February 1985

LRP 227/83

QUASILINEAR THEORY OF UNIFORMLY MAGNETIZED INHOMOGENEOUS PLASMAS : ELECTROMAGNETIC FLUCTUATIONS

F. Yasseen and J. Vaclavik

QUASILINEAR THEORY OF UNIFORMLY MAGNETIZED INHOMOGENEOUS PLASMAS : ELECTROMAGNETIC FLUCTUATIONS

F. Yasseen and J. Vaclavik

Centre de Recherches en Physique des Plasmas
Association Euratom - Confédération Suisse
Ecole Polytechnique Fédérale de Lausanne
CH - 1007 Lausanne / Switzerland

ABSTRACT

The equations of quasilinear theory are derived for a uniformly magnetized inhomogeneous plasma interacting with electromagnetic field fluctuations. The derivation avoids the use of the random phase approximation. The equations are formulated in a way that allows the identification of corrections arizing from a finite Larmor radius to inhomogeneity scale ratio. These equations possess the necessary local conservation properties, and require no assumption on the ratio of the Larmor radius to the fluctuation wavelength.

I. INTRODUCTION

The basic ideas of the quasilinear theory (QLT) were laid down some twenty years ago in the pioneering work of Vedenov, Velikhov and Sagdeev,[1] and Drummond and Pines.[2] The scope of these original papers, however, was set within stringent limitations (i.e., unmagnetized homogeneous plasmas interacting with electrostatic field fluctuations), such that their equations were not applicable to many plasmas of interest. Nevertheless, these authors paved the way for a large body of literature, too large to be reviewed here, mainly concerned either with widening the scope of the theory (e.g., introduction of a magnetic field, removal of the electrostatic approximation),[3,4] or dealing with very specific physical problems.

A survey of the literature on the QLT allows a standard derivation to be identified : It involves the use of the Random Phase Approximation (RPA) applied to the amplitudes of the field fluctuations, and, in general, discounts the non-resonant wave-particle interactions. Moreover, RPA is consistent only with a homogeneous stationary medium.

The present article is the third of a series where the equations of QLT are derived for an inhomogeneous plasma, by means of a correlation function method which avoids the use of RPA.[5] Furthermore, the non-resonant wave-particle interactions are consistently taken into account. The first article of the series dealt with a non-magnetized plasma, interacting with electrostatic field fluctuations only.[6] In the second,[7] we considered a plasma immersed in a uniform magnetic field $\vec{B}_0$, with the inhomogeneity related gradients perpendicular to $\vec{B}_0$, but retained the restriction to electrostatic field fluctuations. It appeared then that the standard expansion[8] in the ratio of the Larmor radius to the inhomogeneity scale, ρ_L/a, was unnecessary.

In the work presented here, we remove the restriction to electrostatic field fluctuations. Also, the role played by the finite Larmor radius is further clarified.

The basic equations are spelled out in Sec. II. They are cast into Fourier space in Sec. III, with the electromagnetic field contributions written in terms of the correlation functions of the electric field fluctuations. This formulation is well suited to the adiabatic approximation,[5] to which Sec. IV is devoted. In Sec. V, we show how to isolate the adiabatic effects due to the finite Larmor radius, and reduce the equations to the lowest order in ρ_L/a. In Sec. VI, the resulting equations are inverted back into real space within the adiabatic approximation. We thus obtain two coupled evolution equations, one for the averaged distribution function of the guiding centers (of each species), the other for the correlation functions of the electric field fluctuations. Finally, in Sec. VII, we show that the coupled equations satisfy (locally) the necessary conservation properties, i.e., those related to the particles, and to their momentum and energy. Simultaneously, we derive explicit expressions for various fluxes.

II. BASIC EQUATIONS

Consider a collisionless plasma immersed in a uniform magnetic field $\vec{B}_0$. For each plasma component, the distribution function $f(\vec{v},\vec{r},t)$ is determined by the Vlasov equation,

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right) f = -\frac{q}{mc}\left(\vec{v} \times \vec{B}_0\right)\cdot\frac{\partial}{\partial \vec{v}} f$$
$$-\frac{q}{m}\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial}{\partial\vec{v}} f, \tag{1}$$

where $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ are electric and magnetic field fluctuations, q and m are respectively the charge and mass of each particle. Following conventional practice, the distribution function is split into an averaged part $\bar{f}$ and a fluctuating part $\tilde{f}$,

$$f = \bar{f}(\vec{v},\vec{r},t) + \tilde{f}(\vec{v},\vec{r},t),$$

with

$$\bar{f} \equiv \langle f \rangle,$$

where the brackets $\langle\ \rangle$ indicate an ensemble average. Furthermore, we assume that the fluctuating quantities are much smaller than the corresponding averaged ones. The fluctuating quantities, of course, average to zero.

The evolution equation for the averaged distribution function $\bar{f}(\vec{v},\vec{r},t)$ is

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\bar{f} = -\frac{q}{mc}(\vec{v}\times\vec{B}_o)\cdot\frac{\partial}{\partial\vec{v}}\bar{f}$$
$$-\frac{q}{m}\left\langle\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial}{\partial\vec{v}}\tilde{f}\right\rangle. \tag{2}$$

The evolution equation for the fluctuating part of the distribution function, once linearized, reads as

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\tilde{f} = -\frac{q}{mc}(\vec{v}\times\vec{B}_o)\cdot\frac{\partial}{\partial\vec{v}}\tilde{f}$$
$$-\frac{q}{m}\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial}{\partial\vec{v}}\bar{f}. \tag{3}$$

If cylindrical coordinates, with the z-axis parallel to $\vec{B}_o$, are introduced in velocity space, Eq.(3) becomes an inhomogeneous first-order differential equation in the polar velocity angle θ,

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\tilde{f} = \omega_c\frac{\partial}{\partial\theta}\tilde{f}$$
$$-\frac{q}{m}\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial}{\partial\vec{v}}\bar{f}, \tag{4}$$

where $\omega_c = qB_o/mc$ is the cyclotron frequency. Equation (4) should be solved for $\tilde{f}$ in terms of $\bar{f}$, and the result substituted in Eq.(2): We can thus eliminate $\tilde{f}$. Closure is obtained by combining this with Maxwell's equations,

$$c^2\,\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) + \frac{\partial^2}{\partial t^2}\vec{E} = -4\pi\frac{\partial}{\partial t}\vec{j}, \tag{5}$$

where the current density $\vec{j}(\vec{r},t)$ is given by

$$\vec{j} = \sum_{\text{species}} q \int d^3v \, \vec{v} \, \tilde{f}.$$

The index denoting the species is implied.

We assume that the averaged distribution function $\bar{f}$ has dependences of the form

$$\bar{f}(\vec{v},\vec{r},t) = F\left(v_\perp, v_\parallel, x + \frac{v_y}{\omega_c}, y - \frac{v_x}{\omega_c}, t\right), \qquad (6)$$

where $F(v_\perp, v_\parallel, x, y, t)$, the distribution function of the guiding centers, has a weak time dependence.[8] In fact, this assumption is equivalent to the lowest order approximation of the expansion in inverse powers of ω_c,[9]

$$\bar{f} = f_0 + \frac{1}{\omega_c} f_1 + O\left(\frac{1}{\omega_c^2}\right), \qquad (7)$$

where f_0 has to satisfy the equilibrium Vlasov equation. In this case,

$$f_0(\vec{v},\vec{r},t) = F\left(v_\perp, v_\parallel, x + \frac{v_y}{\omega_c}, y - \frac{v_x}{\omega_c}, t\right).$$

III. QUASILINEAR EQUATIONS IN FOURIER SPACE

A. Fluctuating part of the distribution function

Casting Eq.(4) into Fourier space, and solving for $\tilde{f}$, we obtain

$$
\begin{aligned}
\tilde{f}(\vec{v},\vec{K}) &= \frac{q}{m\omega_c}\int^{\theta} d\theta' \exp\left[\frac{i}{\omega_c}\xi_{\vec{K}}(\theta') - \frac{i}{\omega_c}\xi_{\vec{K}}(\theta)\right] \\
&\times \int d\!\!\!^{-}Q' \, \frac{E_\nu(\vec{K}-\vec{Q}')}{\omega-\Omega'} \Big\{ \delta_{\nu\gamma}\big[\omega-\Omega' \\
&- (k_\parallel - q'_\parallel)v_\parallel - k_\perp v_\perp \cos(\theta'-\varphi) + q'_\perp v_\perp \cos(\theta'-\psi')\big] \\
&+ (k_\gamma - q'_\gamma)v_\nu \Big\} \frac{\partial}{\partial v_\gamma} \bar{f}(\vec{v},\vec{Q}) , \qquad (8)
\end{aligned}
$$

where the Greek indices denote the Cartesian coordinates, and the phase space variable $\vec{v}$ appearing under the θ'-integral depends on θ'. The Fourier transformation is indicated by the dependence on the wavenumber-frequency vectors $\vec{K} \equiv \{\vec{k},\omega\}$ and $\vec{Q}' \equiv \{\vec{q}',\Omega'\}$. The polar angles associated with $\vec{k}_\perp$ and $\vec{q}'_\perp$ are respectively φ and Ψ', and $d\!\!\!^{-}Q' \equiv d^3q'd\Omega'/(2\pi)^4$. The symbol $\xi_{\vec{K}}(\theta)$ stands for the primitive, with respect to the polar velocity angle, of the free-streaming operator $\partial/\partial t + \vec{v}\cdot\vec{\nabla}$ in Fourier space, and is given by

$$\xi_{\vec{k}}(\theta) = (\omega - k_{\parallel} v_{\parallel})(\theta - \varphi) - k_{\perp} v_{\perp} \sin(\theta - \varphi).$$

Notice that the θ'-integral is evaluated only at the upper bound. This ensures that $\tilde{f}$ is periodic in θ. In order to satisfy causality, we assume that the frequency ω has a small, positive, imaginary part.

