. In particular we have

Corollary 3.9. *If π is the free group on s generators, then*

$$\mathrm{Hom}(\pi, \mathrm{SE}(2)) \simeq \mathbb{T}^s \times \mathbb{C}^s.$$

Proof. This follows from equation (3.9) and the fact that $\mathrm{Hom}(\pi, U(1)) = \mathbb{T}^s$.

□

3.4. **Conjugation by similarities.** Recall that a *similarity* in the plane is the composition of an isometry with a homothety.

Once we identify the euclidean plane with the complex line $\mathbb{C}$, any similarity $g \in \mathrm{Sim}(2)$ can be writen as $g(z) = a \cdot z + b$ where $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. We thus identify $\mathrm{Sim}(2)$ with the following subgroup of $GL_2(\mathbb{C})$:

$$\mathrm{Sim}(2) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \, a, b \in \mathbb{C},\ a \neq 0 \right\}.$$

In particular we have

$$\mathrm{Sim}(2) = \mathbb{R}_+ \rtimes \mathrm{SE}(2) = \mathbb{C}^* \rtimes \mathbb{C}.$$

Definition 3.10. *Two representations $\varphi_1, \varphi_2 : \pi \to \mathrm{SE}(2)$ are* similar *if they are conjugate modulo a similarity.*

Proposition 3.11. *Given a homomorphism $\rho : \pi \to U(1)$ and two cocycles $\tau_1, \tau_2 \in Z^1(\pi, \mathbb{C}_\rho)$, then the representations*

$$\varphi_1 = \begin{pmatrix} \rho & \tau_1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \varphi_2 = \begin{pmatrix} \rho & \tau_2 \\ 0 & 1 \end{pmatrix} \tag{3.13}$$

are similar if and only if there exists a complex number $a \in \mathbb{C}^*$ such that

$$\tau_2 = a\tau_1 \in H^1(\pi, \mathbb{C}_\rho).$$

Proof. The homomorphisms φ_1 and φ_2 are similar if and only if there exists

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{Sim}(2)$$

such that $\varphi_2 = g\varphi_1 g^{-1}$, i.e.

$$\begin{aligned} \begin{pmatrix} \rho & \tau_2 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho & \tau_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \rho & a\tau_1 + b - \rho b \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

This shows that

$$\tau_2 - a\tau_1 = b\,(1-\rho) \in B^1(\pi, \mathbb{C}_\rho).$$

□

For any homomorphism $\varphi : \pi \to \mathrm{SE}(2)$ and any $\lambda \in \mathbb{R}_+$, we can define a new homomorphism $\lambda \cdot \varphi : \pi \to \mathrm{SE}(2)$ by

$$\lambda \cdot \varphi = \begin{pmatrix} \rho & \lambda\tau \\ 0 & 1 \end{pmatrix},$$

This formula defines an action of the multiplicative group $\mathbb{R}_+$ on $\mathcal{R}(\pi, \mathrm{SE}(2))$, and we denote the quotient by

$$\mathcal{SR}(\pi, \mathrm{SE}(2)) = \mathcal{R}(\pi, \mathrm{SE}(2))/\mathbb{R}_+.$$

It follows directly from the definition that

$$\mathcal{SR}(\pi, \mathrm{SE}(2)) = \mathrm{Hom}(\pi, \mathrm{SE}(2))/\,\mathrm{Sim}(2).$$

where $\mathrm{Sim}(2)$ acts by conjugation on $\mathrm{Hom}(\pi, \mathrm{SE}(2))$.

Let us also define

$$\mathcal{SR}^{\mathrm{reg}} = \{[\varphi] = [\rho, \tau] \in \mathcal{SR}(\pi, \mathrm{SE}(2)) \,\big|\, \rho_\varphi \neq id \text{ and } \tau \neq 0\}$$

Corollary 3.12. *If π is a free group on s generators, then*

$$\mathcal{SR}^{reg} \simeq (\mathbb{T}^s \setminus \{id\}) \times \mathbb{CP}^{s-2}$$

Proof. This is an immediate consequence of the previous results, in particular Proposition 3.2 and 3.11.

