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FINITE ELEMENT CALCULATION OF THE ANOMALOUS SKIN EFFECT IN A HOMOGENEOUS, UNMAGNETIZED CYLINDRICAL PLASMA COLUMN

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Abstract

An integro-differential equation which describes the anomalous penetration of an oscillating axial magnetic field into a homogeneous, unmagnetized cylindrical plasma column is solved numerically by a finite element expansion. Test cases show the good convergence properties of the method. Available experimental data are compared with numerical results obtained using the measured parameters.

1. Introduction

An incident electromagnetic field with a frequency ω much below the plasma frequency ω_p ($\omega \ll \omega_p$) is strongly attenuated in an unmagnetized plasma. This skin effect is usually calculated using the linear, local constitutive relation

$$j = \sigma(\omega)E \ , \tag{1}$$

where σ is the scalar conductivity. But when the thermal motion of the electrons during one period is not negligible, this local relationship breaks down and must be replaced by the more general relation[1] :

$$\underline{j}(\underline{r},\omega) = \int d^3r' \ \underline{\underline{\sigma}}(\underline{r}',\underline{r},\omega)\cdot\underline{E}(\underline{r}',\omega) \ , \tag{2}$$

This non-local dependence between $\underline{j}$ and $\underline{E}$ means that the current at one point in the plasma depends on the electric field at all other points due to the thermal motion of the electrons. It modifies the penetration of the electric field giving the so-called anomalous skin effect.

The anomalous skin effect in a semi-infinite, unmagnetized homogeneous plasma with a plane interface, assuming specular reflection of the electrons at the boundary and a normal incidence of the wave has been calculated by E.S. Weibel[2]. Blevin et al.[3] have considered the same problem in a plane and cylindrical geometry but with a gaussian profile of the electrons density assuming the existence of a harmonic, electrostatic potential well to maintain the equilibrium. This choice was motivated by an existing experimental set-up and by the fact that with the gaussian profile the analytical treatment can be carried quite far.

For more general configurations it is necessary to rely on numerical integration of the coupled Vlasov and Maxwell equations. This is done in two steps. First the non-local conductivity tensor $\underline{\underline{\sigma}}(\underline{r}, \underline{r}', \omega)$ is computed by solving Vlasov equation, generally by integration along characteristics. The resulting integro-differential equation which describes the penetration of the electromagnetic field is then solved. In this paper we report an example of such a solution which has been developped specifically to help in interpreting available experimental results.

We consider a straight cylindrical, homogeneous plasma column in an axially-symmetric high frequency B_z field. We assume specular reflection of the electrons on the plasma-wall sheeth. The ion motion is neglected. Collisions are represented by a friction term in Vlasov equation. The non-local conductivity is found by integration along characteristics. Maxwell equations reduce to one equation for the azimuthal component of the electric field which is solved by a finite element expansion, using the weak variational formulation of the equation. In spite of the singularity of the conductivity kernel at $r = r'$ the method converges well. The measurable parameter is the magnetic field B_z which is obtained by differencing the electric field. We observe a linear convergence of B_z in terms of 1/N for $N \geq 15$, already for cases in which B_z varies by two orders of magnitude between the axis and the surface. The generalization to different profiles presents no difficulty as long as the problem remains one-dimensional and the unperturbed trajectories are not overly complicated.

2. The Physical Model

A straight cylindrical plasma column is imbedded in an oscillating axial magnetic field. The penetration of the wave into the plasma column can be described either by its oscillating magnetic field component $B_z(r)\exp(i\omega t)$ or by the oscillating electric field component $E_\theta(r)\exp(i\omega t)$. They are related by

$$B_z(r) = i\frac{c}{\omega r}\frac{d}{dr}(rE_\theta) \quad , \tag{3}$$

The plasma homogeneity implies that the conductivity tensor be diagonal and for our problem we only need the $\sigma_{\theta\theta}$ component.

The radial penetration of the E_θ field is governed by Maxwell's equations

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rE_\theta)\right] = \frac{4\pi i\omega}{c^2}\, j_\theta \ , \tag{4}$$

$$j_\theta = \int_0^R dr\, \sigma_{\theta\theta}(r,r')\, E_\theta(r') \,, \tag{5}$$

with the boundary condition at the origin

$$E_\theta(r = 0) = 0 \,, \tag{6}$$

The conductivity $\sigma_{\theta\theta}$ is obtained by integrating the linearized Vlasov equation for the electron distribution

$$\frac{\partial f}{\partial t} + \nu f + v_r \frac{\partial f}{\partial r} + \frac{v_\theta^2}{r}\frac{\partial f}{\partial v_r} - \frac{v_r v_\theta}{r}\frac{\partial f}{\partial v_\theta} = \frac{e}{m}\frac{\partial f_o}{\partial v_\theta} E_\theta \quad , \tag{7}$$

in which f is the perturbed distribution function and f_o the equilibrium distribution.