The evaluation of the θ'-integral in Eq.(8) is straightforward. The Fourier transform of $\bar{f}$ can be written as

$$\bar{f}(\vec{v}, \vec{Q}') = \exp\left[\frac{i}{\omega_c} (\vec{q}\,' \times \vec{v}) \cdot \vec{e}_{\parallel}\right] F(v_{\perp}, v_{\parallel}, \vec{Q}'), \tag{9}$$

where $\vec{e}_{\parallel}$ is the unit vector along $\vec{B}_0$. Since homogeneity is assumed along the magnetic field, $F(v_{\perp}, v_{\parallel}, \vec{Q}')$ contains by definition a function $\delta(q_{\parallel}')$. Commuting $\partial/\partial v_{\gamma}$ with the exponential yields

$$\frac{\partial}{\partial v_{\gamma}} \bar{f}(\vec{v}, \vec{Q}') = \exp\left[\frac{i}{\omega_c} (\vec{q}\,' \times \vec{v}) \cdot \vec{e}_{\parallel}\right]$$
$$\times \left(i\, \varepsilon_{\parallel \tau \gamma} \frac{q_{\tau}'}{\omega_c} + \frac{\partial}{\partial v_{\gamma}} \right) F(v_{\perp}, v_{\parallel}, \vec{Q}'), \tag{10}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor. Since

$$\frac{1}{\omega_c} (\vec{q}\,' \times \vec{v}) \cdot \vec{e}_{\parallel} = \frac{q_{\perp}' v_{\perp}}{\omega_c} \sin(\theta' - \psi'),$$

we can decompose the exponential in Eq.(10) by using the well-known identity,[10]

$$\exp(a\, i \sin\theta) = \sum_n J_n(a) \exp(i n \theta),$$

where the J_n are Bessel functions of the first kind. The fluctuating part $\tilde{f}$ of the distribution function is then given by

$$\tilde{f}(\vec{v},\vec{K}) = -\frac{iq}{m} \exp\left[\frac{i}{\omega_c}(\vec{k}\times\vec{v})\cdot\vec{e}_\parallel\right]$$

$$\times \sum_j \frac{\exp(-ij\theta)}{\Omega_j(k_\parallel,\omega)} \int dQ'\; E_\beta(\vec{K}-\vec{Q}')\; P_\theta^{j}(\vec{k}_\perp-\vec{q}_\perp{}') \qquad (11)$$

$$\left[A^{D}_{\theta\beta}(\vec{K}-\vec{Q}',\vec{q}_\perp{}') + A_{\theta\beta}(\vec{K}-\vec{Q}')\right] F(v_\perp, v_\parallel, \vec{Q}),$$

where

$$\Omega_j(k_\parallel,\omega) = \omega - k_\parallel v_\parallel - j\omega_c, \qquad (12)$$

$$A^{D}_{\theta\beta}(\vec{K},\vec{q}_\perp{}') = -i\left(\delta_{\theta\parallel}\,\varepsilon_{\parallel\beta\tau}\frac{1}{v_\parallel}\right. \qquad (13)$$

$$\left. + \delta_{\theta\beta}\frac{k_\mu}{\omega}\varepsilon_{\parallel\mu\tau} - \frac{k_\theta}{\omega}\varepsilon_{\parallel\beta\tau}\right)\frac{v_\perp}{\omega_c} q'_\tau,$$

$$A_{\theta\beta}(\vec{K}) = \frac{1}{\omega}\Big[\left(\delta_{\theta\beta}k_\parallel - \delta_{\beta\parallel}k_\theta\right) g(v) \qquad (14)$$

$$+\frac{\omega}{v_\parallel}\delta_{\beta\parallel}\delta_{\theta\parallel} g(v) + \delta_{\theta\beta}\,\omega\frac{\partial}{\partial v_\perp}\Big],$$

and

$$g(v) = v_\perp \frac{\partial}{\partial v_\parallel} - v_\parallel \frac{\partial}{\partial v_\perp}. \tag{15}$$

To obtain Eq.(11), it was necessary to invoke the recurrence and addition properties of the Bessel functions, in particular Graf's addition theorem.[10] The most convenient form for this theorem is

$$\mathcal{J}_j(\vec{k}_\perp - \vec{q}_\perp{}') = \sum_n \mathcal{J}_{j+n}(\vec{k}_\perp)\, \mathcal{J}_n^*(\vec{q}_\perp{}'), \tag{16}$$

where

$$\mathcal{J}_j(\vec{k}_\perp) \equiv J_j\left(\frac{k_\perp v_\perp}{\omega_c}\right) \exp(ij\varphi), \quad \tan\varphi = \frac{k_y}{k_x}, \tag{17}$$

and $\vec{k}_\perp$ and $\vec{q}_\perp'$ are defined with respect to common Cartesian coordinates. In terms of (17), the components of the vector $\vec{P}^j$ are given by

$$\begin{aligned} P_x^j(\vec{k}_\perp) &= \tfrac{1}{2}\left(\mathcal{J}_{j+1}(\vec{k}_\perp) + \mathcal{J}_{j-1}(\vec{k}_\perp)\right), \\ P_y^j(\vec{k}_\perp) &= \tfrac{1}{2}\left(-i\,\mathcal{J}_{j+1}(\vec{k}_\perp) + i\,\mathcal{J}_{j-1}(\vec{k}_\perp)\right), \\ P_\parallel^j(\vec{k}_\perp) &= \frac{v_\parallel}{v_\perp}\mathcal{J}_j(\vec{k}_\perp). \end{aligned} \tag{18}$$

We should mention that the term $A^D_{\theta\beta}$ includes the spatial gradients related to the $\vec{E} \times \vec{B}$ drifts. These gradients are also obtained within the drift kinetic approximation,[9] where finite ρ_L/a effects are neglected.

B. Averaged Vlasov equation

Once the fluctuating part $\tilde{f}$ of the distribution function is explicitely known, it becomes possible to calculate both the RHS of Eq.(2), i.e., the quasilinear (diffusive) term, and the current density $\vec{j}$. The Fourier transform of Eq.(2) is

$$-i\Omega \exp\left[\frac{i}{\omega_c}(\vec{q}\times\vec{v}).\vec{e}_\parallel\right] F(v_\perp, v_\parallel, \vec{Q}) =$$

$$-\frac{q}{m}\int d\kappa \left\langle \left(E_\alpha(\vec{Q}-\vec{\kappa}) + \frac{1}{c}\varepsilon_{\alpha\mu\nu}\, v_\mu\, B_\nu(\vec{Q}-\vec{\kappa})\right) \right. \tag{19}$$

$$\left. \times \frac{\partial}{\partial v_\alpha}\tilde{f}(\vec{v},\vec{\kappa}) \right\rangle .$$

Strictly speaking, the RHS of Eq.(15) should include

$$-\left(i\,\vec{v}\cdot\vec{q} - \omega_c\frac{\partial}{\partial\theta}\right)\frac{1}{\omega_c} f_1(\vec{v},\vec{Q}),$$

the higher order contribution from the expansion (7). The evaluation of f_1 is too difficult to attempt. This contribution may, however, be conveniently integrated out by an adequate averaging procedure; Eq.(19) should be divided by the phase factor $\exp[i(\vec{q}\times\vec{v}).\vec{e}_\parallel/\omega_c]$, then averaged over θ. We thus obtain the equation of motion for the averaged distribution function of the guiding centers,

$$-i\Omega F(v_\perp, v_\parallel, \vec{Q}) = i \frac{q^2}{m^2} \sum_j \int d\vec{K}\, d\vec{Q}'$$

$$\times \left[A^D_{\alpha\lambda}(\vec{K}-\vec{Q}, \vec{q}_\perp) + A'_{\alpha\lambda}(\vec{K}-\vec{Q}) \right] P^{j*}_\lambda(\vec{k}_\perp - \vec{q}_\perp)$$

$$\times \frac{C_{\beta\alpha}\left(\vec{K} - \frac{\vec{Q}+\vec{Q}'}{2}, \vec{Q}-\vec{Q}'\right)}{\Omega_j(k_\parallel, \omega)} P^j_\theta(\vec{k}_\perp - \vec{q}'_\perp) \tag{20}$$

$$\times \left[A^D_{\theta\beta}(\vec{K}-\vec{Q}', \vec{q}'_\perp) + A_{\theta\beta}(\vec{K}-\vec{Q}') \right] F(v_\perp, v_\parallel, \vec{Q}'),$$

where

$$A'_{\alpha\lambda}(\vec{K}) = \frac{1}{\omega}\Big[h(v)\left(\delta_{\alpha\lambda} k_\parallel - \delta_{\alpha\parallel} k_\lambda\right) + h(v)\,\delta_{\alpha\parallel}\delta_{\lambda\parallel}\frac{\omega}{v_\parallel} + \delta_{\alpha\lambda}\frac{\omega}{v_\perp}\frac{\partial}{\partial v_\perp} v_\perp \Big], \tag{21}$$

and

$$h(v) = v_\perp \frac{\partial}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp}\frac{\partial}{\partial v_\perp} v_\perp . \tag{22}$$

One sees that the velocity space operator $A'_{\beta\theta}$ is the adjoint of $A_{\theta\beta}$. The term $C_{\alpha\beta}$, defined by

$$C_{\alpha\beta}(\vec{K}, \vec{Q}) = \left\langle E_\alpha\left(\vec{K} + \frac{\vec{Q}}{2}\right) E^*_\beta\left(\vec{K} - \frac{\vec{Q}}{2}\right) \right\rangle, \tag{23}$$

represents the correlation functions of the electric field fluctuations. The RHS of Eq.(20) then represents the quasilinear wave-particle interaction, the so-called "turbulent collisions".

