July 1976 LRP 110/76

ANOMALOUS SPATIAL DEPLETION OF LOWER-HYBRID CONE THROUGH PARAMETRIC DECAY

Liu Chen[*,**] and R.L. Berger[***]

Centre de Recherches en Physique des Plasmas
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

[*] On leave from Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08540, USA

[**] Work supported by the Swiss National Science Foundation

[***] Work supported by the U.S. Energy Research and Development Administration. Contract E(11-1)-3073

July 1976

LRP 110/76

ANOMALOUS SPATIAL DEPLETION OF LOWER-HYBRID CONE THROUGH PARAMETRIC DECAY

Liu Chen*,**

Centre de Recherches en Physique des Plasmas
Ecole Polytechnique Fédérale de Lausanne
CH-1007 Lausanne, Switzerland

and

R.L. Berger***

Plasma Physics Laboratory, Princeton University
Princeton, New Jersey 08540, USA

ABSTRACT

Analytic solutions for the envelope structures of two nonlinearly coupled lower-hybrid waves propagating along their respective cone trajectories are obtained. The coupling occurs through induced scatterings by particles. The results indicate anomalous spatial pump depletion. Implications to plasma heating are also discussed.

* On leave from Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08540, USA

** Work supported by the Swiss National Science Foundation

*** Work supported by the U.S. Energy Research and Development Administration Contract E(11-1)-3073

1. INTRODUCTION

Recently, there has been active theoretical as well as experimental efforts in studying the r.f. plasma heating scheme using waves in the lower-hybrid frequency range. It is now widely believed that nonlinear effects; such as pump self-induced filamentation and parametric decays, play important roles in this heating scheme [1,2]. One of the major concerns, which motivates the present work, is then how these nonlinear processes affect the transport of r.f. energy into the interior of a plasma. The nonlinear process considered here is the decay instability due to induced scatterings by particles; i.e., nonlinear electron and/or ion Landau dampings [3].

In Section 2, we first describe the theoretical model and approach used in this work. We then derive the set of nonlinearly coupled equations in terms of the action variables. The solutions to these equations are presented in Section 3. Section 4 contains a summary of the theoretical results as well as a discussion on the implications of these results to the plasma heating process.

2. THE NONLINEARLY COUPLED EQUATIONS

The plasma is assumed to be spatially homogeneous and uniformly magnetized with $\underset{\sim}{B} = B\, \underset{\sim}{e}_z$. In the steady state, we assume that the electrostatic potential consists of two parts, $\phi_0(\underset{\sim}{x}, t)$ and $\phi_1(\underset{\sim}{x}, t)$, oscillating at, respectively, frequencies ω_0 and ω_1; i.e.,

$$\phi_j(\underset{\sim}{x},t) = \tilde{\phi}_j(\underset{\sim}{x})\exp(-i\omega_j t) + \text{c.c.}\,; \quad j = 0, 1\,. \tag{1}$$

Here, ω_0 and ω_1 are of the order of lower-hybrid frequency, $\omega_{1h} = \omega_{pi}(1 + \omega_{pe}^2/\omega_{ce}^2)^{-\frac{1}{2}}$ and $\omega_{pe}^2 \sim \omega_{ce}^2$. The notations used here have their standard meanings. ϕ_0 and ϕ_1 are coupled to each other through a low-frequency mode at frequency $\omega_s = \omega_0 - \omega_1 > 0$. Thus, ϕ_0 corresponds to the pump wave excited by an external structure and ϕ_1 is the daughter wave excited by the pump wave ϕ_0 through the decay process. The dominant coupling between ϕ_0 and ϕ_1 comes from the electron density perturbation of the ω_s mode. Let

$$n_{es}(\underset{\sim}{x},t) = \tilde{n}_{es}(\underset{\sim}{x})\exp(-i\omega_s t) + \text{c.c.} \tag{2}$$

