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TRAPPING OF ION WAVES IN CAVITONS

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Abstract

It is shown that ion acoustic waves can be trapped in a density cavity with sufficiently steep walls. Standing waves can be excited during the sudden creation of such a cavity in resonance absorption and could greatly affect the latter process through the enhancement of stimulated Brillouin scattering.

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We consider the following physical process : Intense electromagnetic radiation with p-polarization is incident at an angle θ to the density gradient of an inhomogeneous plasma. The Airy-function swelling[1] of the wave intensity causes it to peak near the reflection layer (where $\omega_p = \omega_o \cos\theta$), and the ponderomotive force creates a density cavity there. Associated with the density minimum is an electrostatic potential, which accelerates ions toward the center of the well. If the bottom of the well is sufficiently flat and the sides sufficiently steep, the trapped ions are two-stream unstable, and the resulting ion waves are trapped within the well. Although the self-consistent solution of this problem including the process of cavity formation is too complex for analytic treatment, the physical ideas involved can be made clear by discussing the various pieces of the whole problem.

1. Ion Waves in a Density Gradient

We consider a one-dimensional plasma without magnetic field. It will be sufficiently accurate to neglect finite ion temperature and deviation from quasineutrality; these approximations are well justified in microwave simulation experiments, where $T_e/T_i \simeq 10$ and $k^2\lambda_D^2 \lesssim 0.01$. Taking $\underline{E} = -\underline{\nabla}\phi$ and neglecting electron inertia, we have for an isothermal electron fluid at temperature T :

$$0 = en\underline{\nabla}\phi - KT\underline{\nabla}n + \underline{F}_{NL} , \tag{1}$$

where F_{NL} is the ponderomotive force of the incident light wave :

$$F_{NL} = -\frac{\omega_p^2}{\omega_0^2}\nabla\frac{\langle E^2\rangle}{8\pi} = -\frac{n}{n_c}\nabla\Phi_{NL}\,, \tag{2}$$

Φ_{NL} being the ponderomotive potential $\langle E^2\rangle/8\pi$ and n_c the critical density $m\omega_0^2/4\pi e^2$. For the present, we neglect the ambipolar field E_0 and assume an equilibrium in which the pressure gradient is balanced by the ponderomotive force in a caviton :

$$KT n_0' + \frac{n_0}{n_c}\Phi_{NL}' = 0, \tag{3}$$

where the prime stands for $\partial/\partial x$.

In first order, we denote perturbations by letters without subscript; for the electron fluid we thus have

$$0 = en_0\phi' - KTn' - \frac{n}{n_c}\Phi_{NL}'\,, \tag{4}$$

the basic assumption being that Φ_{NL} is not perturbed. Eliminating it by Eq. (3), we obtain

$$\phi' = \frac{KT}{e}\frac{n'}{n_0} - \frac{KT}{e}\frac{n}{n_0}\frac{n_0'}{n_0} = \frac{KT}{e}\left(\frac{n}{n_0}\right)'. \tag{5}$$

Defining $$\nu \equiv n/n_0(x)\,, \tag{6}$$

we have $$\phi' = (KT/e)\nu'. \tag{7}$$

The ion equation of motion is

$$M\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\underline{v}\right) = -e\underline{\nabla}\phi, \tag{8}$$

which is trivially satisfied in zero order, since $v_o = \phi'_o = 0$. In first order, for perturbations of the form $\exp(-i\omega t)$, we have

$$-i\omega M v = -e\phi',$$

$$v = -\frac{ie}{M\omega}\phi' = -i\frac{KT}{M\omega}\nu'. \tag{9}$$

The ion equation of continuity gives, in first order,

$$\frac{\partial n}{\partial t} + n_o v' + v n_o' = 0, \tag{10}$$

$$-i\omega\frac{n}{n_o} + v' + v\frac{n_o'}{n_o} = 0. \tag{11}$$

Here n/n_o can be replaced by ν, since $n_i = n_o$ is assumed. Substituting for v from Eq. (9), we have

