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STABILITY OF A DENSE PLASMA CONFINED BY A ROTATING MAGNETIC FIELD

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Abstract.

The stability of a dense plasma (negligible skin depth) confined by a rotating magnetic field which is parallel to the surface is analyzed for plane and cylindrical geometries. This analysis culminates in an integro-differential equation for each mode of deformation. Specific properties of the plasma enter only through the normal acoustic impedance. The concept of average stability is introduced. Very general restrictions on the acoustic impedance of the plasma are introduced which allow the derivation of various stability criteria. These criteria are applied to different cases of interest. The plasma is described successively by fluid models of increasing complexity and a free particle model. Numerical estimates of the frequency needed to achieve stability for each mode are given.

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CONTENTS

Introduction

Many authors (1-6) have proposed the use of alternating magnetic fields, with or without a static field, to confine a plasma. There are two main advantages to this. Firstly, field configurations can be obtained which are impossible to produce with static fields alone, such as a spherical geometry for example (2). Secondly configurations which would be unstable with static fields due to exchange or Rayleigh-Taylor instabilities can be made stable with alternating fields. We are interested here in this second aspect which is not well established at the present time.

Confinements with pure high-frequency alternating fields belong to the field free class of configurations, the skin depth being considered negligible. It has been recognized for some time that in geometries which would be unstable with static fields, alternating fields can provide an average positive restoring force, or as we shall call it : "average stability". In the case of confinement by a static field alone, surface waves propagating in a direction contained in a sector around a direction perpendicular to the field are unstable. The larger the destabilizing force (acceleration, curvature), the wider this sector. We can choose the geometry in such a way that waves propagating along the field lines are stable. By rotating or oscillating the field the sector rotates or oscillates too. A wave in a given direction passes alternatively through stable and unstable phases. The net average result may be stable. This is the fundamental idea of stabilization with alternating fields.

The restoring force is in fact a linear functional of the surface deformation, but having an oscillating time dependence arising from the alternating field. Averaging over a period of the alternating field gives the average restoring force used above. This paper is a study of the influence of the explicit time dependence of the restoring force in situations where there is average stability. It is suspected by analogy with parametric oscillators that average stability does not always imply stability in the usual sense. The frequency of the confining field is the important parameter which is expected to control the stability. This point has been already examined for some particular geometries and plasma models by most of the authors cited above (1, 4, 5, 6, 8). Let us compare their hypotheses and their results.

J. Berkowitz et al (1) investigate the problem of a half-space filled with an incompressible inviscid fluid supported against gravity by a homogeneous rotating field parallel to the plane horizontal interface. There is no average stability for long wavelengths and they find that even in the domain of wavelengths where there is average stability, it is impossible to choose a frequency which provides stability. The addition of a perfectly conducting plane parallel to the interface, at a suitable distance from it produces average stability but does not change Berkowitz results. When we refer herein to Berkowitz's results it should be understood to refer to the case with the conducting plane. It was also found that the growthrate of the instabilities is bounded when the wavelength of the deformation vanishes, which is an improvement over the unbounded growthrate found with a static field only. We can say that the rotating field allows one to control the growthrates of the instabilities.

R.J. Tayler (4) considers a cylindrical discharge confined by a rotating field. The plasma is represented by a compressible fluid model ; transport terms are neglected. Because of the mathematical difficulties he only considers the limit of infinite frequency. He finds, in this limit, that the long wavelengths are unstable. This is due to a lack of average stability for these modes. Furthermore, using the same plasma model as Berkowitz, he finds essentially the same results, viz. there are always instabilities for short wavelengths.

From the abstract it appears that N. Rostoker (5) merely rederives Tayler's results.

E.S. Weibel has stated (7), without proof, as early as 1958, that for a perfectly diamagnetic cylindrical plasma average stability can be obtained by the superposition of an axial and an azimuthal field, of which at least one is oscillating. He proves this in a subsequent paper (6) for the superposition of a static axial field and an alternating azimuthal field. This result also holds true for the rotating field case, but it does not affect the short wavelengths instabilities found by Tayler and Rostoker since for these wavelengths there is always average stability. Weibel attributes the results of Berkowitz, Tayler and Rostoker to an unrealistic choice of a plasma model. In order to prove his point he examines the stability of a Θ-pinch and a superposed alternating ζ-pinch, with a concentric conducting envelope. The plasma is treated as an assembly of non colliding particles. He then proves that any mode can be stabilized with a sufficiently high frequency. However this frequency is an unbounded function of the wave-number. But he proves also that for zero wavelength there is stability for any frequency. Physically then, it is very probable that there must be a finite frequency for which all the modes are stable although this is not proved and no estimation of this frequency is obtained. This loophole is closed in the present paper. In his proof of the possibility of stabilizing any given mode for a sufficiently high frequency the only pronerty needed is that the plasma is dissipative at all frequencies. This leads him to the conjecture that if transport terms (viscosity, thermal conductivity) had been kept in the fluid model used by the previous authors they would have also found that for a sufficiently high frequency there is stability.

J.W. Miles (8) makes such an attempt introducing viscosity into the compressible fluid model. He proceeds to investigate the same problem as in (6). Miles claims that the inclusion of viscosity does not suppress the instabilities. Objections can be raised to this paper on some points. The impedance used corresponds to unrealistic boundary conditions at the interface, that is the stress tensor is not continuous for a non vanishing viscosity and the oscillating terms in the pressure response are assumed to be much smaller than the average term, a condition which is not actually the case since both are of the same order of magnitude. It could nevertheless be argued that, physically, if there are instabilities when the oscillating terms are small then they should become even worse when these terms are large. But even accepting all these premises the mathematical treatment does not appear correct. Judging from the slab problem which is investigated herein, Miles should have found that it was possible to stabilize all the modes with a frequency much smaller (in the ratio of the oscillating terms to the average value of the pressure response) than the cut-off frequency which defines approximately the limit of validity of the model.

The main object of this paper is to shed some light on what appears to be a confused issue and to verify Weibel's conjecture. We shall try to remain general, avoiding as long as possible the choice of a specific plasma model. Some of the results have been available in restricted publications (9) only and in abstract form in (10). The following is a summary of the salient results obtained in this study.

In plane or cylindrical configurations, where all the properties depend only on the distance to the plasma interface, a Fourier analysis of the deformation in normal modes reduces the stability analysis to the examination of an integro-differential equation for each mode. The plasma enters the equation only through its normal acoustic impedance. Four general assumptions are made : there is average stability; the plasma is stable, implying that only surface waves can become unstable; the acoustic impedance has a non negative resistive component for all frequencies and which does not vanish at infinite frequency. Using these assumptions various stability criteria can be derived. A general stability criterion, necessary and sufficient, is given. It is not convenient and other sufficient criteria are given which are of a more direct use. These suffice to show that each mode can be stabilized with a high enough frequency, just as for Weibel's model. This does not prove yet that all modes can be stable using the same frequency. In general two difficulties can arise : the first one can appear at short wavelengths as is the case in (6) and it cannot be solved without having more information about the plasma; a second difficulty may arise for long wavelengths and it is again not possible to draw general conclusions without choosing a specific model of the plasma. We then are compelled to discuss specific models of the plasma in order to answer the stability problem. To help solve the short wavelength problem another simple sufficient stability criterion is derived which requires an additional assumption to be made concerning the impedance.

We first study the same configuration as Berkowitz (with the conducting plane) using as plasma models successively : an inviscid compressible fluid with no acceleration field; the same with an acceleration field; a free particle model without and with an acceleration field, and finally a viscous compressible fluid with no acceleration field. These various models when taken together present all the possible complications which can arise at long and short wavelengths. Their relative simplicity enables one to carry out a complete discussion of these complications. They constitute therefore a good test of the criteria derived in this paper. The results for the collisionless model are very different from those obtained with the fluid models. For the collisionless model there is always stability at a sufficiently high frequency. The minimum frequency for which there is stability is proportional to the acceleration. On the other hand the compressible inviscid fluid model predicts instabilities for a sufficiently high frequency. This last result remains valid when an acceleration field is introduced although the very long wavelengths may become stable depending on the value of γ. Introducing viscosity, it is found that for a frequency of the order of or larger than the collision frequency all the modes are stable.

To estimate the possible effect of the finite thickness of the plasma we choose to study the problem of a slab of plasma confined by the same rotating field on both sides (same means parallel at all times). This geometry is chosen because of its similarity to the cylindrical geometry. The problem divides itself naturally in two parts, namely the problem of symmetric deformations and the problem of antisymmetric deformations which correspond respectively in the cylindrical problem to the modes m even and m odd. Two plasma models only are examined : the free particles model and the viscous compressible fluid model, both without acceleration field. As expected the viscous compressible fluid model does not yield results significantly different from the half-plane. The finite depth of the plasma manifests itself only at long wavelengths whereas it is the short wavelengths which determine the stabilizing frequency. The collisionless model again predicts stability for a sufficiently high frequency, namely a frequency higher than the ion transit frequency.

The case of a cylindrical plasma enclosed by a concentric cylindrical conductor is also treated. It is shown that if the radius of the conductor is less than $\sqrt{3}$ times the plasma radius then there is average stability. The stability problem can easily be solved for the collisionless model. The result is naturally the same as for the slab, except for a numerical factor. For the fluid model the cylindricity is not expected to exhibit any new result; since the impedance for this model is extremely complicated we do not examine it here.

From the above it is clear that the solution of the stability problem is strongly model dependent. In the range where the collisionless model applies there is certainly stability and the minimum frequency is equal within a numerical factor, to the average transit frequency across the plasma. In the case of the fluid models the minimum frequency is of the order of the ion collision frequency if the mean free path is smaller than the depth of the plasma and of the magnetic field. This frequency lies outside the range of strict validity of the models : nevertheless it can be argued that the existence of a cut-off frequency for acoustic waves is independent of the model and thus the result must still be qualitatively correct.

The best solution of course would be to use a model which represents correctly the transition from the fluid region to the free particle region. The formulation of such a model which does more than simply interpolate between the two models is itself an unsolved problem.

II Derivation of equations

2.1. General assumptions

Let us consider a plasma in equilibrium confined by an alternating field or a superposition of such fields, $\vec{B}(t)$. The following is a list of the relevant physical quantities which appear in this paper.

δ = skin depth
λ = mean free path
L = characteristic dimension of the plasma
ε = deformation of the surface about the equilibrium position.
h, n or H = wave number of the deformation
ω = frequency of the magnetic field ; there may be more than one.
υ = sound speed or mean ion velocity

We assume that

(1) $$\delta \ll \varepsilon \ , \qquad \delta \ll \lambda \ , \qquad \delta \ll \frac{1}{H} \ , \quad \frac{\omega\,\delta}{\upsilon} \ll 1$$

These inequalities express the fact that the skin depth is negligible. The assumption of small motions implies that

(2) $$\dot{\varepsilon} \ll \upsilon \ , \quad \varepsilon \ll \frac{1}{H} \quad \text{and} \quad \varepsilon \ll L$$

The frequency is considered to be sufficiently low that displacement currents can be neglected, that is

(3) $$L \ll \frac{c}{\omega}$$

The magnetic field satisfies the equations

(4) $$\text{curl}\ \vec{B} = 0$$
$$\text{div}\ \vec{B} = 0$$

The magnetic field exerts only a normal pressure on the surface and the equality of this magnetic pressure p_m and of the plasma pressure p_G constitutes the equation of motion of the surface. It reads

(5) $$p_G = p_m$$

2.2. Acoustic impedance

For plane (semi-infinite or slabs) or cylindrical plasmas, whose equilibrium properties depend only on the distance to the surface, a double decomposition in normal modes can be performed. The gas pressure p_G at the deformed surface can be written for each mode as

$$(6) \qquad p_G(n,h,t) = p_o + \frac{\partial p_o}{\partial n}\, \varepsilon(n,h,t) - \int_0^t R(n,h,t-t')\, \dot{\varepsilon}(n,h,t')\, dt'$$

where p_o is the initial pressure at the undeformed surface, $\frac{\partial p_o}{\partial n}$ is the normal component of the pressure gradient with the positive normal pointing out of the plasma and n and h are the indices of the mode. R(n,h,t) is the plasma response function which can be related to the normal acoustic impedance (11)in the following manner. The normal acoustic impedance $Z(\Omega)$ is defined as the analytic continuation of $\tilde{R}(n,h,s)$

$$(7) \qquad Z(\Omega) = \tilde{R}(n,h,i\Omega)\,, \quad \Omega \text{ real}$$

where R(n,h,s) is the following Laplace transform

$$(8) \qquad \tilde{R}(n,h,s) = \int_0^\infty R(n,h,t)\, e^{-st}\, dt$$

We shall also make use of the name acoustic impedance to refer to $\tilde{R}(n,h,s)$.

The following assumptions are made regarding $\tilde{R}(n,h,s)$

For $\mathcal{R}e\ s \geqslant 0$

a) $\tilde{R}(n,h,s)$ is holomorphic, continuous on $\mathcal{R}e\ s = 0$, with the possible exception of a pole at the origin.

(9) b) $\left|\tilde{R}(n,h,s)\right| \geqslant$ constant as $s \longrightarrow \infty$

c) $\mathcal{R}e\ \tilde{R}(n,h,s) \geqslant 0$; the equality can be satisfied only for $s = 0$

The first assumption is equivalent to the following statement : the unperturbed plasma is in a stable equilibrium 1). The second assumption expresses the fact that R(n,h,t) is at least as singular as a δ function for t = 0; every realistic model should satisfy this requirement since the velocity of the surface must remain finite when a pressure pulse is applied. The third assumption expresses the fact that the plasma is dissipative (6). Many of the subsequent results will remain true when the inequality (9c) is satisfied only in the open half-plane $\mathcal{R}e\ s > 0$. However the essential results demand that this inequality be satisfied on the imaginary axis. Note that (9c) is satisfied if it holds true along the imaginary axis and for large values of s. The three assumptions (9) thus constitute very general restrictions which any realistic model should satisfy.

Note the crossing relation

$$(10) \quad \tilde{R}(n,h,s^*) = \tilde{R}^*(n,h,s)$$

which follows immediately from the definition (8). The asterisk refers to the complex conjugate.

2.3. Magnetic pressure.

We calculate here the magnetic pressure p_m at the deformed surface for both plane and cylindrical geometries in the case of a rotating magnetic field.

a) Plane geometry

Consider a plane interface (x,y) between the plasma and the rotating magnetic field, which is parallel to the surface, homogeneous and is limited by a perfectly conducting plane located at some distance

1) For n=h=0 there is a pole at the origin if the plasma has a finite thickness. This mode is the only one for which there is a change of volume. After an infinite time therefore the change in plasma pressure becomes proportional to the displacement ε.

D from the plasma. $\vec{B}(t)$ has the form :

(11) $$B_x = B_o \cos \omega t$$
$$B_y = B_o \sin \omega t$$

The indices n,h are defined by

(12) $$\varepsilon(x,y,t) = e^{ihx+iny}\, \varepsilon(n,h,t)$$

The magnetic pressure is given by

(13) $$p_m(n,h,t) = \frac{p_o H \cosh DH}{\sinh DH}(1+ \cos 2\,\omega t)\, \varepsilon(n,h,t) + p_o$$
$$H = \sqrt{h^2+n^2}$$

The time average of this pressure is

(14) $$\langle p_m(n,h,t)\rangle = p_o + \frac{p_o H \cosh DH}{\sinh DH}\, \varepsilon(n,h,t)$$

We note that the average force is a restoring force. Furthermore

(15) $$\frac{p_m(n,h,t) - p_o}{\varepsilon(n,h,t)} \geqslant 0$$

$$\frac{\langle p_m(n,h,t)\rangle - p_o}{\varepsilon(n,h,t)} \geqslant \frac{p_o}{D}$$

These last two equations show the stabilizing effect of the rotating field as compared to the static field.

b) Cylindrical geometry

Consider an infinite cylinder of plasma of radius a confined by a rotating field B_z, B_θ and which is limited by a concentric, perfectly conducting cylinder of radius b. We have

(16) $$B_z = B_o \cos \omega t$$

$$B_\theta = B_o \frac{a}{r} \sin \omega t$$

The normal mode indices n, h are defined through

(17) $$r = a + \varepsilon(n,h,t)\, e^{ihz + in\theta}$$

The magnetic pressure response for any B_θ, B_z is computed in (<u>6</u>). the result is

(18) $$p_m(n,h,t) = p_o - \frac{\varepsilon(n,h,t)}{a} \left(\frac{y_n(h,a)}{a y'_n(h,a)} \quad (B_\theta n + B_z a h)^2 + B_\theta^{\,2} \right)$$

with

(19) $$y_n(h,a) = K'_{|n|}(|h|\, b)\, I_{|n|}(|h|\, a) - I'_n(|h|\, b)\, K_{|n|}(|h|\, a)$$

where $K_n(x)$ et $I_n(x)$ are the modified Bessel functions.