It is easy to show that Eq.(18) is indeed the equation of motion for the distribution function, not of the particles, but of their guiding centers. The θ-average of Eq.(9) is

$$\overline{\frac{1}{2\pi}\int d\theta\, \bar{f}(\vec{v},\vec{Q}) \quad = J_0\left(\frac{q_\perp v_\perp}{\omega_c}\right) F(v_\perp, v_\parallel, \vec{Q})}$$

which differs from $F(v_\perp, v_\parallel, \vec{Q})$ by terms of $O(\rho_L^2/a^2)$ at least. This fact will have significant consequences for the transport equations.

C. Equations for the field fluctuations

Equation (20) governs the evolution of F (the plasma) in terms of $C_{\alpha\beta}$ (the field fluctuations). To make the system self-consistent, another equation is needed, that governs the evolution of $C_{\alpha\beta}$ in terms of F, i.e., a closure equation. This is provided by Maxwell's equations : We Fourier transform Eq. (5), multiply it (dyadically) by $E_\gamma\,(\vec{Q}-\vec{K})$, and finally average over the ensemble to obtain

$$-\frac{1}{\omega^2}\left[c^2 k_\alpha k_\beta + (\omega^2 - c^2 k^2)\delta_{\alpha\beta}\right] C_{\beta\gamma}\left(\vec{k}-\frac{\vec{Q}}{2}, \vec{Q}\right) =$$

$$\sum_{\text{species}} 4\pi \frac{q^2}{m} \sum_j 2\pi \int dv\, d\vec{Q}'\, \frac{v_\perp}{\omega}\, P_\alpha^{j*}(\vec{k}_\perp)$$

$$\times \frac{C_{\beta\gamma}\left(\vec{k}-\frac{\vec{Q}+\vec{Q}'}{2}, \vec{Q}-\vec{Q}'\right)}{\Omega_j(k_\parallel,\omega)} P_\theta^j(\vec{k}_\perp - \vec{q}_\perp')$$

$$\times \left[A^D_{\theta\beta}(\vec{k}-\vec{Q}', \vec{q}_\perp') + A_{\theta\beta}(\vec{k}-\vec{Q}')\right] F(v_\perp, v_\parallel, \vec{Q}'), \tag{24}$$

where $dv \equiv v_\perp\, dv_\perp\, dv_\parallel$. Equation (24) is the closure equation in question. The RHS of Eq.(24) again represents the interaction of the particles with the field fluctuations. The LHS, on the other hand, is purely due to the field fluctuations.

D. The correlation functions.

The correlation functions defined by (23) are simply the transforms of the auto-correlation functions of the electric field fluctuations, written in their center of mass coordinates $(\vec{\rho},\vec{R})$,[5,6]

$$\langle E_\alpha(\vec{R}_1)\, E_\beta(\vec{R}_2)\rangle = \langle E_\alpha\left(\vec{R}+\frac{\vec{\rho}}{2}\right) E_\beta\left(\vec{R}-\frac{\vec{\rho}}{2}\right)\rangle$$

$$\equiv C_{\alpha\beta}(\vec{\rho}, \vec{R}),$$

where

$$\vec{\rho} = \vec{R}_1 - \vec{R}_2 \quad , \quad \vec{R} = \frac{1}{2}(\vec{R}_1 + \vec{R}_2),$$

and the variables $\vec{\rho}$, $\vec{R}$, $\vec{R}_1$, $\vec{R}_2$ represent both space and time. In a homogeneous time-independent system, $C_{\alpha\beta}$ is a function of the difference variable $\vec{\rho}$ only. The inhomogeneity is then represented by $\vec{R}$ (the sum variable) and will be assumed, later on, to be slowly varying.

If $E_\alpha(\vec{K}_1)$ and $E_\beta(\vec{K}_2)$ are the Fourier transforms of $E_\alpha(\vec{R}_1)$ and $E_\beta(\vec{R}_2)$ respectively, then

$$\langle E_\alpha(\vec{K}_1)\, E_\beta^*(\vec{K}_2) \rangle = \langle E_\alpha(\vec{K} + \tfrac{\vec{Q}}{2})\, E_\beta^*(\vec{K} - \tfrac{\vec{Q}}{2}) \rangle$$

$$\equiv C_{\alpha\beta}(\vec{K}, \vec{Q}),$$

where

$$\vec{K} = \frac{1}{2}(\vec{K}_1 + \vec{K}_2) \quad , \quad \vec{Q} = \vec{K}_1 - \vec{K}_2 .$$

The Fourier transformation maps $\vec{\rho}$ into $\vec{K}$ and $\vec{R}$ into $\vec{Q}$. It can be easily shown by using the reality condition $E_\alpha(\vec{K}_1) = E_\alpha^*(-\vec{K}_1)$ that $C_{\alpha\beta}(\vec{K},\vec{Q})$ possesses the following symmetries:

$$C_{\alpha\beta}(\vec{K},\vec{Q}) = C_{\beta\alpha}(-\vec{K},\vec{Q}) = C^{*}_{\beta\alpha}(\vec{K},-\vec{Q})$$
$$= C^{*}_{\alpha\beta}(-\vec{K},-\vec{Q}). \tag{25}$$

We can express in terms of $C_{\alpha\beta}$ quantities that are quadratic in the field fluctuations, and that play important roles in the dynamics of the field fluctuations. For example, the Poynting vector, which represents the energy flux associated with the field fluctuations, is given by

$$S_\alpha(\vec{K},\vec{Q}) = \frac{c}{4\pi}\frac{1}{\omega}\Big[k_\alpha C_{\beta\beta}(\vec{K},\vec{Q})$$
$$- k_\beta C_{\beta\alpha}(\vec{K},\vec{Q})\Big], \tag{26}$$

to lowest order (c.f. Sec. IV). To the same order, the Maxwell stress tensor, which represents the pressure associated with the field fluctuations, takes the form

$$T_{\alpha\beta}(\vec{K},\vec{Q}) = \frac{1}{4\pi}\Big\{C_{\alpha\beta}(\vec{K},\vec{Q}) - \delta_{\alpha\beta}C_{\mu\mu}(\vec{K},\vec{Q})$$
$$+\frac{c^2}{\omega^2}\Big[\varepsilon_{\alpha\lambda\tau}\varepsilon_{\beta\theta\gamma}k_\lambda k_\theta C_{\tau\gamma}(\vec{K},\vec{Q}) \tag{27}$$
$$-\delta_{\alpha\beta}k^2 C_{\mu\mu}(\vec{K},\vec{Q}) + k_\nu k_\mu C_{\nu\mu}(\vec{K},\vec{Q})\Big]\Big\}.$$

The trace of this tensor is

$$T_{\alpha\alpha}(\vec{K},\vec{Q}) = -\frac{1}{8\pi}\left\{ C_{\alpha\alpha}(\vec{K},\vec{Q}) + \frac{c^2}{\omega^2}\left[k^2 C_{\alpha\alpha}(\vec{K},\vec{Q}) - k_\alpha k_\beta C_{\alpha\beta}(\vec{K},\vec{Q})\right]\right\}. \tag{28}$$

Its physical meaning is specified by

$$W^f = -T_{\alpha\alpha}\,, \tag{29}$$

where W^f is the energy density asssociated with the field fluctuations.

IV. ADIABATIC APPROXIMATION

Equations (20) and (24) are complicated integral equations in Fourier space and, as such, are hardly tractable. In particular, they cannot be easily inverted with respect to any of the variables $\vec{Q}$, $\vec{Q}'$, $\vec{K}$. However, if the adiabatic approximation is introduced, [5,11] inversion with respect to $\vec{Q}$, $\vec{Q}'$, the variables of the averaged distribution function F, becomes straightforward. The adiabatic approximation requires that the fluctuating part of any quantity (e.g., $\tilde{f}$) vary on a much faster scale than the corresponding average ($\bar{f}$). It is a necessary condition for QLT to hold. In the notation of Sec. III, this assumption imposes the condition that

$$|\vec{K}| \gg |\vec{Q}'|\,,\ |\vec{Q}|\,, \tag{30}$$

i.e., that the spectrum of F be much narrower than that of $\tilde{f}$. This ensures the existence of a small parameter η

$$\eta \sim |\vec{Q}'|/|\vec{K}|, |\vec{Q}'|/|\vec{K}| \ll 1 \tag{31}$$

in which Eqs.(20) and (24) will be expanded to first order. Henceforth, $\vec{Q}$ and $\vec{Q}'$ will be called slow variables, while $\vec{K}$ will be termed fast.

