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THREE DIMENSIONAL STATIONARY PLASMON DISTRIBUTIONS

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ABSTRACT

Three dimensional stationary states are obtained in which the radiation pressure of plasmons maintains arbitrary density variations of the plasma. The distribution of the plasmons in phase space is a universal function of their energy.

Kingsep et al[1] have proposed a one dimensional model of turbulence in which the turbulent state is built up from a random set of solitons. Two and three dimensional envelope solitons have been described by Kaw et al[2]. In order to avoid the collapse of their solitons they use the nonlinear susceptibility of the form[3]

$$\chi_{NL} = \exp\left[-e^2E^2/4m\omega^2(T_e+T_i)\right] - 1 \quad .$$

The fact that these solitons contain perfectly coherent oscillations probably make them unsuitable for the description of a turbulent state. Kaw et al[4] and Hasegawa[5] have given one dimensional envelope solutions

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for random phase plasmons. Here we present equilibria of both trapped and untrapped random plasma oscillations in three dimensions.

The plasmon distribution function $N(\underline{k},\underline{r})$ for stationary states is governed by the equation[6]

$$\frac{\partial \omega}{\partial \underline{k}} \cdot \frac{\partial N}{\partial \underline{r}} - \frac{\partial \omega}{\partial \underline{r}} \cdot \frac{\partial N}{\partial \underline{k}} = 0. \tag{1}$$

In this equation ω is given by

$$\omega^2(\underline{k},\underline{r}) = \omega_{pe}^2(\underline{r}) + 3V_{Te}^2 k^2$$

where[7] $\omega_{pe}^2 = e^2 n(\underline{r})/m_e$, $v_{Te}^2 = T_e/m_e$. When the plasma is in equilibrium with the radiation pressure of the plasmons its density is

$$n = n_0 \exp\left[-\phi/(T_e + T_i)\right] \tag{2}$$

with the potential

$$\phi = \frac{2e^2}{m_e} \frac{1}{V^2} \sum_k \frac{|E_k|^2}{\omega_k^2} .$$

V is the volume of the cube with periodic boundary conditions. The connection between N and E is found from the equation for the energy density w

$$w = \frac{\hbar}{V} \sum_k N_k \omega_k = \frac{1}{V^2} \sum_k \left(\omega \frac{\partial \varepsilon}{\partial \omega}\right)_{\omega_k} |E_k|^2$$

where ε is the dielectric function. Expressing $|E_k|^2$ in terms of N_k and replacing the sum by an integral one obtains

$$\phi = \frac{\hbar e^2}{m_e \omega_0} \left(\frac{\omega_{pe}(\underline{r})}{\omega_0}\right)^2 (2\pi)^{-3} \int N(\underline{k},\underline{r}) \left(\frac{\omega_0}{\omega(\underline{k},\underline{r})}\right)^2 d^3k . \tag{3}$$

Here we have introduced the frequency $\omega_0 = (e^2 n_0/m_e)^{\frac{1}{2}}$ for the purpose of normalizing our variables.

Any function of $\omega(\underline{k},\underline{r})$ is a solution of (1) so that we may put

$$N = F(\omega^2/\omega_0^2)$$

The problem of determining F is similar, but not identical to the BGK problem of finding the trapped particle distribution[8]. Here it is neither necessary nor useful to distinguish trapped and untrapped plasmons. Combining Eqs.(2) and (3) one finds

$$-\lg \frac{\omega_{pe}^2}{\omega_0^2} = \frac{\hbar e^2}{(2\pi)^3 m_e (T_e+T_i)\,\omega_0} \left(\frac{\omega_{pe}}{\omega_0}\right)^2 \int F\left(\frac{\omega^2}{\omega_0^2}\right)\left(\frac{\omega_0}{\omega}\right)^3 d^3k$$

Putting $\omega_{pe}^2/\omega_0^2 = x$ and changing the variable of integration this equation can be brought into the form

$$-x^{-1}\lg x = (2/A) \int_x^\infty F(y)\, y^{-3/2} (y-x)^{1/2} dy \qquad (4)$$

where

$$A = 24\sqrt{3}\,\pi^2 n_0 \delta_0^3 (T_e + T_i)/\hbar\omega_0$$

and the Debye length δ_0 is referred to the density n_0

$$\delta_0^2 = T_e/e^2 n_0$$

It is important to realize that the right hand side of Eq.(3) is a positive function of x whereas the left hand side is negative for $x > 1$. Thus Eq.(4) has no solution as it stands. Therefore we must replace the left hand side of (4) by

$$L(x) = \begin{cases} -x^{-1} \lg x , & 0 < \alpha \leq x \leq 1 ; \\ 0 , & x < \alpha , \quad 1 < x ; \end{cases}$$

where

$$\alpha = \min[n(\underline{r})/n_0]$$

Thus αn_0 is the minimum density that will be encountered. At the same time we must restrict the argument of the solution to the range $\alpha \leq x \leq 1$. Since the integrand in (4) is positive or zero it follows at once that

$$F(y) = 0 , \qquad y > 1 \tag{5}$$

This is simply reflects the fact that, according to Eq.(2), $n(\underline{r}) \leq n_0$ because $\phi \geq 0$.