Using (16) $p_m(n,h,t)$ becomes

(20) $$p_m(n,h,t) = p_o + \frac{p_o \varepsilon(n,h,t)}{a} \Big\{ f_n(h) + g_n(h) - 1 +$$

$$+ (g_n(h) - f_n(h) + 1) \cos 2\omega t + 2 \sqrt{g_n(h)\, f_n(h)} \sin 2\omega t \Big\}$$

$$g_n(h) = \frac{-n^2 y_n(h,a)}{a y'_n(h,a)}$$

$$f_n(h) = \frac{-a h^2 y_n(h,a)}{y'_n(h,a)}$$

where $y'_n(h,a)$ is the derivative with respect to a. Shifting the origin of time enables us to rewrite (20) in the form

(21) $$p_m(n,h,t) = p_o + \frac{p_o \varepsilon(n,h,t)}{a} \Big\{ f_n(h) + g_n(h) - 1 +$$

$$+ \sqrt{(f_n(h) + g_n(h) - 1)^2 + 4\, g_n(h)} \cos 2\omega t \Big\}$$

The functions $g_n(h)$ and $f_n(h)$ have been studied extensively by Weibel (6) and only the results needed here are quoted. They are both positive definite, $g_n(h)$ is a monotonically increasing function of h and $f_n(h)$ a monotonically decreasing function of h. Furthermore

$$(22) \qquad f_n(h) + g_n(h) - 1 \geqslant \frac{3a^2 - b^2}{b^2 - a^2}$$

We shall assume that $b^2 < 3a^2$ so that the expression (22) is positive. In analogy with the plane geometry we note the inequalities

$$(23) \qquad \frac{\langle p(n,h,t)\rangle - p_o}{\varepsilon(n,h,t)} > \frac{p_o(3a^2-b^2)}{a(b^2-a^2)}$$

$$\frac{p(n,h,t) - p_o}{\varepsilon(n,h,t)} \geqslant -\frac{p_o}{a}\left(\sqrt{(f_n(h) + g_n(h) - 1)^2 + 4g_n(h)} - f_n(h) - g_n(h) + 1\right) \geqslant -\frac{2\,p_o}{a}$$

2.4. Equation of motion

Equating the results for $p_G(n,h,t)$ and $p_m(n,h,t)$ one obtains the equation of motion of the disturbance.

$$(24) \qquad \varepsilon(n,h,t)\left\{X(n,h) + A(n,h)\cos 2\omega t\right\} + \int_0^t R(n,h,t-t')\,\dot{\varepsilon}(t')dt' = 0$$

with the definitions

plane	cylinder
(25) $X(n,h) = \dfrac{p_oH\cosh HD}{\sinh HD} - \dfrac{\partial p_o}{\partial n}$	$X(n,h) = p_o(f_n(h) + g_n(h) - 1) - \dfrac{\partial p_o}{\partial r}$
$A(n,h) = \dfrac{p_oH\cosh HD}{\sinh HD}$	$A(n,h) = p_o\sqrt{(f_n(h)+g_n(h)-1)^2 + 4g_n(h)}$

D, b and a are chosen such that

(26) $X(n,h) \geq X_m > 0$

This implies the following restrictions

(27) $\frac{\partial p_o}{\partial n} < \frac{p_o}{D}, \quad \frac{\partial p_o}{\partial r} < \frac{p_o}{a}\left(\frac{3a^2-b^2}{b^2-a^2}\right)$

We are now ready to discuss the problem of the stability of the equation of motion.

III Stability criteria

3.1. Generalization of the equation

Equation (24) can be slightly generalized so as to cover cases of interest which may occur in other problems, such as those considered by Weibel (6) and Miles (8). This gives

(28) $$(X + \sum_{k=-N}^{+N} a_k e^{ik\omega t})\, \varepsilon(t) + \int_0^t R(t-t')\, \dot{\varepsilon}(t')\, dt' = 0$$

with the restrictions

(29) $X > 0, \quad a^*_k = a_{-k}, \quad a_o = 0$

$\tilde{R}(s)$ satisfies the conditions (9) and the property (10).

3.2. Average stability

Using the symbol $\tilde{\varepsilon}(s)$ to denote the Laplace transform of $\varepsilon(t)$ and choosing $\varepsilon(0)= 1$, the Laplace transform of equation (28) can be written

$$(30) \qquad \tilde{\varepsilon}(s) = \tilde{\varepsilon}_o(s) + g(s) \sum_{-N}^{+N} a_k \varepsilon(s+ik\omega)$$

with

$$(31) \qquad \tilde{\varepsilon}_o(s) = \frac{\tilde{R}(s)}{X+s\,\tilde{R}(s)}$$

$$g(s) = \frac{-1}{X + s\tilde{R}(s)} = \int_o^\infty e^{-st}\, \dot{\varepsilon}_o(t)\, dt$$

The conditions (9) and the restriction (29) are sufficient to guarantee that $g(s)$ is analytic in the half plane and continuous on the imaginary axis. $\tilde{\varepsilon}_o(s)$ has the same property with the possible exception of a pole at the origin. This implies that $\varepsilon_o(t)$ the inverse transform of $\tilde{\varepsilon}_o(s)$, is bounded. This is what we shall call "average stability". To justify this name, we shall show later that the solution of (30) tends toward $\tilde{\varepsilon}_o(s)$ when $\omega \longrightarrow \infty$.

3.3. Iterated equations.

Equation (30) can be iterated any number of times. The equation when iterated k times becomes

$$(32) \qquad \tilde{\varepsilon}(s) = \tilde{\varepsilon}_k(s) + g(s) \sum_{n=-N-kN}^{N+kN} G_n{}^k(s)\, \tilde{\varepsilon}(s+in\omega)$$

$\tilde{\varepsilon}_k(s)$ and $G_n{}^k(s)$ are defined by the recursion relations

$$\tilde{\varepsilon}_k(s) = \tilde{\varepsilon}_{k-1}(s) + g(s) \sum_{n=-kN}^{kN} G_n{}^{k-1}(s)\, \tilde{\varepsilon}_o(s+in\omega)$$

$$(33) \qquad G_n{}^k(s) = \sum_{n=-kN}^{kN} a_{n-m}\, G_m{}^{k-1}(s)\, g(s+im\omega)$$

with

$$(34) \qquad G_n{}^o(s) = a_n$$

Except where otherwise stated s will be limited to the half-plane $\mathcal{R}e\ s \geqslant 0$. To simplify discussion the following definition will be used : a function is called a.c. if it is analytic in the half-plane and continuous on the imaginary axis.

The recurrence relation (33) shows that $G_n^k(s)$ is a.c. since g(s) is a.c.; $\tilde{\varepsilon}_k(s)$ is also a.c. with the additional possibility of its having poles in $s = \pm\, in\omega, |n| \leq kN$

3.4. Simple stability criteria.

Equation (32) can be regarded as a Fredholm equation, the integration being replaced by a sum over the discrete set of points $s+in\omega$. This becomes obvious if we rewrite equation (32) in the following way :

$$(35)\qquad \tilde{\varepsilon}(s+in\omega) = \tilde{\varepsilon}_k(s+in\omega) + g(s+in\omega) \sum_m G^k_{m-n}(s+in\omega)\, \tilde{\varepsilon}(s+im\omega)$$

The variables are n and m; s is only a parameter.

A sufficient condition for the uniform and absolute convergence of the iterated expansion of the solution of (35) is

$$(36)\qquad \sum_m \left| G^k_{m-n}(s+in\omega)\ g\ (s+in\omega) \right| \leq M < 1$$

with M independent of n. If this condition is satisfied for all s (s being always in the half-plane), which means that M is independent of s, the iterated expansion is an uniformly convergent expansion of a.c. functions and is thus also a.c. Again there is the possibility of having the string of poles at $s = \pm\, in\omega$. $\varepsilon(t)$ is therefore bounded and there is stability. The criterion (36) is thus a sufficient condition for stability of the solution of (30)

Let us take as an example k = 1. Condition (36) may be written :

$$(37)\qquad |g(s)| \sum_m \left| \sum_k a_k a_{m-k}\ g(s+ik\omega) \right| \leq M < 1$$

The following simpler condition

$$(38) \qquad \left(\sum_k |a_k|^2\right) \max_{n\neq 0} \left\{ |g(s)\ g(s+in\omega)| \right\} \leq M < 1$$

is also sufficient since it implies (37). It is obvious from (38) and (9b) that $M \to 0$ as $\omega \to \infty$, and $\tilde{\varepsilon}_o(s+i\omega) \longrightarrow 0$ as $\omega \longrightarrow \infty$. One deduces that

$$(39) \qquad \tilde{\varepsilon}(s) \xrightarrow{\omega\to\infty} \tilde{\varepsilon}_o(s)$$

which justifies the expression "average stability" used earlier, since $\varepsilon_o(t)$ appears as the average motion of the surface for very large frequency.

As far as the practical use of criteria (36) and (37) is concerned, note that if they are satisfied on the imaginary axis, they are satisfied everywhere since $g(s)\ G^k(s+i\omega)$ is a.c. and is zero at infinity. The criterion (38) is the one used in (6), where it is derived in a different manner.

Conditions (37) and (38) are simple but very restrictive. They overestimate the frequency necessary to obtain a stable solution. For example when the system has sharp resonances at low frequency, the stabilizing frequency given by (37) is much too high. It is thus necessary to investigate the possibility of deriving a general stability criterion, necessary and sufficient. This is the goal of the next section.

3.5. Method of the determinant

This method is just the original induction method used by Fredholm to solve his equation (12). It is simpler than the original because of the integration on a discrete set but is complicated by the convergence problem due to the infinite range.

Let us replace equation (35) by an infinite system of linear equations for $\tilde{\varepsilon}(s+in\omega)$ and solve it by Cramer's formula

$$(40) \qquad \tilde{\varepsilon}(s) = \frac{A(s)}{D(s)}$$

where D(s) is the infinite determinant

$$(41) \qquad D(s) = \left\| \delta_{m-n} - a_{m-n}\, g(s+im\omega) \right\|$$

and A(s) is the determinant obtained by substituting in D(s) the column m = 0 by the column $\tilde{\varepsilon}_0(s+in\omega)$. Hadamard's formula shows that the determinants A(s) and D(s) converge if

$$(42) \qquad \sum_{n=-\infty}^{n=+\infty} \left| g(s+in\omega) \right|^2 < \infty$$

This condition is satisfied as a consequence of (9b) which shows that for large n the series (42) is majorized by the convergent series.

$$(43) \qquad \text{constant} \sum \frac{1}{n^2} < +\infty$$

From its definition (41) and from the symmetry (10) one obtains :

$$(44) \qquad \begin{aligned} D(s+i\omega) &= D(s) \\ D(s^*) &= D^*(s) \end{aligned}$$

Introducing the usual notation for Fredholm determinants

$$(45) \qquad K\begin{pmatrix} n_1 & n_2 \ldots n_\ell \\ m_1 & m_2 \ldots m_\ell \end{pmatrix} = \begin{vmatrix} 0 & a_{n_2-m_1} & \cdots & a_{n_\ell-m_1} \\ a_{m_2-n_1} & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ & & & 0 \\ a_{m_\ell-n_1} & \cdots & \cdots & \end{vmatrix}$$

D(s) can be expanded

(46) $$D(s) = 1 + \sum_{\ell=2}^{\infty} \frac{(-1)^{\ell}}{\ell!} \sum_{n_1 n_2 .. n_\ell = -\infty} K\begin{pmatrix} n_1 & n_2 .. n_\ell \\ n_1 & n_2 .. n_\ell \end{pmatrix} g(s+in_1\omega)..g(s+in_\ell\omega)$$

This is an expansion in terms of a.c. functions. A direct examination of (46) shows that the expansion converges uniformly and D(s) is thus a.c. The same is true for A(s). $\tilde{\varepsilon}(s)$ is a.c. if D(s) has no zero; then the solution is stable. This gives as the stability condition

(47) $$D(s) \neq 0$$

Note that it is sufficient for stability to verify (47) for $0 \leq \Im ms \leq \frac{\omega}{2}$ as a consequence of the relations (44).

The condition (47) is sufficient but a priori not necessary since the zeros of D(s) could coincide with zeros of A(s). One knows that the resolvent kernel has the same poles as D(s) (12), but it cannot be excluded that the summation over the resolvent kernel, weighted by $\tilde{\varepsilon}_o(s+in\omega)$, makes the pole disappear. This is unlikely but not impossible, especially as $\tilde{\varepsilon}_o(s)$ and g(s) are closely related. We have not been able to show the contrary, except in the limit of high ω. The condition (47) is thus necessary if the zeros of A(s) do not coincide with the zeros of D(s).

The criterion (47) is in general unwieldy because of the slow convergence of the expansion. Nevertheless it can be useful when one is looking for a value of the frequency as low as possible while still having stability, and also in some special cases where there is no other applicable criterion.

It is possible to improve the rate of convergence of the expansions by applying the same method of solution to the iterated equation (32). Letting $D^k(s)$ be the denominator corresponding to the equation iterated k-times, we have (12)

(48) $$D^k\left[g(s)\right] = D\left[g(s)\right] D\left[\alpha g(s)\right] \dots D\left[\alpha^{k-1} g(s)\right]$$

where α is a root of $\alpha^k = 1$. The square brackets indicate that $D^k(s)$ is to be considered as a functional of g(s).

The expanded form is more useful.

(49) $$D^k(s)=1+\sum_{\ell=1}^{+\infty}\frac{(-1)^\ell}{\ell!}\sum_{n_1,n_2\ldots n_\ell=-\infty}^{+\infty} K^{(k)}\begin{pmatrix} n_1 n_2 \ldots n_\ell \\ n_1 n_2 \ldots n_\ell \end{pmatrix} g(s+in_1\omega)\ldots g(s+in_\ell\omega)$$

where

(50) $$K^{(k)}\begin{pmatrix} n_1 n_2 \cdots n_\ell \\ m_1 m_2 \cdots m_\ell \end{pmatrix} = \begin{vmatrix} G^k_{n_1-m_1}(s+im_1\omega) & \cdots & G^k_{n_1-m_\ell}(s+im_\ell\omega) \\ G^k_{n_2-m_1}(s+im_1\omega) & & \vdots \\ \vdots & & \vdots \\ G^k_{n_\ell-m_1}(s+im_1\omega) & \cdots & G^k_{n_\ell-m_\ell}(s+im_\ell\omega) \end{vmatrix}$$

The convergence is faster in (49) than in (46) since the following condition is sufficient for the convergence of (49)

(51) $$\sum_{n=-\infty}^{n=+\infty} |g(s+in\omega)|^{k+1} < \infty$$

We obtain the new A(s), say $A^k(s)$, by multiplying the old A(s) by the same factor as multiplies D(s) in (48). This shows that $D^k(s)$ may contain zeros coming from the factors $D[\alpha^n g(s)]$ which are not poles of the solution. Thus in this case it is necessary to verify if the zeros found for $D^k(s)$ are also zeros of $A^k(s)$. However the condition

(52) $$D^k(s) \neq 0$$

still remains sufficient for stability. We shall not need this condition.

Finally note that the criterion (37) can be deduced from criterion (43). Indeed D(s) can be reexpressed in terms of the traces of the kernel (12)

$$(53)\quad D(s)=\exp\left\{-\sum_{\ell=2}^{\infty}\frac{1}{\ell}\sum_{n_1 n_2 \ldots n_\ell=-\infty}^{+\infty} a_{n_1-n_2}\, a_{n_2-n_3}\ldots a_{n_\ell-n_1}\; g(s+in_1\omega)\ldots g(s+in_\ell\,\omega)\right\}$$

if the expansion converges. The condition (37) insures this convergence and D(s) has therefore no zeros and (47) is satisfied.

3.6. Other forms of stability criteria

One can write

$$(54)\qquad D(s) = D_1(s) + g(s)\, D_2(s)$$

where

$$D_1(s) = 1+\sum_{\ell=2}^{+\infty}\frac{(-1)^\ell}{\ell!}\sum_{\substack{n_1\ldots n_\ell=-\infty \\ n_k\neq 0}}^{+\infty} K\begin{pmatrix} n_1 n_2 \ldots n_\ell \\ n_1 n_2 \ldots n_\ell\end{pmatrix} g(s+in_1\omega)\ldots g(s+in_\ell\,\omega)$$

(55)

$$D_2(s) = -\sum_{\ell=1}^{+\infty}\frac{(-1)^\ell}{\ell!}\sum_{n_1 n_2 \ldots n_\ell=-\infty}^{+\infty} K\begin{pmatrix} o & n_1 \ldots n_\ell \\ o & n_1 \ldots n_\ell\end{pmatrix} g(s+in_1\omega)\ldots g(s+in_\ell\,\omega)$$

The condition (47) can be rewritten as

$$(56)\qquad X + s\tilde{R}(s) - \frac{D_2(s)}{D_1(s)} \neq 0$$

s will be restricted to the domain $0 \leq \mathrm{Im}\ s \leq \frac{\omega}{2}$.

This form is particularly useful when $g(s)$ has poles close to the imaginary axis (weakly damped resonances) and the frequency is high enough so that $D_1(s)$ and $D_2(s)$ may be represented by only a few terms of their expansions. It also shows that when $\omega \to \infty$ the zeros of $D(s)$ coincide with those of $X+s\tilde{R}(s)$, namely the poles of $g(s)$. For ω large one can follow the displacement of these zeros as a function of ω by replacing $D_1(s)$ and $D_2(s)$ by their approximate values.