$$\nu + \frac{i}{\omega}\left(-\frac{iKT}{M\omega}\right)\left(\nu'' + \nu'\frac{n_o'}{n_o}\right) = 0,$$

$$\nu'' + \frac{n_o'}{n_o}\nu' + \frac{\omega^2}{c_s^2}\nu = 0, \tag{12}$$

where

$$c_s^2 \equiv KT/M. \tag{13}$$

The middle term in Eq. (12) describes convective charge transfer in a density gradient and can cause the ion wave to be localized if the gradient is large enough.

2. Trapping of Ion Waves

With the notations

$$g(x) \equiv n_o'/n_o \tag{14}$$

and

$$k_o \equiv \omega/c_s\,, \tag{15}$$

Eq. (12) can be written

$$\nu'' + g(x)\,\nu' + k_o^2\,\nu = 0\,. \tag{16}$$

To put this into WKB form, let

$$y = \nu \exp\left(\tfrac{1}{2}\int g\,dx\right). \tag{17}$$

This leads straightforwardly to

$$y'' + Q(x)\,y = 0\,, \tag{18}$$

where

$$Q(x) = k_o^2 - \tfrac{1}{2}g' - \tfrac{1}{4}g^2. \tag{19}$$

The turning points of this WKB problem are given by $Q(x) = 0$; and if Q is positive between the turning points, one has oscillations there and exponential decay outside. The frequency ω has discrete values given by the quantization condition

$$\int_{x_1}^{x_2} Q^{\frac{1}{2}}(x)\,dx = \left(n+\tfrac{1}{2}\right)\pi. \tag{20}$$

Consider a parabolic density well of the form

$$n_0 = n_{00}\left(1 + \frac{x^2}{x_0^2}\right). \tag{21}$$

We normalize to x_0, defining

$$\xi \equiv \frac{x}{x_0}, \qquad k \equiv \frac{\omega x_0}{c_s}. \tag{22}$$

Since

$$g(\xi) \equiv \frac{n_0'(\xi)}{n_0(\xi)} = \frac{2\xi}{1+\xi^2}, \tag{23}$$

we have

$$\nu'' + \frac{2\xi}{1+\xi^2}\nu' + k^2\nu = 0, \tag{24}$$

where the prime now stands for $\partial/\partial\xi$. From Eq. (19), we find for this case

$$Q(\xi) = k^2 - (1+\xi^2)^{-2}. \tag{25}$$

This shows that Q is smaller at $\xi = 0$ than at the turning points where $Q = 0$. Thus, Q is negative between the turning points, and ion waves are evanescent there while they propagate outside.

Localization is not obtained even if we assume a density hill of the form

$$n = n_{00}\left(1 - \frac{x^2}{x_0^2}\right). \tag{26}$$

It is easily found that in this case

$$Q(\xi) = k^2 + (1-\xi^2)^{-2}, \tag{27}$$

which is also negative at the center. The trouble with a parabolic profile is that, although g(x) and $n_o'(x)$ vanish at x = 0, the gradient in g, namely,

$$g'(\xi) = 2\,\frac{1-\xi^2}{(1+\xi^2)^2} \quad ,$$

does not vanish at $\xi = 0$. To localize ion waves, one must have a caviton with a flatter bottom, and hence steeper walls. It can be shown that cubic or higher order profiles can lead to Q > 0 inside and Q < 0 outside the well.

It should be noted that ion waves could also be trapped on one wall of a caviton, where $g \neq 0$, as long as $g' = 0$ there. For instance, a profile of the form

$$g = a\xi^2 + b, \qquad n_o = n_{oo}\exp\left(b\xi + \tfrac{1}{3}a\xi^3\right)$$

has a $Q(\xi)$ which peaks at $\xi = 0$. However, it is difficult to imagine how standing waves could arise naturally there, and this case will not be discussed further.