□

4. Deformation Theory

4.1. **The Moduli and Teichmüller spaces.** The moduli space of $\Sigma_{g,n}$ is the quotient of the space of conformal structures on $\Sigma_{g,n}$ by the pure diffeomorphism group. Let us be more specific: recall first that a conformal structure is an equivalence class of smooth Riemannian metric m on $\Sigma = \Sigma_{g,n}$, where two Riemannian metrics m_1, m_2 are equivalent if and only if there exists a function $u : \Sigma \to \mathbb{R}$ such that

$$m_2 = e^{2u} m_1.$$

We denote by $\mathrm{Met}(\Sigma)$ the space of all smooth Riemanian metrics on Σ endowed with its natural C^∞ topology and by

$$\mathrm{Conf}(\Sigma) = \mathrm{Met}(\Sigma)/C^\infty(\Sigma)$$

the space of conformal structures. We then define the *moduli space* of $\Sigma_{g,n}$ to be the quotient

$$\mathcal{M}_{g,n} = \mathrm{Conf}(\Sigma)/\,\mathrm{PDiff}^+_{g,n}\,.$$

A point $\mu \in \mathcal{M}_{g,n}$ is concretely represented by a Riemannian metric m on Σ, and two Riemannian metrics m_1, m_2 represent the same modulus point μ if and only if there exists a smooth function $u : \Sigma \to \mathbb{R}$ and a diffeomorphism $h \in \mathrm{PDiff}^+_{g,n}$ such that $m_2 = e^{2u} h^*(m_1)$.

A remark about the smoothness: By definition a point μ in the moduli space is represented by a smooth metric. In particular, the puntures play no role in the definition of the spaces $\mathrm{Met}(\Sigma)$ and $\mathrm{Conf}(\Sigma)$ (but they do in the definition of the moduli space $\mathcal{M}_{g,n}$). However, since only the conformal class of the metric matters, one may also represent μ by a singular metric m as long as this metric is conformally equivalent to a smooth one. In particular we can (and will) represent a point in $\mathcal{M}_{g,n}$ by a metric having conical singularities at the punctures of $\Sigma_{g,n}$, see Remark 1.21

The moduli space is a complicated object, and it is useful to also introduce the simpler space obtaind by considering isotopy classes of conformal structures on $\Sigma_{g,n}$ instead of isomorphism classes: this is the *Teichmüller space* defined as

$$\mathcal{T}_{g,n} = \mathrm{Conf}(\Sigma_{g,n})/\,\mathrm{Diff}^0_{g,n}\,.$$

Let us list some of the basic facts about these spaces:

(1) The Teichmüller space $\mathcal{T}_{g,n}$ is a real analytic variety in a natural way. If $3g - 3 + n > 0$, then it is isomorphic to $\mathbb{R}^{6g-6+2n}$. This space has also a natural complex structure.
(2) The pure mapping class group $\mathrm{PMod}^+_{g,n}$ acts properly and discontinuously on $\mathcal{T}_{g,n}$.
(3) The moduli space is the quotient $\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\,\mathrm{PMod}^+_{g,n}$. It is thus a good orbifold of dimension $6g-6+2n$ with fundamental group $\pi_1(\mathcal{M}_{g,n}) = \mathrm{PMod}_{g,n}$.
(4) There exists a torsion free subgroup $M_0 \subset \mathrm{PMod}^+_{g,n}$ of finite index acting freely on $\mathcal{T}_{g,n}$. The quotient map $\mathcal{T}_{g,n}/M_0$ is a non singular analytic manifold which is a finite cover of the orbifold $\mathcal{M}_{g,n}$.

Proof. Statement (1) is explained in any textbook on Teichmüller theory such as [1]. Statement (2) was first proved by S. Kravetz [18], see also [1]. (3) is a consequence of (1) and (2) and the last statement is discussed in [20, §5.4].

□

4.2. **The deformation space of piecewise flat metrics.** Let us denote by $\mathcal{E}_{g,n}$ the set of all flat metrics on $\Sigma_{g,n}$ with possible conical singularities at the punctures (it is not empty since we have assumed $2g+n-2>0$). To any flat metric $m \in \mathcal{E}_{g,n}$, we associate the following basic invariants : Its conformal class $[m] \in \mathrm{Conf}(\Sigma)$, its area $A = A(m) > 0$ and the order $\beta_i > -1$ of m at the point p_i.

Theorem 4.1. *The map*

$$\begin{aligned} \mathcal{E}_{g,n} &\to \mathrm{Conf}(\Sigma) \times \mathbb{R}_+^n \\ m &\mapsto ([m], (s_1, ..., s_n)), \end{aligned}$$

where $s_i = A\,(1+\beta_i) > 0$, is a bijection.

Proof. This is just a reformulation of Theorem 1.22.