(57) $$D_1(s) \sim 1$$

$$D_2(s) \sim -\sum_K |a_k|^2 \, g(s+ik\omega)$$

The condition (56) and the relations (57) show that the position of the zeros of $D(s)$ depend as much on the asymptotic behaviour of $g(s)$ for large s as on its behaviour near the pole. This shows that Miles's method (8), which approximates $g(s)$ by the contribution of the pole and its conjugate is not applicable for large frequencies.

The Nyquist diagram method is the best way of making practical use of this criterion (56). If ω is large enough for $D_2(s)/D_1(s)$ to be a.c., then the function $G(s)$ defined by

(58) $$G(s) = X + s\tilde{R}(s) - \frac{D_2(s)}{D_1(s)}$$

is also a.c. Consider the contour $\mathcal{C}$: $+\infty$, 0, $+\frac{i\omega}{2}$, $+\frac{i\omega}{2}+\infty$ shown in fig. 1. A necessary and sufficient condition for (56) to be true is that the curve representing $G(s)$, where s describes $\mathcal{C}$, does not encircle the origin or that if it passes through the origin, the origin corresponds to a purely imaginary value of s. If, for very large ω, there are no roots of $D(s)$, then, because of (44) and (9c), roots can only appear, as ω decreases, by moving through the imaginary axis.

All the criteria which will be used in this paper and which do not demand additional assumptions on the behaviour of $\tilde{R}(s)$ have now been derived.

A useful application of the results of this section will now be presented.

3.7. Case where g(s) has a pair of poles on the imaginary axis.

An important special case is the one where g(s) has a pair of simple poles on the imaginary axis, say at $\pm$ i.

We have

$$(59) \qquad \tilde{R}(i) = iX$$

From (9c) which is assumed to be still valid everywhere else, we obtain

$$(60) \qquad \mathcal{I}m\, \tilde{R}'(i) \equiv \mathcal{I}m\, \frac{d\tilde{R}(s)}{ds}\bigg|_{s=i} = 0$$

$$\tilde{R}'(i) > 0$$

Let us use the method presented in the preceding section to follow the behaviour of the zero of D(s) which goes to +i as $\omega \to \infty$. We assume that ω is sufficiently large so that the forms (57) can be used. Let s be the root of D(s). Then to first order, one has

$$(61) \qquad s-i \approx \frac{-i\,\dfrac{D_2(i)}{D_1(i)}}{X+\tilde{R}'(i)} \approx \frac{i\sum_k |a_k|^2\, g(i+ik\omega)}{X+\tilde{R}'(i)}$$

from (60) and (61)

$$(62) \qquad \mathcal{R}e(s-i) \approx \frac{-1}{X+\tilde{R}'(i)} \sum_{k>0}^{N} |a_k|^2 \left\{ \mathcal{I}m\, g(i+ik\omega) - \mathcal{I}m\, g(ik\omega-i) \right\}$$

Inequality (9c) implies that $\mathcal{I}m\, g(i+ik\omega) > 0$ and (9b) that $\mathcal{I}m\, g(iy) \to 0$ as $y \to \infty$. This means that if $\mathcal{I}m\, g(iy)$ decreases monotonically as $y \to \infty$ then $\mathcal{R}e(s-i) > 0$ and there is always instability. In other words there exists a frequency Ω such that for all $\omega \geqslant \Omega$ there is instability.

This is certainly a surprising result. If $\Im m\, g(iy)$ decreases while oscillating, $\Re e(s-i)$ changes sign and the system is alternatively stable and unstable. This shows that it is impossible to find a Ω such that there is stability for all $\omega > \Omega$, except if $\Im m\, g(iy) = 0$. This exception is precisely what happens with the Mathieu equation for which

$$(63) \qquad g(s) = \frac{-1}{s^2+1}$$

In this case it can be seen directly from their definitions that $D_1(iy)$ and $D_2(iy)$ are real and thus that the zero of $D(s)$ stays on the imaginary axis. There is stability for large ω-a well known result indeed!

3.8. Stability criterion for a special case.

There is a particular form of the impedance for which one can establish another criterion of stability. This is for

$$(64) \qquad R(t) = \delta(t) + Q(t)$$

where $Q(t)$ is a function of bounded variation.

We shall use the notation $V[Q]$ for the total variation of Q for $0 \leq t < \infty$. To simplify the formulas we define

$$(65) \qquad f(t) = \sum_k a_k e^{ik\omega t}$$

Equation (28) now reads

$$(66) \qquad \left\{X + f(t)\right\} \varepsilon(t) + \dot{\varepsilon}(t) = - \int_0^t Q(t-t') \, \dot{\varepsilon}(t') \, dt'$$

Defining the auxiliary function $E(t)$ by

$$(67) \qquad E(t) = \exp\left\{-Xt - \int_0^t f(t') \, dt' - Q(o)\, t\right\}$$

and denoting the period of f(t) by T, we obtain

$$(68) \quad E(t+T) = e^{-(X+Q(0))T} E(t)$$

On applying the method of variation of constants to equation (66) we obtain the following equation

$$(69) \quad \varepsilon(t) = \varepsilon_0(t) + \int_0^t S(t,t')\, \varepsilon(t')\, dt'$$

where

$$(70) \quad \varepsilon_0(t) = E(t) \left\{ 1 + \int_0^t Q(t')\, E^{-1}(t')\, dt' \right\}$$

$$S(t,t') = -E(t) \int_{t'}^{t} \dot{Q}(t''-t')\, E^{-1}(t'')\, dt''$$

We shall assume that

$$(71) \quad X + Q(0) > 0$$

Introducing the positive function $\Phi(t)$

$$(72) \quad \Phi(t) = E(t) \int_0^t E^{-1}(t')\, dt'$$

and using the relation (68) we obtain

$$(73) \quad \Phi(t+nT) = \frac{\int_0^T \frac{E(t+T)}{E(t+t')}\, dt' - e^{-nT(X+Q(0))} \int_0^T E(t)\, E^{-1}(t')\, dt'}{1 - \exp\{-(X+Q(0))T\}}$$

from which one deduces the inequality

$$(74) \quad \Phi(t+nT) \leq \Phi_0(t) = \frac{\int_0^T \frac{E(t+T)}{E(t+t')}\, dt'}{1 - \exp\{-(X+Q(0))T\}}$$

This shows that $\Phi(t)$ is bounded by the periodic function $\Phi_o(t)$. It follows that $\varepsilon_o(t)$ is also bounded since

(75) $$\varepsilon_o(t) \leq E(t) + \Phi_o(t)\ \mathrm{Max}\left\{Q(t)\right\}$$

We have also that

(76) $$\int_0^t \left|S(t,t')\right| dt' \leq \Phi_o(t) \int_0^{\rho} \left|\dot{Q}(t')\right| dt' = V[Q]\Phi_o(t)$$

If the condition

(77) $$V[Q]\mathrm{Max}\left\{\Phi_o(t)\right\} < 1$$

is satisfied the iterated expansion of equation (69) converges uniformly and since $\varepsilon_o(t)$ is bounded, $\varepsilon(t)$ is also bounded. The condition (77) is thus a sufficient criterion for stability. Its main advantage is its simplicity.

If

(78) $$X + Q(0) + f(t) > -P \qquad P > 0$$

then

(79) $$\Phi_o(t) \leq \frac{(1-\alpha)T + \alpha T e^{\alpha P T}}{1 - \exp\left\{-(X+Q(0))T\right\}}$$

where α is the fraction of a period during which the exponent of $E(t)$ is positive. Using this upper bound, the condition (77) becomes

(80) $$T V[Q] \frac{1-\alpha+\alpha e^{\alpha P T}}{1 - \exp\left\{-(X+Q(0))T\right\}} < 1$$

This condition can be satisfied only if

(81) $$V[Q] < X + Q(0)$$

If this inequality is satisfied then (80) is satisfied for $T < T_m$ where T_m is the solution of the equation

$$(82) \qquad T_m = \frac{1-e^{-(X+Q(0))\,T_m}}{V[Q]\,(1-\alpha+\alpha e^{\alpha P T_m})}$$

In particular if $V[Q] \ll X+Q(0)$ and $\alpha P \ll V[Q]$ it gives

$$(83) \qquad T_m \approx \frac{1}{V[Q]}$$

The criterion (81) will subsequently be shown to be very useful in the problem of the stabilization of short wavelengths.

3.9. Problem of the stabilization of all the modes.

Let us return to the original equation (24) with its explicit dependence on n and h (or H). We have $a_k = 0$ for $k \neq \pm 1$ and $a_1 = a_{-1} = A(n,h)/2$ and 2ω is the applied frequency[2].

For a given n,h the stability criterion (38) is always satisfied for a high enough frequency. An estimation of the frequency needed to achieve stability can be obtained in the following manner. Let us determine Y such that $A(n,h)\,|g(iy)| \leq 1$ for $|y| \geq Y$. Introducing the definitions

$$(84) \qquad g_{max}(n,h) = A(n,h)\ \mathrm{Max}\left\{|g(iy)|\right\}$$

$$R(n,h) = \mathrm{Min}\left\{\mathcal{R}e\ \tilde{R}(n,h,iy)\right\}, |y| \geq Y$$

---

2) We could choose $a_2 = a_{-2} = A(n,h)/2$ with $a_k = 0$, $k \neq \pm 2$ and keep ω as the applied frequency. We should obtain the same result. This is not obvious in the expansion of D(s) but, if we look back at its definition (41), we see that $D_2(s) = D_1(s)\,D_1(s+i\omega)$ where $D_2(s)$ is D(s) used in this footnote and $D_1(s)$ is the D(s) used in the text. $D_1(s+i\omega)$ appears also as a factor in A(s), giving us finally the same result as above.

the criterion (38) is satisfied if

$$(85) \qquad \omega \geqslant \Omega(n,h) = \frac{A^2(n,h)\ g_{max}(n,h)}{2R(n,h)}$$

It should be kept in mind that the frequency so determined is not necessary the best one (the lowest one). For better determinations of $\Omega(n,h)$ the other criteria, in particular (40), (56) and (80) may give better estimates. Nevertheless the value given in (85) is adequate for the discussion of the dependence on n,h and also for a presentation of the problems which arise.

Stability for all modes can only be realized if $\Omega(n,h)$ is bounded. For any given n,h $\Omega(n,h)$ is finite. Thus there are only two possibilities for obtaining an unbounded $\Omega(n,h)$. Firstly the case when $n,h \to \infty$; this is what we shall call the short wavelengths problem. Secondly the case when $n,h \to 0$; this is the long wavelengths problem.

Consider first the case of the short wavelengths. From (25) it follows that $A(n,h) \to \infty$ as $n,h \to \infty$. $\Omega(n,h)$ as given by (85) will thus be bounded only if

$$(86) \qquad \frac{A^2(n,h)\ g_{max}(n,h)}{R(n,h)} < +\infty \qquad \text{as} \quad n,h \to \infty$$

This is a very restrictive condition which is not always satisfied. Furthermore condition (86) is not necessary, since the criterion (38) is not. It may happen that (86) is violated and yet $\Omega(n,h)$ be bounded. This happens for instance with the collisionless model. In conclusion the short wavelengths problem cannot be solved without looking in detail at the model and eventually by using some other stability criteria.

The long wavelengths problem is not obvious and it does not even exist for all models. If it does exist it arises from a non uniformity in the convergence of $\tilde{R}(n,h,s)$ when $n,h \to 0$ around $s = 0$.

This is obvious in the case of a finite plasma where $\tilde{R}(o,o,s)$ has a pole at the origin while $\tilde{R}(n,h,s)$ with $n,h \neq 0$ does not. This lack of uniformity can have as counterpart a corresponding lack of uniformity of $g_{max}(n,h)$. Here is a simple example to help visualize the problem. Take

$$(87) \qquad \tilde{R}(H,s) = R + \frac{s}{s^2+H^2} \qquad\qquad R > 0$$

This $\tilde{R}(H,s)$ satisfies all the conditions (9) and develops a pole at the origin for $H = 0$. Choosing $X = 1$ in order to simplify the notation, we have

$$(88) \qquad g(H,s) = \frac{-(s^2+H^2)}{2s^2+H^2+Rs(s^2+H^2)}$$

we can assume $H \ll 1$. $g(s)$ has a pair of poles located at

$$(89) \qquad s \approx \pm \frac{iH}{\sqrt{2}} - \frac{RH^2}{2}$$

This gives

$$(90) \qquad g_{max}(H) = \frac{\sqrt{2}}{RH}$$

This is not bounded as $H \rightarrow 0$ while $g_{max}(o) = \frac{1}{2}$. Fig. 2 represents qualitatively the behaviour of $g(H,s)$. This is an example where $\Omega(n,h)$ is not bounded. It is not always so. It should also be remarked that an unbounded $g_{max}(n,h)$ does not automatically mean that $\Omega(n,h)$ is unbounded, but as in the case for short wavelengths, that the criterion (38) is not adequate to solve the problem. The criterion (56) is the one to be used here. As is the case for the short wavelengths the answer to the stability problem cannot be given without knowing more about the model than the very general restrictions (9).

IV. Simple models

The conclusion of the last section was that the stability problem could not be solved without having more information regarding the acoustic impedance of the plasma than just the very general restrictions (9). As a first step we shall examine some simple models in detail in order to gain some feeling for the long and short wavelengths problems and furthermore to glean some information on what frequency is necessary to stabilize all the modes.

We shall consider essentially only two classes of models : fluid models and collisionless models. Since they represent the two limits between which the correct representation of the plasma should be found, it is hoped that the results are relevant and that reality may be approached by interpolation.

4.1. Half space geometry

Consider an isothermal plasma filling a half space. There may be an acceleration field normal to the surface. Let the plasma be confined by a rotating magnetic field given by (11). We shall look into the stability of this confinement for various models of the plasma.

a) One fluid model without dissipation

We describe the plasma by the usual one fluid model with no acceleration field. p_o and ρ_o are respectively the pressure and density of the uniform gas. If υ is the sound speed, an acoustic wave can be described by

$$(91) \qquad p = P(H,s)\, e^{k_{\ell} z + iHx + st}$$

where p is the pressure increment, z the coordinate in the direction normal to the surface and x a direction parallel to the surface.

k_ℓ is given by the dispersion relation

$$(92) \quad k_\ell^2 = H^2 + \frac{s^2}{\upsilon^2}$$

The equation of motion at the surface gives immediately the impedance

$$(93) \quad R(H,s) = \frac{-P(H,s)}{\mathcal{L}[\dot{\varepsilon}(t)]} = \frac{\rho_0 s}{k_\ell}$$

The branch of the square root for k_ℓ is determined by the condition that $k_\ell > 0$ for s real positive. $\tilde{R}(H,s)$ is analytic in the full plane except for a cut attached to the two singular points $s_r = \pm i\upsilon H$. The cut can be chosen in such a way that the conditions (9) are satisfied for $\Re e\ s > 0$. The condition (9b) is satisfied for all directions since

$$(94) \quad \tilde{R}(H,s) \xrightarrow{s\to\infty} Z_0 = \rho_0 \upsilon$$

Along the imaginary axis $\tilde{R}(H,s)$ is analytic for $s \neq s_r$. Choosing $s = iy$, y real, we have

$$(95) \quad \begin{array}{ll} 0 \le |y| \le H\upsilon & \tilde{R}(H,iy) = i \dfrac{y}{\sqrt{H^2\upsilon^2 - y^2}} Z_o \\ |y| > H\upsilon & \tilde{R}(H,iy) = \dfrac{y}{\sqrt{y^2 - H^2\upsilon^2}} Z_o \end{array}$$

We see that the impedance is purely reactive (inductive) for $|y| \le H\upsilon$ and purely resistive for $|y| > H\upsilon$. The region $|y| < H\upsilon$ is the inertial region where the plasma acts only as a mass. The second region $|y| > H\upsilon$ is the acoustic region where acoustic waves carry into the plasma the motion of the surface. This explains the relations (95). These relations replace (9c) on the imaginary axis.

g(s) has a pair of poles on the imaginary axis at $s_\pm$

$$(96) \quad s_\pm = \pm i \frac{X(H)}{\sqrt{2}\, Z_o} \sqrt{\sqrt{1 + \frac{4H^2\upsilon^2 Z_o^2}{X^2(H)}} - 1}$$

These poles are located between the origin and $\pm iH\upsilon$. When $H \to 0$ these poles move towards the origin where they cancel each other since for H = 0, $g(s) = \frac{-1}{X+sZ_o}$. For $H \to \infty$, $s_{\pm} \sim \pm i\sqrt{\sqrt{1+4\gamma^2}-1}\, uH/\sqrt{2}\gamma$

Let us consider now the stability problem. The impedance (93) has all the properties of the example considered in (3.7) with a monotonic behaviour for $|g(iy)|$ for large y. It has been shown in (3.7) that for high enough frequency there is always instability. High enough implies that

(97) $\omega \gg \mathrm{Max}\ (\upsilon H, \frac{\upsilon}{D})$

For $HD \ll 1$, that is in the limit of long wavelengths, the formula (62) gives for the rate of growth of the instability

(98) $\tau \approx \frac{(H\upsilon)^3}{8\omega^2\gamma}\ (HD)$

For $H\upsilon \gg \frac{\upsilon}{D}$, the growth rate becomes

(99) $\tau \approx \frac{H^3\upsilon^3}{6\omega^2\gamma^3}\left\{\sqrt{1+4\gamma^2} - 1 - \frac{2}{1+4\gamma^2}\right\}$

This is a completely different type of instability from that discussed by Berkowitz (1) since it occurs for a frequency larger than the resonance frequency ($|s_{\pm}|$) of the system, while in the incompressible model considered by Berkowitz it is stable for a frequency much larger than the resonance frequency. For a frequency less than the resonance frequency, the criteria derived do not give an immediate answer, but from the results obtained with the incompressible model it is almost certain that there will also be instabilities.