The collisions are represented by the relaxation term νf in eq. (7). We model the unperturbed electronic distribution function by a Maxwellian

$$f_o(v_r,v_\theta) = f_o(v_\perp^2 = v_r^2 + v_\theta^2) = \frac{n_o}{\pi v_{th}^2} \exp\left(- \frac{v_\perp^2}{v_{th}^2}\right) \tag{8}$$

where v_{th} denotes the thermal velocity of the electrons

$$v_{th} = \sqrt{2kT/m} \,, \tag{9}$$

and n_o is the number density. The ion motion is neglected.

Integrating eq. (7) along the unperturbed trajectories of the electrons, we can express the distribution function f(r) in terms of $E_\theta(r')$. The current density j_θ, given by

$$j_\theta = -en_o \iint dv_r dv_\theta v_\theta f \,, \tag{10}$$

is then expressed in terms of E_θ, thus giving $\sigma_{\theta\theta}$. Carrying out the calculations the result can be written as

$$\sigma_{\theta\theta}(\omega,r',r) = \frac{c^2}{4\pi R^3\omega} K(r,r')$$

$$K(r',r) = \frac{4\lambda s r_m}{\pi r} \int_o^{\pi/2} d\alpha \frac{\sin^2\alpha}{\sqrt{p^2-\sin^2\alpha}} \sum_{+-} \{\Psi[sr_m(\sqrt{p^2-\sin^2\alpha} \pm \cos\alpha), \varepsilon]$$

$$+\sum_{n=o}^{\infty} \Psi[2s(n+1)\sqrt{1-r_m^2\sin^2\alpha} \pm sr_m(\sqrt{p^2-\sin^2\alpha} \pm \cos\alpha), \varepsilon]\} \tag{11}$$

where λ, s and ε are dimensionless parameters defined by

$$\lambda = \frac{R^2}{\delta_o^2} = \frac{R^2\omega_p{}^2}{c^2}\left(1 + \frac{\nu^2}{\omega^2}\right)^{-1/2}$$

$$s = R\left(\frac{\omega^2+\nu^2}{v_{th}{}^2}\right)^{1/2}$$

$$\varepsilon = tg^{-1}\frac{\nu}{\omega} \tag{12}$$

R is the radius of the plasma. The functions r_m, p and Ψ appearing in eq. (11) are respectively

$$r_m = \min(r',r)/R \tag{13}$$

$$p = \frac{\max(r',r)}{\min(r',r)} \tag{14}$$

$$\Psi(x,y) = \int_0^\infty dt\, t^2 e^{-t^2 - \frac{ix}{t}} \exp(-iy) \tag{15}$$

The first sum in eq. (11) means that we have to sum over all possible combinations of the plus and minus signs. The parameter s defined in eqs. (12) can be regarded as the factor of anomaly of the skin-effect : s becomes infinite if the thermal effects are neglected. Note that the kernel $\sigma_{\theta\theta}(r,r')$ is symmetric and has a logarithmic singularity at $r = r'$ $(p = 1)$. Substituting eq. (11) into (5) and (4) we obtain an integro-differential equation for $E_\theta(r)$, with the boundary condition (6) and the condition that $B_z(R)$ is known.

3. The Weak Form of the Electro-magnetic Equation

To solve numerically eq. (4) we first derive its weak form, by multiplying eq. (4) with $rg(r)$, where $g(r)$ is an arbitrary trial function of class C and such that $g(0) = 0$, and integrate over r.

This gives

$$\int_0^R dr\, rg(r) \frac{d}{dr}\frac{1}{r}\frac{d}{dr}(rE_\theta) = i/R^3 \int_0^R dr \int_0^R dr' rg(r)\; K(r,r')\; E_\theta(r')\;, \tag{16}$$

By partial integration of the left-hand side we obtain

$$\int_0^R \frac{dr}{r}\frac{d}{dr}[rg(r)]\, \frac{d}{dr}[rE_\theta(r)] + \frac{i}{R^3}\int_0^R dr \int_0^R dr' rg(r)\; K(r,r')\; E_\theta(r') =$$

$$- \frac{i\omega R}{c}\, g(R)\; B_z(R) \tag{17}$$

Thus the problem now consists in determining E_θ such that the equality (17) holds for all g in G, where G is some test space of C1 functions defined in {O,R} such that g(0) = 0. Apart from some mathematical subtleties this problem is equivalent to our initial differential problem (4). This weak form (17) is the starting point of our computational scheme.

4. Numerical Discretization

For convenience we normalize r to 1, by replacing R = 1 into eq. (17). Since all other quantities are dimensionless they remain unchanged.

