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A USEFUL FORM OF THE VARIATIONAL PRINCIPLE OF IDEAL
MHD FOR ONE-DIMENSIONAL NUMERICAL
STABILITY CALCULATIONS

K. Appert, D. Berger and R. Gruber

Centre de Recherches en Physique des Plasmas

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

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A b s t r a c t

We treat the variational problem of ideal one-dimensional MHD stability by the method of Ritz-Galerkin ^[4]. Finite elements are taken as the basic functions. In order to find the best approximating elements we have performed a radius dependent transformation on the eigenvectors. The matrices of the transformed eigenvalue problem are given explicitly.

This report serves as an appendix to published papers [2, 5, 6].

1. Normal mode analysis

We study weak perturbation of a cylindrical ideal MHD equilibrium. The perturbation is described by the normal modes [1, 2, 3]

$$\vec{\xi}(t, r, \theta, z) = \vec{\xi}(r) \exp [i(\omega t + m\theta + kz)] \quad \text{with}$$

$$\vec{\xi}(r) = (\xi_r, \xi_\theta, \xi_z) = (\xi_1, -i\xi_2, -i\xi_3).$$

The linearized equations of motion can be brought into the variational form of Hamilton's principle

$$\omega^2 \delta k = \delta W_p \quad (1)$$

where the potential and the kinetic energy of the plasma are given by W_p and $\omega^2 k$ respectively. W_p and $\omega^2 k$ are quadratic forms of $\vec{\xi}(r)$. Takeda et al. [1] have shown how the problem (1) can be treated by the method of finite elements. In this approximation problem (1) reduces to an algebraic eigenvalue problem.

Takeda et al. [1] approximate the real physical displacement $\vec{\xi}(r)$ by linear admissible functions. However, it may be more appropriate to expand a transformed displacement

$$\vec{\eta}(r) = U(r) \vec{\xi}(r) \quad (2)$$

in a series of finite elements [2].

The transformation taken in [2] has given better results for the displacements near the axis. In [2] it was shown that the singularities induced by the cylindrical coordinates may give rise to numerical difficulties near the axis. Further it was shown that these difficulties may be removed by a special choice of U. Then there was a difficulty of a different character, which could not be removed at that time. One class of solutions was badly represented, because the internal condition ($\text{div } \vec{\xi} = 0$) characterising this class, could not be exactly fulfilled with the linear elements. So we proposed to use constant elements [6] together with another transformation U which allows to satisfy the internal condition. Therefore we have chosen the transformation

$$U(r) = \begin{pmatrix} r^{-\alpha} & 0 & 0 \\ br^{\beta} & \frac{1}{c}r^{-\gamma} & 0 \\ 0 & 0 & dr^{-\delta} \end{pmatrix} \quad (3)$$

where $\alpha, \beta, \gamma, \delta, b, c$ and d are constants. Note that ξ_2 is not mixed up with the other components, because the regularity conditions at the axis [2] have no influence on ξ_2 . Because (1) contains only derivatives on ξ_r the form of U will introduce only derivatives on η_1 . So U is a simple generalisation of the transformations used in [2] and [6]. The expansion of $\vec{\eta}(r)$ in finite elements is given by

$$\vec{\eta}(r) = \begin{pmatrix} \eta_1(r) \\ \eta_2(r) \\ \eta_3(r) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N_r} R_i r_i(r) \\ \sum_{i=1}^{N_\theta} \theta_i t_i(r) \\ \sum_{i=1}^{N_z} z_i z_i(r) \end{pmatrix} \quad (4)$$

where the elements are denoted by r_i , t_i and z_i , the expansion coefficients by R_i , Q_i and Z_i . Note that the numbers N_r, N_θ, N_z may differ by 1, if a mixture of linear and constant finite elements [6] is used.

Let us define

$$\vec{X}^T = (R_1, \theta_1, z_1, R_2, \theta_2, z_2, \dots) \equiv (x_1, \dots, x_m) \quad (5)$$

where $m = N_r + N_\theta + N_z$. Then (1) can be written in the form

$$\omega^2 \sum_{k=1}^m \left(\int_0^1 R_{pR}(r) dr \right) x_k = \sum_{k=1}^m \left(\int_0^1 C_{pR}(r) dr \right) x_k, \quad p=1, \dots, m \quad (6)$$

Let us denote the total number of integration intervals by N and a specific interval by i . Then the matrices R and C may be written in terms of partial matrices (\tilde{R}_i) and (\tilde{C}_i) which are for $U = \mathbf{1}$ and linear finite elements the same as in [1]. They are given by