Berkowitz has considered the problem of a superposed constant acceleration field $g > 0$, normal to the surface. This can also be done with our model.

The acoustic impedance becomes in this case

(100) $\widetilde{R}(H,s) = s\,\frac{\Omega+k_L}{s^2+H^2\upsilon^2}\ Z_o$

with the dispersion relation

$$k_\ell = \frac{-\gamma\Omega}{2} + \sqrt{s^2+H^2\upsilon^2 + \frac{\gamma^2\Omega^2}{4} + (\gamma-1)\,\frac{H^2\upsilon^2\Omega^2}{s^2}}$$

(101)

$$\Omega = \frac{g}{\upsilon}$$

For any γ we will have $\tilde{R}(h,s) \to Z_o$ as $s \to \infty$. $\gamma = 2$ is a singular case for which the expression (100) simplifies to

$$\tilde{R}(H,s) = Z_o\sqrt{\frac{s^2+\Omega^2}{s^2+H^2\upsilon^2}} \tag{102}$$

$\tilde{R}(H,s)$ has two pairs of branch points in $\pm\, i\Omega$, $\pm\, iH\upsilon$, which can be connected by two cuts located in the left half plane. $\tilde{R}(H,s)$ is thus analytic in the right half-plane. Note that $\tilde{R}(\frac{\Omega}{\upsilon},s) = Z_o$. Along the imaginary axis, $\tilde{R}(H,iy)$ is purely resistive for $|y| > \text{Max}\,(H\upsilon,\Omega)$ and $|y| < \text{Min}\,(H\upsilon,\Omega)$ and purely imaginary for $\text{Min}\,(H\upsilon,\Omega) < |y| < \text{Max}\,(H\upsilon,\Omega)$. In the domain where it is purely imaginary, it is inductive (of the same sign as y) for $\Omega < H\upsilon$ and capacitive for $H\upsilon < \Omega$. This means that g(s) possesses a pair of poles on the imaginary axis located between $\pm\,(i\Omega, iH\upsilon)$ for $H\upsilon > \Omega$ only.

For $\gamma < 2$ which is the interesting case, $\tilde{R}(H,s)$ still has two branch points along the imaginary axis given by

$$s^2 = \frac{-\upsilon^2H^2-\gamma^2\,\frac{\Omega^2}{4} \pm\sqrt{(\upsilon^2H^2+\gamma^2\,\frac{\Omega^2}{4})^2 - 4(\gamma-1)\,\Omega^2\upsilon^2H^2}}{2} \tag{103}$$

and a pair of poles in $\pm\, iH\upsilon$. The poles are located between the branch points on the same side of the origin. The branch points never coincide. If $\pm\, iy_1$ are the branch points nearest to the origin and $\pm\, iy_2$ the others, $\tilde{R}(H,iy)$ is purely imaginary for $y_1 < |y| < y_2$, inductive if $|y| < H\upsilon$ and capacitive for $|y| > H\upsilon$. For $|y| > y_2$ or $|y| < y_1$, the impedance has a resistive component. In particular for $|y| > y_2$

$$(104)\qquad \Re e\ \tilde{R}(H, iy) \gg \frac{|y|}{\sqrt{y^2 - H^2 \upsilon^2}}\ Z_o \gg Z_o$$

If

$$(105)\qquad H^2 \upsilon^2 > (\gamma - 1)\ \Omega^2 - \frac{2-\gamma}{2}\ \Omega \left\{ \frac{X(H)}{Z_o} + \frac{\Omega(2-\gamma)}{2} - \frac{\Omega^2 (\gamma - 1)\ Z_o}{X(H)} \right\}$$

then g(s) has a pair of poles on the imaginary axis at s_r given by

$$(106)\qquad s_r^2 = \Delta - \sqrt{\Delta^2 + \frac{H^2 \upsilon^2 X^2(H)}{Z_o^2}}$$

where $\Delta = \frac{X^2(H)}{2Z_o^2} + \frac{\Omega X(H)}{2Z_o}\ (2-\gamma) + (1-\gamma)\ \frac{\Omega^2}{2}$

This formula, as well as (105) remains true for $\gamma = 2$. Let us return to the stability problem. The condition for having average stability can be written

$$(107)\qquad \Omega < \frac{\upsilon}{\gamma D}$$

If it is satisfied the quantity in brackets in (105) is always positive and g(s) has always poles on the imaginary axis for $H^2 \upsilon^2 > (\gamma - 1)\ \Omega^2$. If ω satisfies the restrictions (97), the results (3.7) show that the modes given by (105) are unstable. Applying the formula (61) to the case where $\Omega \ll \frac{\upsilon}{\gamma D}$ and $HD \ll 1$ gives for the growth rate

$$(108)\qquad \tau \approx \frac{H\upsilon}{8\omega^2} \left\{ \Omega\ \frac{(2-\gamma)}{2}\ H\upsilon + \frac{HD}{\gamma}\ (H^2 \upsilon^2 - \Omega^2) \right\}$$

It may easily be verified that $\tau > 0$ in the range defined by (105) and $\tau < 0$ outside.

For $H^2 \nu^2 \gg \text{Max}\left\{ \frac{\upsilon\Omega(2-\gamma)\gamma}{2D}, \Omega^2 \right\}$ τ is still given by (98) or (99).

Summarising, the addition of an acceleration field has not changed the conclusions as far as instabilities and growth rates are concerned. However at long wavelengths (of the order of the depth of the plasma) and depending on the value of γ there may be stability.

b) Collisionless model

This model is described in detail in (6). Here are the essential points. The plasma is considered to behave as an ensemble of non-colliding particles (electrons and ions) having a Maxwellian distribution for both species, at the same temperature T; the electron contribution to the impedance is neglected[3]; ions are reflected specularly at the surface. These three assumptions constitute the very essence of this model which is at the opposite extreme to the one fluid model used previously.

The equilibrium ion distribution function $fo(v)$ is

$$\text{(109)} \qquad fo(v) = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}}$$

where m is the ion mass and n the ion density. A deformation of the surface $\varepsilon(t)$ causes a perturbation in the distribution $\delta f(v_z, v_x, t)$ at the surface. The assumptions of specular reflection and no collisions gives

$$\text{(110)} \qquad \delta f\,(v_z, v_x, t) = 0 \qquad v_z > 0$$

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3)This point will be discussed later. In (6) for the mode n=h=0 the contribution of the electrons for $t = \infty$ is kept. In our case it is zero because of the infinite depth of the plasma. The assumption of identical temperature for the ions and electrons does not play any role here, but it does in the justifications of the model.

(110) $$\delta f(v_z, v_x, t) = -2 \frac{\partial f_o}{\partial v_z} \dot{\varepsilon}(t) \qquad v_z < 0$$

The acoustic impedance is thus simply

(111) $$\tilde{R}(H,s) = \frac{2}{\sqrt{\pi}} \, nm\,\upsilon, \qquad \upsilon = \sqrt{\frac{2kT}{m}}$$

or introducing $Z_o = \rho_c \upsilon$

(112) $$\tilde{R}(H,s) = \frac{2}{\sqrt{\pi}} Z_o$$

The impedance is a resistance, independent of H and s. The stability problem is immediately solved, since the solution to equation (28) is

(113) $$\varepsilon(t) = \exp \left\{ - \frac{\sqrt{\pi}}{2Z_o} \left(X(H)t - \frac{A(H)}{2\omega} \sin 2\omega t \right) \right\}$$

This solution is decaying and there is always stability. The inclusion of a constant acceleration field $g > 0$, modifies the calculation in the following way. If x is a coordinate in the surface plane, the boundary condition and the fact that the distribution is constant along a trajectory give the relations

(114)
$$\delta f(v_x, v_z, t, x) = \delta f(v_x - v_z, t, x) + \frac{2m}{kT} v_z \, fo(v) \, \dot{\varepsilon}(t) =$$

$$= \delta f\left(v_x, v_z, t + \frac{2v_z}{g}, x + \frac{2v_x v_z}{g}\right) + \frac{2m}{kT} v_z \, fo(v) \, \dot{\varepsilon}(t)$$

After Laplace transformation and a mode decomposition we obtain

(115)
$$\delta\tilde{f}(v_x, v_z, s, H) = \frac{+\frac{2m}{kT} v_z \, fo(v)}{1 - e^E} \mathcal{L}[\dot{\varepsilon}(t)]$$

$$E = \frac{2v_z s}{g} + 2iH \frac{v_x v_z}{g}$$

where $\delta\tilde{f}(v_x, v_z, s, H)$ is the Laplace transform of $\delta f(v_x, v_z, t, H)$
The impedance becomes[4)]

(116) $\tilde{R}(H,s) = \frac{m^2}{kT} \int d^3v \, f_o(v) \, v_z^3 \coth E$

Expanding coth E and transforming back to t, we obtain

(117) $R(H,t) = \frac{2}{\sqrt{\pi}} Z_o \left[\delta(t) + \frac{2g}{\upsilon} e^{\frac{-H^2\upsilon^2 t^2}{4}} Q_1\left(\frac{gt}{2}\right) \right]$

with

(118) $Q_1(t) = \sum_{n=1}^{\infty} \frac{t^3}{n^4} e^{-\frac{t^2}{n^2}}$

$Q_1(t)$ is represented in fig. 3. Note that $Q_1(t)$ is positive. Also

(119) $Q_1(\infty) = \frac{\sqrt{\pi}}{4}$

By numerical computation we find, for any $\delta > 0$

(120) $V\left[e^{-\delta t^2} Q_1(t)\right] < 1.14$

The average stability condition (27) gives

(121) $\frac{g}{\upsilon} < \frac{\upsilon}{D}$

The condition (81) of applicability of the stability criterion (80) for $H \sim 0$ may be written

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4) The adiabatic contribution of the electrons has been omitted because it does not alter anything to the stability analysis but merely complicates the writing of equations. It contributes only to the mode H = 0 which presents no problem in any case. The same omission is carried out for all cases without further mention.

(122) $\frac{3,7g}{v} < \frac{v}{D}$

If this is satisfied (82) will give a frequency Ω_m such that for $\omega \geqslant \Omega_m$ all the modes are stable by virtue of (120). If $g \ll \frac{v^2}{3D}$ one finds

(123) $\Omega_m \approx 2,3\ \pi\ \frac{g}{v}$

A value of Ω_m sufficient for stability, for any D satisfying (121), is

(124) $\Omega_m = 3\pi\frac{g}{v}$

We shall prove this as follows. Consider the case where $D \leq \frac{v^2}{6g}$ We have

(125) $\frac{V[Q(t)e^{-h^2t^2}]}{X(H)} \leq .515$

$\alpha \leq .093$

By substitution one verifies that (80) is satisfied. For $\frac{v^2}{g} \geqslant D > \frac{v^2}{6g}$, we subdivide the range of values of H into two. Consider the case where $vH \coth DH \geqslant \frac{6g}{v}$. The inequalities (125) are still true and (80) is thus satisfied. There remains the range $vH \coth DH < \frac{6g}{v}$.

We use the Nyquist diagram method explained in (3.5). For a frequency larger than or equal to Ω_m the approximations (57) are very good (within a couple of percents at worst). The following properties of the impedance can be established by a combination of analytical and numerical calculations.

$$y\, \mathfrak{Im}\, \tilde{R}(H,iy) < 0 \quad \text{for } |y| \leq y_r(H)$$

$$y_r(H) \geqslant y_r(o) > 1.15\ g/v$$

$$\mathfrak{Re}\, \tilde{R}(H,iy_r(H)) > \mathfrak{Re}\, \tilde{R}(H,1.15\tfrac{g}{v}) = .06\sqrt{\pi}\, Z_o$$

$$\mathcal{I}m\ y\tilde{R}(H,iy) > 0 \quad \text{for } |y| \geqslant \Omega_m \tag{126}$$

$$\mathcal{R}e\ \tilde{R}(H,iy) \geqslant \frac{\sqrt{\pi}}{2} Z_o \quad \text{for } |y| \geqslant \Omega_m$$

where $y_r(H)$ is the value of $|s|$ for which $\mathcal{I}m\ \tilde{R}(H,s) = 0$

It follows that

$$\left|\mathcal{I}m\ \frac{D_2}{D_1}\right| < \frac{4}{3\pi^3}\sqrt{\pi}\ |y| \approx 4.3\ 10^{-2}\sqrt{\pi}\ Z_o\ |y| \tag{127}$$

$$\mathcal{R}e\ \frac{D_2}{D_1} < \frac{40}{9\pi^3}\ X(H) \approx .141\ X(H)$$

Note that the last inequality does not involve absolute values. The left hand side becomes negative when $D \rightarrow \frac{u^2}{g}$. Fig. 4 shows Nyquist diagrams for a given H. Two curves are shown which represent the image of $\mathcal{C}$ for $\omega = \infty$ (for which $G(s) = X + s\tilde{R}(s)$) and the image of $\mathcal{C}$ with D_2/D_1 replaced by the bounds (127). A and A' are the images of the point, namely $iy_r(H)$, where the impedance passes from capacitive to inductive. From the diagram it is clear that there is stability if X' stays on the positive real axis, if A' stays in the first quadrant and if the entire curve between these two points remains in the right half plane. All these conditions are satisfied by virtue of (126) and (127) and thus there is indeed stability.

Summarizing, we have shown that for a frequency higher than or equal to Ω_m, as given in (124), there is stability for any D so long as there is average stability (121). This result is in sharp contrast with the result obtained with the fluid model without transport terms where there are always instabilities for any frequency. Weibel (6) (for a cylindrical geometry, but the physics remains the same) has suggested that this difference is due to the neglect of all dissipative terms in the fluid model. To examine this idea we shall add the dissipative terms due to viscosity to the one fluid model equations. Thermal conductivity is certainly as important but it complicates the algebra without changing the results. It only adds dissipation and therefore should help stabilisation. Its effect may be estimated by putting $\gamma = 1$ in the results, which corresponds to infinite thermal conductivity. In order to keep the algebra manageable only the case without acceleration field is examined.

c) Viscous one fluid model

We expect viscosity to provide two stabilizing influences. Firstly, for long wavelengths, the dissipation should suppress the instabilities found in (4.1) since for small H the growthrates are only proportional to H^4, while dissipation should give a damping of order H^2. Secondly, for short wavelengths, the particulate nature of the plasma should start playing a role by, hopefully, providing already strong dissipation in the inertia region.

The linearized equations for the model are

$$\frac{\partial p}{\partial t} = -\gamma \operatorname{div} \vec{v} \tag{128}$$

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} \vec{v}$$

$$\frac{\partial \vec{v}}{\partial t} = -\frac{u^2}{\gamma}\,\overrightarrow{\text{grad}}\; p + \frac{\nu}{3}\,\overrightarrow{\text{grad}}\,\operatorname{div} \vec{v} + \nu \nabla^2 v$$

where p and ρ are the relative values of the pressure and density of the perturbation and $\vec{v}$ the velocity. ν is the kinematic viscosity.

The dispersion equation for plane waves decouples into two modes, corresponding respectively to a longitudinal acoustic mode k_L and an incompressible mode k_T.

$$(129)\qquad k_L^{\,2} = H^2 + \frac{s^2/u^2}{1+\frac{4\alpha s}{3u}} \qquad\qquad \alpha = \frac{\nu}{u}$$

$$k_T^{\,2} = H^2 + \frac{s}{\alpha u}$$

where α is a length of the order of the mean free path.

As an additional boundary condition at the surface we take that the tangential component of the stress vanishes at the surface, that is $p_{\parallel} = 0$. With this condition the acoustic impedance is

$$(130)\qquad \tilde{R}(H,s) = Z_o u \left\{ \frac{4\alpha^2 H^2 (k_T^{\,2} - k_T k_L)}{s k_L} + \frac{s}{u^2 k_L} \right\}$$

$\tilde{R}(H,s)$ is still analytic in a suitably cut s plane. There are two branch points in s_T and s_o

$$(131)\qquad s_T = -\alpha H^2 u \quad , \quad s_o = -\frac{3u}{4\alpha}$$

and a pair of branch points s_L, where $\tilde{R}(H,s)$ is infinite

$$(132)\qquad s_L = -\frac{2\alpha H^2 u}{3} \pm iHu\sqrt{1-\frac{4\alpha^2 H^2}{9}}$$

For $\alpha H \ll 1$, this becomes

$$(133)\qquad s_L \approx \pm iHu - \frac{2\alpha H^2 u}{3}$$

As $\alpha \rightarrow 0$ these singularities become identical with those of (93). Since s_o, s_T, s_L are located in the left half plane, the condition (9a) is satisfied. (9b) is also satisfied because of

$$(134)\qquad \tilde{R}(H,s) \xrightarrow{s\rightarrow\infty} 2\sqrt{\frac{\alpha s}{3u}}\, Z_o$$

Condition (9c) is satisfied by each of the two terms in the square brackets in (130).