□

Definition 4.2. *We will endow the set $\mathcal{E}_{g,n}$ with the topology for which this map is a homeomorphism.*

A metric $m_2 \in \mathcal{E}_{g,n}$ is said to be a *deformation* of the metric $m_1 \in \mathcal{E}_{g,n}$ if the two metrics differ by a homothety and an isotopy fixing the punctures, i.e. if there exists $h \in \mathrm{PDiff}^0_{g,n}$ and $\lambda > 0$ such that $m_2 = \lambda\, h^*(m_1)$. We denote by $\mathcal{DE}_{g,n}$ the deformation space of flat metrics on $\Sigma_{g,n}$ with possible conical singularities at the punctures:

$$\mathcal{DE}_{g,n} = \mathcal{E}_{g,n}/(\mathbb{R}_+ \times \mathrm{PDiff}^0_{g,n}).$$

Corollary 4.3. *This space is homeomorphic to $\mathbb{R}^{6g+3n-7}$. In fact we have the following canonical identification:*

$$\mathcal{DE}_{g,n} = \mathcal{T}_{g,n} \times \Delta,$$

where $\mathcal{T}_{g,n}$ is the Teichmüller space and $\Delta \subset \mathbb{R}^n$ is defined by

$$\Delta = \{\vec{\beta} = (\beta_1, ...\beta_n) \in \mathbb{R}^n \,\big|\, \beta_i > -1 \text{ and } \sum_i \beta_i = 2g-2\}.$$

Let us fix an element $\vec{\beta} = (\beta_1, ...\beta_n) \in \Delta$ and denote by $\mathcal{E}_{g,n}(\vec{\beta})$ the space of singular flat metrics with a conical singularity of order β_i at p_i $(i = 1, ..., n)$. We also introduce the corresponding deformation space: $\mathcal{DE}_{g,n}(\vec{\beta}) = \mathcal{E}_{g,n}(\vec{\beta})/(\mathbb{R}_+ \times \mathrm{PDiff}_{g,n})$. The previous corollary gives us the identification

$$\mathcal{DE}_{g,n}(\vec{\beta}) = \mathcal{T}_{g,n}.$$

4.3. **Revisiting the development and the holonomy.** Consider the punctured surface $\Sigma_{g,n} = \widehat{\mathbb{U}}/\Gamma$ as in section 2.2, and fix a flat metric m_0 with conical singularity of order β_j at p_j $(j = 1, \dots, n)$. If f_0 is a germ of an isometry near a point $\tilde{z}_0$, to the euclidean plane (identified with $\mathbb{C}$), then we obtain a map $f : \mathbb{U} \to \mathbb{C}$ by analytic continuation from f_0. This map is a local isometry for the metric m_0 on $\mathbb{U}$ and the canonical metric on $\mathbb{C}$ (indeed, the set of points where a map f between two flat surfaces is an isometry is easily seen to be both open and closed). The map f extends by continuity to $\widehat{\mathbb{U}}$. The resulting map $f : \widehat{\mathbb{U}} \to \mathbb{C}$ is the developing map which we already met in Section 1.3. The associated holonomy is the unique homomorphism $\varphi : \Gamma \to \mathrm{SE}(2)$ such that

$$f(\gamma u) = \varphi(\gamma) f(u).$$

Theorem 4.4. *The following properties of the development and its holonomy are satisfied:*

(1) $f : \widehat{\mathbb{U}} \to \mathbb{C}$ *is surjective;*
(2) $f(\gamma u) = \varphi(\gamma) f(u)$ *for all* $\gamma \in \Gamma$*;*
(3) *for any* $y \in P^{-1}(p_i) \subset \Upsilon$*, the isometry* $\varphi(c_y) \in \mathrm{SE}(2)$ *is a rotation of angle* $\theta_i = 2\pi(\beta_i + 1)$*;*
(4) *if* β_i *is not an integer, then* $f(y)$ *is the unique fixed point of* $\varphi(c_y)$*.*

Proof. The first assertion has been proved in Proposition 1.10, the second is the definition of the holonomy and the last two assertions are contained in Proposition 1.15.

□

Corollary 4.5. *If* $\beta_i \notin \mathbb{Z}$ *for any* $i = 1, ..., n$*, then the restriction of* f *to the set* Υ *is determined by the holonomy.*

Proof. Fix $y \in \Upsilon$ and let $c_y \in \Gamma$ be the corresponding group element given by the map (2.7). Then $f(y) \in \mathbb{C}$ is the fixed point of the rotation $\varphi(c_y) \in \mathrm{SE}(2)$. This fixed point is given explicitly by

$$f(y) = \frac{\tau_y}{(1 - \rho_y)}, \tag{4.14}$$

where $\rho_y \in U(1)$ is the rotation part and $\tau_y \in \mathbb{C}$ is the translation part of $\varphi(c_y)$. □