$(\tilde{C}_i) =$

$f_{11}^{a_{11}+2f_{14}^{a_{14}}+f_{44}^{a_{44}}}$	$f_{12}^{a_{12}+f_{24}^{a_{24}}}$	$f_{13}^{a_{13}+f_{34}^{a_{34}}}$	$f_{11}^{a_{15}+f_{14}^{(a_{18}+a_{54})}+f_{44}^{a_{48}}}$	$f_{12}^{a_{16}+f_{24}^{a_{24}}}$	$f_{13}^{a_{17}+f_{34}^{a_{34}}}$
$f_{12}^{a_{21}+f_{24}^{a_{24}}}$	$f_{22}^{a_{22}}$	$f_{23}^{a_{23}}$	$f_{12}^{a_{25}+f_{24}^{a_{24}}}$	$f_{22}^{a_{26}}$	$f_{23}^{a_{27}}$
$f_{13}^{a_{31}+f_{34}^{a_{34}}}$	$f_{23}^{a_{32}}$	$f_{33}^{a_{33}}$	$f_{13}^{a_{35}+f_{34}^{a_{34}}}$	$f_{23}^{a_{36}}$	$f_{33}^{a_{37}}$
$f_{11}^{a_{51}+f_{14}^{(a_{81}+a_{45})}+f_{44}^{a_{84}}}$	$f_{12}^{a_{52}+f_{24}^{a_{24}}}$	$f_{13}^{a_{53}+f_{34}^{a_{34}}}$	$f_{11}^{a_{55}+2f_{14}^{a_{58}}+f_{44}^{a_{88}}}$	$f_{12}^{a_{46}+f_{24}^{a_{24}}}$	$f_{13}^{a_{47}+f_{34}^{a_{34}}}$
$f_{12}^{a_{61}+f_{24}^{a_{24}}}$	$f_{22}^{a_{62}}$	$f_{23}^{a_{63}}$	$f_{12}^{a_{64}+f_{24}^{a_{24}}}$	$f_{22}^{a_{66}}$	$f_{23}^{a_{67}}$
$f_{13}^{a_{71}+f_{34}^{a_{34}}}$	$f_{23}^{a_{72}}$	$f_{33}^{a_{73}}$	$f_{13}^{a_{74}+f_{34}^{a_{34}}}$	$f_{23}^{a_{76}}$	$f_{33}^{a_{77}}$

(7)

and

$g_{11} \mathcal{S}^{a_{11}}$	$g_{12} \mathcal{S}^{a_{12}}$	0	$g_{11} \mathcal{S}^{a_{15}}$	$g_{12} \mathcal{S}^{a_{16}}$	0
$g_{12} \mathcal{S}^{a_{21}}$	$g_{22} \mathcal{S}^{a_{22}}$	0	$g_{12} \mathcal{S}^{a_{25}}$	$g_{22} \mathcal{S}^{a_{26}}$	0
0	0	$g_{33} \mathcal{S}^{a_{33}}$	0	0	$g_{33} \mathcal{S}^{a_{37}}$
$g_{11} \mathcal{S}^{a_{51}}$	$g_{12} \mathcal{S}^{a_{52}}$	0	$g_{11} \mathcal{S}^{a_{55}}$	$g_{12} \mathcal{S}^{a_{56}}$	0
$g_{12} \mathcal{S}^{a_{61}}$	$g_{22} \mathcal{S}^{a_{62}}$	0	$g_{12} \mathcal{S}^{a_{65}}$	$g_{22} \mathcal{S}^{a_{66}}$	0
0	0	$g_{33} \mathcal{S}^{a_{73}}$	0	0	$g_{33} \mathcal{S}^{a_{77}}$

 $(\tilde{R}_i) =$

(8)

1
5
1

where:

$$P_{11} = r^{2\alpha+1} \left\{ \frac{1}{r^2} (B_z^2 + \Gamma_p) \left[\alpha+1 - bcmr^{\beta+\delta} \right]^2 + \left[(\alpha-1) \frac{B_\theta}{r} + bckB_z r^{\beta+\delta} \right]^2 + \left(\kappa B_z + \frac{mB_\theta}{r} \right)^2 - 4 \left(\frac{B_\theta}{r} \right)^2 - 2 \frac{B_\theta}{r} r \frac{d}{dr} \left(\frac{B_\theta}{r} \right) \right\} \quad (9)$$

$$P_{12} = cr^{\delta+\alpha+1} \left\{ \frac{m}{r^2} (B_z^2 + \Gamma_p) \left[\alpha+1 - bcmr^{\beta+\delta} \right] - B_z \kappa \left[(\alpha-1) \frac{B_\theta}{r} + bckB_z r^{\beta+\delta} \right] \right\}$$