We may thus apply the method of section (3.9) for any mode. Let us investigate first the long wavelengths problem, using the criterion (56). For $\alpha H \ll 1$ we have on the imaginary axis

$$(135)\quad \begin{aligned} &\mathcal{R}e\ \tilde{R}(H,iy) \geq 2(\alpha H)\ Z_o && |y| \leq Hu \\ &\mathcal{R}e\ \tilde{R}(H,iy) \geq Z_o && |y| \geq Hu \end{aligned}$$

We again assume that ω is large, that is,

$$(136)\quad \omega \gg \mathrm{Max}\ (Hu, \frac{u}{D})$$

in order that we may use the asymptotic forms of (57). We shall assume also for now that $\alpha \ll D$.

We have

$$(137)\quad \begin{aligned} &\left|\mathcal{J}m\ \frac{D_2(iy)}{D_1(iy)}\right| \leq \frac{A^2(H)}{6\omega^2 Z_o}\ |y| \\ &\mathcal{R}e\ \frac{D_2(iy)}{D_1(iy)} \leq \frac{A^2(H)X(H)}{6\omega^2 Z_o^2} \ll X(H) \end{aligned}$$

We again use the Nyquist diagram method. Let us draw first the image of $\mathcal{C}$ for $\omega = \infty$, that is with $D_2(s) = 0$ (fig. 5). This curve is in the upper half plane because of (9c). For $\omega < \infty$ the curve is displaced horizontally by $-\mathcal{R}e\ \frac{D_2(s)}{D_1(s)}$ which is much smaller than X(H) (137) and downward by $-\mathcal{J}m\ \frac{D_2(s)}{D_1(s)}$ The new curve is represented qualitatively in fig. 5. It is seen that the stability problem reduces to a comparison of the respective sizes of OA and AB. The example illustrated corresponds to a stable case. The value of s, say s_r, at point A is still given by (96), neglecting terms of higher order in H. We deduce that

$$(138)\quad OA \geq |s_r|\ 2(\alpha H)\ Z_o + \mathcal{J}m\ \frac{Z_o s_r^2}{\sqrt{H^2u^2 + s_r^2 + \frac{4}{3}\alpha\frac{s_r^3}{u}}}$$

The second term is always positive or zero. From equation (137)

$$(139)\quad AB \leq \frac{A^2(H)}{6\omega^2 Z_o}\ |s_r|$$

There is stability if OA $\geqslant$ AB. Using (138) and (139) we will find sufficient conditions for stability.

A sufficient condition is

(140) $$\omega^2 > \frac{u^2 H}{4\alpha} \; \frac{1}{1+3H^2 Z_o{}^2/X^2(H)}$$

For HD $\gg$ 1, this condition becomes

(141) $$\omega^2 > \frac{u^2 H}{12\gamma^2 \alpha}$$

For HD $\ll$ 1 it becomes

(142) $$\omega^2 \gtrsim \frac{u^2 H}{4\alpha}$$

This last formula shows that the very long wavelengths ($H < \frac{\alpha}{D^2}$) are always stable if $\omega > \frac{u}{D}$. We have also shown that, when $H > \frac{\alpha}{D^2}$ there is stability if $\omega \gtrsim u\sqrt{\frac{H}{\alpha}}$ within a numerical factor which changes from 1/2 when HD$\ll$1 to $1/\gamma\sqrt{12}$ for HD $\gg$1. This frequency remains in the range of validity of the model. The reason for the stability of the long wavelengths may be found by comparing (98) and (133). The growthrate of the instabilities for $\alpha = 0$ goes as H^4 while the damping goes as H^2, thus dominating for sufficiently small H.

For HD$\gg$1 the condition (136) and (140) show that any given mode can be made stable. However these conditions give us an unbounded value as $H \to \infty \, (\sim \frac{1}{\alpha})$ thus bringing us to the short wavelengths problem. In the strict frame of validity of the model the problem of the stability cannot be answered for the following reason. Let us choose a ω such that $\frac{u}{D} \ll \omega \ll \frac{u}{\alpha}$; there exists a H such that

(143) $$uH \ll \omega \ll uH\left(\frac{u}{\alpha}\right)$$

The condition (136) is satisfied but (141) is not. Since the condition (141) is not necessary but only sufficient this does not imply instability Nevertheless, the estimates of OA and AB, (138) and (139), which were used to derive these conditions are certainly correct in their dependence on

H and α, even though the coefficients may not be the best ones. This means that when (143) is satisfied, AB>>OA and there will be instability. The only solution is to raise the frequency until $\omega \sim \frac{u}{\alpha}$; this assures stability for all modes up to $\alpha H \sim 1$. For $\alpha H > 1$ we may expect dissipation to be sufficiently strong to provide stability. In order to verify this the criterion (38) will be used. We need two properties of the impedance, namely

$$(144) \quad \Re e\, \check{R}(H,iy) \geq \frac{\alpha H}{2} Z_o \quad \text{for all } H$$

$$\Im m\, \frac{\tilde{R}(H,iy)}{y} \leq 0 \qquad \text{for } \alpha H > 1, \ |y| \leq Hu$$

They imply that for $\alpha H \geq 2$, $g(iy)$ reaches its maximum value at the origin, namely $g_{max} = 1$. The criterion (38) is thus satisfied for any ω and these modes are always stable. A more careful analysis shows that if $\gamma \leq 2$ $g_{max}=1$ when $\alpha H \geq .70$. It is of interest to locate approximately the value of H for which the minimum frequency required by the criterion (38) is a maximum. Numerically one finds that $\alpha H \approx .4$ for $\gamma = 1$ which corresponds to a sufficient frequency ω of about $\omega \approx .25\frac{u}{\alpha}$ For $\gamma > 1$ this frequency is found to be even lower.

In conclusion, for the case where $D \gg \alpha$, we have shown that all the modes are stable for a frequency $\omega \gtrsim .25\frac{u}{\alpha}$. No attempt has been made to improve on this value. Such an attempt could be tried, using the general criterion (47).

For $D \lessapprox \alpha$ the situation is not so clear. Naturally for $\omega \gtrsim \frac{u}{D}$ all the modes are stable, but is it really necessary to go to such a high frequency or would indeed a lower value of the order of $\frac{u}{\alpha}$ suffice ? This question cannot really be answered, since, of necessity, we are forced to choose such a high frequency in order to use the asymptotic forms (57). Without the use of (57) the long wavelengths problem could not be solved by the criterion (56) for which there is no known substitute.

4.2. Slab geometry

In order to study the influence of the finite thickness of the plasma on the results obtained in the preceding section, we choose the following geometry : a slab of plasma of thickness 2a confined by the same rotating field on both sides. The problem of the stability of this geometry can be reduced to the simpler one of a slab of thickness a having the field applied on one surface only, the other being fixed. Two different sets of boundary conditions have to be imposed at the fixed surface corresponding respectively to symmetric and antisymmetric deformations of the slab.

a) Viscous one fluid model.

Keeping the same notation as before, the boundary conditions at the fixed surface are

$$(145)\qquad \begin{array}{lll} (S) & v_z = 0 & \dfrac{\partial v_x}{\partial z} = 0 \\ (A) & v_x = 0 & \dfrac{\partial v_z}{\partial z} = 0 \end{array}$$

(S) designates the symmetric case and (A) the antisymmetric one. The acoustic impedances are given by

(146)

$$(S)\qquad \tilde{R}(H,s) = Z_o u \left\{ 4\alpha^2 H^2 \frac{k_T^2}{s}\left(\frac{\mathrm{ch}\, k_L a}{k_L \mathrm{sh} k_L a} - \frac{\mathrm{ch}\, k_T a}{k_T \mathrm{sh} k_T a}\right) + \frac{s\ \mathrm{ch}\, k_L a}{u^2 k_L \mathrm{sh} k_L a} \right\}$$

$$(A)\qquad \tilde{R}(H,s) = Z_o u \left\{ 4\alpha^2 H^2 \frac{k_T^2}{s}\left(\frac{\mathrm{sh}\, k_L a}{k_L \mathrm{ch} k_L a} - \frac{\mathrm{sh}\, k_T a}{k_T \mathrm{ch} k_T a}\right) + \frac{s\ \mathrm{sh}\, k_L a}{u^2 k_L \mathrm{ch} k_L a} \right\}$$

These impedances reduce to the half-plane impedance (30) when

$$(147)\qquad \mathcal{R}e\ k_L a \gg 1 \quad \text{and}\ \mathcal{R}e\ k_T a \gg 1$$

The only singularities of the impedance are poles given by the equations

(148)
$$s_L = \frac{-2\alpha u H^{*2}}{3} \pm H^* u \sqrt{\frac{4\alpha^2}{9} H^{*2} - 1}$$
$$s_T = -\alpha u H^{*2}$$

where

(149)
$$\text{(S)} \qquad H^{*2} = H^2 + \frac{n^2\pi^2}{a^2}$$
$$\text{(A)} \qquad H^{*2} = H^2 + \frac{(n+\frac{1}{2})^2\pi^2}{a^2}$$

The poles s_L are located either on a circle centered at $s = -\frac{3u}{4\alpha}$ and tangent to the imaginary axis or on the part of the real negative axis defined by $s \leq -\frac{3u}{4\alpha}$. For $\alpha H \geqslant 3/2$ there are no more poles on the circle. The poles s_T are distributed along the negative imaginary axis. For $\alpha H^{*2} \ll 1$

(150) $$s_L \approx \pm iH^* u - \frac{2}{3}\alpha H^{*2} u$$

All the singularities are in the left half-plane and therefore (9a) is satisfied. For s sufficiently large the inequalities (147) are satisfied and the asymptotic behaviour of the impedance is thus the same as for the half-plane (134). Condition (9b) is therefore also satisfied. Because of the dissipation due to viscosity, statement (9c) is also true. The stability criteria derived can then be applied to this problem.

In the following considerations we shall assume that $a \gg \alpha$. For $\alpha H \ll 1$, we have along the imaginary axis

(151) $$\mathcal{R}e\; k_L \geqslant \sqrt{2}(\alpha H)\, H$$

The inequalities (147) are therefore satisfied if $H \gg \frac{1}{\sqrt{\alpha a}}$ and the impedances (146) then reduce to the half-plane impedance (130). More precisely, for $H = \frac{1}{\sqrt{\alpha a}}$, we find

$$(152) \qquad |\coth k_L a - 1| < .07, \qquad |\operatorname{th} k_L a - 1| < .07$$

This shows that for $H = \frac{1}{\sqrt{\alpha a}}$ the half plane impedance is already a reasonably good approximation to the slab impedances (146). We shall take this value as the dividing line between short and long wavelengths.

The stability problem at short wavelengths is thus covered by the results of section (4.1c).

The problem of the long wavelengths is now considered using the stability criterion (56). Along the imaginary axis the inequalities (152) are satisfied for $|s| \geq \frac{2u}{\sqrt{\alpha a}}$. This implies

$$(153) \qquad \mathcal{R}e\, \tilde{R}(H,iy) > .9\, Z_o\,, \qquad |y| \geq \frac{2u}{\sqrt{\alpha a}}$$

Let us consider a definite wavelength. In order that we may use the asymptotic forms (57) we assume that

$$(154) \qquad \omega > \mathrm{Max}\left\{ \frac{X(H)}{Z_o}\,,\ \frac{2u}{\sqrt{\alpha a}} \right\}$$

Henceforth we shall drop the index H in X(H) and we shall assume that $D \gg \alpha$. It follows that

$$(155) \qquad \left| \mathcal{I}m\, \frac{D_2(iy)}{D_1(iy)} \right| < \frac{X^2}{6 Z_o \omega^2}\, |y|$$

$$\mathcal{R}e\, \frac{D_2(iy)}{D_1(iy)} < \frac{X}{2}$$

Consider now the equation

$$(156)\qquad X + s\tilde{R}(s) - \frac{D_2(s)}{D_1(s)} = 0$$

For $\omega = \infty$ all the roots of this equation lie in the left half-plane. As ω decreases some of these roots move towards the imaginary axis and the limit of instability will be reached when one of these roots reaches the imaginary axis. We shall determine a necessary condition for (156) to have roots on the imaginary axis part of the contour $\mathcal{C}$. To simplify the writing we introduce the following notation

$$(157)\qquad (S)\qquad I_m = \mathfrak{Im}\,\frac{\coth k_L a}{k_L}\ ;\quad R_m = \mathfrak{Re}\,\frac{\coth k_L a}{k_L}\ ;\quad Z_\tau = \coth k_T a$$

$$(A)\qquad R_m = \mathfrak{Re}\,\frac{\mathrm{th}\, k_L a}{k_L}\ ;\quad I_m = \mathfrak{Im}\,\frac{\mathrm{th}\, k_L a}{k_L}\ ;\quad Z_\tau = k_T\,\mathrm{th}\, k_T a$$

From its definition $I_m \leq 0$. Taking the real and imaginary part of (156) and after an obvious substitution we obtain

$$(158)\qquad \begin{aligned} \delta X &= X^* \,\mathrm{tg}\,\theta - 4\alpha y H^2 \left\{ R_m - I_m \,\mathrm{tg}\,\theta \right\} + 4\alpha^2 H^2 u\, \mathfrak{Im}\, Z_\tau \\ R_m(y^2 - 4\alpha^2 H^4 u^2) &= uX^* - 4\alpha H^2 y u I_m - 4\alpha^2 H^2 u^2 \mathfrak{Re}\, Z_\tau \end{aligned}$$

where

$$(159)\qquad \delta X = \mathfrak{Im}\,\frac{D_2(iy)}{Z_o D_1(iy)}\ ;\quad X^* = \frac{X}{Z_o} + \mathfrak{Re}\,\frac{D_2(iy)}{Z_o D_1(iy)} > \frac{X}{2Z_o}$$

For $y < \frac{u}{\alpha}$ the last term of the second equation (158) is negligible. From the second equation, we find that $y^2 > 4\alpha^2 H^4 u^2$ since $R_m > 0$ for $y^2 < 4\alpha^2 H^4 u^2$. This implies that $\mathrm{tg}\,\theta \leq 0$. We have to consider the two cases separately.

We consider first the symmetric case which exhibits precisely the non uniform behaviour mentioned in 3.9 when $H \rightarrow 0$. The range of y is divided into two sections : the inertial range $0 \leq y \leq uH$ and the acoustic range $uH \leq y \leq \omega$.

We shall first prove that there cannot be any roots of (158) in the acoustic range if ω satisfies the condition (154). Consider two cases

Case 1) : $\frac{n\pi u}{a} \leq \Im m\ k_L \leq \frac{(n+\frac{1}{2})\ \pi u}{a}$; $n = 0, 1, 2, \ldots$

In this case the second equation of (158) gives

$$(160) \qquad \delta X \leq - \frac{\Im m\ k_L}{\Re e\ k_L} X^*$$

Case 2) : $\frac{(n+\frac{1}{2})\ \pi u}{a} \leq \Im m\ k_L \leq \frac{(n+\frac{1}{2})\ \pi u}{a}$

Using both equations (158), we obtain the condition

$$(161) \qquad \delta X \leq - \text{Min} \left\{ \frac{X^*}{2} , \frac{\alpha a y X^{*2}}{12 Z_o^2 u^2} \right\}$$

Collecting the two results (160) and (161) and using the inequality (155) we obtain the following necessary condition for (158) to have roots in the acoustic range

$$(162) \qquad \omega < \text{Max} \left\{ \frac{2X}{3Z_o} , \frac{\sqrt{2}\, u}{\sqrt{\alpha a}} \right\}$$

This condition is incompatible with (154). Therefore there cannot be any roots of (158) in the acoustic range, thus proving our assertion. In the inertial region $y \leq Hu\ \Re e\ k_L \geq \Im m\ k_L$, we again consider two cases

Case 1) : $\Re e\ k_L \leq \frac{1}{a}$

The following inequalities hold

$$(163) \qquad tg\ \theta \leq - \frac{\Im m\ k_L}{\Re e\ k_L} \quad ; \quad R_m \leq \frac{1.3}{a \Re e^2 k_L}$$

Using these relations in the system (158) we find

$$(164) \quad \delta X \leq - \frac{\alpha a y X^2}{8 Z_o^2 u^2}$$

This implies the necessary condition

$$(165) \quad \omega < \sqrt{\frac{4}{3}} \frac{u}{\sqrt{\alpha a}}$$

which is again incompatible with the assumption (154). Therefore no root can cross this section of the imaginary axis. For $aH \leq 1$, we can thus say that a frequency satisfying (154) is sufficient to stabilize these modes in the symmetric case.

For $aH > 1$ we have to consider the second case namely

case 2) : $\mathcal{R}e\ k_L > \frac{1}{a}$

The inequalities (163) are replaced by

$$(166) \quad \mathrm{tg}\ \theta \leq .9 \frac{\mathcal{I}m\ k_L}{\mathcal{R}e k_L} \quad ; \quad R_m \leq \frac{1.3}{\mathcal{R}e\ k_L}$$

Proceeding in the same manner as above we obtain the necessary condition

$$(167) \quad \omega < \sqrt{\frac{3}{2}} \quad u \sqrt{\frac{H}{\alpha}}$$

keeping in mind that the necessary condition for a root to cross the imaginary axis is the opposite of a sufficent condition for stability, we can compare this last result with the results for the half-plane (141 or 142). We see that they have the same dependence on H and α but the numerical coefficients differ. The similarity is comprehensibl since, for $\mathcal{R}e\ k_L \geq \frac{1}{a}$, the slab impedance is already quite close to the half-plane impedance. The difference in the coefficient is not important and could have been reduced to a large extent by more careful bounds in (155) and a better use of (166).