Theorem 4.6. (A) *There is a well defined map*

$$hol : \mathcal{DE}_{g,n} \to \mathcal{SR}(\pi_{g,n}, \mathrm{SE}(2)),$$

such that $hol([m])$ *is the conjugacy class of the holonomy homomorphism* $\varphi_m : \pi_{g,n} \to \mathrm{SE}(2)$*.*

(B) *The map* $hol : \mathcal{DE}_{g,n} \to \mathcal{SR}(\pi_{g,n}, \mathrm{SE}(2))$ *is continuous.*

(C) *There are natural actions of* $\mathrm{PMod}_{g,n}$ *on* $\mathcal{DE}_{g,n}$ *and* $\mathrm{POut}(\pi_{g,n})$ *on* $\mathcal{SR}(\pi, \mathrm{SE}(2))$*, and the map hol is* Φ*-equivariant where* Φ *is the Dehn-Nielsen-Baer isomorphism (2.8).*

(D) *The map hol is locally injective.*

Remarks. 1.) The map $hol : \mathcal{DE}_{g,n} \to \mathcal{SR}(\pi_{g,n}, \mathrm{SE}(2))$ is called the *holonomy mapping.*
2.) A more elaborate investigation would show that the holonomy mapping is in fact real analytic, see [36]. The proof below is perhaps not optimal from the point of view of rigour, but we have tried to emphasize the geometric point of view.

Proof. (A) To any flat metric m on $\Sigma_{g,n}$ with conical singularities at the punctures, we have associated a holonomy homomorphism $\varphi_m : \pi_{g,n} \to \mathrm{SE}(2)$ which depends on the choice of a developing map f_m, but changing the developing map does not affect the conjugacy class of φ_m (see Proposition 1.14). On the other hand, it is clear that if two flat metrics m, m' on $\Sigma_{g,n}$ are similar, the associated holonomies $\varphi_m, \varphi_{m'}$ are also similar. In short, to any deformation class of flat metric $[m] \in \mathcal{DE}_{g,n}$ with conical singularities on $\Sigma_{g,n}$ we associate a well defined element $[\varphi_m] = hol(m) \in \mathcal{SR}(\pi_{g,n}, \mathrm{SE}(2))$.

(B) The developing map f_m is not uniquely associated to a flat metric m, but it is well defined modulo $SE(2)$ (two developing maps for the same metric differ by postcomposition with an isometry). The $SE(2)$ orbit of the developing map f_m varies continuously with the metric m and therefore it is also the case for the associated holonomy class. Hence the map $hol : \mathcal{DE}_{g,n} \to \mathcal{SR}(\pi_{g,n}, \mathrm{SE}(2))$ is continuous.

(C) Any diffeomorphism h of $\Sigma_{g,n}$ fixing the punctures acts on $\mathcal{E}_{g,n}$ by pulling back the metric $(m \mapsto h^*m)$. If h is isotopic to the identity, it acts trivially on $\mathcal{DE}_{g,n}$; we thus have a well defined action of $\mathrm{PMod}_{g,n}$ on $\mathcal{DE}_{g,n}$.

Similarly, any automorphism of $\pi_{g,n}$ acts on $\mathrm{Hom}(\pi_{g,n}, \mathrm{SE}(2))$, and inner automorphisms act trivially on the representation spaces $\mathcal{R}(\pi_{g,n}, \mathrm{SE}(2))$ and $\mathcal{SR}(\pi_{g,n}, \mathrm{SE}(2))$. We thus have a natural action of $\mathrm{POut}(\pi_{g,n})$ on these spaces. It is clear from the construction of the isomorphism $\Phi : \mathrm{PMod}_{g,n} \xrightarrow{\sim} \mathrm{POut}(\pi_{g,n})$ (see the proof of Lemma 2.3) that the map hol is equivariant.

(D) To prove the local injectivity of hol, we consider two nearby flat metrics m, m' with conical singularities on $\Sigma_{g,n} = \widehat{\mathbb{U}}/\Gamma$ and we assume that they have the same holonomy φ. Since the holonomy around a conical singularity p_i is a rotation of angle $\theta_i = 2\pi(\beta_i + 1)$, it is clear that both metrics m and m' have the same singularity order (the holonomy only controls the cone angle modulo 2π, but since m and m' are nearby metrics, they actually have equal cone angles).

