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**Asymptotics of $q$-difference equations. (English summary)**

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Asymptotics of solutions of difference equations with a parameter are shown to lead to a well-defined and computable exponential growth rate in terms of a relative entropy of the characteristic polynomial of the $q$-difference equation. This result is based on the assumption that the eigenvalues are non-vanishing and non-colliding.

The $q$-difference equation considered originates from an extension of a topological invariant function used for distinguishing knots. The colored Jones function of a knot $K$ in 3-space is a sequence of Laurent polynomials $J_K\colon \mathbb{N}\to\mathbb{Z}[q^{\pm}]$ in which the first term, $J_K(1)$, is the Jones polynomial of $K$. The Jones polynomial is defined for each knot and is also linked through Witten's reformulation [V. F. R. Jones, Ann. of Math. (2) **126** (1987), no. 2, 335–388; MR0908150 (89c:46092); E. Witten, Comm. Math. Phys. **121** (1989), no. 3, 351–399; MR0990772 (90h:57009)] to the expectation value of a quantum field theory. The colored Jones function also connects two different views of knots, namely quantum field theory and Riemannian geometry. Its behavior at infinity gives the Hyperbolic Volume Conjecture (HVC):

$$\lim_{n\to\infty}\frac{\log|J_K(n)(e^{\frac{2\pi i}{n}})|}{n}=\frac{1}{2\pi}\mathrm{vol}(S^3-K),$$

where $\mathrm{vol}(S^3-K)$ is the volume of a complete hyperbolic metric in the knot complement $S^3-K$. More generally, S. Gukov [Comm. Math. Phys. **255** (2005), no. 3, 577–627; MR2134725 (2006f:58029)] formulated the Generalized HVC (GHVC), in which the above limit is identified with known hyperbolic invariants. The difficulty is that, if the terms in the colored Jones function are in some sense random, then it is hard to expect that the limit at infinity exists and that it can be computed.

Rather than trying to directly compute the limit, studying the relation between the terms of the colored Jones function through a difference equation involving some adjacent terms of the series is one way to proceed in order to extract some information about its asymptotical behavior, and thus giving some information about the rightness of the HVC. The $q$-difference equation

$$\sum_{j=0}^{d} b_j(q^n,q)J_K(n+j)=0 \qquad (1)$$

leads to the characteristic polynomial $P(v,\lambda)=\sum_{j=0}^{d} b_j(v,1)\lambda^j$ whose roots are its eigenvalues $\lambda_1,\dots,\lambda_d$. Because the asymptotics of (1) are governed by the magnitude of these eigenvalues, the unit circle is partitioned into a finite union of closed arcs $S^1=\bigcup_{p\in\mathcal{P}} I_p$ such that the magnitude of the eigenvalues does not change in each arc; that is, there is a permutation $\sigma_p(\cdot)$ of the set $\{1,\dots,d\}$ such that $|\lambda_{\sigma_p(1)}(v)|\geq\cdots\geq|\lambda_{\sigma_p(d)}(v)|$ for all $v\in I_p$.

It is then shown that a local fundamental set of solutions of the $q$-difference equation (1) is a set $\{\psi_1, \dots, \psi_q\}$ such that for every solution $\psi$ and every $p \in \mathcal{P}$ and every $m = 1, \dots, q$, there exist smooth functions $c_m^p$ for which $\psi(k,q) = \sum_{i=1}^d c_i^p(q)\psi_{\sigma_p(i)}(k,q)$. The following interpolation property holds: There exist smooth functions $\Phi_m$ that, although not uniquely determined, asymptotically give the behavior of the fundamental set of solutions. More precisely, (i) $\Phi_m$ admits a uniform asymptotic expansion $\Phi_m(x,\varepsilon) \sim_{\varepsilon\to 0} \sum_{s=0}^\infty \varphi_{m,s}(x)\varepsilon^s$ where $\varphi_{m,s} \in C^\infty([0,1])$; (ii) for every $m$ and $(k,d)$ such that $q^k \in I_p$, $\psi_m(k, e^{\frac{2\pi i\alpha}{n}}) = \exp(\frac{n}{\alpha}\Phi_m(\frac{k\alpha}{n}, \frac{\alpha}{n}))$; (iii) $\Phi_m$ has leading term $\varphi_{m,0}(x) = \int_0^x \log(\lambda_m(e^{2\pi it}))dt$.

This means that asymptotically the GHVC holds in the sense that for every $\alpha \in [0,1]$,

$$\lim_{n\to\infty} \frac{\log|\psi_m(m, e^{\frac{2\pi i\alpha}{n}})|}{n} = \int_0^1 \log|\lambda_m(e^{2\pi i\alpha t})|dt.$$

The structure of the term on the right then calls for the definition of the $S$-entropy. Fix a collection $S = \{S_p | \, p \in \mathcal{P}\}$ of subsets of $\{1, \dots, d\}$. Then define the map $\sigma_S(\alpha) = \int_0^1 \log\chi_S(e^{2\pi i\alpha t})dt$ with $\chi_S(v) = \max_{j\in S_p}|\lambda_{\sigma_p(j)(v)}|$ if $v \in I_p$. This definition can be adapted to a knot by defining its $J$-entropy as the entropy associated to its $q$-difference equation. It is denoted by $\{\sigma_{S,K}^J\colon [0,1] \to \mathbb{R} | \, S \subset \{1, \dots, q\}\}$. Finally, the main result is given as

$$\lim_{n\to\infty} \frac{\log|f(n)e^{\frac{2\pi i\alpha}{n}}|}{n} = \sigma_S(\alpha).$$

The proof of these results essentially rests on showing that a fundamental set of solutions $\{\psi_1, \dots, \psi_d\}$ exists for the $q$-difference equation (1). The $q$-difference equation is converted to an $\varepsilon$-difference equation by setting $q = e^{2\pi i\varepsilon}$, where $\varepsilon$ is a small nonnegative number (because only the asymptotic behavior is considered). Then, this $\varepsilon$-difference equation is shown to admit a fundamental set of solutions in three steps: (a) There exists a solution $\psi_1$ with the stated properties in which $\lambda_1$ is an eigenvalue with maximum magnitude. (b) This solution is used to reduce by one the degree of the $\varepsilon$-difference equation. (c) The constructed set of solutions is shown to be a locally fundamental set of solutions. The main argument consists in defining a solution through formal series and showing their convergence. Here, the instrumental property that the eigenvalues are non-vanishing and non-colliding is used.

After defining the $A$-polynomial of a knot and its corresponding $S$-entropy written as $\sigma_{S,K}^A$, the paper ends with two examples of knots, namely the trefoil $3_1$ and figure eight $4_1$, for which the $q$-difference equations of the colored Jones function are given. The former is not hyperbolic, and the latter is. For the trefoil, $\sigma_{S,3_1}^A(1) = \mathrm{vol}_{3_1} = 0$, and for the figure eight, $\sigma_{\{1,3\},4_1}^A(1) = \mathrm{vol}(4_1) = 2.029883$ and $\sigma_{\{1,2,3\},4_1}^A(1) = 2\mathrm{vol}_{4_1} = 4.05977$. These results are obtained by direct computations. Hence, the HVC is true for the $4_1$ knot. However, the eigenvalues of the associated $q$-difference equation have the property of collision, resonance, and vanishing. These issues and other related difficulties will be addressed in forthcoming publications.

The authors also stress that some of the work performed has already appeared elsewhere in communications by several different authors, and one purpose of the paper under review has been to gather and compound these contributions into a single paper.

{For the entire collection see MR2271293 (2007f:57003)}

Reviewed by *Philippe A. Müllhaupt*