It is then straightforward to derive the following set of coupled equations for $\tilde{\phi}_0$ and $\tilde{\phi}_1$;

$$(\underset{\sim}{\nabla}_\perp \cdot \epsilon_{\perp 0} \underset{\sim}{\nabla}_\perp - \nabla_z \epsilon_{z0} \nabla_z)\tilde{\phi}_0 = -i\frac{4\pi e}{\omega_0} \underset{\sim}{\nabla} \cdot (\tilde{n}_{es} \underset{\sim}{u}_1), \tag{3}$$

$$(\underset{\sim}{\nabla}_\perp \cdot \epsilon_{\perp 1} \underset{\sim}{\nabla}_\perp - \nabla_z \epsilon_{z1} \nabla_z)\tilde{\phi}_1 = -i\frac{4\pi e}{\omega_1} \underset{\sim}{\nabla} \cdot (\tilde{n}_{es}^* \underset{\sim}{u}_0)\,; \tag{4}$$

where with j = 0,1

$$\epsilon_{\perp j} = 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega_j^2}\,, \tag{5}$$

$$\epsilon_{zj} = \frac{\omega_{pe}^2}{\omega_j^2} - 1 \simeq \frac{\omega_{pe}^2}{\omega_j^2} \quad , \tag{6}$$

and $\underset{\sim}{u}_j$ is mainly due to the electron $\underset{\sim}{E} \times \underset{\sim}{B}$ drift,

$$\underset{\sim}{u}_j \simeq -\frac{c}{B}(\underset{\sim}{\nabla}_\perp \tilde{\phi}_j \times \underset{\sim}{e}_z) \quad . \tag{7}$$

Also, we note that $\underset{\sim}{\nabla}_\perp = \frac{\partial}{\partial x}\underset{\sim}{e}_x + \frac{\partial}{\partial y}\underset{\sim}{e}_y$ and $\nabla_z = \frac{\partial}{\partial z}$.

For toroidal devices, x, y and z correspond to the (minor) radial, poloidal and toroidal directions, respectively. Furthermore, in deriving Eqs.(3) and (4) we have ignored the thermal effects; which is justified so long as $|v_i^2 \nabla^2|, |v_e^2 \nabla_z^2| << |\omega_j^2|$; $|v_e^2 \nabla_\perp^2| << \omega_{ce}^2$ and ω_j is not close to ω_{lh}. For regions near the plasma surface, these conditions are easily satisfied in typical experiments.

As to the low-frequency ω_s mode, the dominant coupling is due to the parallel ponderomotive force, $-m_e(\underset{\sim}{u}\cdot\underset{\sim}{\nabla})u_z$, produced by ϕ_0 and ϕ_1; which acts on the electrons along $\underset{\sim}{B}$. Denoting the corresponding potential be $\tilde{\psi}_p$; i.e.,

$$\nabla_z \tilde{\psi}_p = -(m_e/e)[(\underset{\sim}{u}_0\cdot\underset{\sim}{\nabla})u_{z1}^* + (\underset{\sim}{u}_1^*\cdot\underset{\sim}{\nabla})u_{z0}] \,, \tag{8}$$

we obtain, using Eq.(7) and noting $u_{zj} = (ie/m_e\omega_j)\nabla_z \tilde{\phi}_j$,

$$\tilde{\psi}_p(\underset{\sim}{x}) = -\frac{ic}{B\omega_0}[(\underset{\sim}{\nabla}_\perp\tilde{\phi}_1^* \times \underset{\sim}{\nabla}_\perp\tilde{\phi}_0)\cdot\underset{\sim}{e}_z] \,. \tag{9}$$

Because $\omega_s \ll \omega_{ce}$, electron dynamics is mainly along $\underset{\sim}{B}$. We then can use $\tilde{\psi}_p$ along with the self-consistent potential $\tilde{\phi}_s$ in the electron one-dimensional (in $\underset{\sim}{B}$ direction) Vlasov equation to obtain the perturbed electron distribution function $\tilde{f}_{es}$;

$$(-i\omega_s + v_z \nabla_z)\,\tilde{f}_{es}(\underset{\sim}{x}, v_z) = -\frac{e}{m_e}\frac{df_{oe}}{dv_z}\nabla_z(\tilde{\phi}_s + \tilde{\psi}_p)\,. \tag{10}$$