Furthermore, the adiabatic approximation entails that the growth rates (which affect the averages) are adiabatically smaller than the frequencies of the fluctuations,

$$|\omega_i| \ll |\omega_r|,$$

where ω_r and ω_i are the real and imaginary parts of the frequency.

To be consistent with the results of linear theory, the resonant contributions from integrals involving the denominator Ω_j also have to be adiabatically smaller than the non-resonant contributions. Symbolically,

$$\delta(\Omega_j) \ll \left|\frac{\mathcal{P}}{\Omega_j}\right|, \tag{32}$$

with

$$\eta \sim \delta(\Omega_j)\left|\frac{\mathcal{P}}{\Omega_j}\right|^{-1}, \tag{33}$$

where

$$\Omega_j \equiv \omega_r - k_\parallel v_\parallel - j\omega_c , \tag{34}$$

and $\mathcal{P}$ indicates the principal value.

Once the expansion is performed, inversion with respect to the variables $\vec{Q}$ and $\vec{Q}'$ can be done by following a straightforward prescription : Replace

$$\{\vec{q}',\Omega'\} \quad \text{by} \quad i\{-\vec{\nabla},\partial/\partial t\}$$

acting on $F(v_\perp, v_\parallel, \vec{R})$, and

$$\{\vec{q},\Omega\} \quad \text{by} \quad i\{-\vec{\nabla},\partial/\partial t\}$$

acting on both $F(v_\perp, v_\parallel, \vec{R})$ and $C_{\alpha\beta}(\vec{K},\vec{R})$. The slow dependence of the correlation function is in real space, while the rapid dependence remains in Fourier space.

V. SMALL LARMOR RADIUS APPROXIMATION

Before applying the adiabatic approximation, it is necessary to take a closer look at the functions $\mathcal{J}_j$, where the three different scalelengths (Larmor radius $v_\perp/\omega_c$, wavelength $k_\perp^{-1}$, and inhomogeneity scale $q_\perp^{-1}$) interact.

It can be easily seen that assumption (30) implies that

$$\varrho_L / a \sim \frac{q_\perp' v_\perp}{\omega_c} << 1, \tag{35}$$

since the terms obtained from an expansion of the RHS of Eq.(16) in the parameter ρ_L/a correspond order by order to those obtained from an adiabatic expansion of its LHS. These first order terms can be separated into adiabatic contributions on the one hand, and corrections due to the finite Larmor radius with respect to the inhomogeneity scale on the other. For example, the former will appear in the equation of motion for the correlation functions, while the latter provide corrections to the dielectric tensor. These corrections have been obtained previously by several authors.[8,12]

It is possible to explicitly separate these two contributions by using Graf's addition theorem. If we write the terms involving the Bessel functions in the "center of mass" coordinates (c.f.Sec.III), the averaged Vlasov equation (20), and the closure equation (24), become

$$\begin{aligned}
-i\Omega F(v_\perp, v_\parallel, \vec{Q}) &= \frac{i q^2}{m^2} \sum_{j,n,m} \int dk\, dQ' \left[A'_{\alpha\lambda}(\vec{k}-\vec{Q}) \right. \\
&+ \left. A^{D}_{\alpha\lambda}(\vec{k}-\vec{Q}, \vec{q}_\perp) \right] P_\lambda^{j+n*}\left(\vec{k}_\perp - \frac{\vec{q}_\perp + \vec{q}_\perp'}{2}\right) J_n\left(\frac{\vec{q}_\perp - \vec{q}_\perp'}{2}\right) \\
&\times \frac{C_{\beta\alpha}\left(\vec{k} - \frac{\vec{Q}+\vec{Q}'}{2}, \vec{Q}-\vec{Q}'\right)}{\Omega_j(k_\parallel, \omega)} J_m^*\left(\frac{\vec{q}_\perp' - \vec{q}_\perp}{2}\right) P_\theta^{j+m}\left(\vec{k}_\perp - \frac{\vec{q}_\perp + \vec{q}_\perp'}{2}\right) \qquad (36) \\
&\times \left[A^{D}_{\theta\beta}(\vec{k}-\vec{Q}', \vec{q}_\perp') + A_{\theta\beta}(\vec{k}-\vec{Q}') \right] F(v_\perp, v_\parallel, \vec{Q}'),
\end{aligned}$$

and

$$-\frac{1}{\omega^2}\left[c^2 k_\alpha k_\beta + (\omega^2 - c^2 k^2)\delta_{\alpha\beta}\right] C_{\beta\gamma}(\vec{k}-\tfrac{\vec{Q}}{2},\vec{Q}) =$$

$$\sum_{\text{species}} 4\pi \frac{q^2}{m} \sum_{j,n,m} 2\pi \int dv\, d\vec{Q}'\, \frac{v_\perp}{\omega}\, P_\alpha^{j+n^*}(\vec{k}_\perp - \tfrac{\vec{q}_\perp'}{2})$$

$$\times\, \mathcal{J}_n(-\tfrac{\vec{q}_\perp'}{2})\, \frac{C_{\beta\gamma}(\vec{k}-\frac{\vec{Q}+\vec{Q}'}{2}, \vec{Q}-\vec{Q}')}{\Omega_j(k_\parallel,\omega)}\, \mathcal{J}_m^*(\tfrac{\vec{q}_\perp'}{2})$$

(37)

$$\times\, P_\theta^{j+m}(\vec{k}_\perp - \tfrac{\vec{q}_\perp'}{2})\left[A^D_{\theta\beta}(\vec{k}-\vec{Q}',\vec{q}_\perp') + A_{\theta\beta}(\vec{k}-\vec{Q}')\right] F(v_\perp, v_\parallel, \vec{Q}').$$

The adiabatic approximation can now be applied to Eqs.(36) and (37) without any ambiguity. However, they cannot be inverted using the procedure outlined in the previous section, without an expansion in ρ_L/a of the functions $\mathcal{J}_m$ and $\mathcal{J}_n$. These functions contain the non-adiabatic finite Larmor radius corrections. To lowest order in ρ_L/a, Eqs.(36) and (37) reduce to

$$-i\Omega F(v_\perp, v_\parallel, \vec{Q}) = i\frac{q^2}{m^2}\sum_j \int d\vec{k}\, d\vec{Q}'$$

$$\times\left[A'_{\alpha\lambda}(\vec{k}-\vec{Q}) + A^D_{\alpha\lambda}(\vec{k}-\vec{Q},\vec{q}_\perp)\right]$$

$$\times\, P_\lambda^{j^*}(\vec{k}_\perp - \tfrac{\vec{q}_\perp+\vec{q}_\perp'}{2})\, \frac{C_{\beta\alpha}(\vec{k}-\frac{\vec{Q}+\vec{Q}'}{2},\vec{Q}-\vec{Q}')}{\Omega_j(k_\parallel,\omega)}\, P_\theta^{j}(\vec{k}_\perp - \tfrac{\vec{q}_\perp+\vec{q}_\perp'}{2}) \qquad (38)$$

$$\times\left[A^D_{\theta\beta}(\vec{k}-\vec{Q}',\vec{q}_\perp') + A_{\theta\beta}(\vec{k}-\vec{Q}')\right] F(v_\perp, v_\parallel, \vec{Q}')$$

and

$$-\frac{1}{\omega^2}\left[c^2 k_\alpha k_\beta + (\omega^2 - c^2 k^2)\delta_{\alpha\beta}\right] C_{\beta\gamma}\left(\vec{K}-\frac{\vec{Q}}{2}, \vec{Q}\right) =$$

$$\sum_{\text{species}} 4\pi \frac{q^2}{m} \sum_j 2\pi \int d\vec{Q}'\, dv\, \frac{v_\perp}{\omega}\, P_\alpha^{j\,*}\left(\vec{k}_\perp - \frac{\vec{q}_\perp}{2}\right)$$

$$\times \frac{C_{\beta\gamma}\left(\vec{K}-\frac{\vec{Q}+\vec{Q}'}{2}, \vec{Q}-\vec{Q}'\right)}{\Omega_j(k_\parallel, \omega)}\, P_\theta^j\left(\vec{k}_\perp - \frac{\vec{q}_\perp'}{2}\right) \tag{39}$$

$$\times \left[\overset{D}{A}_{\theta\beta}(\vec{K}-\vec{Q}', \vec{q}_\perp') + A_{\theta\beta}(\vec{K}-\vec{Q}')\right] F(v_\perp, v_\parallel, \vec{Q}').$$

In what follows, we shall consider this approximation only.