We now turn to the antisymmetric case. This case is regular when $H \to 0$. For $H = 0$ and $y \ll \frac{u}{a}$ the impedance reduces to

(168) $\tilde{R}(0,y) \approx (\rho_0 a)\, y$

which is the impedance of a rigid slab of density ρ_0 and thickness a. This shows that at low frequencies the slab moves as a rigid body.

We shall treat the antisymmetric case in a manner analogous to that already applied to the symmetric case. In the acoustic region and when $\mathcal{I}m\, k_L > \frac{1}{a}$, everything is identical to the symmetric case except that the two regions are interchanged. The final result is the following necessary condition for a root to cross the imaginary axis.

(169) $\omega < \frac{2u}{\sqrt{\alpha a}}$

which is incompatible with (154).

For $\mathcal{I}m\, k_L \leqslant \frac{1}{a}$ we find that

(170) $\operatorname{tg} \theta \leqslant -\ .60\ (\mathcal{R}e\, k_L)\ (\mathcal{I}m\, k_L)$

$R_m < \frac{1.55}{a}$

giving the necessary condition

(171) $\omega < \frac{1.6\, u}{\sqrt{\alpha a}}$

which is incompatible with (154).

In the inertial region[5] we consider again two cases :

case 1) : $\mathcal{R}e\, k_L \leqslant \frac{1}{a}$

---

5) The distinction between inertial and acoustic regions is arbitrary in this case since the impedance has no singularity at the transition point.

We have

$$(172)\qquad \mathrm{tg}\ \theta \leq -.44\ (\mathcal{R}e\ k_L)\ (\mathcal{I}m\ k_L)$$

$$R_m \leq \frac{1}{a}$$

This leads again to the condition (171). If $aH \leq 1$, the final result is that for a frequency which satisfies (154) there is stability. This is the same result as for the symmetric case.

For $aH > 1$ we have the additional possibility

case 2) : $\mathcal{R}e\ k_L > \frac{1}{a}$

In this case

$$(173)\qquad \mathrm{tg}\ \theta \leq -.44\ \mathcal{I}m\ k_L / \mathcal{R}e\ k_L$$

$$R_m \leq \frac{1}{\mathcal{R}e\ k_L}$$

These imply the condition

$$(174)\qquad \omega < \frac{3u}{2}\sqrt{\frac{H}{\alpha}}$$

This last condition has the same form as (167) with a different numerical coefficient. The same remarks made in the symmetric case after (161) apply here.

We can summarize the preceeding results as follows. In assuming that $D = \infty$ we have found the following sufficient conditions for stability

$$\omega \geqslant \frac{2}{\sqrt{\alpha a}} \qquad \text{for } aH \leq 9/16$$

$$(175)\qquad \omega \geqslant \frac{3u}{2}\sqrt{\frac{H}{\alpha}} \qquad \text{for } \frac{9}{16} \leq aH \leq \sqrt{\frac{a}{\alpha}}$$

$$\omega \geqslant \frac{u}{\gamma\sqrt{12}}\sqrt{\frac{H}{\alpha}} \qquad \text{for } \frac{a}{\alpha} \gg aH > \sqrt{\frac{a}{\alpha}}$$

This last condition is the condition (141) which remains valid for the short wavelengths.

If $D > \sqrt{\alpha a}$ conditions (175) still hold. If $D < \sqrt{\alpha a}$ but $HD > 1$ there remains just the last condition in (175). For $D < \sqrt{\alpha a}$ and $HD \leq \frac{\alpha}{D}$ only the condition (154) remains, namely $\omega > \frac{u}{D}$ and finally for $\frac{\alpha}{D} \leq HD \leq 1$ there remains the second condition in (175) and the condition (142) which replaces the third one in (175).

The introduction of a finite depth into the semi-infinite problem (4.2) has only affected the results at long wavelengths, $aH \lesssim 1$ and changed the numerical factor in the conditions (141) and (142). The dependence on H and α is not affected. Furthermore the difference in the numerical coefficient is probably meaningless, the bounds used in (141) and (142) being better determined than those used in (175) All this justifies the use of the semi-infinite problem as a test of any improved plasma model which one chooses to investigate.

For $D \lesssim \alpha$ the remarks made in the semi-infinite problem are still valid. We have also assumed $a \gg \alpha$. This seems like a reasonable restriction if the model is to be correct. For $a \ll \alpha$ the collisionless model gives a more realistic idea of the solution. For $a \sim \alpha$ we would require an intermediate model in order to follow the transition from the fluid model to the collisionless model.

b) Collisionless model

This is the same model as that used in the half-space case (4.1b). We shall keep the same notation as in (4.1b).

If $f(v_x, v_z, t)$ is the perturbation of the distribution function at the fixed surface, the boundary conditions corresponding respectively to the symmetric and antisymmetric cases are

$$\text{(S)} \quad f(v_x, v_z, t) = f\,(v_x, -v_z, t)$$

(176)

$$\text{(A)} \quad f(v_x, v_z, t) = -f(v_x, -v_z, t)$$

Let us call $\delta f(v_x, v_z, t, x)$ the perturbation of the distribution functic at the free surface. From the specular reflection hypothesis and by integration along a trajectory we obtain the relations

(S)

(177)

$$\delta f(v_x, v_z, t, x) = \delta f(v_x, -v_z, t, x) + \frac{2m}{kT} v_z\, f_o(v)\dot{\varepsilon}(t) = $$
$$= \delta f\left(v_x, v_z, t + \frac{2a}{v_z},\ x + \frac{2av_x}{v_z}\right) + \frac{2m}{kT}\, v_z\, f_o(v)\, \dot{\varepsilon}(t)$$

(A)

$$\delta f(v_x, v_z, t, x) = \delta f(v_x, -v_z, t, x) + \frac{2m}{kT} v_z\, f_o(v)\, \dot{\varepsilon}(t) = $$
$$= -\delta f\left(v_x, v_z, t + \frac{2a}{v_z},\ x + \frac{2av_x}{v_z}\right) + \frac{2m}{kT} v_z\, f_o(v)\, \dot{\varepsilon}(t)$$

After a normal mode expansion in x and a Laplace transformation in t we obtain

$$\text{(S)} \quad \widetilde{\delta f}(v_x, v_z, s, H) = \frac{\frac{2m}{kT}\, v_z\, f_o(v)\mathcal{L}[\dot{\varepsilon}(t)]}{1 - e^{E}}$$

(178)

$$\text{(A)} \quad \widetilde{\delta f}(v_x, v_z, s, H) = \frac{\frac{2m}{kT}\, v_z\, f_o(v)\mathcal{L}\,[\dot{\varepsilon}(t)]}{1 + e^{E}}$$

where

(179) $$E = \frac{2as}{v_z} + 2iHa\,\frac{v_x}{v_z}$$

The two impedances are given by

$$\text{(S)}\quad \tilde{R}_S(H,s) = \frac{m^2}{kT}\int d^3v\, v_z^3\, f_o(v)\, \frac{1+e^{-E}}{1-e^{-E}}$$

(180)

$$\text{(A)}\quad \tilde{R}_A(H,s) = \frac{m^2}{kT}\int d^3v\, v_z^{\,3}\, f_o(v)\, \frac{1-e^{-E}}{1+e^{-E}}$$

Transforming back to t, R(H,t) can be conveniently split in two parts

$$(181)\quad \tilde{R}_{\substack{S\\A}}(H,s) = \frac{2}{\sqrt{\pi}}\, Z_o \left\{ \delta(t) + \frac{2u}{a}\, e^{\frac{-u^2H^2t^2}{4}}\, Q_{\substack{S\\A}}\left(\frac{ut}{2a}\right)\right\}$$

where

$$(182)\quad Q_S(t) = \sum_{n=1}^{\infty} \frac{n^4}{t^5}\exp\left(-\frac{n^2}{t^2}\right)$$

$$Q_A(t) = \sum_{n=1}^{\infty} (-1)^n\, n^4/t^5\, \exp\left(-\frac{n^2}{t^2}\right)$$

$Q_S(t)$ and $Q_A(t)$ are represented in fig. 6.

Note that $\lim_{t\to\infty} Q_S(t) = \frac{3\sqrt{\pi}}{8}$

and, for any $\delta>0$

$$V\left[Q_S(t)\, e^{-\delta t^2}\right] \leq 1.70$$

(183)

$$V\left[Q_A(t)\, e^{-\delta t^2}\right] \leq 1.92$$

Returning to the stability problem, we can employ the same methods as in (4.1b). Note the similarity between $Q_S(t)$ and $Q_1(t)$. This shows that the acceleration field acts essentially as a velocity dependent specularly reflecting wall. If the following inequalities are satisfied

$$(184)\quad \begin{array}{ll} (S) & a > 3D \\ (A) & a > 3.4D \end{array}$$

Then the criterion (80) is applicable for all modes and there exists a frequency Ω_m given by (82) such that for $\omega \geqslant \Omega_m$ there is stability. If $a \gg 3D$, Ω_m is given by (83), namely

$$(185)\quad \begin{array}{ll} (S) & \Omega_m \approx 3.4\ \pi\ \frac{u}{a} \\ (A) & \Omega_m \approx 3.9\ \pi\ \frac{u}{a} \end{array}$$

Proceeding in the same manner as for the half-plane problem we shall show that for any frequency higher than Ω_m given by

$$(186)\quad \Omega_m = 5\pi\frac{u}{a}$$

there is stability. In the range $D \leqslant .15a$ criterion (80) is satisfied. For $D > .15a$ we again divide the problem into two : firstly for $.15aH \coth HD \geqslant 1$ criterion (80) is still satisfied ; for $.15aH \coth HD < 1$ we have to use criterion (56). The two cases (S) and (A) have to be considered separately since the impedances behave quite differently at low frequencies. The symmetric case is essentially identical to the half-plane problem (4.1b). We examine it first. The following properties are the analogous of (126)

$$(187)\quad \begin{array}{l} \mathfrak{Im}\ \tilde{R}(H,iy_r(H)) = 0 \\ y_r(H) \geqslant y_r(o) > 3.8\frac{u}{a} \\ y\ \mathfrak{Im}\ \tilde{R}(H,iy) \leqslant 0 \quad \text{for } |y| \leqslant y_r(H) \\ \mathcal{Re}\ \tilde{R}(H,iy_r) \geqslant \mathcal{Re}\ \tilde{R}(H,i\ 3.8\frac{u}{a}) > .095\sqrt{\pi}\ Z_o \\ \mathcal{Re}\ \tilde{R}(H,iy) \geqslant .45\sqrt{\pi}\ Z_o \quad \text{for } |y| \geqslant \Omega_m \end{array}$$

These properties have been found by a combination of analytical and numerical calculations. Of the properties enumerated in (126) only one is missing here, namely the impedance does not remain inductive above $y_r(H)$, but this is irrelevant. The relations (187) guarantee the validity of the asymptotic formulas (57) in using (56). We have, in analogy to (127)

$$\mathcal{R}e\,\frac{D_2(iy)}{D_1(iy)} < \frac{X}{2}$$

(188)

$$\left|\mathcal{J}m\,\frac{D_2(iy)}{D_1(iy)}\right| < .021\,\sqrt{\pi}\,Z_o|y|$$

For the given H the Nyquist diagrams corresponding respectively to $\omega = \infty$ and $\omega = \Omega_m$ are represented schematically in fig. 7. We see that they are essentially identical to the diagrams for the half-plane case (fig. 4). There is stability, since by virtue of (187) and (188) the point A', image of $y_r(H)$, is in the first quadrant and X', image of the origin, remains to the right to the origin.

For the antisymmetric case the impedance is not capacitive at low frequencies. We have indeed

(189) $$\check{R}(H,iy) \approx i\,(am)\,y$$

This is the same result as with the fluid model, since am is the mass of the half slab per unit area. The behaviour of the resistive component of the impedance at low frequency is crucial for the stability. The following inequalities hold

$$\mathcal{R}e\,\tilde{R}(H,iy) \geq .20\,\sqrt{\pi}\,Z_o y^4\,\frac{a^4}{u^4} \quad \text{for } |y|<\frac{u}{a}$$

(190)

$$\mathcal{J}m\,\tilde{R}(H,iy)\,y < Z_o\,\frac{a}{u}\,y^2 \quad \text{for } |y|<\frac{u}{a}$$

$$\mathcal{R}e\,\tilde{R}(H,iy) > .20\,\sqrt{\pi}\,Z_o \quad \text{for } |y|\geq\frac{u}{a}$$

$$\mathcal{R}e\,\tilde{R}(H,iy) > .45\,\sqrt{\pi}\,Z_o \quad \text{for } |y|\geq\Omega_m$$

We have that, as before, the asymptotic representation (57) is valid to a very good accuracy. Furthermore

$$\mathcal{R}e\,\frac{D_2(iy)}{D_1(iy)} < \frac{X}{2}$$

(191)

$$\left|\mathcal{J}m\,\frac{D_2(iy)}{D_1(iy)}\right| \lesssim \frac{A^2\,a^2}{90\pi^{5/2}Z_o u^2}\,|y|$$

Fig. 8 represents very schematically the Nyquist diagrams corresponding to $\omega = \wp$ and to the worst case obtained by replacing D_2/D_1 by the bounds (191). There is stability since the point X' is on the positive real axis and A' is above the origin. Let us verify this last point. The value of y, say y_r, corresponding to A' satisfies

$$(192)\qquad |y_r| \geqslant \mathrm{Min}\left\{\sqrt{\frac{A(H)u}{2Z_o a}}\ ,\ \frac{u}{a}\right\}$$

by virtue of (191). This means that

$$(193)\qquad OA' > |y|\,\mathrm{Min}\left\{.20\sqrt{\pi}\,Z_o y_r^{\,4}\,\frac{a^4}{u^4},\ .20\sqrt{\pi}\,Z_o\right\} - \frac{A^2a^2}{90\pi^{5/2}Z_o u^2}\,|y|$$

By substituting we obtain

$$(194)\qquad OA' > |y|\ \mathrm{Min}\left\{ .0496\sqrt{\pi}\,\frac{A^2a^2}{Z_o u^2}\ ;\ \ .184\sqrt{\pi}\,Z_o\right\} > 0$$

We have thus completed the proof. In conclusion if $\omega \geqslant 5\pi\frac{u}{a}$, there is stability for all modes and any D. Note that as far the half-plane problem it is the case of small D which fixes the frequency. From (194) it is clear that a much smaller frequency suffices to stabilize in the cases of large D or small H.

4.3. Cylindrical geometry

Collisionless model.

The cylindrical geometry, as defined in 2.3b, is the most interesting configuration from the point of view of experimentation. We do not expect results very different from the slab problem except for numerical coefficients. Since the algebra for the viscous fluid model is almost intractable we shall only consider the collisionless model used previously. The impedance has been already calculated in (6) and we shall therefore use this result with the same notation as in 2.3b and 4.2b, we have

$$(195) \quad R(n,h,t) = \frac{2}{\sqrt{\pi}} Z_o \left\{ \delta(t) + \frac{2u}{a} \; Q_n\left(\frac{ut}{2a}\right) e^{\frac{-u^2h^2t^2}{4}} \right\}$$

where

$$(196) \quad Q_n(t) = \frac{1}{\sqrt{\pi}\, t^6} \sum_{\lambda=1} (-1)^{n\lambda} \lambda^5 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \cos^8\alpha \cos(2n\lambda\alpha) \, e^{-\frac{\lambda^2\cos^2\alpha}{t^2}}$$

This can also be written as

$$(197) \quad Q_n(t) = \frac{2\sqrt{\pi}}{t^6} \frac{d^4}{d(1/t^2)^4} \sum_{\lambda=1}^{\infty} \frac{e^{-\lambda^2/2t^2}}{\lambda^3} \quad I_{n\lambda}\,(\lambda^2/2t^2)$$

$Q_n(t)$ for $0 \leq n \leq 3$ are shown in fig. 9. These curves are quite similar to the curves in fig. 3, 6. From these curves we find that for any $\delta > 0$

$$(198) \quad \begin{aligned} V\left[Q_o(t)\; e^{-\delta t^2} \right] &\leq 2.22 \\ V\left[Q_1(t)\; e^{-\delta t^2} \right] &\leq 1.80 \\ V\left[Q_2(t)\; e^{-\delta t^2} \right] &\leq 1.00 \\ V\left[Q_3(t)\; e^{-\delta t^2} \right] &\leq .50 \end{aligned}$$

We see that the variation of $Q_n(t)$ decreases monotonically as n increases.