3. Solution for a Particular Profile

For a density well symmetric about x = 0 which can localize ion waves, it is clear that we need

$$g(0) = 0, \qquad g'(0) = 0, \qquad g(-x) = -g(x), \tag{28}$$

where $g(x) = n_o'/n_o$. The simplest function satisfying these conditions is

$$g = 4(x/x_o)^3 = 4\xi^3. \tag{29}$$

This implies a density profile of the form

$$n_o = n_{oo}\, e^{x^4/x_o^4} = n_{oo}\, e^{\xi^4}. \tag{30}$$

Eq. (16) becomes

$$\nu'' + 4\xi^3\nu' + k^2\nu = 0. \tag{31}$$

With the transformation

$$y = \nu \exp\left(\tfrac{1}{2}\xi^4\right), \tag{32}$$

we obtain

$$y'' + (k^2 - 6\xi^2 - 4\xi^6)\,y = 0, \tag{33}$$

where

$$Q(\xi) = k^2 - 6\xi^2 - 4\xi^6. \tag{34}$$

The functions $n_o(\xi)$ and $g(\xi)$ are shown on Fig. 1; the density well has very steep sides. The shape of Φ_{NL} required to produce this profile is given by Eq. (3) :

$$\Phi_{NL} = -\int n_c KT\, g(\xi)\, d\xi,$$

or, from Eq. (29),

$$\frac{\langle E^2\rangle}{8\pi n_c KT} = C - \xi^4. \tag{35}$$

The value of the constant C is immaterial; the shape of this curve is also shown in Fig. 1. It is seen that the curve is flatter-topped than an Airy function; however, in the dynamic process of caviton formation the steady state implied by Eq. (3) does not necessarily obtain.

The turning points of Eq. (33) are easily found.
Let

$$z \equiv x^2. \tag{36}$$

Then Q = 0 implies

$$4z^3 + 6z - k^2 = 0. \tag{37}$$

This is a cubic of the form

$$z^3 + az + b = 0, \tag{38}$$

with

$$a = \frac{3}{2}, \qquad b = -k^2/4. \tag{39}$$

Since the discriminant

$$D \equiv \frac{b^2}{4} + \frac{a^3}{27} = \frac{k^4}{64} + \frac{1}{8} > 0, \tag{40}$$

there is only one real root, given by

$$z = (-\tfrac{1}{2}b + D^{1/2})^{1/3} + (-\tfrac{1}{2}b - D^{1/2})^{1/3}. \tag{41}$$

Thus the turning points are given by

$$\xi_j = \pm\left[\tfrac{1}{2}\left(k^2+\sqrt{k^4+8}\right)^{1/3} + \tfrac{1}{2}\left(k^2-\sqrt{k^4+8}\right)^{1/3}\right]^{1/2}, \tag{42}$$

with $\xi \equiv x/x_o$ and $k \equiv \omega x_o/c_s$. The half-width ξ_j is a function of k (or ω) and is plotted as a function of k on the left half of Fig. 1. The shape of $Q(\xi)$ also depends on k and is shown in Fig. 1; more precisely, we have shown $Q^{\frac{1}{2}}$, which is the local wavenumber k_{local}, as a function of ξ for k = 3, 5, 10, and 15. The possible frequencies are fixed by

$$\int_{\xi_1}^{\xi_2} \left(k^2 - 6\xi^2 - 4\xi^6\right)^{1/2} d\xi = \left(n+\tfrac{1}{2}\right)\pi, \tag{43}$$

where $k = \omega x_o/c_s = k_o x_o$. Thus, for a given mode number n, k is fixed, so that $\lambda_o = 2\pi c_s/\omega$ scales with the caviton width x_o.