It follows that both metrics are isometric near the singularities: we can thus find an isotopy h_1 of the surface such that $m = h_1^* m'$ near the singularities. Hence we can simply assume without loss of generality that $m = m'$ near the singularities; it is therefore possible to divide the surface in $n + 1$ parts

$$\Sigma_{g,n} = D \cup E_1 \cup \cdots \cup E_n,$$

where $D \subset \Sigma'$ is a compact region and E_i is a neighbourhood of the puncture p_i such that $m = m'$ on E_i. We also assume that the E_i are pairwise disjoint disks. We denote by $\widehat{E}_i = P^{-1}(E_i) \subset \widehat{\mathbb{U}}$ and $\widehat{D} = P^{-1}(D) \subset \widehat{\mathbb{U}}$ the lifts of E_i and D on the universal branched cover $P : \widehat{\mathbb{U}} \to \Sigma_{g,n}$. The set $\widehat{E} = \cup_i \widehat{E}_i \subset \widehat{\mathbb{U}}$ is a neighbourhood of $\Upsilon = P^{-1}(\{\text{punctures}\})$.

Let f_m and $f_{m'}$ be the developing maps of m and m'. By Corollary 4.5, the two maps coincide on Υ. Because the two metrics coincide on E_i, the map f_m and $f_{m'}$ coincide up to a rotation on each component of $\widehat{E}_i$; we can thus find a second isotopy h_2 of Σ, which is a rotation near the punctures and is the identity on D and such that $f_{m'} \circ \widehat{h}_2 = f_m$ on $\widehat{E}$.

Replacing m' with $h_2^* m'$, we can thus assume that both developing maps coincide on $\widehat{E}$.

To any point $x \in \mathbb{U}$, we associate the set

$$\Lambda(m, m', x) = f_m^{-1}(f_{m'}(x)) \subset \widehat{\mathbb{U}}.$$

Since f_m is a local diffeomorphism, $\Lambda(m, m', x)$ is a discrete set. It varies continuously with m, m'.

Claim: *If m is close enough to m', then for any point $x \in U$, there exists a unique point $y = Q(x) \in \Lambda(m, m', x)$ which is the nearest point for the hyperbolic distance. The map $x \mapsto Q(x)$ is Γ-equivariant.*

Indeed, if $x \in \widehat{E}$, then the claim is clear: since $f_m(x) = f_{m'}(x)$, we have $Q(x) = x$. For any point, the claim is clear if $m = m'$ (and in this case $Q(x) = x$). For points in $\widehat{D}$, and m' close to m the claim follows from the compactness of D and the discreteness and continuity of $\Lambda(x, m, m')$.

For $t \in [0, 1]$, we denote by $Q_t(x)$ the point on the hyperbolic segment $[x, Q(x)]$ such that $d_H(x, Q_t(x)) = t d_H(x, Q(x))$ (observe that if $x \in \widehat{E}$, then $Q_t(x) = x$ for any t). This is a Γ-equivariant isotopy of U from the identity to Q. It extends as the identity on Υ.

We now define an isotopy $h_t : \Sigma \to \Sigma$ by $h_t(x) = P(Q_t(P^{-1}(x)))$. It is a well defined isotopy such that $h_1^* m' = m$, since we clearly have $f_m = f_{m'} \circ Q$.

We thus have proved that two metrics with the same holonomy are isotopic provided they are close enough. In other words, the map *hol* is locally injective.

□

5. The Main Theorem

We are now in position to prove the main result. First recall the statement:

Theorem 5.1. *Given a punctured surface $\Sigma_{g,n}$ such that $2g+n-2 > 0$ and $\vec{\beta} \in \Delta$ such that no β_i is an integer, there is a well defined group homomorphism*

$$\Phi : \mathrm{PMod}_{g,n} \to \mathcal{G} = \mathrm{Aut}(\mathbb{T}^{2g}) \times \mathrm{PGL}_{2g+n-2}\,\mathbb{C},$$

and a Φ-equivariant local homeomorphism

$$\mathcal{H} : \mathcal{T}_{g,n} \to \Xi = \mathbb{T}^{2g} \times \mathbb{CP}^{2g+n-3}.$$

The theorem says that $\mathcal{M}_{g,n} = \mathcal{T}_{g,n} / \mathrm{PMod}_{g,n}$ is a good orbifold with a $(\mathcal{G}, \Xi)$-structure.

Proof The group homomorphism Φ is given by the Dehn-Nielsen-Baer isomorphism and the map $\mathcal{H}$ is essentially given by the holonomy mapping of the previous theorem. We divide the proof of the theorem in 5 steps:

Recall that holonomy splits in a rotation part $\rho_m : \pi \to U(1)$ (the character) and a translation part τ_m. The character depends only on the conjugacy class of φ_m.

Step 1: *τ_m is not identically zero.*

Indeed, if $\tau_m \equiv 0$, then the holonomy group $\varphi_m(\pi_{g,n})$ is a pure rotation group in the plane. This is impossible by Proposition 1.13.

