Kinetic and Fluid Ballooning Stability with Anisotropic Energetic Electron Layers

W. A. Cooper

Ecole Polytechnique Fédérale de Lausanne
Centre de Recherches en Physique des Plasmas
Association Euratom-Confédération Suisse
CH-1015 Lausanne, Switzerland

Abstract. A kinetic ballooning mode theory is developed from the gyrokinetic equation in the frequency range for which the ions are fluid, the thermal electron response is adiabatic and the hot electrons are non-interacting due to their large drift velocity. Trapped particle effects are ignored. The application of the quasineutrality condition together with the parallel and binormal components of Ampere's Law reduces the gyrokinetic equation to a second order ordinary differential equation along the equilibrium magnetic field lines. The instability dynamics are dominated by the pressure gradients of the thermal species in the fluid magnetohydrodynamic limit. The resulting equation combines features of both the Kruskal-Oberman energy principle [1] and the rigid hot particle energy principle proposed by Johnson *et al*. [2] to model the Astron device.

Keywords: gyrokinetic, ballooning, quasineutrality, Ampere's Law
PACS: 52.35.Py

INTRODUCTION

Ballooning mode theory [3, 4] applied to magnetohydrodynamics (MHD) yields criteria that give useful indications about the stability properties of a magnetically confined system, but these criteria tend to be pessimistic because they ignore finite diamagnetic drift and Larmor radius corrections. Diamagnetic drifts already play an important role in the local stability properties of stellarators [5]. For conditions where energetic particle species are strongly non-Maxwellian, the pressure moment anisotropy that is driven can be significant. Fluid theories have been developed that treat the hot particle contributions to the pressure and current as fully interacting [6] (based on the Kruskal-Oberman energy principle [1]) or treat the hot particle drift frequency to be much larger than typical mode growth rates so that energetic particle layers are effectively rigid and noninteracting [7] (based on an energy principle derived by Johnson *et al*. [2]).

THE GYROKINETIC EQUATION

A kinetic ballooning mode theory has previously been applied to consider energetic ion species [8]. In this article, we consider the impact of an energetic electron species on the kinetic ballooning stability of a magnetically confined plasma configuration in the intermediate frequency regime and in which trapped particle effects are ignored. The linearised gyrokinetic equation valid for particle species in which the unperturbed distribution function depends on the energy E and the magnetic moment μ is [9]

$$v_{\parallel}(b\cdot\nabla)\hat{g}_j - i(\omega-\omega_{dj})\hat{g}_j = i\omega\frac{e_j}{M_j}\left(\frac{\partial F_{0j}}{\partial E} - \frac{M_j B\times k_{\perp}\cdot\nabla F_{0j}}{e_j\omega B^2}\right) \qquad (1)$$
$$[J_0(\alpha_j)(\hat{\phi}-v_{\parallel}\hat{A}_{\parallel}) + (v_{\perp}/k_{\perp})J_1(\alpha_j)\delta\hat{B}_{\parallel}],$$

where the perturbed distribution function $\hat{f}_j$ is expressed as

$$\begin{aligned}\hat{f}_j &= \hat{g}_j exp(iL_j) + \frac{e_j}{M_j}\left(\frac{\partial F_{0j}}{\partial E} + \frac{1}{B}\frac{\partial F_{0j}}{\partial \mu}\right)\hat{\phi} \qquad (2)\\ &- \frac{e_j}{M_j B}\frac{\partial F_{0j}}{\partial \mu}\left[J_0(\alpha_j)\hat{\phi} + \frac{v_{\perp}}{k_{\perp}}J_1(\alpha_j)\delta\hat{B}_{\parallel}\right] - \frac{e_j}{M_j B}\frac{\partial F_{0j}}{\partial \mu}[1-J_0(\alpha_j)exp(iL_j)]v_{\parallel}\hat{A}_{\parallel}.\end{aligned}$$