The wave amplitude can be found by integrating Eq. (33) for y. However, the density perturbation is

$$n/n_o = \nu = y \exp\left(-\tfrac{1}{2}\xi^4\right), \tag{44}$$

according to Eq. (32). Thus the envelope of the wave amplitude is peaked at the center of the well. The wavelength, however, is constant throughout most of the well, becoming longer near the walls, as shown by the plots of k_{local} vs. ξ in Fig. 1.

4. Excitation of Standing Waves

We now consider the zero-order electric field. When the caviton is first formed, electrons are ejected by the ponderomotive force, and a potential hill is created to eject the ions. Because of Debye shielding, a potential difference of about ½KT/e will exist between the center of the caviton and its sides, and any excess potential will occur in a sheath at the caviton walls. In a collisionless cold-ion plasma, this ambipolar potential gives rise to an ion velocity distribution that is very peaked near the sheaths[2]. In fact, the ion distribution arriving at a sheath edge is already marginally ion-acoustic unstable[2]. After the requisite number of ions has been ejected, a quasi-steady state is reached in which the outward ponderomotive pressure balances the inward plasma pressure, and the ambipolar potential vanishes (to order T_i/T_e); this is the state assumed in the previous sections. If now the pump field is reduced, the potential must reverse so that the resulting electric field helps to balance the inward plasma pressure, according to Eq. (1). We now have a potential well. In particular, if the pump field is switched off suddenly (relative to an ion transit time), Eq. (1) predicts that the potential will be simply related to the density profile by the Boltzmann relation

$$n_0 = n_{00} \exp (e\phi_0/KT). \tag{44}$$

For a square density profile such as that shown in Fig. 1, this means that the potential distribution will be nearly sheath-like. The previously accelerated ions are trapped and oscillate in the potential well. In addition, as the caviton fills in, new ions fall into the well suffering the full potential drop $e\phi_{max}$. Thus, the ion distribution near the caviton walls consists of two nearly monoenergetic streams with velocities $\pm v_o = \pm(2e\phi_{max}/M)^{\frac{1}{2}}$, plus a number of slower ions which were near the caviton center at the time the ambipolar potential was reversed and therefore did not receive the full acceleration. The ion distribution near the caviton center is nearly the same as that at the sheath edge if the cavity bottom is nearly flat (see Fig. 1).

We approximate this situation by a four-component plasma consisting of Maxwellian electrons with density n_{oe}, cold ions with density n_{o3}, and two cold ion streams with densities $n_{o1} = n_{o2}$ and velocities $\pm v_o$. The variation of v_o with x and the effect of n_o' on the ion motion are neglected in the square-well approximation. The effect of the boundaries is neglected for the moment, and we look for perturbations of the form $\exp i(kx-\omega t)$.

The first-order ion equations of motion, from Eq. (8), are

$$\left.\begin{aligned} (\omega - kv_0)\, v_1 &= (ek/M)\,\phi \\ (\omega + kv_0)\, v_2 &= (ek/M)\,\phi \\ \omega\, v_3 &= (ek/M)\,\phi\,, \end{aligned}\right\} \tag{45}$$

while the electrons satisfy

$$e\phi/KT_e = n_e/n_{oe} = (n_1+n_2+n_3)/n_{oe} . \tag{46}$$

The ion continuity equations give

$$\left.\begin{aligned}(\omega-kv_o)n_1 &= kn_{o1}v_1\\ (\omega+kv_o)n_2 &= kn_{o2}v_2\\ \omega\, n_3 &= kn_{o3}v_3 .\end{aligned}\right\} \tag{47}$$

Defining
$$\delta \equiv n_{o3}/n_{oe} , \tag{48}$$

so that
$$n_{o1} = n_{o2} = \tfrac{1}{2}(1-\delta)\,n_{oe} , \tag{49}$$

we may substitute Eqs. (45) and (46) into (47) to obtain the set

$$\left.\begin{aligned}(\omega-kv_o)^2 n_1 - \tfrac{1}{2}(1-\delta)k^2c_s^2\,(n_1+n_2+n_3) &= 0\\ (\omega+kv_o)^2 n_2 - \tfrac{1}{2}(1-\delta)k^2c_s^2\,(n_1+n_2+n_3) &= 0\\ \omega^2 n_3 - \quad \delta\, k^2c_s^2\,(n_1+n_2+n_3) &= 0 .\end{aligned}\right\} \tag{50}$$