As to the effectively unmagnetized ions ($\omega_s^2 \gg \omega_{ci}^2$), they only respond linearly to $\tilde{\phi}_s$. Hence, we have

$$(-i\omega_s + \underset{\sim}{v}\cdot\underset{\sim}{\nabla})\,\tilde{f}_{is}(\underset{\sim}{x}, \underset{\sim}{v}) = \frac{e}{m_i}\frac{\partial f_{io}}{\partial \underset{\sim}{v}}\cdot\underset{\sim}{\nabla}\tilde{\phi}_s\,. \tag{11}$$

Poisson's equation

$$-\nabla^2\tilde{\phi}_s = 4\pi e(\tilde{n}_{is} - \tilde{n}_{es}) \tag{12}$$

then relates $\tilde{\phi}_s$ and, hence, $\tilde{n}_{es} = n_0\int dv_z\,\tilde{f}_{es}$ to ψ_p. Equations (3), (4), (9), (10), (11) and (12), thus, provide a complete description of the nonlinearly coupled system.

To proceed further analytically, we make the following WKB approximation

$$\tilde{\phi}_j(\underset{\sim}{x}) = \Phi_j(\epsilon x, \epsilon z)\exp(i\underset{\sim}{k}_j\cdot\underset{\sim}{x})\;;\; j = 0, 1 \tag{13}$$

and

$$[\tilde{n}_{es}, \tilde{\psi}_p, \tilde{\phi}_s](\underset{\sim}{x}) = [N_{es}, \Psi_p, \Phi_s](\epsilon x, \epsilon z)\exp(i\underset{\sim}{k}_s\cdot\underset{\sim}{x})\,. \tag{14}$$

Here, $\epsilon \ll 1$, $\underset{\sim}{k}_0 = k_{x0}\, \underset{\sim}{e}_x + k_{z0}\, \underset{\sim}{e}_z$, $\underset{\sim}{k}_1 = k_{x1}\, \underset{\sim}{e}_x + k_{y1}\, \underset{\sim}{e}_y + k_{z1}\, \underset{\sim}{e}_z$, $\underset{\sim}{k}_s = \underset{\sim}{k}_0 - \underset{\sim}{k}_1$ and $(\omega_j, \underset{\sim}{k}_j)$ satisfies the lower-hybrid dispersion relation

$$\omega_j = \omega_{lh} \left(1 + k_{zj}^2 m_i / k_{\perp j}^2 m_e \right)^{1/2} \; ; \; j = 0, 1 \; . \tag{15}$$

Note, in Eqs.(13) and (14), because we are mainly interested in the spatial structures in the x-z plane, we have assumed the envelopes to be independent of y. Equations (3) and (4) then become

$$\left[k_x \epsilon_\perp \frac{\partial}{\partial x} - k_z \epsilon_z \frac{\partial}{\partial z} \right]_0 \Phi_0 = \lambda N_{es} \Phi_1 \; , \tag{16}$$

$$\left[k_x \epsilon_\perp \frac{\partial}{\partial x} - k_z \epsilon_z \frac{\partial}{\partial z} \right]_1 \Phi_1 = -\lambda N_{es}^* \Phi_0 \; , \tag{17}$$

where

$$\lambda = \frac{2\pi e c}{B \omega_0} \left[(\underset{\sim}{k}_1 \times \underset{\sim}{k}_0) \cdot \underset{\sim}{e}_z \right] \; . \tag{18}$$