We have applied the Coulomb gauge $\nabla\cdot\hat{A}=0$ for the vector potential $\hat{A}$, while b is the unit vector along unperturbed magnetic field lines, B is the equilibrium magnetic field strength, $\hat{g}_j$ is the nonadiabatic component of the perturbed distribution function, the phase factor $L_j \equiv b\times k_{\perp}\cdot v_{\perp}/\Omega_j$, $k_{\perp}$ is the wave vector, $\Omega_j \equiv e_j B/M_j$ is the gyrofrequency of a particle of the $j-th$ species, ω_{dj} is the drift frequency written as $\omega_{dj} = -(\omega_{\kappa}v_{\parallel}^2 + \omega_B v_{\perp}^2/2)/\Omega_j$, $\omega_{\kappa} = b\times k_{\perp}\cdot\kappa$ is the curvature component of the drift while $\omega_B = B\times k_{\perp}\cdot\nabla B/B^2$ corresponds to the grad-B component, $v_{\perp}$ is the particle velocity field perpendicular to the field lines, $v_{\parallel}$ is the parallel component of the velocity with respect to the equilibrium magnetic field, F_{0j}, e_j and M_j represent the unperturbed distribution function, the charge and the mass of the $j-th$ species of the plasma, respectively. The perturbed field quantities are the electrostatic potential $\hat{\phi}$, the parallel component of the vector potential $\hat{A}_{\parallel}$ and the field compression $\delta\hat{B}_{\parallel}$. $J_0(\alpha_j)$ and $J_1(\alpha_j)$ are Bessel functions with argument $\alpha_j = k_{\perp}v_{\perp}/\Omega_j$. The application of Faraday's law and the constraint that the modes be MHD-like (parallel electric field vanishing) imples that we can write $i\omega\hat{A}_{\parallel} = b\cdot\nabla\hat{\psi}_{\parallel}$. The evaluations of the quasineutrality condition $(\sum_j e_j\int d^3v\hat{f}_j = 0)$ as well as of the parallel and binormal components (normal to the magnetic field and the wave vector) of Ampere's law $(\nabla\times\nabla\times\hat{A}=\hat{j})$ yield moments of the nonadiabatic component of the perturbed distribution function in terms of $\hat{\phi}$, $\hat{\psi}_{\parallel}$ and $\delta\hat{B}_{\parallel}$.

THE FREQUENCY ORDERING

The frequency ordering we adopt considers the thermal electrons as "adiabatic", namely $\omega-\omega_{de} \ll \omega_{be}, \omega_{te}, k_{\parallel}v_{the}$. The ionic species are treated as "fluid", namely $\omega_{bi}, \omega_{ti}, k_{\parallel}v_{thi} \ll \omega-\omega_{di}$. The energetic electrons are modelled as "rigid", namely $\omega \ll \omega_{bh}, \omega_{th}, k_{\parallel}v_h < \omega_{dh}$. Here the subscripts of ω indicated by b and t identify bounce and transit frequencies, while the subscripts *e*, *i* and *h* correspond to the thermal electrons, ion species and hot electrons, respectively. v_{the} (v_{thi}) is the thermal electron (ion) velocity. This is referred to as the intermediate frequency regime. The gyrokinetic equation is solved order by order. For the thermal electrons, we obtain at zero order that

$$\hat{g_{e0}}^{+} + \hat{g_{e0}}^{-} = 2J_0(\alpha_e)\frac{e}{M_e}\left[\frac{\partial F_{0e}}{\partial E} + \frac{M_e B \times k_\perp \cdot \nabla F_{0e}}{e\omega B^2}\right]\hat{\psi}_\|, \tag{3}$$

where $\hat{g}_j^{\pm} \equiv \hat{g}_j(\pm|v_\||)$. The other relevant expression required are

$$\hat{g_{e0}}^{+} - \hat{g_{e0}}^{-} = \hat{g_{e1}}^{+} + \hat{g_{e1}}^{-} = 0, \tag{4}$$