VI. <u>QUASILINEAR EQUATIONS</u>

A. Equation of motion for F

To avoid any ambiguity, let the operators in time, space, and velocity space act on everything they precede, while operators in Fourier space act only within the nearest brackets or parantheses. The operator $\overset{D}{A}_{\theta\beta}$ becomes

$$A^{D}_{\theta\beta}(\vec{k},\vec{\nabla}) = \frac{1}{\omega}\Big(- \delta_{\theta\parallel}\, \varepsilon_{\parallel\beta\tau}\, \frac{\omega}{v_\parallel} \tag{40}$$

$$- \delta_{\theta\beta}\, k_\mu\, \varepsilon_{\parallel\mu\tau} + k_\theta\, \varepsilon_{\parallel\beta\tau} \Big) \frac{v_\perp}{\omega_c} \nabla_\tau \, .$$

To simplify the notation, let us introduce the term $D^j_{\beta\alpha}$, defined by

---

$$D^{j}_{\beta\alpha}(\vec{k},\vec{R}) = \pi\, \delta(\Omega_j)\, C_{\beta\alpha}(\vec{k},\vec{R})$$

$$+ \frac{1}{2}\frac{\mathcal{P}}{\Omega_j}\left\{ \left[\frac{\partial}{\partial\omega} C_{\beta\alpha}\right]\frac{\partial}{\partial t} + \frac{\partial}{\partial t}\left[\frac{\partial}{\partial\omega} C_{\beta\alpha}\right]\right\}. \tag{41}$$

With this definition, Eq.(38), expanded and inverted according to the prescription of Sec. IV, may be written as

---

$$\frac{\partial}{\partial t} F(v_\perp, v_\parallel, \vec{R}) = \frac{q^2}{m^2} \sum_j \int d\kappa \left[A'_{\alpha\lambda}(\vec{\kappa}) + A^D_{\alpha\lambda}(\vec{\kappa},\vec{\nabla}) \right]$$

$$\times \; P_\lambda^{j*}(\vec{k}_\perp)\, D^j_{\beta\alpha}(\vec{\kappa},\vec{R})\, P^j_\theta(\vec{k}_\perp)\; (A^D_{\theta\beta} + A_{\theta\beta})F$$

$$- \frac{1}{2}\frac{q^2}{m^2} \sum_j \mathcal{P} \int d\kappa \; (A'_{\alpha\lambda} + A^D_{\alpha\lambda}) \frac{1}{\Omega_j}$$

$$\times \left\{ \left[\frac{\partial}{\partial k_\nu} P_\lambda^{j*} C_{\beta\alpha}(\vec{\kappa},\vec{R}) P^j_\theta\right] \nabla_\nu + \nabla_\nu \left[\frac{\partial}{\partial k_\nu} P_\lambda^{j*} C_{\beta\alpha} P^j_\theta\right]\right\}$$

$$\times \; (A^D_{\theta\beta} + A_{\theta\beta})\, F$$

$$(42)$$

$$+ \frac{q^2}{m^2} \sum_j \mathcal{P} \int d\kappa \; (A'_{\alpha\lambda} + A^D_{\alpha\lambda}) \frac{1}{\Omega_j} P_\lambda^{j*} C_{\beta\alpha} P^j_\theta$$

$$\times \left[\left(\frac{\partial}{\partial t}\frac{\partial}{\partial \omega} - \nabla_\nu \frac{\partial}{\partial k_\nu}\right)(A^D_{\theta\beta} + A_{\theta\beta})\right] F$$

$$+ \frac{q^2}{m^2} \sum_j \mathcal{P} \int d\kappa \; \left[\left(\frac{\partial}{\partial t}\frac{\partial}{\partial \omega} - \nabla_\nu \frac{\partial}{\partial k_\nu}\right)(A'_{\alpha\lambda} + A^D_{\alpha\lambda})\right]$$

$$\times \; \frac{1}{\Omega_j} P_\lambda^{j*} C_{\beta\alpha} P^j_\theta \; (A^D_{\theta\beta} + A_{\theta\beta})\, F \quad .$$

where some obvious dependences have been dropped. Equation (42), which is a diffusion equation in phase space, constitutes the equation of motion for the averaged distribution function. It involves both the resonant and non-resonant particles. With some algebra, one can show that Eq.(42) is real.

B. Closure equation

Equation (39) becomes, when submitted to the same treatment as Eq.(38),

$$-\frac{1}{\omega^2}\left[c^2 k_\alpha k_\beta + (\omega^2 - c^2 k^2)\,\delta_{\alpha\beta}\right]\left(1 - \frac{i}{2}\frac{\partial}{\partial t}\frac{\partial}{\partial \omega} + \frac{i}{2}\nabla_\nu \frac{\partial}{\partial k_\nu}\right) C_{\beta\gamma}(\vec{K},\vec{R}) =$$

$$- i \sum_{\text{species}} \frac{4\pi q^2}{m}\, 2\pi \sum_j \int dv\, \frac{v_\perp}{\omega}\, P_\alpha^{j*}(\vec{k}_\perp)\, D^j_{\beta\gamma}(\vec{K},\vec{R})\, P_\theta^j(\vec{k}_\perp)$$

$$\times \left[A^D_{\theta\beta}(\vec{K},\vec{v}) + A_{\theta\beta}(\vec{K})\right] F(v_\perp, v_\parallel, \vec{R})$$

$$+ \frac{i}{2} \sum_{\text{species}} \frac{4\pi q^2}{m}\, 2\pi \sum_j \mathcal{P}\int dv\, \frac{v_\perp}{\omega}\, \frac{1}{\Omega_j}\left[\left(\frac{\partial}{\partial k_\nu} P_\alpha^{j*} C_{\beta\gamma} P_\theta^j\right)\nabla_\nu\right. \tag{43}$$

$$\left. + \nabla_\nu P_\alpha^{j*}\left(\frac{\partial}{\partial k_\nu} C_{\beta\gamma}\right) P_\theta^j\right]\left(A^D_{\theta\beta} + A_{\theta\beta}\right) F$$

$$- i \sum_{\text{species}} 4\pi \frac{q^2}{m}\, 2\pi \sum_j \mathcal{P}\int dv\, \frac{v_\perp}{\omega}\, P_\alpha^{j*} C_{\beta\gamma} P_\theta^j\, \frac{1}{\Omega_j}$$

$$\times \left[\left(\frac{\partial}{\partial t}\frac{\partial}{\partial \omega} - \nabla_\nu \frac{\partial}{\partial k_\nu}\right)\left(A^D_{\theta\beta} + A_{\theta\beta}\right)\right] F .$$

Equation (43) is both tensorial and complex, and can be uniquely separated into Hermitean and anti-Hermitean parts.

C. Dispersion relation

To lowest order in ρ_L/a, the Hermitean part of Eq.(42) can be written as

$$\varepsilon^{ND}_{\alpha\beta}(\vec{K},\vec{R})\, C_{\beta\gamma}(\vec{K},\vec{R}) = 0 , \tag{44}$$

where $\varepsilon^{ND}_{\alpha\beta}(\vec{K},\vec{R})$ is the non-dissipative (Hermitean) part of the dielectric tensor. It is given by

$$\varepsilon^{ND}_{\alpha\beta}(\vec{K},\vec{R}) = \frac{1}{\omega^2}\left[c^2 k_\alpha k_\beta + (\omega^2 - c^2 k^2)\,\delta_{\alpha\beta}\right]$$

$$+ \sum_{\text{species}} 4\pi \frac{q^2}{m}\, 2\pi \sum_j \mathcal{P} \int dv\, \frac{v_\perp}{\omega^2}\, P^{j*}_\alpha(k_\perp)\, P^j_\beta(\vec{k}_\perp)\, \frac{1}{\Omega_j} \tag{45}$$

$$\times \left[A_{\theta\beta}(\vec{K}) - \delta_{\theta\beta} \frac{v_\perp}{\omega\,\omega_c}\, \varepsilon_{\parallel\mu\nu}\, k_\mu \nabla_\nu \right] F(v_\perp, v_\parallel, \vec{R}).$$

As expected, $\varepsilon^{ND}_{\alpha\beta}(\vec{K},\vec{R})$ involves only the non-resonant particles. Equation (44) can be used to deduce algebraic relations between the components of the correlation function, giving the different $C_{\alpha\beta}$ in terms of, say, C_{xx} and the components of the dielectric tensor. Furthermore, it implies that

$$\det \, \varepsilon_{\alpha\beta}^{ND}(\vec{k},\vec{R}) = 0 \, , \tag{46}$$

and

$$\det \, C_{\beta\gamma}(\vec{k},\vec{R}) = 0 \, . \tag{47}$$

Equation (46) is then the dispersion relation.