For the induced scattering processes considered here, $\omega_s \sim k_s v_i$ or $k_{zs} v_e$ and the low-frequency mode is non-resonant. Thus, we can replace $\underset{\sim}{\nabla}$ by $i\underset{\sim}{k}_s$ in Eqs.(10), (11) and (12). We then obtain

$$4\pi e N_{es} = \beta \Psi_p \; , \tag{19}$$

where

$$\beta(\omega_s, \underset{\sim}{k}_s) = \frac{k_s^2 \chi_e (1 + \chi_i)}{\epsilon_s} \; , \tag{20}$$

$$\epsilon_s = 1 + \chi_e + \chi_i \ , \tag{21}$$

$$\chi_e = -\frac{\omega_{pe}^2}{k_s^2} \int dv_z \frac{df_{oe}/dv_z}{v_z - \omega_s/k_{zs}} \ , \tag{22}$$

$$\chi_i = -\frac{\omega_{pi}^2}{k_s^2} \int dv \frac{df_{oi}/dv}{v - \omega_s/k_s} \ , \tag{23}$$

and, from Eq.(9),

$$\psi_p = -\frac{ic}{B\omega_0} [(\underset{\sim}{k}_1 \times \underset{\sim}{k}_0) \cdot \underset{\sim}{e}_z] \Phi_0 \Phi_1^* \ . \tag{24}$$

Substituting Eq.(19) into Eqs.(16) and (17) and expressing in terms of the action variables $I_j = k_j^2 |\Phi_j|^2/\omega_j$ with j = 0,1, we obtain

$$\underset{\sim}{\nabla} \cdot (\underset{\sim}{v}_{g0} I_0) = -\alpha I_0 I_1 \ , \tag{25}$$

$$\underset{\sim}{\nabla} \cdot (\underset{\sim}{v}_{g1} I_1) = \alpha I_0 I_1 \ , \tag{26}$$

where

$$\alpha = \frac{1}{2} \left(\frac{c k_{y1}}{k_1 B}\right)^2 \left(1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}\right)^{-1} \mathrm{Im}\, \beta > 0 \ , \tag{27}$$

and

$$\underset{\sim}{v}_{gj} = \left(\frac{\partial \omega}{\partial k_x} \underset{\sim}{e}_x + \frac{\partial \omega}{\partial k_z} \underset{\sim}{e}_z\right)_j \ ; \quad j = 0, 1 \tag{28}$$

is the group velocity. Note Eqs.(25) and (26) exhibit the conservation of action flux; i.e., $\sum_{j=0,1} \underset{\sim}{\nabla} \cdot (\underset{\sim}{v}_{gj} I_j) = 0$. Solutions to this set of non-linearly coupled equations are discussed in the next section.

3. SOLUTIONS OF THE COUPLED EQUATIONS

Equations (25) and (26) can be written as

$$\frac{\partial G_0}{\partial x} + C_0 \frac{\partial G_0}{\partial z} = - G_0 G_1 \quad , \tag{25'}$$

$$\frac{\partial G_1}{\partial x} + C_1 \frac{\partial G_1}{\partial z} = G_0 G_1 \quad . \tag{26'}$$

Here, $G_0 = \alpha I_0/V_{gx1}$, $G_1 = \alpha I_1/V_{gx0}$ and $C_j = (V_{gz}/V_{gx})_j$ with j = 0,1. Equations (25)' and (26)' indicate that if nonlinear coupling is absent, $G_0(I_0)$ and $G_1(I_1)$ just propagate along there respective cone trajectories; i.e., G_0 and G_1 are only functions of, respectively, $z - C_0x$ and $z - C_1x$. Nonlinear couplings, thus, make G_0 and G_1 functions of both $z - C_0x$ and $z - C_1x$. There exists a class of exact solutions to Eqs.(25)' and (26)' and is given by [4]

$$G_0(\tau, \xi) = \frac{T_\tau(\tau)}{Z(\xi) - T(\tau)} \quad , \tag{29}$$

$$G_1(\tau, \xi) = \frac{Z_\xi(\xi)}{Z(\xi) - T(\tau)} \quad , \tag{30}$$

where

$$\tau = -(z - C_0x)/V \; , \tag{31}$$

$$\xi = (z - C_1x)/V \; , \tag{32}$$

and

$$V = C_0 - C_1 \quad . \tag{33}$$

Subscripts τ and ξ denote derivatives. Formally, T and Z are two arbitrary, differentiable functions; which, however, are uniquely defined by the boundary conditions at x = 0. Denoting $G_0(x = 0, z) = \bar{G}_0(z)$ and $G_1(x = 0, z) = \bar{G}_1(z)$, we then have