Following the procedure used in the slab problem, we first use criterion (80) in the limit $b \to a$, that is when $X(n,h) \to \infty$ and $\alpha \to 0$. Using the formula (83) we thus find that all modes are stable if

$$(199) \quad \omega \geqslant 4{,}5\ \pi\ \frac{u}{a}$$

It is the mode n = 0, $h \sim 0$ which gives the worst problem. By analogy with the slab case we expect that, for any $a \leq b \leq \sqrt{3}a$, all modes will be stable if

$$(200) \quad \omega \geqslant 5{,}5\ \pi\ \frac{u}{a}$$

This frequency is not very different from the value found for the slab. In both cases the frequency is essentially determined by the case when $b \sim a$. The frequency appears to be unnecessarily high if one considers only the long wavelengths. Numerical calculations may shed some light on whether or not the value of the frequency given in (83) is really necessary since the analytical problem is very difficult.

5. Discussion

This study of various plasma models leaves many questions unanswered. The difficulties in the stability analysis are localized in two regions. Namely in the limit of long wavelengths and in the limit of short wavelengths. In all the models studied these difficulties are solved. Nevertheless, the reasons behind the successful solution of the difficulties vary from model to model. For example, in the long wavelengths limit, it may be because the plasma is capacitive when dissipation is insufficient or because the resistive component is large enough; in the limit of short wavelengths, because the impedance reduces to a constant resistive component at high frequency or because the impedance grows as fast as the magnetic pressure response. This variety of underlying reasons leaves the feeling that the choice of another model may give a negative answer. The hypothesis that the long wavelengths are always stable for any plasma model would be physically satisfying, but has not been proved. The problem of the limit of short wavelengths will always constitute the real difficulty and this is quite comprehensible mathematically. In this limit A(h,n) and X(n,h) become infinite. Dividing through the equation (24) by X(n,h) it can be written in the form

$$\left\{1 + A/X \cos 2\omega t\right\} \varepsilon + \frac{1}{X} \int_0^t R(t-t') \, \dot{\varepsilon}(t') \, dt' = 0$$

We see that we cannot go to the limit $X \to \infty$. This is analogous to the problem of a differential equation with a very small coefficient for the highest derivative. (This is more than an analogy since the case of the incompressible fluid leads precisely to such an equation since $R(t) = \delta'(t)$, see also (4.1b)). We therefore cannot hope for a definitive answer to this problem.

Another question is how much confidence can we place on the free particle model. In this model the electron contribution has been neglected except for its adiabatic contribution. This approximation which is based on an argument of mass ratio is reasonable, but it would be satisfying to have definite answers on this problem. The half-plane problem would provide an adequate testing ground. Such an attempt is already under way. The hypothesis of specular reflection is yet another approximation used. A relaxation of this condition should in fact help stability by giving an additional phase mixing at the boundary (diffuse reflection) and therefore an enhanced dissipation.

Turning to the fluid models, the case where $\varepsilon < \lambda$ is of great practical interest. It is not known if the impedance determined with the implicit assumption $\varepsilon \gg \lambda$ still represents correctly this case. Physically it appears that the impedance used is correct at frequencies lower than the collision frequency. At higher frequencies the impedance is probably just equal to Z_0, the impedance of a semi-infinite collisionless plasma. Only a calculation with an intermediate model could give a definitive answer to this problem.

Two extreme models, the fluid model and the free particle model have been studied. Because of the wide differences in the results which were obtained it would be of interest to have at our disposal an intermediate model which would provide a link. This model would have to be more than a simple interpolation between the two extreme ones. The Bhatnagar-Gross-Krook (14) model does not appear to be the answer, according to the results of the careful analysis of L. Sirovich (13).

As usual the present stability analysis is limited to small amplitude deformations. The non-linear regime is probably beyond reach. It may be that non linear limitation of the growth of the instabilities is effective. This would imply that a frequency just sufficient to stabilize the long wavelengths may be used in experiments, the unstable short wavelengths being limited before they reach the wall. This is not an unreasonable hypothesis.

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FIGURE CAPTIONS

Fig. 1 The contour $\mathcal{C}$ used in the Nyquist diagram method.

Fig. 2 Example of non uniform behaviour in $g_{max}(H)$ along the imaginary axis. The values of H correspond to the ordering $H_1 > H_2 > H_3 > 0$.

Fig. 3 Behaviour of $Q_1(t)$ in the semi-infinite problem with an acceleration field.

Fig. 4 Nyquist diagrams for the semi-infinite problem with an acceleration field : free particle model. The curve 1 is the image of $\mathcal{C}$ for $\omega = \infty$. The curve 2 is the image of $\mathcal{C}$ with $D_2(s)/D_1(s)$ replaced by the bounds (127). These curves are not drawn to scale.

Fig. 5 Nyquist diagrams for the semi-infinite problem with no acceleration field : viscous compressible fluid model and $\alpha H \ll 1$. The curve 1 is the image of $\mathcal{C}$ for $\omega = \infty$. The curve 2 is the image of $\mathcal{C}$ for a finite frequency for which there is still stability. These curves are not drawn to scale.

Fig. 6 Curves representing $Q_S(t)$ and $Q_A(t)$ in the slab problem.

Fig. 7 Nyquist diagrams for the slab problem : free particle model, symmetric case. The curve 1 is the image of $\mathcal{C}$ for $\omega = \infty$. The curve 2 is the image of $\mathcal{C}$ for $\omega = \Omega_m$. (Not drawn to scale).

Fig. 8 Nyquist diagrams for the slab problem : free particle model, antisymmetric case. The curve 1 is the image of $\mathcal{C}$ for $\omega = \infty$. The curve 2 is the image of $\mathcal{C}$ for $\omega = \Omega_m$. (Not drawn to scale).

Fig. 9 Curves representing $Q_n(t)$ in the cylindrical problem for $n = 0, 1, 2, 3$. The values of n are shown on each curve.