Let the interval {0,1} be subdivided into N intervals

$$[(j-1)h, jh] \qquad j = 1, ..,N \tag{18}$$

$$\text{where } h = 1/N \tag{19}$$

The principle of the method is to replace the infinite dimensional space G by a finite dimensional subspace S^N of functions which are linear over each intervals {(j-1)h, jh}, continuous at the modes r = jh, and zero at r = 0. Thus S^N can be defined by

$$S^N = \text{Span}\left\{\Psi_1^N, \ldots, \Psi_N^N\right\} \tag{20}$$

where Ψ_j^N equals one at the particular mode r = jh and vanishes at the others (fig. 1).

The trial function g(r) and the approximate function E_θ are supposed to lie in S^N, i.e.

$$g(r) = \Psi_m^N(r) \qquad m = 1, \ldots, N \tag{21}$$

$$E_\theta(r) = \sum_{n=1}^{N} e_n \Psi_n^N(r) \tag{22}$$

Note that by the particular definition of the roof-function Ψ_n^N, e_n is the value of E_θ at the mode r = nh.

The discretized form of eq. (17) then reads

$$\sum_{n=1}^{N} (A_{mn} + iB_{mn})\, e_n = \frac{-i\omega R}{c}\, B_z(1)\, \delta_{mN} \tag{23}$$

where

$$A_{mn} = \int_o^1 \frac{1}{r} \left(\frac{d}{dr} r\Psi_m^N\right) \left(\frac{d}{dr} r\Psi_n^N\right) dr \tag{24}$$

$$B_{mn} = \int_o^1 \int_o^1 dr'drrK(r',r)\, \Psi_n^N(r')\, \Psi_m^N(r) \tag{25}$$

and δ_{mn} is the Kronecker symbol. Note that the matrices A_{mn}, B_{mn} are symmetric. Only A_{mn} can be sparse. We remark also that the right hand side of eq. (23) is a column vector with all elements but the last being zero. Thus the solution e_n is given by the last column of the inverse of the matrix $(A_{mn} + i\, B_{mn})$.

The B_z field is deduced from eq. (3) by using the finite difference approximation, i.e.

$$\left(\frac{1}{r} \frac{d}{dr} rE_\theta\right)_{r=(j-1)h} = \frac{e_j - e_{j-1}}{h} + \frac{e_j + e_{j-1}}{2h(j-\frac{1}{2})} \qquad j = 1, \ldots, N \tag{26}$$

5. Results

The integral in eq. (24) can be easily calculated. But the double integral in eq. (25) must be computed by some numerical method. We have chosen the nine-points formula[5]. The computation of the matrix B, which is full, takes a rather long machine time, even if the symmetry is used. The maximum number of intervals N we have taken is 40 when running with the machine CDC Cyber 70.

Fig. 2 shows a convergence study of the solution B_z for two values of λ. The value of the B_z amplitude at the axis is represented as a function of 1/N. The straight line corresponds to linear convergence. We note that there is already linear convergence for $N \geq 15$. This can be expected since the E_θ field is approximated by piecewise linear function in the weak variational form and the B_z field is calculated by numerical differentiation of the E_θ field, using the finite difference expression (26). In principle it is possible to extrapolate the computed values of B_z to $N = \infty$. The global convergence of B_z is demonstrated if Fig. 3 and Fig. 4 where B_z varies by two orders of magnitude.

The approach to the Cold Plasma Approximation solution is shown in Fig. 5 where the amplitude of the B_z field versus the radius is drawn for increasing values of s. We have noted that for s > 20 the Cold Plasma Approximation and our model give practically the same result. We note also the approach is not monotonic.

In Fig. 6 some numerical results are drawn for different values of s. The comparison with experimental results taken in[6] is shown in Fig. 7. We note that our simplified model (the density profile of the electrons and the current of the discharge are ignored in our calculations) agree quite well with the experiments.

6. Conclusion

The results of this numerical calculation show that the finite element method is suitable to solve the anomalous skin effect equations. The model presented was convenient for the integration of Vlasov equation but with the finite element technique the direct integration along the orbits should present no difficulty in a more general case.

The comparison between experimental and computed results show discrepancies in some cases which are due to the model. The inclusion of a weak magnetic field, which is suspected to be the cause of the small discrepancy is feasible with the two steps technique described here.

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Figure Captions

Figure 1 Piecewise linear basis function

Figure 2 Convergence study. The amplitude of the B_z field at the axis is plotted versus 1/N for two values of λ. The straight line corresponds to linear convergence.

Figure 3 Amplitude of B_z versus the radius for different values of the number of intervals N. The broken line is calculated by using the Cold Plasma Approximation.

Figure 4 Amplitude of B_z at the axis (r=0) as a function of λ for different values of the number of intervals N.

Figure 5 Amplitude of B_z versus the radius for increasing values of s. The curve for $s = \infty$ is calculated from the Cold Plasma Approximation.

Figure 6 Amplitude of B_z versus the radius for different values of s.

Figure 7 Comparison with the experimental results represented by the dots. The broken line is calculated from the Cold Plasma Approximation.