$$T(\tau) = -\frac{1}{2} - \frac{1}{V} \int_0^{-V\tau} ds\, \bar{G}_0(s) \exp\left[\int_0^s dr\, G_t(r)/V \right] , \tag{34}$$

$$Z(\xi) = \frac{1}{2} + \frac{1}{V} \int_0^{V\xi} ds\, \bar{G}_1(s) \exp\left[\int_0^s dr\, G_t(r)/V \right] , \tag{35}$$

with $G_t(z) = \bar{G}_0(z) + \bar{G}_1(z)$.

To illuminate the physical meanings of the solutions, let us take the following simple boundary conditions

$$\bar{G}_0(z) = \begin{cases} A_0 \ , & |z| \le a \\ 0 \ , & |z| > a \end{cases} \tag{36}$$

and, similarly,

$$\bar{G}_1(z) = \begin{cases} A_1 \ , & |z| \le a \\ 0 \ , & |z| > a \end{cases} \quad . \tag{37}$$

Let $A_t = A_0 + A_1$, the solution then is;

for $|z - C_0 x|$, $|z - C_1 x| \leq a$,

$$G_0(\tau,\xi) = \frac{A_t A_0}{A_0 + A_1 \exp(A_t x)} , \tag{38}$$

$$G_1(\tau,\xi) = \frac{A_t A_1 \exp(A_t x)}{A_0 + A_1 \exp(A_t x)} ; \tag{39}$$

for $|z - C_0 x| \leq a$, $|z - C_1 x| > a$,

$$G_0(\tau,\xi) = \frac{A_t A_0}{A_0 + A_1 \exp[A_t(\tau + a/v)]} , \tag{40}$$

$$G_1(\tau,\xi) = 0 ; \tag{41}$$

and for $|z - C_0 x| > a$, $|z - C_1 x| \leq a$,

$$G_0(\tau,\xi) = 0 , \tag{42}$$

$$G_1(\tau,\xi) = \frac{A_t A_1 \exp[A_t(\xi + a/v)]}{A_0 + A_1 \exp[A_t(\xi + a/v)]} . \tag{43}$$

The above solution, thus, suggests a typical scale length of pump depletion (or growth of the decay wave) in x given by

$$x_d = 1/A_t = [\alpha(I_{00}/V_{gx1} + I_{10}/V_{gx0})]^{-1} . \tag{44}$$

Here, I_{j0} is the value of I_j at $x = 0$ for $j = 0,1$. Note, from the definition of α in Eq.(27), x_d has a simple physical interpretation; i.e., x_d = (effective group velocity in x)/(effective parametric growth rate). To estimate x_d, we assume $v_{gx0} \sim v_{gx1}$ and $I_{00} \gtrsim I_{10}$, then x_d becomes approximately, using Eq.(27),

$$x_d \simeq \omega_0 V_{gx0} / \alpha |E_0|^2$$

$$\simeq \left|\frac{cE_0}{BC_s}\right|^{-2} \left[\frac{2}{n_{z0}} \left(\frac{\omega_0^2}{\omega_{lh}^2} - 1\right)^{3/2} \frac{c/\omega_{lh}}{\omega_0/\omega_{lh}} \left(\frac{m_e}{m_i}\right)^{1/2} \right]. \tag{45}$$

Here, $|E_0| = |k_0 \Phi_0|$, $n_{z0} = Ck_{z0}/\omega_0$, C_s is the ion-acoustic speed and we have assumed $\omega_s \simeq k_{zs}v_e$ or $k_s v_i$ for maximum coupling. For typical tokamak experiments, $n_{z0} \simeq$ 2-3, $\omega_0/\omega_{1h} \sim 3$ and $\omega_{1h} \sim 10^{10}$, we have

$$x_d \simeq \left|\frac{CE_0}{BC_s}\right|^{-2} \text{(cm)} \quad . \tag{46}$$

Currently, lower-hybrid heating experiments are running with $|CE_0/BC_s|^2 \lesssim$ 0(1), thus, the depletion length is in the centimeter range. The anomalous pump depletion, however, does not occur until the pump wave penetrates a typical distance x_p into the plasma; which is given by