where the subscripts 0 and 1 refer to the lowest and first order, respectively. For the thermal ion species, the zero order in the $\omega_{bi}/max(\omega, \omega_{di})$ expansion yields

$$\hat{g_{i0}}^{+} + \hat{g_{i0}}^{-} = -\frac{2\omega}{\omega - \omega_{di}}\frac{Z_i e}{M_i}\left[\frac{\partial F_{0i}}{\partial E} - \frac{M_i B \times k_\perp \cdot \nabla F_{0i}}{Z_i e\omega B^2}\right]\left[J_0(\alpha_i)\hat{\phi} + \frac{v_\perp}{k_\perp}J_1(\alpha_i)\delta\hat{B}_\|\right], \tag{5}$$

where Z_i refers to the atomic number of the $i-th$ ionic species. The other relevant contributions are

$$\hat{g_{i0}}^{+} - \hat{g_{i0}}^{-} = \hat{g_{i1}}^{+} + \hat{g_{i1}}^{-} = 0. \tag{6}$$

First order expressions for $\hat{g_{e1}}^{+} - \hat{g_{e1}}^{-}$ and $\hat{g_{i1}}^{+} - \hat{g_{i1}}^{-}$ need not be evaluated. The examination of the gyrokinetic equation for the hot electrons reveals that the nonadiabatic component of the perturbed distribution function vanishes to zero order. To first order we obtain $\hat{g_{h1}}^{+} + \hat{g_{h1}}^{-} = 0$ and

$$\hat{g_{h1}}^{+} - \hat{g_{h1}}^{-} = 2i\frac{e}{M_e}\left[\frac{\partial F_{0h}}{\partial E} + \frac{M_e B \times k_\perp \cdot \nabla F_{0h}}{e\omega B^2}\right]\frac{J_0(\alpha_h)|v_\||}{\omega_{dh}} b \cdot \nabla\hat{\psi}_\|. \tag{7}$$

THE KINETIC BALLOONING MODE EQUATION

Substituting the relevant contributions for the nonadiabatic component of the perturbed distribution function of each species in the quasineutrality condition and in the binormal component of Ampere's law secures expressions of the electrostatic potential $\hat{\phi}$ and perturbed parallel magnetic field $\delta\hat{B}_\|$ in terms of $\hat{\psi}_\|$. The next step is to expand the $\sum_j e_j \int d^3vexp(iL_j)$ moment of the gyrokinetic equation to derive the kinetic ballooning mode equation in the intermediate frequency regime with a strongly drifting energetic electron layer. The resulting equation is

$$(B \cdot \nabla)\left[\frac{k_\perp^2}{B^2}(1+\Gamma)(B \cdot \nabla)\hat{\psi}_\|\right] = \mathscr{K}\hat{\psi}_\|, \tag{8}$$

where we have assumed that the thermal species are represented by Maxwellian distribution functions,

$$\Gamma \equiv \frac{4\pi}{k_\perp^2}\sum_j \frac{e_j^2}{M_j B}\int_0^\infty dE\int_0^{E/B} d\mu B|v_\||[1 - J_0^2(\alpha_j)]\frac{\partial F_{0j}}{\partial \mu}, \tag{9}$$

and ignoring contributions of $O(\beta^2)$,

$$\begin{aligned}
\mathscr{K} &= \frac{eN_e\omega}{B}(\omega_\kappa+\omega_B)\Big(1-\frac{\omega_{*pe}}{\omega}\Big) \\
&- \frac{e^2N_e^2\omega^2}{p_e}\Big(1-\frac{\omega_{*e}}{\omega}\Big)\Bigg\{1-\frac{\Big(1-\frac{\omega_{*e}}{\omega}\Big)}{[1+\sum_i\tau_i(Q_i+U_i)+\tau_hU_h]}\Bigg\} \\
&- \frac{e^2N_e^2\omega^2}{B^2}\Bigg\{\Big(1-\frac{\omega_{*pe}}{\omega}\Big)-\Big(1-\frac{\omega_{*e}}{\omega}\Big)\frac{\sum_i z_i(Q_i'+U_i')-z_hU_h'}{[1+\sum_i\tau_i(Q_i+U_i)+\tau_hU_h]}\Bigg\}^2.
\end{aligned} \tag{10}$$