The condition that the determinant of the coefficients vanish is

$$\begin{Vmatrix} (\omega-kv_o)^2-\tfrac{1}{2}(1-\delta)k^2c_s^2 & -\tfrac{1}{2}(1-\delta)k^2c_s^2 & -\tfrac{1}{2}(1-\delta)k^2c_s^2 \\ -\tfrac{1}{2}(1-\delta)k^2c_s^2 & (\omega+kv_o)^2-\tfrac{1}{2}(1-\delta)k^2c_s^2 & -\tfrac{1}{2}(1-\delta)k^2c_s^2 \\ -\delta k^2c_s^2 & -\delta k^2c_s^2 & \omega^2-\delta k^2c_s^2 \end{Vmatrix} = 0. \tag{51}$$

Substracting the third column from the first and second columns, we obtain

$$\left\|\begin{matrix} (\omega-kv_0)^2 & 0 & -\frac{1}{2}(1-\delta)k^2c_s^2 \\ 0 & (\omega+kv_0)^2 & -\frac{1}{2}(1-\delta)k^2c_s^2 \\ -\omega^2 & -\omega^2 & \omega^2-\delta k^2c_s^2 \end{matrix}\right\| = 0 . \tag{52}$$

Expansion in minors of the bottom row yields

$$-\omega^2(\omega+kv_0)^2\tfrac{1}{2}(1-\delta)k^2c_s^2 - \omega^2(\omega-kv_0)^2\tfrac{1}{2}(1-\delta)k^2c_s^2 + \\ + (\omega^2-\delta k^2c_s^2)(\omega^2-k^2v_0^2)^2 = 0 . \tag{53}$$

Defining

$$W = \omega^2 \\ K \equiv k^2c_s^2 \\ V \equiv k^2v_0^2 , \tag{54}$$

we obtain the cubic equation

$$W^3 - (K-2V)W^2 + [V-K(1-3\delta)]VW - \delta KV^2 = 0 . \tag{55}$$

To see the nature of the dispersion relation, assume that the population of non-oscillating ions is small, so that $\delta \ll 1$ and can be treated as a perturbation. To zero-order in δ, we have

$$W_0^2 - (K+2V)W_0 + V(V-K) = 0, \tag{56}$$

$$2W_o = K + 2V \pm [(K+2V)^2 - 4V(V-K)]^{1/2}. \tag{57}$$

Unstable solutions with zero frequency exist if we take the minus sign and the discriminant is sufficiently large. This requires

$$(K+2V)^2 < (K+2V)^2 - 4V(V-K)$$

$$0 < 4V(K-V)$$

$$\therefore \quad V < K, \quad \text{or} \quad v_o^2 < c_s^2. \tag{58}$$

Thus there is instability if v_o is sufficiently small. This illogical conclusion is typical of fluid two-stream instabilities and is, as usual, reversed by kinetic effects, which will require v_o to be larger than the ion thermal velocity. Note that finite-frequency modes are not unstable in this case because the discriminant is positive definite.

If we now rewrite Eq. (55) with W_o in the small term containing δ, we obtain

$$2W = K + 2V \pm [(K+2V)^2 - 4V\{V - K + \delta K(3 - \frac{V}{W_o})\}]^{1/2}. \tag{59}$$

Now the condition for purely growing modes is

$$V < K[1 - \delta(3 + \frac{V}{|W_o|})], \tag{60}$$

where we have taken W_o to be negative. We see that the range of instability (and the growth rate) have been decreased. This is simply because the ion streams must move more plasma than just themselves. In addition, it is seen from Eq. (59) that finite-frequency modes can be excited if δ is sufficiently large, since the δ term is negative and can make the discriminant negative (and $W = \omega^2$ complex).