$$x_p = x_d \ln (I_{00} V_{gx0} / I_{10} V_{gx1}). \tag{47}$$

Thus, x_p depends on the wave intensity of the decay wave, I_{10}. To estimate (I_{10} / I_{00}), we note that the results of both theoretical [5] as well as numerical [6] investigations on the nonlinear saturation of the decay instability has shown that, in general, $I_{10} \sim I_{00}$ at saturation such that the parametric growth rate is balanced by the linear and nonlinear damping rates. x_p is then estimated to be

$$x_p \sim O(x_d) \quad . \tag{48}$$

The above estimates show that for typical tokamak parameters and lower-hybrid wave heating experiments with $|CE_0/BC_s|^2 \lesssim O(1)$, the pump penetration and depletion lengths in x(the minor radius) are in the centimeter range. As pointed out to us by F.W. Perkins, the most pessimistic observation is that the scaling law (which, from Eq.(45), depends mainly on density and electron temperature) does not improve with the size of the device. It, therefore, indicates that for present (e.g. ATC and PLT) and future generations of tokamaks, nonlinear decay processes will anomalously deplete the pump wave energy near the surface of the plasma.

4. SUMMARY AND DISCUSSION

In this paper, we consider the effects of parametric decays on the transport of lower-hybrid pump wave energy into the interior of a plasma. Using the WKB approximation, a set of equations describing two nonlinearly coupled lower-hybrid waves propagating along their respective cone trajectories are derived in terms of the action variables. The two waves correspond to, respectively, the pump and the decay waves. The nonlinear couplings considered here are due to nonlinear electron and/or ion Landau dampings. Exact solutions are then constructed from the boundary conditions. Using simple but not totally unrealistic boundary conditions, scale lengths for pump depletion and penetration are obtained; which for typical tokamak parameters and pump power $|CE_0/BC_s|^2 \lesssim 0(1)$, are of the order of centimeters. The results, thus, indicate that for present and future large-scale tokamaks, the lower-hybrid pump wave tend to anomalously depletes its energy close to the plasma surface.

We now briefly discuss the implications of the above results to the plasma heating process. We note that the depleted pump wave energy goes into the decay wave, which is parametrically amplified. However, due to the fact that the decay wave has a lower frequency and generally a finite k_y component, its cone trajectory in the x-z plane is closer to the plasma surface than that of the pump wave; i.e., $C_1 > C_0$ using the notation of Eqs.(25)' and (26)'. This observation suggests that the decay wave will have further problems in penetrating into the plasma and will deposit most of its energy near the surface; which implies, therefore, anomalous heating at the plasma surface.

Finally, we remark that the analysis presented here can be extended to include the effects due to plasma inhomogeneities. However, since the scale lengths due to nonlinear effects are shown to be of the order of centimeters and are generally comparable or shorter than those associated with inhomogeneities, the essential features of the results obtained here, therefore, are expected to remain valid.

ACKNOWLEDGMENT

One of us (L.C.) acknowledges useful discussions with Professor E.S. Weibel.

REFERENCES

[1] G. Morales and Y.C. Lee, Phys.Rev.Lett. 35, 930 (1975)

[2] R.L. Berger, L. Chen, C. Cheng, P.K. Kaw and F.W. Perkins, Proceedings of Annual Meeting on Theoretical Aspects of Controlled Thermonuclear Research, Madison, Wisconsin, USA (1975)

[3] M. Porkolab, Phys.Fluids 17, 1432 (1974)

[4] F.Y.F. Chu, Phys.Lett. 51A, 129 (1975). Also, F.Y.F. Chu and C.F.F. Karney, to be published

[5] A. Hasegawa and L. Chen, Phys.Fluids, 18, 1321 (1975); A. Rogister, Phys.Rev.Lett. 34, 80 (1975)

[6] L. Chen and R.L. Berger, Princeton University Report No PPL-MATT-1169 (1975)