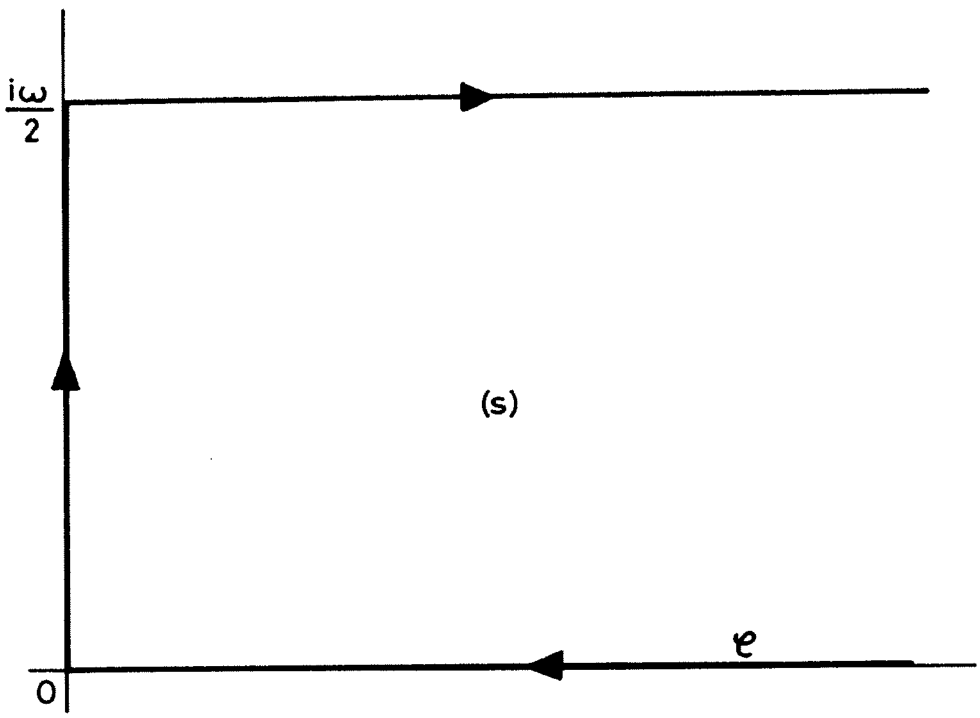

Fig. 1

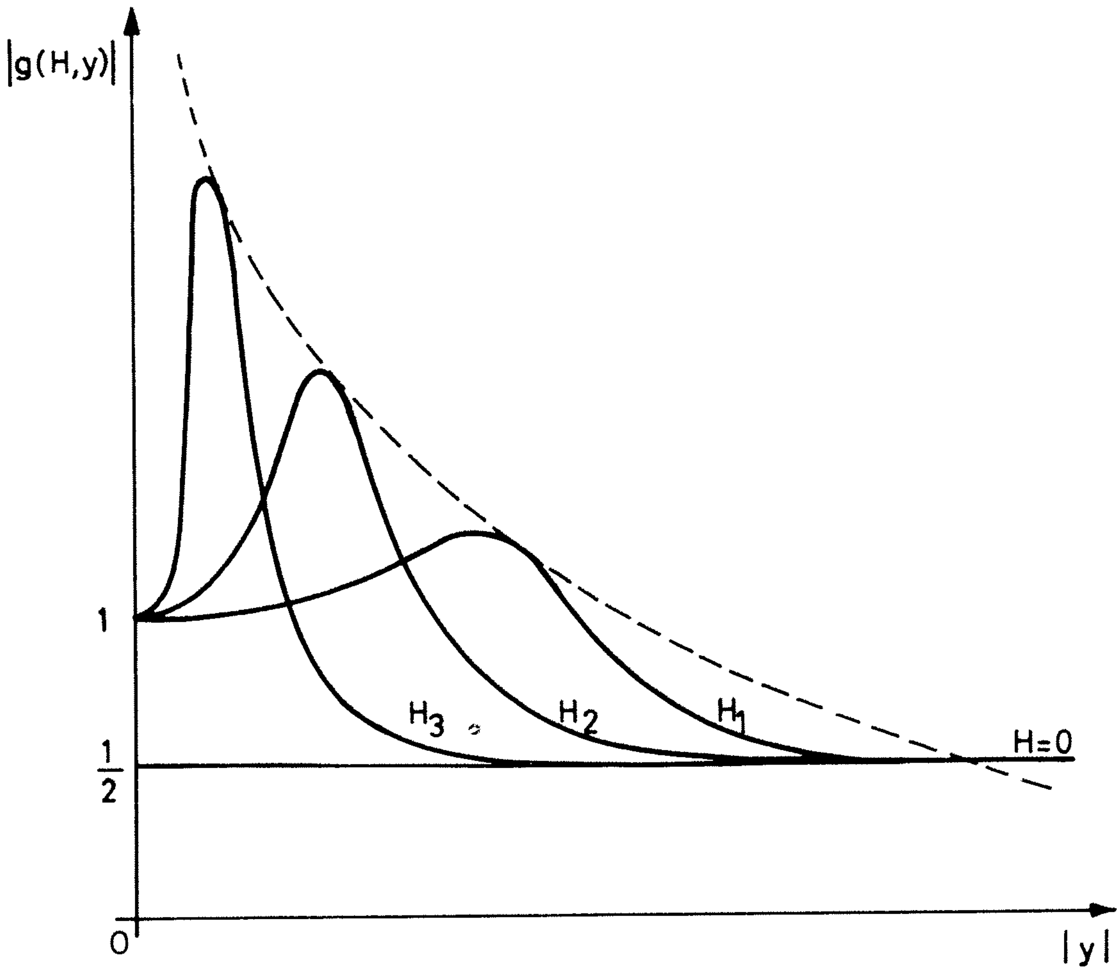

Fig. 2

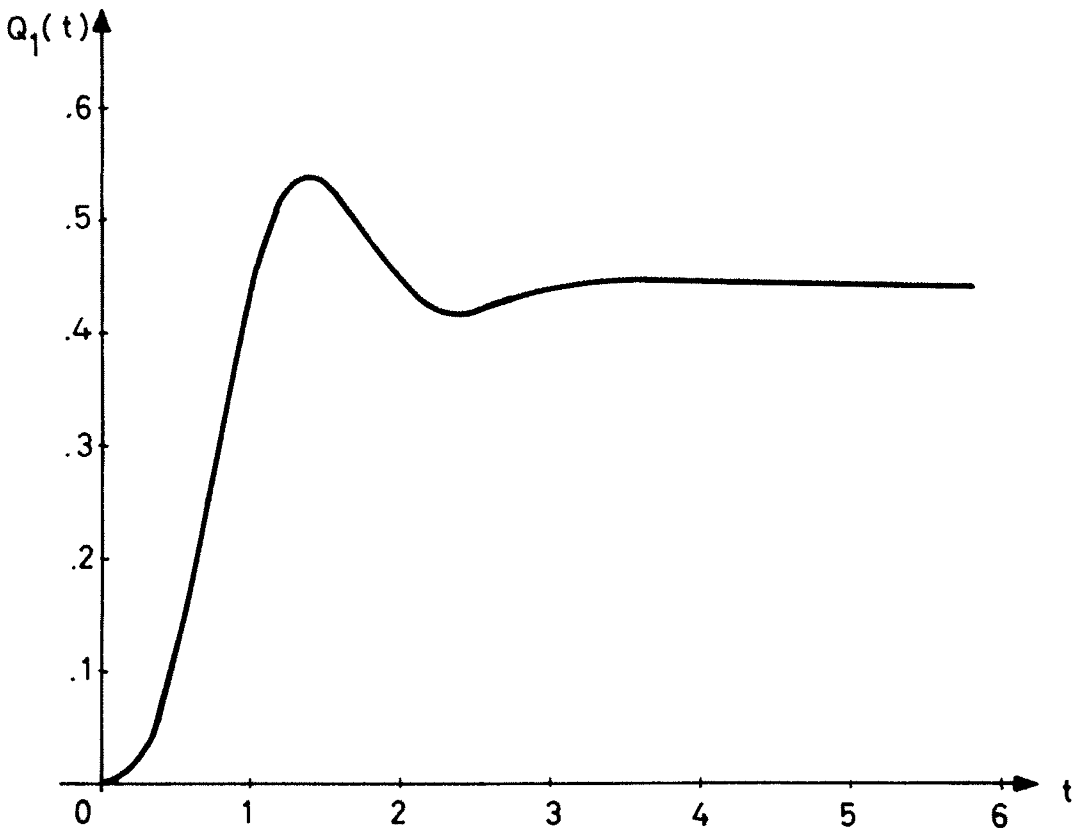

Fig. 3

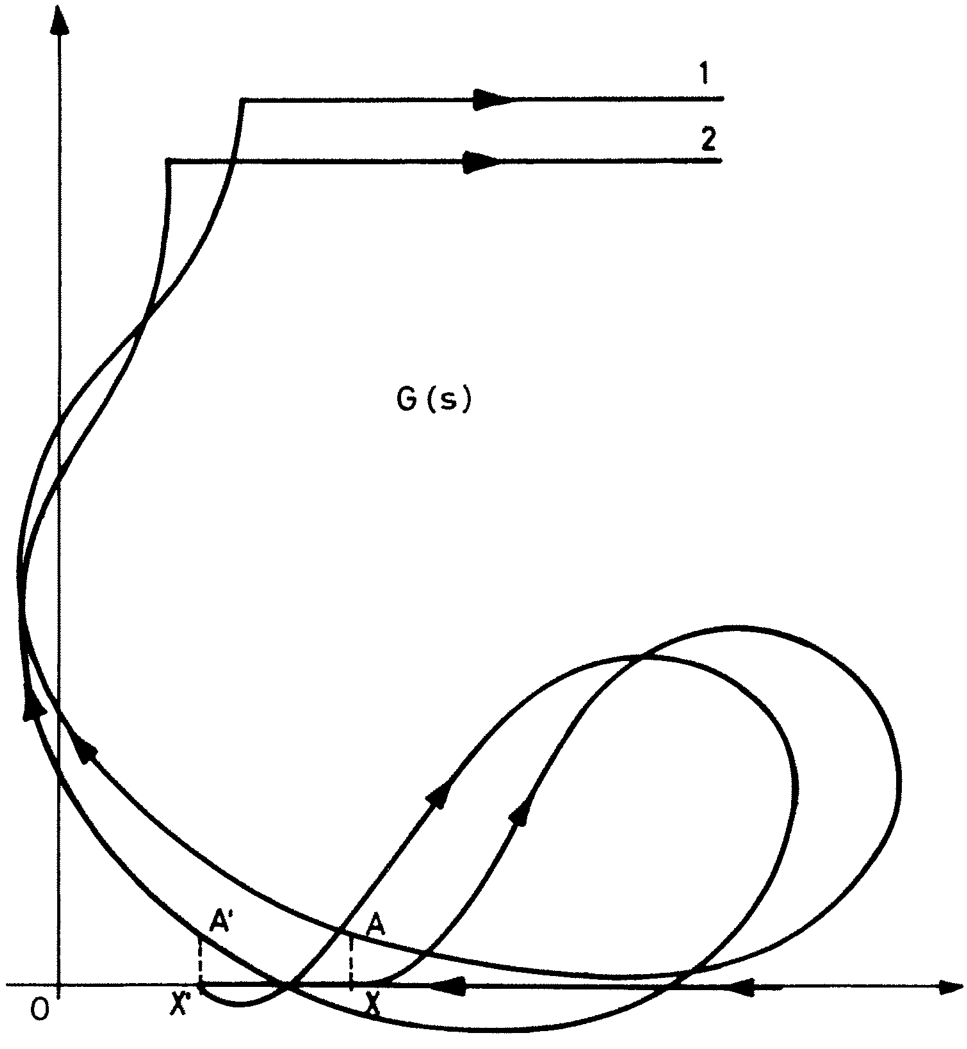

Fig. 4

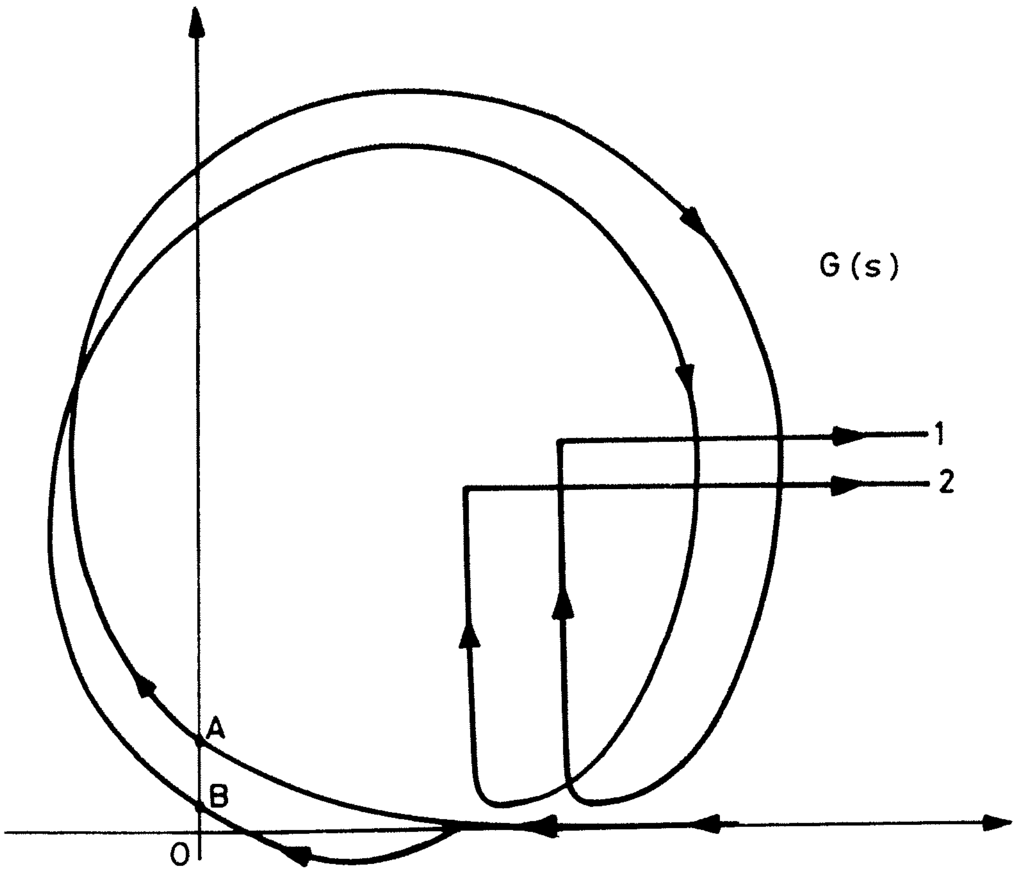

Fig. 5

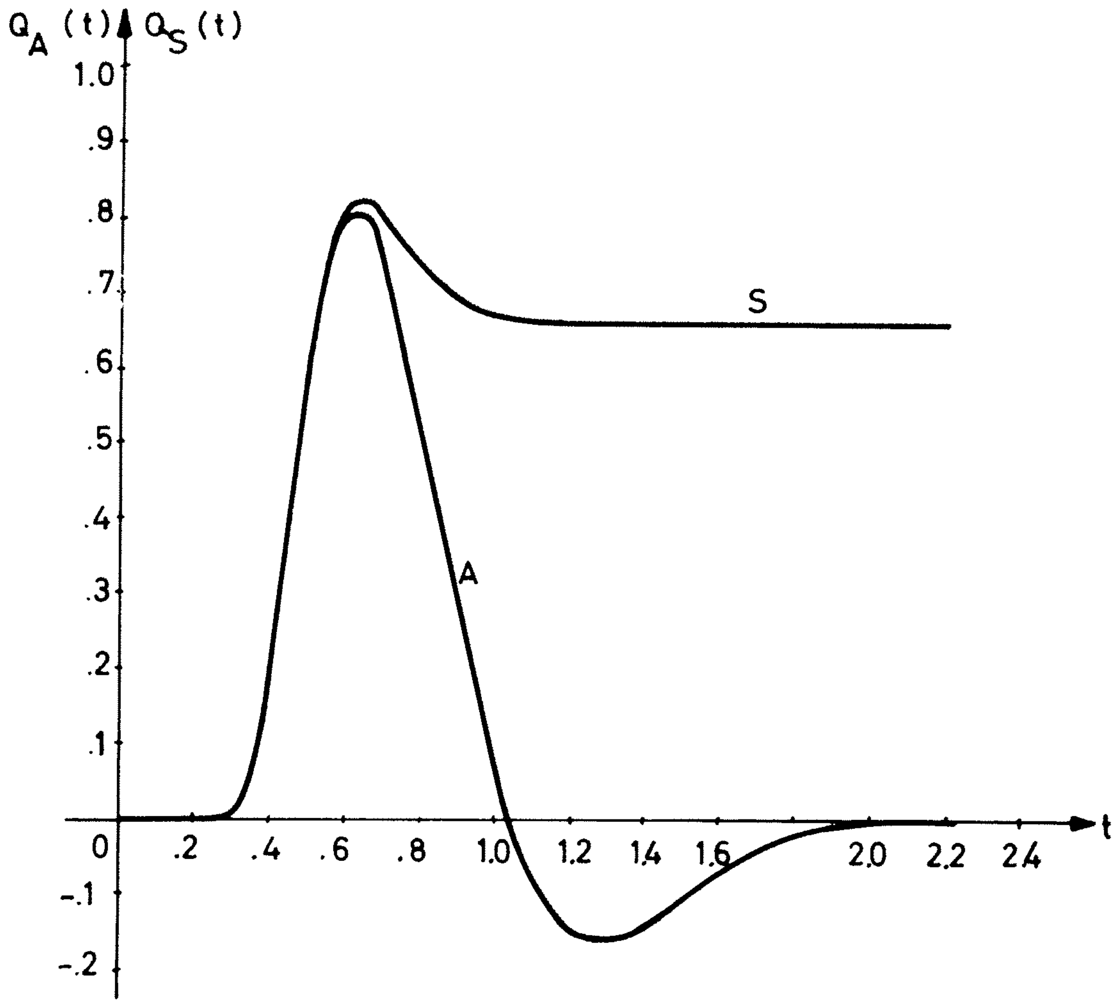

Fig. 6

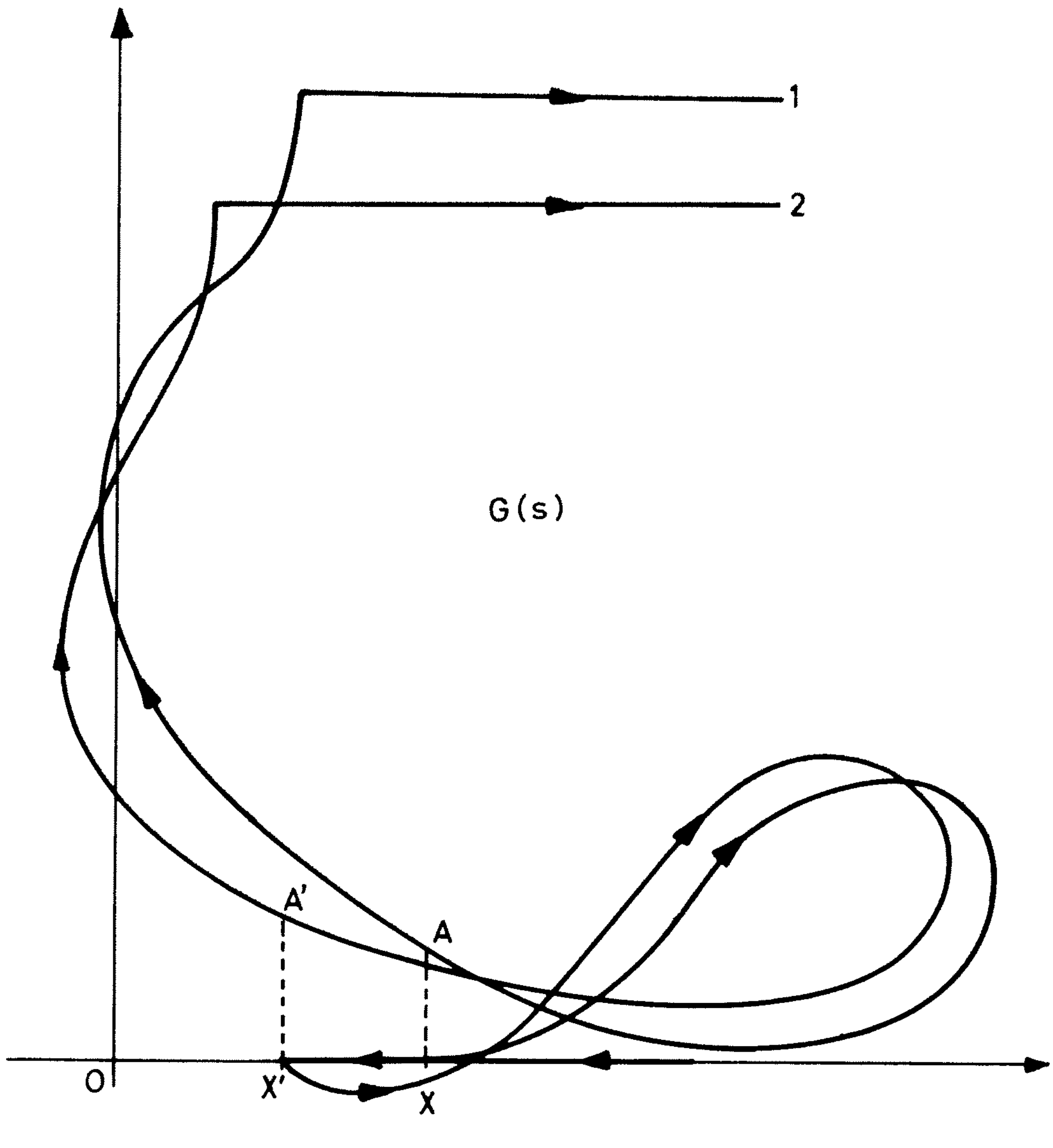

Fig. 7

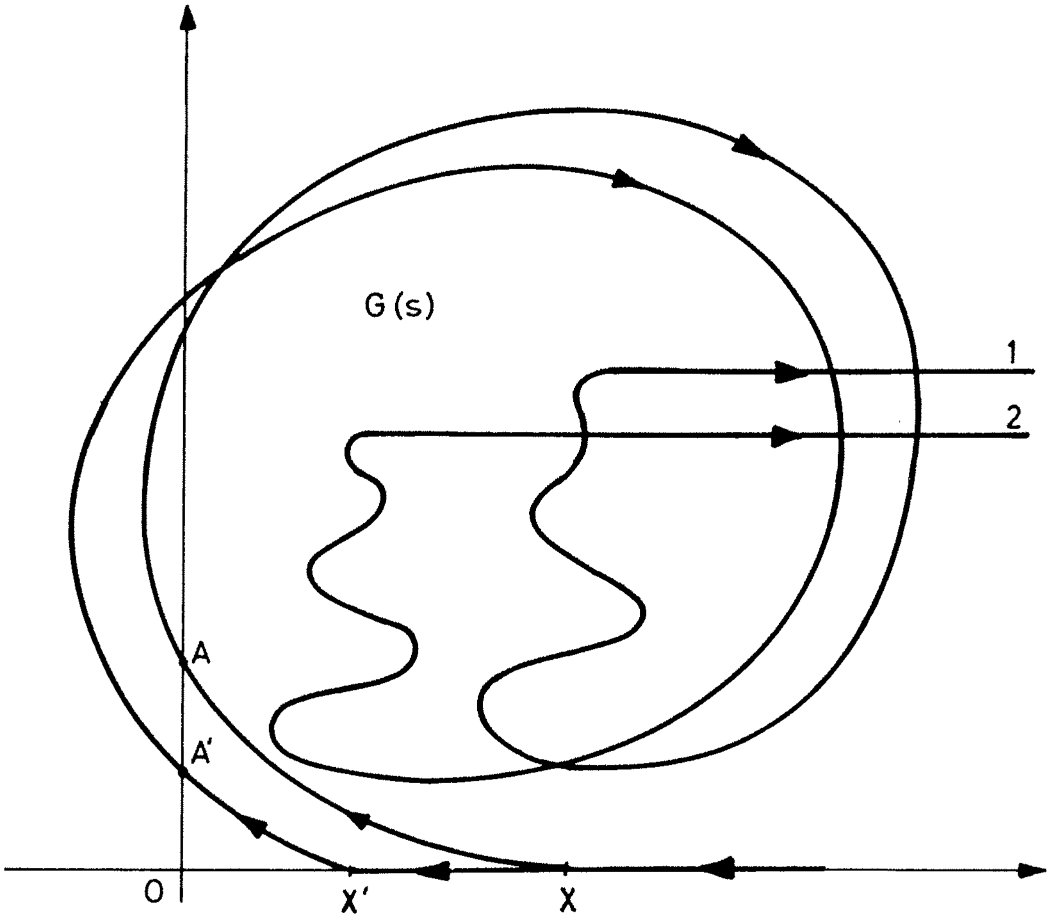

Fig. 8

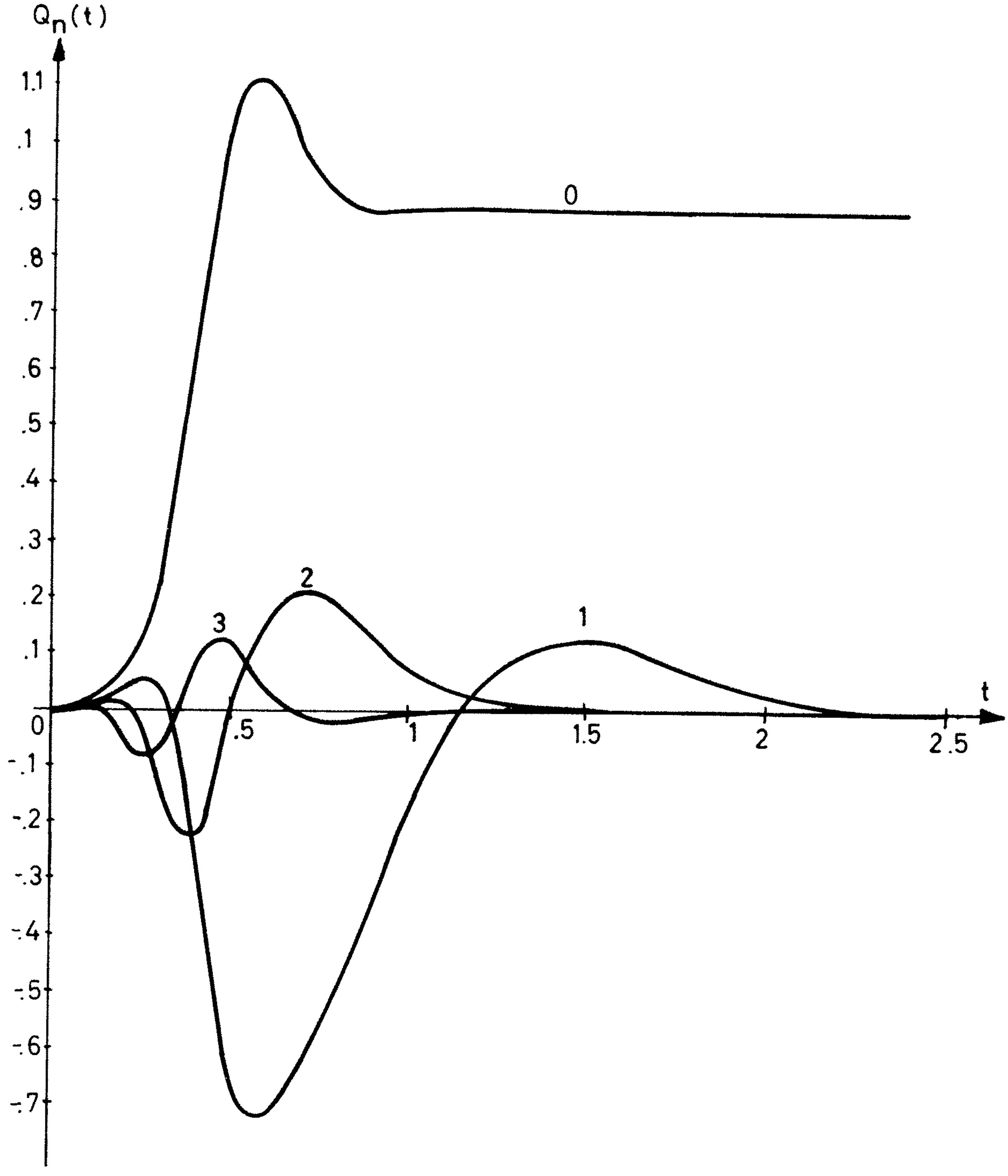

Fig. 9