$$P_{13} = dr^{\alpha+\delta+2} \left\{ \frac{1}{r^2} \left(\Gamma_p \kappa - \frac{B_\theta}{r} m B_z \right) \left[\alpha+1 - bcmr^{\beta+\delta} \right] + \frac{B_\theta}{r} \kappa \left[(\alpha-1) \frac{B_\theta}{r} + bckB_z r^{\beta+\delta} \right] \right\}$$

$$2P_{14} = r^{2\alpha+2} \left\{ \frac{1}{r^2} (B_z^2 + \Gamma_p) \left[\alpha+1 - bcmr^{\beta+\delta} \right] + \frac{B_\theta}{r} \left[(\alpha-1) \frac{B_\theta}{r} + bckB_z r^{\beta+\delta} \right] \right\}$$

$$P_{22} = c^2 r^{2\delta-1} \left\{ \Gamma_p m^2 + (\kappa^2 r^2 + m^2) B_z^2 \right\}$$

$$P_{23} = cd r^{\delta+\delta} \left\{ \Gamma_p m \kappa - (\kappa^2 r^2 + m^2) \frac{B_\theta}{r} B_z \right\}$$

$$P_{24} = cr^{\delta+\delta} \left\{ \Gamma_p m - (\kappa r B_\theta - m B_z) B_z \right\}$$

$$P_{33} = d^2 r^{2\delta+1} \left\{ \Gamma_p \kappa^2 + (\kappa^2 r^2 + m^2) \left(\frac{B_\theta}{r} \right)^2 \right\}$$

$$P_{34} = dr^{\alpha+\delta+1} \left\{ \Gamma_p \kappa + (\kappa r B_\theta - m B_z) \frac{B_\theta}{r} \right\}$$

$$P_{44} = r^{2\alpha+1} \left\{ B_z^2 + B_\theta^2 + \Gamma_p \right\}$$

and

$$\begin{aligned}
 g_{11} &= r^{2\alpha+1} \left\{ 1 + c^2 b^2 r^{2\beta+2\gamma} \right\} \\
 g_{12} &= -bc^2 r^{\alpha+\beta+2\gamma+1} \\
 g_{22} &= c^2 r^{2\gamma+1} \\
 g_{33} &= d^2 r^{2\delta+1}
 \end{aligned} \tag{10}$$

The factors $a_{\nu\mu}$ are the contributions of the basic functions and are determined by:

$$a_{\nu\mu} = e_\nu e_\mu \quad , \quad \nu, \mu = 1, \dots, 8 \tag{11}$$

where $\vec{e} = (r_i, t_i, z_i, r_{i+1}, t_{i+1}, z_{i+1}, r_{i+1})$ and $\frac{d}{dr} = ' .$

2. Boundary and regularity conditions

We want to solve the eigenvalue problem

$$(C - \omega^2 R) \cdot \vec{X} = 0 \tag{12}$$

with C, R symmetric and R positive definite.

ω^2 is the eigenvalue and \vec{X} the corresponding eigenfunction.

The boundary and the regularity conditions are

$$\xi_r (r=1) = 0 \tag{13}$$

$$r \vec{\xi}(r=0) = 0 \quad (14)$$

The regularity condition (14) is satisfied, if the matrix elements of (7) and (8) are finite at $r=0$. The transformation used in [2] corresponds to the parameters

$$b=c=d=1, \alpha=|m-1|, -\beta=\gamma=\alpha+2\delta_{m1}, \delta=|m| \quad (15)$$

where δ_{m1} is the Kronecker symbol.

This transformation was specifically chosen to describe the asymptotic behaviour of the displacements in a θ pinch at $r=0$.

For the case $m > 0$, we propose here a weaker condition to fulfil the regularity condition (14):

$$\beta+\gamma=0, \alpha=0, \delta=0, bcm=1 \quad (16)$$

In stability calculations $\text{div } \vec{\xi} = \frac{1}{r}(r\xi_1)' + \frac{m}{r}\xi_2 + R\xi_3$ is for some eigenmodes very small [5, 6] or even zero [2, 6]. In order to describe this fact we have to choose the elements properly. The parameters $b=1$ and $c=\frac{1}{m}$ correspond to the transformation

$$\begin{aligned} \eta_1 &= \xi_1 \\ \eta_2 &= \frac{\xi_1 + m\xi_2}{r} \\ \eta_3 &= \xi_3 \end{aligned} \quad (17)$$

Taking for η_1 linear and for η_2 and η_3 constant elements, we are able to describe the divergence exactly.

It was this choice of transformation and elements which made it possible to calculate degenerated modes [6], localised modes [5] and even singular modes with a continuous spectrum.

A c k n o w l e d g m e n t s

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