For $y \leq 1$ the function F is obtained by the Abel[9] inversion formula

$$F(y) = -\pi^{-1} A y^{3/2} \frac{d}{dy} \int_y^1 \frac{(1 - \lg x)\, dx}{x^2 (x-y)^{1/2}} \tag{6}$$

At $y = 0$ and $y = 1$ this function has the integrable singularities

$$F(y) \approx \pi^{-1} A (1-y)^{-1/2} \tag{7}$$

$$F(y) \approx -\pi^{-1} A \lg y \tag{8}$$

Apart from the factor A, the function F(y) is a universal function, valid for trapped and untrapped plasmons. This is a very different situation from the one encountered in the BGK problem.

The plasmon distribution function

$$N(\underline{k},\underline{r}) = F\left(n(\underline{r})/n_0 + 3\delta_0^2 k^2\right) \tag{9}$$

represents a spatial variations of the plasma density held in equilibrium by the radiation pressure of the Langmuir oscillations. Any density distribution $n(\underline{r})$ can be assumed provided $n(\underline{r}) \leq n_0$. Obviously $n(\underline{r})$ must not be allowed to become very much less than n_0 lest $\delta_0 k$ become large enough for Landau damping. Therefore Eq.(7) is a reasonable approximation to F for physically realizable cases.

If $n_{max} < n_0$ the plasmons with

$$\omega_0^2 \gg \omega_{pe}^2 + 3V_{Te}^2 k^2 > \left(\omega_{pe}^2\right)_{max}$$

are untrapped; nevertheless they are distributed according to Eq.(9), with F given by Eqs.(5) and (6).

The question immediately arises as to why Kaw et al[2] have not found this universal distribution. The essential difference between their treatment and ours is that they include charge separation of ions and electrons which leads to an integro-differential equation, analogous to BGK rather then our Abel equation. It is the differential term that introduces the additional liberty. However for stationary states, this term is negligible since the density variations must necessarily be of very much larger scale then the Debye length. It can only become important for perturbations moving very close to the speed of sound.

It is interesting to calculate the number of plasmons $\nu(\underline{r})$, irrespective of $\underline{k}$

$$\nu(\underline{r}) = 4\pi \int N k^2 dk$$

For shallow densities variations using again Eq.(7), one obtains the approximate result

$$\nu(\underline{r}) = (2\pi)^3 (T_e + T_i)(n_0 - n(\underline{r})) / \hbar\omega_0$$

which is a pressure balance equation. It tells us that in this equilibrium the average plasmon exerts a pressure of $\hbar\omega_0/(2\pi)^3$ rather than T as for a particle.

The spatially averaged spectrum

$$S(\underline{k}) = \int N d^3r$$

cannot be evaluated unless $n(\underline{r})$ is given. It is, however, instructive to calculate $S(\underline{k})$ for density well, or "bubble", of the shape

$$n(\underline{r}) = \begin{cases} n_0 (\alpha + r^2/r_0^2) , & r \leq R \\ n_0 , & r > R \end{cases}$$

where

$$R = (1-\alpha)^{1/2} r_0$$

is the radius and $n_1 = \alpha n_0$ is the lowest density reached in its center. Using again Eq.(7) for F one obtains

$$S(\underline{k}) = \begin{cases} \pi A R^3 (1 - n_1/n_0)^{-3/2} (1 - n_1/n_0 - 3\delta_0^2 k^2) , & \delta_0 k \leq \eta \\ 0 , & \delta_0 k > \eta \end{cases}$$

where

$$\eta = \left[\frac{1}{3}\left(1 - n_1/n_0\right)\right]^{1/2} \leq 3^{-1/2}$$

It is pleasing to note that the spectrum is cut off below the Debye wave number, inspite of the fact that Landau damping is not contained in the model. The form of the spectrum and the cut off are independent of the radius of the bubble.

It is remarkable that the equilibrium states presented here result from a universal plasmon distribution function but admit of arbitrary density variation. The simplicity of this result is due to the neglect of wave-particle and wave-wave interaction which will have to be taken into account in future extensions of this work. The ion acoustic dynamics should also be included in order to treat states varying with time and to examine the stability of the equilibria. One of the principal aims will be the description of a turbulent state.

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$$\phi'' + \phi'/x - \phi/x^2 + [1-\sigma - \exp(-\phi^2)]\phi = 0$$

have the same general properties. The two equations simply apply to two different polarizations in cylindrical geometry: $u \sim E_z$ while $\phi \sim E_\theta$.

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