Step 2: *There is a canonical isomorphism*

$$\begin{array}{ccc} \mathrm{Hom}(\pi_{g,n}, U(1)) & \simeq & \mathrm{Hom}(\pi_{g,0}, U(1)) \times \mathrm{Hom}(\pi_{0,n}, U(1)). \\ \rho_m & \mapsto & (\rho', \rho'') \end{array}$$

Furthermore $\rho'' \in \mathrm{Hom}(\pi_{0,n}, U(1))$ is given by

$$\rho''(c_i) = e^{\theta_i},$$

where c_i is the homotopy class of a loop traveling once around the puncture p_i and $\theta_i = 2\pi(\beta_i + 1)$ is the total angle at the cone point p_i.

This splitting easily follows from the identity (2.3) and the fact that $U(1)$ is abelian.

Let us now fix an element $\vec{\beta} = (\beta_1, ...\beta_n) \in \Delta$ and set

$$\mathcal{SR}_{\vec{\beta}}(\pi_{g,n}, \mathrm{SE}(2)) = \left\{\varphi \in \mathcal{SR}(\pi_{g,n}, \mathrm{SE}(2)) \,\middle|\, \rho''(c_i) = e^{\theta_i}, i = 1, ..., n\right\}$$

and

$$\mathcal{SR}^{reg}_{\vec{\beta}} = \mathcal{SR}^{reg} \cap \mathcal{SR}_{\vec{\beta}}.$$

Step 3: *If at least one β_i is not an integer, then we have*

$$\mathcal{SR}^{reg}_{\vec{\beta}}(\pi_{g,n}, \mathrm{SE}(2)) \simeq \Xi = \mathbb{T}^{2g} \times \mathbb{CP}^{2g+n-3}$$

Indeed, it follows from Step 2 and the results of Section 3 that any $\varphi \in \mathcal{SR}^{reg}_{\vec{\beta}}$ is characterized by $\rho' \in \mathrm{Hom}(\pi_{g,0}, U(1)) \simeq \mathbb{T}^{2g}$ and the projective class of $\tau \in H^1(\pi_{g,n}, \mathbb{C}_\rho) \simeq \mathbb{C}^{2g+n-2}$ (because $\pi_{g,n}$ is isomorphic to the free group on $s = 2g+n-1$ generators).

Step 4: *The group* $\mathrm{POut}(\pi_{g,n})$ *acts naturally on* Ξ *and thus we have a natural homomorphism* $\Phi : \mathrm{PMod}_{g,n} \to \mathcal{G} = \mathrm{Aut}(\mathbb{T}^{2g}) \times \mathrm{PGL}_{2g+n-2}\,\mathbb{C}$.

This is clear from Step 3 and part (C) of Theorem 4.6.

Step 5: *The map $\mathcal{H}$ given by the composition:*

$$\mathcal{T}_{g,n} \overset{\simeq}{\to} \mathcal{DE}_{g,n}(\vec{\beta}) \overset{hol}{\to} \mathcal{SR}^{reg}_{\vec{\beta}} \overset{\simeq}{\to} \Xi$$

is well defined, continuous, locally injective and Φ*-equivariant.*

Indeed, the fact that no β_i is integer, together with Step 1, implies that $hol : \mathcal{DE}_{g,n} \to \mathcal{SR}(\pi, \mathrm{SE}(2))$ maps $\mathcal{DE}_{g,n}(\vec{\beta})$ into $\mathcal{SR}^{reg}_{\vec{\beta}}$. The map $\mathcal{H} : \mathcal{T}_{g,n} \to \Xi$ is therefore well defined. It follows from Theorem 4.6 that $\mathcal{H}$ is continuous, locally injective and Φ-equivariant.

It remains only to show that $\mathcal{H}$ is a local homeomorphism, but since $\mathcal{T}_{g,n}$ and Ξ are both manifolds of dimension $6g-6+2n$, the conclusion follows from Brouwer's Theorem on invariance of dimension.

□

5.1. **The case of the sphere.** Suppose that $g = 0$, i.e. Σ is a sphere, choose n numbers ($n \geq 2$) $\beta_1, \beta_2, \ldots, \beta_n$ such that $2+\sum_i \beta_i = 0$, and denote by $\mathcal{M}$ the space of flat metrics on S^2 having n conical singularities of order $\beta_1, \beta_2, \ldots, \beta_n$.