Although the frequency ordering previously applied to consider energetic ions [8] differs from that applied for the hot electrons investigated in this work, the resulting kinetic ballooning mode equation derived is the same for both cases. The diamagnetic drift frequencies are defined as

$$\omega_{*j}\equiv-\frac{p_{\perp j}B\times k_\perp\cdot\nabla\Phi}{Z_jeB^2N_j^2}\frac{\partial N_j}{\partial\Phi}\Big|_B; \qquad \omega_{*pj}\equiv\frac{B\times k_\perp\cdot\nabla\Phi}{Z_jeB^2N_j}\frac{\partial p_{\perp j}}{\partial\Phi}\Big|_B \tag{11}$$

where $2\pi\Phi$ is the toroidal magnetic flux, while $p_{\perp j}$ and N_j represent the perpendicular pressure and density of the $j-th$ species, respectively. The radial derivatives (with respect to Φ) are evaluated at fixed B. However, for the Maxwellian thermal species, the corresponding pressures and densities are constant on the flux surfaces. The other definitions that apply are

$$Q_j \equiv \frac{p_{\perp j}}{\rho_{mj}N_j}\int d^3v\Big(\frac{\omega}{\omega-\omega_{dj}}\Big)J_0^2(\alpha_j)\Big[\frac{\partial F_{0j}}{\partial E}-\frac{M_jB\times k_\perp\cdot\nabla F_{0j}}{e_j\omega B^2}\Big], \tag{12}$$

$$Q_j' \equiv \frac{p_{\perp j}}{\rho_{mj}N_j}\int d^3v\Big(\frac{\omega}{\omega-\omega_{dj}}\Big)\frac{\partial J_0^2(\alpha_j)}{\partial b_j}\Big[\frac{\partial F_{0j}}{\partial E}-\frac{M_jB\times k_\perp\cdot\nabla F_{0j}}{e_j\omega B^2}\Big], \tag{13}$$

$$U_j \equiv -\frac{p_{\perp j}}{\rho_{mj}N_j}\int d^3v\Big[\frac{\partial F_{0j}}{\partial E}+\frac{1-J_0^2(\alpha_j)}{B}\frac{\partial F_{0j}}{\partial\mu}\Big], \tag{14}$$

$$U_j' \equiv \frac{p_{\perp j}}{\rho_{mj}N_jB}\int d^3v\frac{\partial J_0^2(\alpha_j)}{\partial b_j}\frac{\partial F_{0j}}{\partial\mu}. \tag{15}$$

The mass density $\rho_{mj}\equiv M_jN_j$, the Larmor radius parameter $b_j\equiv k_\perp^2p_{\perp j}/(\Omega_j^2\rho_{mj})$, while $\tau_j\equiv Z_j^2N_j^2p_e/(p_{\perp j}N_e^2)$, $z_j\equiv Z_jN_j/N_e$ and p_e is the electron thermal pressure.

THE MHD APPROXIMATION

To obtain the MHD limit, an expansion of Q_i and Q_i' in ω_{di}/ω that removes the drift resonances, of the Bessel functions in the smallness of their argument α_i and the approximation $b_i<1$ must be undertaken. This yields $z_i(Q_i'+U_i')-z_hU_h'\approx Z_iN_i(1-\omega_{*pi}/\omega)/N_e$. Using the condition $U_i=1$, $U_i'=0$ valid for Maxwellian thermal ions and $\tau_hU_h\sim$