This is an extremely simplified treatment which neglects two types of damping : Landau damping and density-gradient damping. Landau damping cannot be treated without a detailed knowledge of the ion distribution function, but the trapped density-gradient modes can, in principle, be found by the technique of Sec. 3. We may neglect the variation of v_o across the well, but we must retain the density gradient and the x-derivative of the perturbed quantities.

The first-order ion equations of motion and continuity for the three species are

$$\left.\begin{aligned} -i\omega v_1 + v_0 v_1' &= -(e/M)\,\phi' \\ -i\omega v_2 - v_0 v_2' &= -(e/M)\,\phi' \\ -i\omega v_3 \qquad\quad &= -(e/M)\,\phi' \end{aligned}\right\} \tag{61}$$

$$\left.\begin{aligned} -i\omega n_1 + n_{01} v_1' + v_1 n_{01}' + v_0 n_1' &= 0 \\ -i\omega n_2 + n_{02} v_2' + v_2 n_{02}' - v_0 n_2' &= 0 \\ -i\omega n_3 + n_{03} v_3' + v_3 n_{03}' \qquad &= 0 \end{aligned}\right\} \tag{62}$$

The variable ϕ' can be eliminated by the electron Boltzmann relation {Eq. (46)}, and n'_{o1} and n'_{o2} can probably be neglected by the sharp-sheath approximation, since v'_o has been neglected anyway. Nonetheless, we are left with six simultaneous differential equations in six unknowns, which is too complicated a system to solve analytically.

5. Discussion

Although the thresholds can only be obtained by computation, it is clear from the above that trapped ion modes can exist in a sufficiently square density well and can be excited by a sufficiently large and monoenergetic population of trapped ions. The most unstable wavelength will be determined by v_o and will lie between two extremes : short λ's near $2\pi\lambda_D$ will be Landau damped, and long λ's near $2\pi x_o$ will be damped by the density gradient. In between, a number of discrete wavelengths (and corresponding frequencies) can be simultaneously unstable. Note that Landau damping is likely to be smaller than in a Maxwellian plasma. Ions with energies larger than the well depth $e\phi_{max}$ are not trapped, and if $e\phi_{max}$ is comparable to KT, there is an absence of trapped ions faster then the ion wave. As for transit ions that pass through the well, there is also a dearth of these that have the proper velocity, since they must have near-zero velocity outside the well, and therefore have near-zero flux.

The mechanism described here yields features resembling those of ion oscillations observed in microwave experiments[3]: the localization of coherent standing waves in a caviton, the tendency for k to be less than ω/c_s, the tendency for the mode number n to be large, the existence of steep cavity walls, the tendency for waves to be excited after a sharp turn-off of the pump, and the simultaneous occurrence of discrete frequency bands at low pump power.

If this effect really occurs, it can have important consequences for resonance absorption in laser fusion. It is currently thought that at high powers profile steepening would suppress stimulated Brillouin scattering (SBS). However, if ion waves can be trapped in a cavity dug at the reflection layer, the SBS becomes an absolute instability instead of a convective one and is likely to prevent laser energy from reaching the resonance layer, where it can be linearly converted to plasma waves. Of course, the direct conversion to ion energy at the reflection layer is of some benefit, but the reflection coefficient should become very large once the ion waves reach a reasonable amplitude.