Note that expression (45) does not satisfy the Onsager relation relative to the inversion of the magnetic field, [13]

$$\varepsilon_{\alpha\beta}^{ND}(\vec{B}_o) = \varepsilon_{\beta\alpha}^{ND}(-\vec{B}_o) \, . \tag{48}$$

We can separate the dielectric tensor given by Eq.(45) into a part $\varepsilon_{\alpha\beta}^{Hom}$, that coincides with the dielectric tensor of a homogeneous plasma, and another $\varepsilon_{\alpha\beta}^{Inhom}$ that consists of gradient terms,

$$\varepsilon_{\alpha\beta}^{ND} = \varepsilon_{\alpha\beta}^{Hom} + \varepsilon_{\alpha\beta}^{Inhom} \, .$$

While $\varepsilon_{\alpha\beta}^{Hom}$ satisfies Eq.(48), $\varepsilon_{\alpha\beta}^{Inhom}$ does not. We have in fact

$$\varepsilon_{\alpha\beta}^{Inhom}(\vec{B}_o) = -\varepsilon_{\beta\alpha}^{Inhom}(-\vec{B}_o) \, ,$$

since the gradients carry the sign of ω_c.

It should be mentioned that Eq.(37) can be used to calculate the finite Larmor radius corrections due to the inhomogeneity in a systematic way . To do so, one should neglect the adiabatic corrections, and expand the functions $\mathcal{J}_m$ and $\mathcal{J}_n$ in powers of their argument. We have verified that the expressions obtained in this rather easy way agree with those given in standard references.[8]

D. Equation of motion for $C_{\alpha\beta}$

To lowest order in ρ_L/a, the anti-Hermitean part of Eq.(43) yields

$$\frac{\partial}{\partial t}\left[\varepsilon^{ND}_{\alpha\beta}(\vec{k},\vec{R})\frac{\partial}{\partial\omega}C_{\beta\gamma}(\vec{k},\vec{R})\right]+\left(\frac{\partial}{\partial\omega}C_{\beta\gamma}\right)\frac{\partial}{\partial t}\varepsilon_{\alpha\beta}$$

$$-\nabla_\nu\left(\varepsilon^{ND}_{\alpha\beta}\frac{\partial}{\partial k_\nu}C_{\beta\gamma}\right)-\frac{\partial}{\partial k_\nu}\left(C_{\beta\gamma}\nabla_\nu\varepsilon^{ND}_{\alpha\beta}\right)$$

$$-\varepsilon^{D}_{\alpha\beta}(\vec{k},\vec{R})\,C_{\beta\gamma}= \tag{49}$$

$$-C_{\beta\gamma}\;2\sum_{\text{species}}\frac{4\pi q^2}{m}\,2\pi\sum_j\mathcal{P}\int dv\,\frac{v_\perp}{\omega}\;P^{j*}_\alpha(\vec{k}_\perp)P^{j}_\theta(\vec{k}_\perp)$$

$$\times\frac{1}{\Omega_j}\left\{\left(\frac{\partial}{\partial t}\frac{\partial}{\partial\omega}-\frac{1}{2}\nabla_\nu\frac{\partial}{\partial k_\nu}\right)\left[A^{D}_{\theta\beta}(\vec{k},\vec{v})+A_{\theta\beta}(\vec{k})\right]\right\}F(v_\perp,v_\parallel,\vec{R})$$

where $\overset{D}{\varepsilon}_{\alpha\beta}(\vec{K},\vec{R})$ is the dissipative part of the dielectric tensor. Not surprisingly, it involves the resonant particles,

---

$$\varepsilon^{D}_{\alpha\beta}(\vec{K},\vec{R}) = -\pi \sum_{\text{species}} 4\pi \frac{q^2}{m} 2\pi \sum_{j} \int dv$$

$$\times \frac{v_\perp}{\omega^2} P_\alpha^{j*}(\vec{k}_\perp) P^{j}(\vec{k}_\perp)\, \delta(\Omega_j) \tag{50}$$

$$\times \left[A_{t,} \; \cdots \; \varepsilon_{\nu\beta} \frac{1}{\omega_c} \varepsilon_{\parallel\mu\nu} k_\mu \nabla_\nu \right] F(v_\perp, v_\parallel, \vec{R}).$$

Equation(49) constitutes the equation of motion for the correlation functions.

Before closing this section, we should draw attention to two points. Firstly, Eqs.(42) and (49) govern the self-consistent evolution of the distribution function F and the tensor of correlation functions $C_{\alpha\beta}$. For practical purposes, however, one need not follow the evolution of all the components of $C_{\alpha\beta}$. Instead, within the limits of the ordering used here, it suffices to follow the distribution function F and only one component of $C_{\alpha\beta}$; the others can be determined from Eq.(44), as outlined earlier.

Secondly, it is not generally possible to write all the non-resonant terms in Eq.(49) as a combination of $\vec{K}$ and $\vec{R}$-derivatives of $C_{\alpha\beta}$ and $\varepsilon^{ND}_{\alpha\beta}$, since the term on the RHS of Eq.(43) cannot be related to such derivatives of $\varepsilon^{ND}_{\alpha\beta}$.

The underlying reason for this is that one cannot separate the slow from the fast dependence in the unexpanded dielectric tensor, which has three explicit dependences : one on a fast scale ($\vec{K}$), another on a slow scale ($\vec{Q}'$), and the last mixing the slow and fast ($\vec{K}-\vec{Q}'$). Symbolically;

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(\vec{K}, \vec{K}-\vec{Q}', \vec{Q}') .$$

As an illustration for this point, we consider the behavior of the conductivity tensor, which is akin to that of the dielectric tensor. The general expression for Ohm's law in an inhomogeneous time-varying medium is

$$j_\alpha(\vec{\rho}) = \int dR' \, \sigma_{\alpha\beta}(\vec{\rho}, \vec{R}') \, E_\beta(\vec{R}') .$$

Whenever it is possible to separate the dependences of the conductivity tensor $\sigma_{\alpha\beta}$ into a fast dependence, and a slow one, the natural variables for this tensor would be the "center of mass" coordinates, i.e.,

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}\left(\vec{\rho} - \vec{R}', \frac{\vec{\rho} + \vec{R}'}{2}\right) , \qquad (51)$$

where the difference variable fluctuates rapidly, and the sum variable changes only slowly.

Equation (11) can be used to calculate an expression for Ohm's law in Fourier space explicitely. Symbolically, it takes the form

$$j_\alpha(\vec{k}) = t_\alpha(\vec{k}) \int d\vec{Q}' \, g_\beta(\vec{k}-\vec{Q}') \, f(\vec{Q}') \, E_\beta(\vec{k}-\vec{Q}') ,$$

which becomes

$$j_\alpha(\vec{\rho}) = \int dR' dR \, t_\alpha(\vec{\rho}-\vec{R}) \, g_\beta(\vec{R}-\vec{R}') \, f(\vec{R}) \, E_\beta(\vec{R}')$$

in real space. By identification, the conductivity tensor can be written as

$$\sigma_{\alpha\beta}(\vec{\rho}, \vec{R}') = \int dR \, t_\alpha(\vec{\rho}-\vec{R}) \, f(\vec{R}) \, g_\beta(\vec{R}-\vec{R}'),$$

i.e., it involves a convolution over $\vec{R}$. Nothing can be said about the variables $\vec{\rho}$ and $\vec{R}'$, one can only state that $\vec{R}$ is a slow variable. It is, of course, always formally possible to write $\sigma_{\alpha\beta}$ in the form given by (51). In the present case, however, due to the convolution over $\vec{R}$, one is not entitled to say that the difference variable is fast and the sum variable slow.

V. CONSERVATION PROPERTIES, TRANSPORT EQUATIONS, AND FLUXES

A. Conservation properties

Equations (42) and (49) govern the evolution of a closed system : A plasma interacting self-consistently with electromagnetic field fluctuations. Since no external sources are involved, the equations must satisfy (locally) the conservation of particles, of momentum, and of energy.

The evolution equations for the densities particles, momentum and energy (for each species) are provided by the appropriate velocity moments of Eq.(42). These are the transport equations.

The equations of particle transport adopt, of course, a conservative form, since there is no creation and annihilation of particles. On the other hand, the transport equations for the momentum and energy of a given plasma component are not individually conservative : They include sources that represent the quasilinear interaction between the particles and the field fluctuations (i.e., the "turbulent collisions"). This is understandable since, within the framework of QLT, particles and waves may exchange momentum and energy.

However, the transport equations for the momentum and energy associated with the plasma as a whole are conservative : Equation (49) may be used to write the sum of the sources related to each component in a conservative form, related only to the field fluctuations, e.g., the divergence of the Maxwell stress tensor or the Poynting vector.