$O(N_h/N_e)^2$, we can reduce $1+\tau_i(Q_i+U_i)+\tau_h U_h \approx 1-\omega_{*e}/\omega-\tau_h\omega_{*h}/\omega+(1-\omega_{*pi}/\omega)\tau_i[b_i+p_i(\omega_\kappa+\omega_B)/(\rho_{mi}\Omega_i\omega)]$. Finally $1+\Gamma\approx\sigma$, where σ is the firehose stability criterion parameter corresponding to $1+(p_\perp-p_\parallel)/B^2$ and $p_\perp$ ($p_\parallel$) is the total perpendicular (parallel) pressure. Therefore, the driving and inertial terms of the ballooning mode equation in the MHD approximation become

$$\begin{aligned}\mathscr{K} = & - \Big[\frac{\partial(p_\parallel-p_{\parallel h})}{\partial\Phi}\Big|_B+\frac{\sigma}{\tau}\frac{\partial(p_\perp-p_{\perp h})}{\partial\Phi}\Big|_B\Big]\Big(\frac{B\times k_\perp\cdot\nabla\Phi}{B^2}\Big)\frac{B\times k_\perp\cdot\kappa}{B^2} \\ & + \Big[\frac{\partial(p_\perp-p_{\perp h})}{\partial\Phi}\Big|_B\Big]\Big(\frac{B\times k_\perp\cdot\nabla\Phi}{B^3}\Big)^2\frac{\partial p_{\perp h}}{\partial\Phi}\Big|_B \\ & - \frac{k_\perp^2\omega^2}{B^2}\Big[\sum_i M_iN_i\Big(1-\frac{\omega_{*pi}}{\omega}\Big)-\Big(\frac{M_eN_h}{b_h}\Big)\frac{\omega_{*h}}{\omega}\Big].\end{aligned} \tag{16}$$

The first term on the right hand side corresponds to the interactions of essentially the thermal pressure gradients with the magnetic field line curvature, the second term represents the interaction of thermal and hot particle pressure gradients which is formally small and the last term identifies the plasma inertia modified by thermal ion and hot electron diamagnetic drift effects.

COMPARISON WITH FLUID BALLOONING EQUATIONS

It is instructive to compare the kinetic ballooning mode equation in the MHD limit derived in the previous section with fully fluid derivations of the ballooning mode equation as represented by the minimisations of the Kruskal-Oberman energy principle [1] (fully interacting energetic particle model) and the rigid hot particle energy principle proposed by Johnson *et al*. [2] The ballooning mode equation resulting from the rigid hot particle model of Johnson *et al*. can be expressed as [7]

$$\begin{aligned}& (B\cdot\nabla)\Big[\frac{k_\perp^2}{B^2}(B\cdot\nabla\chi)\Big]+\Big\{\Big[\frac{\partial(p_\parallel-p_{\parallel h})}{\partial\Phi}+\sigma\frac{\partial(p_\perp-p_{\perp h})}{\partial\Phi}\Big]\Big(\frac{B\times k_\perp\cdot\nabla\Phi}{B^2}\Big)\frac{B\times k_\perp\cdot\kappa}{B^2} \\ - & \frac{\partial(p_\perp-p_{\perp h})}{\partial\Phi}\Big(\frac{B\times k_\perp\cdot\nabla\Phi}{B^3}\Big)^2\Big[\frac{\partial p_{\perp h}}{\partial\Phi}\Big|_B+(\tau-1)\Big(\frac{B\times k_\perp\cdot\nabla B^2}{2B\times k_\perp\cdot\nabla\Phi}\Big)\Big]\Big\}\chi \\ - & \rho_m\gamma^2\frac{k_\perp^2}{B^2}\chi=0.\end{aligned} \tag{17}$$