References

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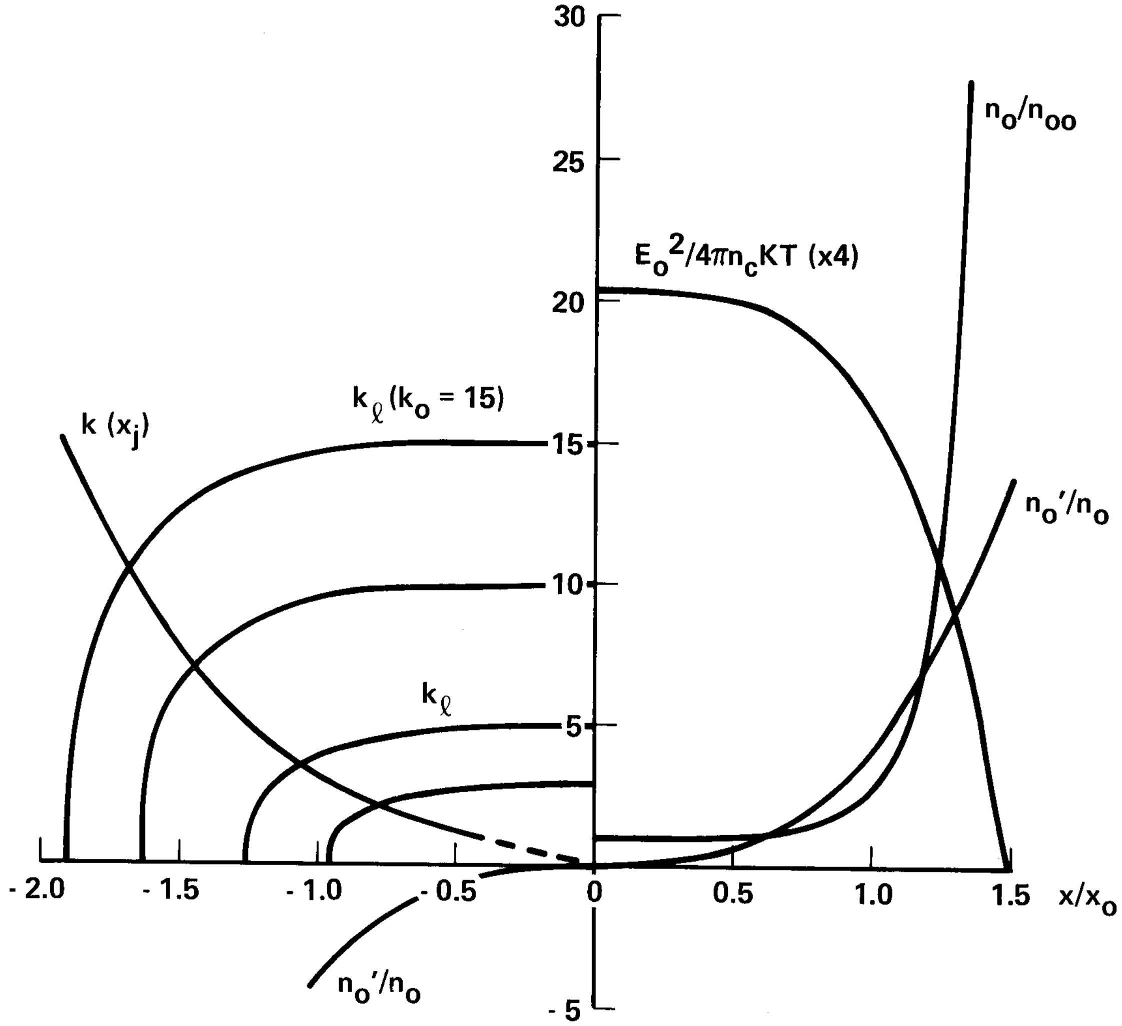
30
25
20
15
10
5
- 5
n_o/n_{oo}
$E_o^2/4\pi n_c KT$ (x4)
$k_\ell (k_o = 15)$
$k\ (x_j)$
n_o'/n_o
k_ℓ
- 2.0
- 1.5
- 1.0
- 0.5
0
0.5
1.0
1.5
x/x_o
n_o'/n_o

Figure 1.