B. Particle transport

Equation (42), integrated over the velocities, takes the form of an equation of continuity,

$$\frac{\partial}{\partial t} n(\vec{R}) + \nabla_\nu \Phi_\nu(\vec{R}) = 0 , \tag{52}$$

where n, defined by

$$n(\vec{R}) = 2\pi \int dv \, F(v_\perp, v_\parallel, \vec{R}) , \tag{53}$$

is the density of the guiding centers associated with the particles of the plasma component : Up to 1st order in ρ_L/a, n corresponds to the density of the particles (c.f. Sec.III). The guiding center flux Φ_ν is perpendicular to the magnetic field. With the definition

$$\begin{aligned}
\overline{\Phi}_\nu(v_\perp, v_\parallel, \vec{R}) = {} & \frac{qc}{mB_0} \varepsilon_{\parallel\mu\nu} \sum_j \int dK \, k_\mu \frac{v_\perp}{\omega} \\
& \times \Big\{ P_\alpha^{j*}(\vec{k}_\perp) \, D_{\beta\alpha}^j(\vec{K}, \vec{R}) \, P_\theta^j(\vec{k}_\perp) \left[A_{\theta\beta}^D(\vec{K}, \vec{\nabla}) + A_{\theta\beta}(\vec{K}) \right] \\
& - \frac{1}{2} \left[\frac{\partial}{\partial k_\nu} P_\alpha^{j*} C_{\beta\alpha}(\vec{K}, \vec{R}) P_\theta^j \frac{\mathcal{P}}{\Omega_j} \left(A_{\theta\beta}^D + A_{\theta\beta} \right) \right] \nabla_\nu \\
& - \frac{1}{2} \nabla_\nu \left(\frac{\partial}{\partial k_\nu} P_\alpha^{j*} C_{\beta\alpha} P_\theta^j \right) \frac{\mathcal{P}}{\Omega_j} \left(A_{\theta\beta}^D + A_{\theta\beta} \right) \\
& + P_\alpha^{j*} C_{\beta\alpha} P_\theta^j \frac{\mathcal{P}}{\Omega_j} \left[\left(\frac{\partial}{\partial t} \frac{\partial}{\partial \omega} - \frac{1}{2} \nabla_\nu \frac{\partial}{\partial k_\nu} \right) \left(A_{\theta\beta}^D + A_{\theta\beta} \right) \right] \Big\} F(v_\perp, v_\parallel, \vec{R}) \\
& - \frac{q^2}{m^2} \sum_j \mathcal{P} \int dK \left[\left(\frac{\partial}{\partial t} \frac{\partial}{\partial \omega} - \nabla_\nu \frac{\partial}{\partial k_\nu} \right) A_{\alpha\lambda}^D \right] \\
& \times P_\lambda^{j*} C_{\beta\alpha} P_\theta^j \frac{1}{\Omega_j} \left(A_{\theta\beta}^D + A_{\theta\beta} \right) F ,
\end{aligned} \tag{54}$$

the flux may be written as

$$\Phi_\nu(\vec{R}) = 2\pi \int dv \; \overline{\Phi}(v_\perp, v_\parallel, \vec{R}) \; . \tag{55}$$

The equations of transport for the total mass and charge, including the relevant fluxes, can be easily obtained from Eq.(41): We simply multiply by the mass (or charge) of each particle, and sum over the species. For the charge, we obtain

$$\frac{\partial}{\partial t} \rho(\vec{R}) + \nabla_\nu j_\nu(\vec{R}) = 0, \tag{56}$$

where ρ is the charge density, defined by

$$\rho(\vec{R}) = \sum_{\text{species}} q\, n(\vec{R}), \tag{57}$$

and j_ν is the current density,

$$j_\nu(\vec{R}) = \sum_{\text{species}} q\, \Phi_\nu(\vec{R}). \tag{58}$$

By contracting Eq.(43), and substituting into the RHS of Eq.(58), we obtain another expression for j_ν (we use Eq.(43) rather than Eq.(49) in order to simplify the algebra),

$$\begin{aligned} j_\nu(\vec{R}) = \frac{c}{8\pi B_0} \int d\kappa \; \frac{1}{\omega^2} \Big\{ & 2c^2 \left[k^2 \delta_{\alpha\beta} - k_\alpha k_\beta \right] \frac{k_\mu}{\omega} \frac{\partial}{\partial t} \\ & - \left[c^2 k_\alpha k_\beta + (\omega^2 - c^2 k^2)\, \delta_{\alpha\beta} \right] \nabla_\mu \\ & - c^2 2 k_\mu \left[k_\alpha \nabla_\beta - k_\theta \nabla_\theta \, \delta_{\alpha\beta} \right] \Big\} C_{\beta\alpha}(\vec{K}, \vec{R}). \end{aligned} \tag{59}$$

A straightforward implication from Eq.(59) is that the current vanishes if the correlation functions are homogeneous and time-independent.

C. Energy transport

For each plasma component, the equation of energy transport is

$$\frac{\partial}{\partial t} W(\vec{R}) + \nabla_\nu T_\nu(\vec{R}) = \beta(\vec{R}), \tag{60}$$

which is obtained by multiplying Eq.(42) by the kinetic energy of each particle $m(v_\perp{}^2+v_\parallel{}^2)/2$ and integrating over the velocities. The energy density W of the plasma component is defined by

$$W(\vec{R}) = \frac{m}{2} 2\pi \int dv\, (v_\perp^2+v_\parallel^2)\, F(v_\perp, v_\parallel, \vec{R}), \tag{61}$$

while the term T_ν, given by

$$\begin{aligned} T_\nu(\vec{R}) &= 2\pi \int dv\, \frac{m}{2} (v_\perp^2+v_\parallel^2)\, \Phi(v_\perp, v_\parallel, \vec{R}) \\ &- \frac{1}{2}\frac{q^2}{m} 2\pi \sum_j \mathcal{P} \int dv\, dK \left[\frac{\partial}{\partial k_\nu} P_\alpha^{j*}(\vec{k}_\perp)\, P_\theta^{j}(\vec{k}_\perp)\right] \frac{1}{\Omega_j} \\ &\times C_{\beta\alpha}(\vec{K},\vec{R}) \left[A^{D}_{\theta\beta}(\vec{K},\vec{v}) + A_{\theta\beta}(\vec{K})\right] F(v_\perp, v_\parallel, \vec{R}), \end{aligned} \tag{62}$$

may be interpreted as an energy flux. Equation (60) is not conservative because some energy is exchanged between the particles of the plasma component and the field fluctuations. This energy is represented by β. With the definition

$$
\begin{aligned}
\Lambda^{j}_{\alpha\gamma}(v_\perp, v_\parallel, \vec{K}, \vec{R}) = {} & P_\alpha^{j*}(\vec{k}_\perp)\Big\{ D^{j}_{\beta\gamma}(\vec{K},\vec{R}) P_\theta^{j}(\vec{k}_\perp) \\
& \times \left[A^{D}_{\theta\beta}(\vec{K},\vec{\nabla}) + A_{\theta\beta}(\vec{K}) \right] - \frac{1}{2}\nabla_\nu \left[\frac{\partial}{\partial k_\nu} C_{\beta\gamma}(\vec{K},\vec{R}) \right] \\
& \times P_\theta^{j} \frac{\mathcal{P}}{\Omega_j} \left(A^{D}_{\theta\beta} + A_{\theta\beta} \right) + C_{\beta\gamma} P_\theta^{j} \frac{\mathcal{P}}{\Omega_j} \\
& \times \left[\left(\frac{\partial}{\partial t}\frac{\partial}{\partial \omega} - \nabla_\mu \frac{\partial}{\partial k_\mu} \right) \left(A^{D}_{\theta\beta} + A_{\theta\beta} \right) \right] \Big\} F(v_\perp, v_\parallel, \vec{R}) \\
& - \frac{i\mathcal{P}}{2\Omega_j} \left[\frac{\partial}{\partial k_\nu} P_\alpha^{j*} C_{\beta\gamma} P_\theta^{j} \right] \nabla_\nu \left(A^{D}_{\theta\beta} + A_{\theta\beta} \right) F,
\end{aligned}
\tag{63}
$$

we may write β as

$$
\beta(\vec{R}) = \frac{q^2}{m} 2\pi \sum_j \int dv\, dK\, \frac{v_\perp}{\omega} \Lambda^{j}_{\alpha\alpha}(v_\perp, v_\parallel, \vec{K}, \vec{R})
\tag{64}
$$

However, the energy exchanged between the plasma as a whole and the field fluctuations may be written in a conservative form. By using Eq.(43), we can show that

$$\sum_{\text{species}} \beta(\vec{R}) = \frac{\partial}{\partial t} T_{\alpha\alpha}(\vec{R}) - \nabla_\nu S_\nu(\vec{R}), \tag{65}$$

where $T_{\alpha\alpha}$ and S_ν are given by

$$T_{\alpha\alpha}(\vec{R}) = \frac{1}{8\pi} \int d\vec{K} \, \frac{1}{\omega^2} \left[c^2 k_\alpha k_\beta - (\omega^2 + c^2 k^2) \delta_{\alpha\beta} \right] C_{\beta\alpha}(\vec{K}, \vec{R}) \tag{66}$$

and

$$S_\nu(\vec{R}) = \frac{c}{4\pi} \int d\vec{K} \, \frac{1}{\omega} \left(k_\nu \delta_{\alpha\beta} - k_\alpha \delta_{\nu\beta} \right) C_{\beta\alpha}(\vec{K}, \vec{R}) . \tag{67}$$

These terms can be identified with the help of Eqs.(26) and (28) : $T_{\alpha\alpha}$ is the trace of the Maxwell stress tensor, while S_ν is the Poynting vector which represents the energy flux due to the field fluctuations. As mentionned earlier, the physical meaning of $T_{\alpha\alpha}$ is given by

$$T_{\alpha\alpha}(\vec{R}) = -W^f(\vec{R}), \tag{68}$$

where W^f is the energy density of the field fluctuations. Thus the equation for energy transport takes the form

$$\frac{\partial}{\partial t}\left[W^p(\vec{R}) + W^f(\vec{R})\right] + \nabla_\nu \left[T_\nu^p(\vec{R}) + S_\nu(\vec{R})\right] = 0, \quad (69)$$

where W^p and $T_\nu{}^p$ are the energy density and energy flux of the plasma particles respectively. The superscript p indicates a sum over the species. The meaning of Eq.(69) is clear. The energy of the plasma and the self-consistent field fluctuations is locally conserved; the energy lost by the plasma is gained by the fluctuations, and vice versa.