(in the fluid limit $\omega\rightarrow i\gamma$) while the ballooning mode equation obtained from the Kruskal-Oberman energy principle is [6]

$$\begin{aligned}(B\cdot\nabla)\Big[\frac{\sigma k_\perp^2}{B^2}(B\cdot\nabla\chi)\Big] & + \Big(\frac{\partial p_\parallel}{\partial\Phi}\Big|_B+\frac{\sigma}{\tau}\frac{\partial p_\perp}{\partial\Phi}\Big|_B\Big)\Big(\frac{B\times k_\perp\cdot\nabla\Phi}{B^2}\Big)\frac{B\times k_\perp\cdot\kappa}{B^2}\chi \\ & - \rho_m\gamma^2\frac{k_\perp^2}{B^2}\chi=0,\end{aligned} \tag{18}$$

Note that $\rho_m\approx\sum_i M_iN_i$ and $\tau\equiv 1+(1/B)(\partial p_\perp/\partial B)$ is the mirror stability criterion parameter.

SUMMARY AND DISCUSSION

In summary, a kinetic ballooning mode equation has been derived with an energetic electron species population in the intermediate frequency regime where the thermal ions are fluid, the thermal electrons are adiabatic and the hot electrons are noniteractive due to their large drift velocity. Trapped particle effects have been ignored. A moment of the gyrokinetic equation in the ballooning approximation, the quasineutrality condition and the parallel and binormal components of Ampere's law constitute the formal equations that govern the derivation of the kinetic ballooning mode equation. Although the final expression is the same as that obtained for energetic ion species [8], the intermediate steps differ due to the frequency ordering assumed. The MHD limit of the kinetic ballooning mode equation combines features of the fluid ballooning equations derived from the Kruskal-Oberman theory and the rigid hot particle energy principle of Johnson *et al*. Specifically, the ballooning driving term of the kinetic equation with large hot electron drift velocity is more closely aligned with that obtained from the rigid model of Johnson *et al*. with some minor discrepancies such as the extra term proportional to $(\tau-1)B\times k_{\perp}\cdot\nabla B^2$ and a factor $1/\tau$ missing from one of the terms multiplying the field line curvature. However, the field line bending term is identical to that obtained from the Kruskal-Oberman energy principle.

ACKNOWLEDGMENTS

This research was partially sponsored by the Fonds National Suisse de la Recherche Scientifique and by Euratom.

REFERENCES

1. M. D. Kruskal and C. R. Oberman, *"On the Stability of Plasma in Static Equilibrium"*, *Phys. Fluids* **1**, 275 (1958).
2. J. L. Johnson, R. M. Kulsrud and K. E. Weimer, *"Application of the Energy Principle to Astron-Type and Other Axisymmetric Devices"*, *Plasma Phys.* **11**, 463 (1969).
3. J. W. Connor, R. J. Hastie and J. B. Taylor, *"High Mode Number Stability of an Axisymmetric Toroidal Plasma"*, *Proc. R. Soc. London Ser. A* **365**, 1 (1979).
4. R. L. Dewar and A. H. Glasser, *"Ballooning mode spectrum in general toroidal systems"*, *Phys. Fluids* **26**, 3038 (1983).
5. B. F. McMillan and R. L. Dewar, *"Stellarator Stability with Respect to Global Kinetic Ballooning Modes"* *Nucl. Fusion* **46**, 477 (2006).
6. W. A. Cooper, *"MHD Stability in 3D Anisotropic Pressure Plasmas"*, *Proc. Varenna-Lausanne International Workshop on Theory of Fusion Plasmas*, p. 311, Editrice Compositori (1992).
7. W. A. Cooper, *"Variational Formulation of the Linear MHD Stability of 3D Plasmas with Noninteracting Hot Electrons"*, *Plasma Phys. Control. Fusion* **34**, 1011 (1992).
8. W. A. Cooper, *" Kinetic and Fluid Ballooning Stability in Anisotropic Ion Tokamaks"* *Phys. Fluids* **26**, 1830 (1983).
9. P. J. Catto, W. M. Tang and D. E. Baldwin, *"Generalized Gyrokinetics"*, *Plasma Phys.* **23**, 639 (1981).