Such a metric $m \in \mathcal{M}$ can be uniformized as follows : identify Σ with $\mathbb{C} \cup \infty$, and write m as

$$m = C \cdot \prod_{i=1}^{n} |z - p_i|^{2\beta_i} |dz|^2\,,$$

where $p_1, p_2, \ldots, p_n$ is the set of conical singularities and C is a positive constant representing a dilation factor. It is easy to see that $\mathcal{M}$ is homeomorphic to the quotient

$$\{(p_1, p_2, \ldots, p_n) \in (\mathbb{C} \cup \infty)^n : p_i \neq p_j \text{ if } i \neq j\}/PSL_2(\mathbb{C})\,,$$

$\mathcal{M}$ is thus a complex manifold of dimension $n-3$, its fundamental group is the pure braid group PB_n.

Applying the main theorem, we obtain a representation

$$\Phi : \mathrm{PMod}_{0;n} = PB_n \to \mathrm{PGL}_{n-2}(\mathbb{C})$$

and a Φ equivariant, local homeomorphism

$$\mathcal{H} : \mathcal{T}_{0,n} \to \mathbb{CP}^{n-3}.$$

In fact, a finer analysis shows that the image of Φ is contained in $PU(1, n-3) \subset \mathrm{PGL}_{n-2}(\mathbb{C})$. Furthermore, when the orders satisfy some arithmetical conditions, the image of Φ is a lattice in $PU(1, n-3)$:

Theorem 5.2. *Assume that $-1 < \beta_1, \beta_2, ..., \beta_n < 0$, $\sum_i \beta_i = -2$ and suppose that*

$$\beta_i + \beta_j > -1 \;\Rightarrow\; (1 + \beta_j + \beta_i)^{-1} \in \mathbb{N}, \tag{5.15}$$

then $\Phi(PB_n)$ is a lattice in $PU(1, n-3)$.
These lattices are quotients of the braid group. Some of them are non arithmetic.

This Theorem was first proved by Schwartz (1873) for $n = 4$ and by Picard (1888) for $n = 5$ in their study of the monodromy of the hypergeometric equations. It has been generalized for any n by P. Deligne and G. Mostow in 1986, see [10]. These authors use the cohomology with coefficients in flat vector bundle on an algebraic curve.

In the paper [31], W. Thurston obtain the same result by studying a deformation space of piecewise flat triangulations on the sphere (this nice paper is a 1987 preprint of W. Thurston, which has been rewritten and appeared in electronic form in 1998). It is worthwile to quote also the related papers [4, 13, 25, 28].

Our approach can be seen as a bridge between the approach of Thurston and that of Deligne-Mostow.

Observe that the moduli space $\mathcal{M} = \mathcal{T}_{0,n}/PB_n$ carries a complex hyperbolic metric (depending upon the choice of the β_i's). It is not complete as a Riemannian manifold and it carries a natural completion $\overline{\mathcal{M}}$. Thurston shows that $\overline{\mathcal{M}}$ is a complex hyperbolic manifold with singularities of conical type. This cone-manifold has finite volume.

Furthermore, when the β_i's satisfy the condition (5.15), then $\overline{\mathcal{M}}$ is an orbifold. It is thus possible to construct complete complex hyperbolic orbifolds $\overline{\mathcal{M}}$ of finite volume.

References

[1] W. Abikoff, *The real analytic theory of Teichmüller space*, Lecture Notes in Mathematics, 820. Springer, Berlin (1980).

[2] J. Ambjorn, M. Carfora, A. Marzuoli, *The geometry of dynamical triangulations* Lecture Notes in Physics, New Series m50, Springer (1997).

[3] B. H. Bowditch. Singular Euclidean structures on surfaces. Journal of the London Mathematical Society **44** (1991), pp 553-565.

[4] C. Bavard and E. Ghys, *Polygones du plan et Polyèdres Hyperboliques*, Geom. Dedicata 43 (1992), **2**, 207–224.

[5] J. S. Birman, *Braids, Links and Mapping Class Groups*, Annals of Math. Studies, No. 82, Princeton University Press, Princeton, New Jersey, 1974.

[6] D. Burago, Yu Burago and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics. vol. 33, American Mathematical Society

[7] R.D. Canary, D.B. Epstein, D. B. A. and P. Green, *Notes on notes of Thurston*, In: London Math. Soc. Lecture Note Ser. 111, Cambridge University Press, Cambridge, 1987.

[8] M. Carfora, C. Dappiaggi, and A. Marzuoli, *The modular geometry of random Regge triangulations.* Classical Quantum Gravity **19** (2002), no. 20, 5195–5220.

[9] S. Choi, *Geometric structures on orbifolds and holonomy representations,* Geom. Dedicata 104 (2004), 161–199.