D. Momentum transport

Due to the presence of the magnetic field, it is the quantity p_ν, defined by

$$p_\nu = m\left(v_\nu + \omega_c \, \varepsilon_{\nu \parallel \mu} \, r_\mu \right), \quad (70)$$

that is conserved for each particle. The momentum density is defined by

$$P_\nu(\vec{R}) = \int d^3v \; p_\nu \; F(v_\perp, v_\parallel, \vec{R}). \quad (71)$$

Since F is axisymmetric in velocity space, the parallel momentum density is

$$P_{\parallel}(\vec{R}) = m\, 2\pi \int dv\; v_{\parallel}\, F(v_{\perp}, v_{\parallel}, \vec{R}), \qquad (72)$$

and the perpendicular momentum density reduces to

$$P_{\nu}(\vec{R}) = m\,\omega_c\, \varepsilon_{\nu\parallel\mu}\, r_{\mu}\, n(\vec{R})\,, \quad \nu = x, y. \qquad (73)$$

The evolution equation for $P_{\parallel}$, i.e., the transport equation for parallel momentum, is obtained by multiplying Eq.(42) by $mv_{\parallel}$, and integrating over the velocities. For each component, we have

$$\frac{\partial}{\partial t} P_{\parallel}(\vec{R}) + \nabla_{\nu}\, \Pi_{\parallel\nu}(\vec{R}) = \zeta_{\parallel}(\vec{R}). \qquad (74)$$

The term $\Pi_{\parallel\nu}$, given by

$$\Pi_{\parallel\nu}(\vec{R}) = 2\pi m \int dv\; v_{\parallel}\, \bar{\Phi}_{\nu}(v_{\perp}, v_{\parallel}, \vec{R})$$

$$- \frac{q^2}{m}\, 2\pi \sum_{j} \mathcal{P} \int dv\, d^3k \left[\left(\delta_{\alpha\lambda} k_{\parallel} - \delta_{\alpha\parallel} k_{\lambda} \right) \frac{v_{\perp}}{\omega} + \frac{v_{\perp}}{v_{\parallel}}\, \delta_{\lambda\parallel}\delta_{\alpha\parallel} \right] \qquad (75)$$

$$\left[\frac{\partial}{\partial k_{\nu}} P_{\lambda}^{j*}(\vec{k}_{\perp})\, P_{\theta}^{j}(\vec{k}_{\perp}) \right] \frac{1}{\Omega_j}\, C_{\beta\alpha}(\vec{k}, \vec{R})$$

$$\times \left[\overset{D}{A}_{\theta\beta}(\vec{k}, \vec{v}) + A_{\theta\beta}(\vec{k}) \right] F(v_{\perp}, v_{\parallel}, \vec{R}),$$

may be interpreted as a part of the partial pressure tensor due to the plasma component under consideration. As in the case of energy transport, Eq.(74) is not conservative; the term $\eta_{\parallel}$ may be interpreted as the momentum exchanged between the field fluctuations and the plasma component. This term is given by

$$\eta_{\parallel}(\vec{R}) = -\frac{q^2}{m^2} 2\pi \sum_j \int dv\, d\vec{k} \left[\frac{v_\perp}{\omega} \left(\delta_{\alpha\lambda} k_{\parallel} - \delta_{\alpha\parallel} k_\lambda \right) + \frac{v_\perp}{v_\parallel} \delta_{\lambda\parallel} \delta_{\alpha\parallel} \right] \Lambda^j_{\lambda\alpha} \left(v_\perp, v_\parallel, \vec{k}, \vec{R} \right)$$

$$- \frac{\partial}{\partial t} \frac{q^2}{m} \sum_j 2\pi\, P \int dv\, d\vec{k}\, \frac{v_\perp}{\omega^2} \left(\delta_{\alpha\lambda} k_\parallel - \delta_{\alpha\parallel} k_\lambda \right) \tag{76}$$

$$\times P^{j*}_\lambda(\vec{k}_\perp)\, C_{\beta\alpha}(\vec{k},\vec{R})\, P^j_\theta(\vec{k}_\perp) \frac{1}{\Omega_j}$$

$$\times \left[A^D_{\theta\beta}(\vec{k},\vec{\nabla}) + A_{\theta\beta}(\vec{k}) \right] F(v_\perp, v_\parallel, \vec{R}) .$$

However, the momentum exchanged between the field fluctuations and the plasma as a whole may be written in a conservative form,

$$\sum_{\text{species}} \eta_\parallel(\vec{R}) = -\frac{1}{c^2} \frac{\partial}{\partial t} S_\parallel(\vec{R}) - \nabla_\nu T_{\parallel\nu}(\vec{R}) , \tag{77}$$

where $T_{\parallel\nu}$, given by

$$T_{\parallel\nu}(\vec{R}) = \frac{1}{4\pi} \int d\vec{k} \left[C_{\nu\parallel}(\vec{k},\vec{R}) \right. \tag{78}$$

$$\left. + \frac{c^2}{\omega^2} \left(k_\parallel k_\beta C_{\beta\nu} - k_\parallel k_\nu C_{\beta\beta} + k_\nu k_\beta C_{\beta\parallel} - k^2 C_{\nu\parallel} \right) \right],$$

is the portion of the Maxwell stress tensor whose divergence represents the force exerted by the field fluctuations that is parallel to the magnetic field. The transport equation for parallel momentum is therefore

$$\frac{\partial}{\partial t}\left[P_{\parallel}^{P}(\vec{R}) + \frac{1}{c^2} S_{\parallel}(\vec{R})\right] + \nabla_{\nu} \left[\Pi_{\parallel\nu}^{P}(\vec{R}) + T_{\parallel\nu}(\vec{R})\right] = 0 \, . \tag{79}$$

Equation (79) shows that the total parallel momentum is locally conserved, and can only be exchanged between the plasma and the field fluctuations.

In the case of perpendicular momentum, it is possible to write the evolution equation for P_{ν} (Eq.(73)) in a conservative form, which is related to the continuity equation (52).[7] However, the resulting equation lacks a clear physical meaning for the following reason. In order to describe the transport of perpendicular momentum, it is necessary to include the first order term of the expansion (7). This is the lowest order term which depends on the polar velocity angle. Since Eq.(42) describes the evolution of a distribution function averaged over this angle (c.f.Sec.III), the study of perpendicular momentum lies outside the scope of this work.

VIII. CONCLUSION

The quasilinear equations have been generalized to describe the evolution of an inhomogeneous magnetized plasma interacting with self-consistent field fluctuations of an electromagnetic nature. These equations were obtained without introducing the RPA, and are valid within the adiabatic approximation, and for a small Larmor radius to inhomogeneity scale ratio. The equations consistently take into account both resonant and non-resonant wave-particle interactions. Furthermore, the transport equations derived from them show that particles, momentum, and energy are locally conserved. The transport equations provide explicit expressions for quantities of physical relevance, e.g., the current density, energy flux, and pressure tensor.

In spite of the restrictions imposed by the small Larmor radius to inhomogeinity scale ratio, the equations derived in this paper retain a large degree of generality, and thus should be applicable to a wide range of specific problems.

ACKNOWLEDGEMENT

This work was supported by the Ecole Polytechnique Fédérale de Lausanne, by the Fonds National Suisse de la Recherche Scientifique, and by Euratom. The authors thank Dr. Alan Turnbull who carefully read and criticized the manuscript.

REFERENCES

1 A.A. Vedenov, E.P. Velikhov, and R. Sagdeev, Nucl. Fusion 1, 82 (1961).

2 W.E. Drummond and D. Pines, Nucl. Fusion Suppl. Pt. 3, 1049 (1962).

3 V.L. Yakimenko, Sov. Phys. JETP 17, 1032 (1963).

4 C. F. Kennel and F. Engelmann, Phys. Fluids 9, 2377 (1966).

5 J. Vaclavik and K. Appert, Phys. Fluids 23, 1801 (1980).

6 E.S. Weibel and J. Vaclavik, Phys. Fluids 24, 413 (1981).

7 F. Yasseen and J. Vaclavik, Phys. Fluids 26, 468 (1983).

8 A.B. Mikhailovskii, in Reviews of Plasma Physics, edited by A.M. Leontovich (Consultants Burau, New York, 1966), Vol. 3.

9 T.F. Volkov, in Reviews of Plasma Physics, edited by M.A. Leontovich (Consultant Bureau, New York, 1966), Vol. 4.

10 I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Series, and Products (Academic Press, New York, 1965).

11 B.B. Kadomtsev, Plasma Turbulence (Academic Press, New York, 1965).

12 H.L. Berk, J. Plasma Phys. 20, 205 (1978).

13 L.D. Landau and E.M. Lifschitz, Statistical Mechanics, (Pergamon Press, London, 1958).