[10] P. Deligne and G. Mostow *Monodromy of hypergeometric functions and non lattice integral monodromy*, Publ. I.H.E.S. **63** 1986 .
[11] D.B.A. Epstein and R.C. Penner, *Euclidean decompositions of noncompact hyperbolic manifolds*. J. Differ. Geom. **27**, No.1, 67-80 (1988).
[12] A. Eremenko, *Metrics of positive curvature with conic singularities on the sphere*, Proc. Amer. Math. Soc. 132 (2004), no. 11, 3349–3355
[13] F. Fillastre, *From Spaces of Polygons to Spaces of Polyhedra Following Bavard, Ghys and Thurston*, Paper on arXiv:math.MG/0308187
[14] W. Goldman, *Geometric structures on manifolds and varieties of representations*, Geometry of group representations (Boulder, CO, 1987), 169–198, Contemp. Math., 74, Amer. Math. Soc., Providence, RI, 1988.
[15] M. Heins, *On a class of conformal metrics*, Nagoya Math. J., 21 (1962) 1–60.
[16] D. Hulin and M. Troyanov, *Sur la courbure des surfaces ouvertes*, C.R. Acad. Sci. Paris, **310** (1990) 203-206.
[17] D. Hulin and M. Troyanov, *Prescribing Curvature on Open Surfaces*, Math. Annalen **293** (1992) 277-315 .
[18] S. Kravetz *On the geometry of Teichmüller spaces and the structure of their modular groups*, Ann. Acad. Sci. Fenn., Ser. A1, 278, (1959)
[19] C. Indermitte, T. M. Liebling, H. Clémençon and Marc Troyanov, *Voronoi Diagrams on Piecewise Flat Surfaces and an Application to Biological Growth*, Theoretical Computational Science **263** (2001) 263-274 .
[20] N. V. Ivanov, *Mapping Class Groups*, in Handbook of geometric topology, 523–633 (edited by R. J. Daverman and R. B. Sher.) North-Holland, Amsterdam, 2002.
[21] S. Katok, *Fuchsian groups*, Chicago Lectures in Mathematics, 1992
[22] S. Kojima, *Complex hyperbolic cone structures on the configuration spaces*, Rend. Istit. Mat. Univ. Trieste 32 (2001), suppl. 1, 149–163 (2002).
[23] R. McOwen, *Point singularities and conformal metrics on Riemann surfaces*, Proc. AMS, 103 (1988) 222-224.
[24] S. Morita, *Introduction to mapping class groups of surfaces and related groups*, this book.
[25] G. Mostow, *Braids, hypergeometric functions, and lattices*, Bull. Amer. Math. Soc. **16** (1987), no. 2, 225–246.
[26] É. Picard, *De l'intégration de l'équation* $\Delta u = e^u$ *sur une surface de Riemann fermée*, J. reine angew. Math., 130 (1905) 243–258.
[27] I. Rivin *Combinatorial optimization in geometry*, Advances in Applied Mathematics archive, Volume 31, 1 (2003).
[28] S. Kojima, *Complex hyperbolic cone structures on the configuration spaces*, Rend. Istit. Mat. Univ. Trieste 32 (2001), suppl. 1, 149–163 (2002).
[29] G. Shimura *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; and Princeton University Press, Princeton, N.J., 1971.
[30] W. Thurston *Three-dimensional geometry and topology*, Vol. 1. Princeton University Press, Princeton, NJ, 1997
[31] W. Thurston *Shapes of polyhedra and triangulations of the sphere*, The Epstein birthday schrift, 511–549 (electronic), Geom. Topol. Monogr., 1, Publ., Coventry, 1998.
[32] M. Troyanov, *Les Surfaces Euclidiennes à singularités coniques*, Ens. Math. **32** 79–94 1986 .
[33] M. Troyanov, *Prescribing Curvature on Compact Surfaces with Conical singularities*, Trans. Amer. Math. Soc. **324** (1991) 793-821.
[34] M. Troyanov, *Surfaces Riemanniennes à Singularités Simples*, Proceedings of Symposia Pure Math. **54** part 2 (1993) 619-628.
[35] M. Umehara and K. Yamada, *Metrics of constant curvature 1 with three conical singularities on 2-sphere*, Illinois J. Math., 44 (2000) 72–94.
[36] W.A. Veech, *Flat Surfaces*, Amer. J. Math. **115** (1993) no 3, 589–689.
[37] H. Zieschang, E. Vogt, H.-D. Coldeway, *Surfaces and planar discontinuous groups*, Lecture Notes in Math., No. 835, Springer-Verlag, 1980.

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